New Bounds for Odd Colourings of Graphs

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Abstract

Given a graph G, a vertex-colouring σ of G, and a subset $X \subseteq V(G)$, a colour $x \in \sigma(X)$ is said to be *odd* for X in σ if it has an odd number of occurrences in X. We say that σ is an *odd colouring* of G if it is proper and every (open) neighbourhood has an odd colour in σ. The odd chromatic number of a graph G, denoted by $\chi_o(G)$, is the minimum $k \in \mathbb{N}$ such that an odd colouring $\sigma\colon V(G) \to [k]$ exists. In a recent paper, Caro, Petruševski and Škrekovski conjectured that every connected graph of maximum degree $\Delta \geqslant 3$ has odd-chromatic number at most $\Delta + 1$. We prove that this conjecture holds asymptotically: for every connected graph G with maximum degree Δ, $\chi_o(G) \leqslant \Delta + O(\ln \Delta)$ as $\Delta \to \infty$. We also prove that $\chi_o(G) \leqslant \lfloor 3\Delta/2 \rfloor + 2$ for every Δ. If moreover the minimum degree δ of G is sufficiently large, we have $\chi_o(G) \leqslant \chi(G) + O(\Delta \ln \Delta/\delta)$ and $\chi_o(G) = O(\chi(G) \ln \Delta)$. Finally, given an integer $h \geqslant 1$, we study the generalisation of these results to h-odd colourings, where every vertex v must have at least $\min\{\deg(v), h\}$ odd colours in its neighbourhood. Many of our results are tight up to some multiplicative constant.

Mathematics Subject Classifications: 05C15, 05C35

1 Introduction

All graphs considered here are finite, undirected, and simple. Let $[k]$ denote the set of first k positive integers. For a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. A hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is a generalisation of a graph; its (hyper-)edges are subsets of $V(\mathcal{H})$ of arbitrary positive size. A vertex *k-colouring* of a graph or a hypergraph G is an assignment $\sigma\colon V(G) \to [k]$, whose images are referred to as *colours*. Motivated by a frequency assignment problem in cellular networks, Even, Lotker, Ron, and Smorodinsky [12] introduced the notion of conflict-free colourings of hypergraphs. A colouring σ of a hypergraph $\mathcal{H}$ is *conflict-free* if for every edge $e \in E(\mathcal{H})$ there exists a colour appearing exactly once in e. Pach and Tardos [20] studied this notion and proved that every hypergraph with fewer than $\binom{s}{2}$ edges (for some integer s) has a

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conflict-free colouring with fewer than s colours. Kostochka, Kumbhat, and Luczak [17] further studied conflict-free colouring for uniform hypergraphs.

1.1 A motivation coming from proper conflict-free colourings

A colouring σ of a graph G is *proper* if $\sigma(u) \neq \sigma(v)$ for all $uv \in E(G)$. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum integer k such that there is a k-colouring of G. One can observe that a conflict-free colouring of a hypergraph $\mathcal{H}$ is in particular a proper colouring of the graph formed by the hyper-edges of size 2 in $\mathcal{H}$. Given a graph G and a hypergraph $\mathcal{H}$ on the same vertex set V, we say that $(G, \mathcal{H})$ is a *graph-hypergraph pair*. A *proper conflict-free k-colouring of* $(G, \mathcal{H})$ (*pcf k-colouring* for short) is a mapping $\sigma\colon V \to [k]$ that is both a proper colouring of G and a conflict-free colouring of $\mathcal{H}$. By the previous observation, this can be seen as a conflict-free colouring of the hypergraph $(V, E(\mathcal{H}) \cup E(G))$. There are many classical constraints for a proper colouring of a given graph G that are in particular satisfied by a pcf-colouring of $(G, \mathcal{H})$, for some hypergraph $\mathcal{H}$ carefully chosen. For instance, if $E(\mathcal{H})$ contains a maximum independent set of every even cycle in G, then a pcf-colouring of $(G, \mathcal{H})$ is in particular an acyclic colouring of G. As another example, if $E(\mathcal{H})$ contains all $(\beta+1)$-subsets of every neighbourhood in G, then a pcf-colouring of $(G, \mathcal{H})$ is in particular a β-frugal colouring of G.

There has been a specific focus on the special case of pcf-colourings σ of $(G, \mathcal{H})$ when $\mathcal{H}$ is the neighbourhood-hypergraph of G, i.e. $E(\mathcal{H}) = \{N(v) : v \in V(G), \deg(v) > 0\}$. In that case, we say that σ is a pcf-colouring of G (so when we omit $\mathcal{H}$, it is implicitly the neighbourhood-hypergraph of G). In other words, a *pcf-colouring* of G is a proper colouring of G such that for every non-isolated vertex v, there is a colour that appears exactly once among the neighbours of v. We let $\chi_{\text{pcf}}(G)$ be the smallest integer k such that a pcf k-colouring of G exists. This notion is the combination of proper colouring and the pointed conflict-free chromatic parameter introduced by Cheilaris [4].

The notion of pcf-colourings of graphs was formally introduced by Fabrici, Lužar, Rindošová, and Soták [13], where they investigated the pcf-colourings of planar and outerplanar graphs, among many other related variants of a proper conflict-free colouring. They proved that $\chi_{\text{pcf}}(G) \leqslant 8$ for all planar graphs and $\chi_{\text{pcf}}(G) \leqslant 5$ for all outerplanar graphs. Further studies in pcf-colourings of sparse graphs can be found in [3, 6, 13, 14, 18].

Given a (hyper-)graph $\mathcal{H}$, the *degree* of a vertex v, denoted by $\deg_{\mathcal{H}}(v)$, is the number of edges of $\mathcal{H}$ containing v. We denote by $\delta(\mathcal{H})$ and $\Delta(\mathcal{H})$ the minimum and maximum degrees of $\mathcal{H}$, respectively. We denote $\epsilon(\mathcal{H})$ the minimum size of an edge in $\mathcal{H}$. Given a graph G, we let $\delta^*(G)$ denote the *degeneracy* of G, that is $\delta^*(G) = \max_{H \subseteq G} \delta(H)$. Caro, Petruševski, and Škrekovski [3] proposed the following conjecture about pcf-colourings.

Conjecture 1 (Caro, Petruševski, Škrekovski [3, Conjecture 6.4])**.** If G is a connected graph of maximum degree $\Delta \geqslant 3$, then $\chi_{\text{pcf}}(G) \leqslant \Delta + 1$.

As a first step toward their conjecture, Caro, Petruševski, and Škrekovski [3] proved that for such a graph G, $\chi_{\text{pcf}}(G) \leqslant \lfloor 2.5\Delta \rfloor$. Recently, it has been observed by Cranston and Liu [11] that $\chi_{\text{pcf}}(G) \leqslant \Delta(G) + \delta^*(G) + 1$ (they actually more generally proved that there always exists a pcf $(\Delta(\mathcal{H}) + \delta^*(G) + 1)$-colouring of any given pair $(G, \mathcal{H})$). They further reduced the gap to Conjecture 1 by proving that $\chi_{\text{pcf}}(G) \leqslant \left\lceil 1.6550826\Delta + \sqrt{\Delta} \right\rceil$, given that Δ is large enough. Very recently, Liu and Reed [19] asymptotically proved that $\chi_{\text{pcf}}(G) \leqslant \Delta + 100057\Delta^{\frac{2}{3}} \log \Delta$ for sufficiently large Δ.

We would also like to mention several related works on some specific classes of graphs. Cho, Choi, Kwon, and Park [8] proved that if G is a graph such that every neighbourhood $N_G(v)$ is a union of at most ℓ cliques, then it holds that $\chi_{\text{pcf}}(G) \leqslant \frac{2\ell-1}{\ell}\Delta + 2$. As a corollary, they proved that $\chi_{\text{pcf}}(G) \leqslant \frac{3}{2}\Delta + 2$ for quasi-line graphs. Later, Theorem 1 has been solved up to a constant additive error term for some specific classes of graphs [15]. It was proved that $\chi_{\text{pcf}}(G) \leqslant \Delta + 6$ for claw-free graphs and $\chi_{\text{pcf}}(G) \leqslant \Delta + 4$ for quasi-line graphs, and this was done using constructive polynomial-time algorithms.

1.2 Odd colourings

In [5], Cheilaris, Keszegh, and Pálvölgyi introduced a weakening of conflict-free colourings. An *odd colouring* σ of a hypergraph $\mathcal{H}$ satisfies the constraint that in every edge $e \in E(\mathcal{H})$ there is a colour x with an odd number of occurrences in σ; we say that x is an *odd colour* of e in σ. It is straightforward that a conflict-free colouring of $\mathcal{H}$ is in particular an odd colouring of $\mathcal{H}$. Petruševski and Škrekovski [22] later considered that notion applied to the neighbourhood-hypergraph of a graph G. An *odd colouring* of G is a proper colouring of G with the additional constraint that each non-isolated vertex has a colour appearing an odd number of times in its neighbourhood. The *odd chromatic number* of G, denoted by $\chi_o(G)$, is the minimum integer k such that an odd k-colouring of G exists. Since odd colourings are a weakening of pcf-colourings, it always holds that $\chi_o(G) \leqslant \chi_{\text{pcf}}(G)$. In the last couple of years, there has been some interest in determining the extremal value of χ_o in various classes of graphs.

In [22], Petruševski and Škrekovski showed that $\chi_o(G) \leqslant 9$ for every planar graph G with a proof that relies on the discharging method (note that this is weaker than the result about pcf-colourings of planar graphs from [6] stated earlier, which was proved a couple of months later). Furthermore, they conjectured that this bound may be reduced to 5. If true, this would be tight, since $\chi_o(C_5) = 5$. Recently there has been considerable attention in odd colourings of planar graphs [2, 7, 9, 10, 21].

Caro, Petruševski, and Škrekovski [2] also studied various properties of the odd chromatic number of general graphs; in particular, they proved the following facts: every graph of maximum degree three has an odd 4-colouring; every graph, except for C_5, of maximum degree Δ has an odd 2Δ-colouring. Moreover, they proposed a conjecture for general graphs, which is a weaker form of Conjecture 1.

Conjecture 2 (Caro, Petruševski, Škrekovski [2, Conjecture 5.5])**.** If G is a connected graph of maximum degree $\Delta \geqslant 3$, then $\chi_o(G) \leqslant \Delta + 1$.

Our main result states that Conjecture 2 holds asymptotically as $\Delta \to \infty$.

Theorem 3. *For every graph G of maximum degree Δ,*

$$\chi_o(G) \leqslant \Delta + \lceil 4(\ln\Delta + \ln\ln\Delta + 3) \rceil .$$

For small values of Δ, we provide another bound on $\chi_o(G)$ that is derived from a relatively simple (deterministic) colouring procedure.

Theorem 4. *For every graph G of maximum degree Δ,*

$$\chi_o(G) \leqslant \left\lfloor \frac{3\Delta}{2} \right\rfloor + 2.$$

1.3 General hypergraphs

The proofs of our results stated in Theorem 3 and Theorem 4 rely highly on the structure of neighbourhood hypergraphs. In a more general setting, we could wonder how the odd colouring problem behaves on any graph-hypergraph pair $(G, \mathcal{H})$. We were able to extend Theorem 3 to that more general setting, at the cost of requiring a lower bound on the minimum hyper-edge size $\epsilon(\mathcal{H})$ in $\mathcal{H}$.

Theorem 5. *There exists a universal constant C such that, for every graph-hypergraph pair $(G, \mathcal{H})$, if $\epsilon(\mathcal{H}) \geqslant C \log \Delta(\mathcal{H})$ then there is an odd k-colouring of $(G, \mathcal{H})$, where*

$$k \leqslant \Delta(G) + C \log \Delta(\mathcal{H}).$$

With that extra condition on $\epsilon(\mathcal{H})$, we can actually derive an upper bound on $\chi_o(G, \mathcal{H})$ that mainly depends on $\chi(G)$ rather than $\Delta(G)$. Moreover, we show with a construction that this bound is tight up to the precise value of the constant C.

Theorem 6. *There exists a universal constant C such that, for every graph-hypergraph pair $(G, \mathcal{H})$, if $\epsilon(\mathcal{H}) \geqslant C \log \Delta(\mathcal{H})$ then there is an odd k-colouring of $(G, \mathcal{H})$, where*

$$k \leqslant \chi(G) \cdot C \log \Delta(\mathcal{H}).$$

When $\epsilon(\mathcal{H})$ gets closer to $\Delta(G)$, we show that the difference $\chi_o(G, \mathcal{H}) - \chi(G)$ gets relatively small.

Theorem 7. *There exists a universal constant C such that, for every graph-hypergraph pair $(G, \mathcal{H})$, there is an odd k-colouring of $(G, \mathcal{H})$, where*

$$k \leqslant \chi(G) + C \, \frac{\Delta(G) \log \Delta(\mathcal{H})}{\epsilon(\mathcal{H})}.$$

A direct consequence of Theorem 7 is that for quasi-regular graphs G (that is, the ratio $\Delta(G)/\delta(G)$ is bounded by a uniform constant), the difference $\chi_o(G) - \chi(G)$ is small, namely $O(\log \Delta(G))$. This contrasts with the general case where that difference can be much larger: if G is the 1-subdivision of the complete graph on $\Delta + 1 \geqslant 5$ vertices, then $\chi(G) = 2$ while $\chi_o(G) = \Delta + 1$.

1.4 Requiring more than one odd colour in neighbourhoods

We finish our work by considering a natural strengthening of odd colourings. For a positive integer h, an *h-odd colouring* σ of a hypergraph $\mathcal{H}$ satisfies the constraint that every edge $e \in E(\mathcal{H})$ has at least $\min\{h, |e|\}$ colours with an odd number of occurrences. For a graph-hypergraph pair $(G, \mathcal{H})$, an *h-odd k-colouring* of $(G, \mathcal{H})$ is a mapping $\sigma\colon V \to [k]$ that is both a proper colouring of G and an h-odd colouring of $\mathcal{H}$. The least k for which G is h-odd k-colourable is in turn called the *h-odd chromatic number* of $(G, \mathcal{H})$ and we denote it by $\chi_o^h(G, \mathcal{H})$. We say that σ is an *h-odd k-colouring* of G if σ is an h-odd k-colouring of $(G, \mathcal{H})$ when $\mathcal{H}$ is the neighbourhood-hypergraph of G. We denote $\chi_o^h(G)$ the *h-odd chromatic number* of G, respectively. We note that these notions parallel the strengthening of pcf colourings proposed in [7], that is *proper h-conflict-free colourings*, where "an odd number of occurrences" is replaced with "a unique occurrence". The associated chromatic parameter is denoted χ_{pcf}^h, and has been studied in a setting similar to ours in [16], where they prove the following.

Theorem 8 (Kamyczura, Przybyło, 2024)**.** *For every integers Δ and $h \geqslant 20 \ln \Delta$, every graph G of maximum degree Δ and minimum degree $\delta \geqslant 75h$ has*

$$\chi_{\mathrm{pcf}}^h(G) \leqslant \Delta + \frac{30\Delta}{\delta}\, h.$$

In particular, if G is quasi-regular and $h \geqslant 20 \ln \Delta$, then $\chi_{\mathrm{pcf}}^h(G) \leqslant \Delta + O(h)$. We first prove that a similar bound holds for χ_o^h without the requirement that G is quasi-linear.

Theorem 9. *There exists a universal constant C such that, for every graph G of maximum degree Δ and minimum degree δ, if $\delta/2 \geqslant h \geqslant C \log \Delta$, then*

$$\chi_o^h(G) \leqslant \Delta(G) + C \cdot h.$$

With the same hypotheses, we also show that χ_{pcf}^h is not too large compared to χ, as follows.

Theorem 10. *There exists a universal constant C such that, for every graph G of maximum degree Δ and minimum degree δ, if $\delta/2 \geqslant h \geqslant C \log \Delta$, then*

$$\chi_o^h(G) \leqslant C \cdot h\chi(G),$$

and this is tight up to the multiplicative constant.

An interesting consequence of Theorem 9 is that $\chi_o^h(G) = O(\Delta)$ if G is Δ-regular and $h \leqslant \Delta/2$. One could wonder what happens when $\Delta/2 < h < \Delta$, given that $\chi_o^\Delta(G) = \chi(G^2)$ can be as large as $\Delta^2 + 1$ (e.g. if G is the incidence graph of a projective plane). We show that there is a smooth progression from the linear to the quadratic regime.

Theorem 11. *There exists a universal constant C such that, for every Δ-regular graph G, if $h \geqslant C \log \Delta$ and writing $h = \Delta + 1 - t$, then*

$$\chi_o^h(G) \leqslant C\, \frac{\Delta^2}{t},$$

and this is tight up to the multiplicative constant.

1.5 Organisation of the paper

The paper is organised as follows. In Section 2, we provide probabilistic tools that we will rely on in our proofs. In Section 3, we prove Theorem 4. In Section 4, we prove Theorem 3. In Section 5, we analyse odd colourings for general hypergraphs with additional constraints on the edge cardinalities. Finally, in Section 6, we extend our results from Section 5 to h-odd colourings, a generalisation of odd colourings that consists in requiring that every hyper-edge e contains at least $\min\{|e|, h\}$ odd colours, for a given integer $h \geqslant 1$.

2 Probabilistic tools.

The first probabilistic result that we need is the following lopsided version of the Symmetric Lovász Local Lemma (LLL for short) (see e.g. [1]).

Lemma 12 (Lopsided Lovász Local Lemma). *Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a finite set of random (bad) events, and let d be a fixed integer. Suppose that, for every $i \in [n]$, there is a set $\Gamma(i) \subseteq [n]$ of size at most d such that, for every $Z \subseteq [n] \setminus \Gamma(i)$,*

$$\mathbb{P}\left[B_i \,\middle|\, \bigcap_{j\in Z} \overline{B_j}\right] \leqslant p.$$

If $epd \leqslant 1$, then $\mathbb{P}\left[\bigcap_{i\in[n]} \overline{B_i}\right] > 0$.

Many random variables we analyse in this paper are highly concentrated around their mean. This is a consequence of Chernoff's bounds as stated hereafter.

Lemma 13 (Chernoff's bounds). *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be i.i.d. $(0,1)$-valued random variables, and let $\mathbf{S}_n \coloneqq \sum_{i=1}^n \mathbf{X}_i$. Let us write $\mu \coloneqq \mathbb{E}[\mathbf{S}_n]$. Then*

(i) *for every $0 \leqslant \eta < \mu$,*

$$\mathbb{P}[\mathbf{S}_n \leqslant \mu - \eta] \leqslant e^{-\frac{\eta^2}{2\mu}}, \text{ and}$$

(ii) *for every $\eta > 0$,*

$$\mathbb{P}[\mathbf{S}_n \geqslant \mu + \eta] \leqslant e^{-\frac{\eta^2}{2(\mu+\eta)}}.$$

Finally, we will need to analyse a specific Markovian process as described in the following lemma. First we observe that, using the well-known Stirling bounds

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \leqslant n! \leqslant \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$$

for every integer $n \geqslant 1$, it is straightforward to derive that

$$\frac{n!}{\left(\frac{n}{2}\right)!} \leqslant \sqrt{2}\left(\frac{2n}{e}\right)^{n/2} \tag{1}$$

for every even integer $n \geqslant 2$. We will also use the classical upper bound $\binom{n}{k} \leqslant \left(\frac{ne}{k}\right)^k$.

Lemma 14. *Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be a sequence of binary random variables with values in $\{-1, 1\}$, let $S_0 \in \mathbb{N}$ be a non-negative integer, and let $\mathbf{S}_i \coloneqq S_0 + \sum_{j=1}^{i} \mathbf{X}_j$ for every integer $i \geqslant 0$. If there exists a real number $\tau > 0$ such that, for every $i \geqslant 0$, it holds that $\mathbb{P}[\mathbf{X}_i = -1 \mid \mathbf{X}_1 = X_1, \dots, \mathbf{X}_{i-1} = X_{i-1}] \leqslant s/\tau$ where $s = S_0 + \sum_{j=1}^{i-1} X_j$, then*

$$\mathbb{P}[\mathbf{S}_n \leqslant k] \leqslant \sqrt{2}\binom{n}{k}\left(\frac{2n-2k}{e\tau}\right)^{\frac{n-k}{2}},$$

for every integers $0 \leqslant k \leqslant n$.

Proof of Lemma 14. Given a possible outcome $(X_1, \dots, X_n)$ of $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ that yields that $\mathbf{S}_n \leqslant k$, let $I \coloneqq \{i : X_i = -1\}$. Noting that $\mathbf{S}_n = S_0 + n - 2|I|$ conditioned on $\mathbf{X}_1 = X_1, \dots, \mathbf{X}_n = X_n$, we infer that $|I| \geqslant \frac{S_0+n-k}{2} \geqslant \frac{n-k}{2}$. Moreover, if i_j is the j-th last element in I, then it deterministically holds that $S_0 + \sum_{\iota=1}^{i_j-1} X_\iota \leqslant k + j$, and hence

$$\mathbb{P}\left[\mathbf{X}_{i_j} = -1 \mid \mathbf{X}_1 = X_1, \dots, \mathbf{X}_{i_j-1} = X_{i_j-1}\right] \leqslant \frac{k+j}{\tau}.$$

We therefore have the following (crude) upper-bound:

$$\begin{aligned}\mathbb{P}[\mathbf{X}_1 = X_1, \dots, \mathbf{X}_n = X_n] &\leqslant \prod_{i\in I} \mathbb{P}[\mathbf{X}_i = -1 \mid \mathbf{X}_1 = X_1, \dots, \mathbf{X}_{i-1} = X_{i-1}] \\ &\leqslant \prod_{j=1}^{\frac{n-k}{2}} \frac{k+j}{\tau} = \frac{\left(\frac{n+k}{2}\right)!}{k!\,\tau^{\frac{n-k}{2}}}.\end{aligned}$$

Since there are at most $\binom{n}{\frac{n-k}{2}}$ possible choices for the $\frac{n-k}{2}$ last elements of I, we have

$$\begin{aligned}\mathbb{P}[\mathbf{S}_n \leqslant k] &\leqslant \binom{n}{\frac{n-k}{2}} \cdot \frac{\left(\frac{n+k}{2}\right)!}{k!\,\tau^{\frac{n-k}{2}}} = \frac{n!}{k!\left(\frac{n-k}{2}\right)!\,\tau^{\frac{n-k}{2}}} = \binom{n}{k}\frac{(n-k)!}{\left(\frac{n-k}{2}\right)!\,\tau^{\frac{n-k}{2}}} \\ &\leqslant \sqrt{2}\binom{n}{k}\left(\frac{2n-2k}{e\tau}\right)^{\frac{n-k}{2}} \qquad \text{by (1)} \\ &\leqslant \sqrt{2}\left(\frac{ne}{k}\right)^{k}\left(\frac{2n-2k}{e\tau}\right)^{\frac{n-k}{2}}.\end{aligned}$$

□

Let G be a graph, and $\mathscr{C}(G) \subseteq [k]^{V(G)}$ a prescribed set of k-colourings of G. Given a colouring $\sigma \in \mathscr{C}(G)$ and a vertex $v \in V(G)$, we let $L_\sigma(v)$ be the set of colours x such that, if we redefine $\sigma(v) \leftarrow x$, we still have $\sigma \in \mathscr{C}(G)$. For instance, if $\mathscr{C}(G)$ is the set of proper k-colourings of G, then $L_\sigma(v) = [k] \setminus \sigma(N(v))$.

Lemma 15. *Let G be a graph, and $\mathscr{C}(G)$ a set of colourings of G. Let $\boldsymbol{\sigma}$ be drawn uniformly at random from $\mathscr{C}(G)$. For a given subset of vertices $X \subseteq V(G)$, let $B_{\boldsymbol{\sigma}}$ be the bad event that X has no odd colour in $\boldsymbol{\sigma}$, and let $M \subseteq X$ be a subset of size $m \leqslant |X|$. If* *there exists an integer τ such that we deterministically have $|L_{\boldsymbol{\sigma}}(v)| \geqslant \tau$ for every vertex $v \in M$, then for every possible realisation σ_0 of $\boldsymbol{\sigma}_{|V(G)\setminus M}$, we have*

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}} \;\middle|\; \boldsymbol{\sigma}_{|V(G)\setminus M} = \sigma_0\right] \leqslant \sqrt{2}\left(\frac{2m}{e\tau}\right)^{m/2}.$$

Proof. Let $M = \{u_1, \ldots, u_m\}$ be a fixed subset of X. Let σ_0 be a possible realisation of $\boldsymbol{\sigma}_{|V(G)\setminus M}$. Let $\boldsymbol{\sigma}_0$ be drawn uniformly at random from the extensions of σ_0 to $\mathscr{C}(G)$. For every $1 \leqslant i \leqslant m$, we let $\boldsymbol{\sigma}_i \in \mathscr{C}(G)$ be obtained from $\boldsymbol{\sigma}_{i-1}$ by resampling the colour of u_i uniformly at random from $L_{\boldsymbol{\sigma}_{i-1}}(u_i)$. For every $i \leqslant m$, let $\mathbf{S}_i$ be the number of odd colours of $X \setminus \{u_{i+1}, \ldots, u_m\}$ in $\boldsymbol{\sigma}_i$. For every $i \geqslant 1$, we have $\mathbf{S}_i = \mathbf{S}_{i-1} - 1$ if $\boldsymbol{\sigma}_i(u_i)$ is one of the $\mathbf{S}_{i-1}$ odd colours of X in $\boldsymbol{\sigma}_{i-1}$; since there are at least τ choices for $\boldsymbol{\sigma}_i(u_i)$ this happens with probability at most k/τ if $\mathbf{S}_{i-1} = k$. Otherwise, we have $\mathbf{S}_i = \mathbf{S}_{i-1} + 1$. So the sequence $(\mathbf{S}_i)_{i \leqslant m}$ satisfies the hypotheses of Lemma 14, hence by setting $k := 0$ we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}_m}\right] = \mathbb{P}\left[\mathbf{S}_m = 0\right] \leqslant \sqrt{2}\left(\frac{2m}{e\tau}\right)^{m/2}.$$

Since we resample the colours uniformly at random, the random colourings $(\boldsymbol{\sigma}_i)_{i \leqslant m}$ are identically distributed. Therefore, if $\boldsymbol{\sigma}$ is drawn uniformly at random from $\mathscr{C}(G)$, we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}} \;\middle|\; \boldsymbol{\sigma}_{|V(G)\setminus M} = \sigma_0\right] = \mathbb{P}\left[B_{\boldsymbol{\sigma}_0}\right] = \mathbb{P}\left[B_{\boldsymbol{\sigma}_m}\right],$$

and the conclusion follows. □

3 A greedy bound

Given a proper k-colouring $\sigma\colon V(G) \to [k]$, and a vertex $u \in V(G)$, we denote $\mathcal{U}_\sigma(u)$ the set of odd colours of $N_G(u)$ in σ. So σ is an odd k-colouring if $|\mathcal{U}_\sigma(u)| \geqslant 1$ for every vertex $u \in V(G)$. If $\mathcal{U}_\sigma(u) = \{x\}$, we say that u is *σ-critical*, and that x is its *witness colour*; we denote it $w_\sigma(u) := x$.

Proof of Theorem 4. Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of G. We let $H_i := G[\{v_1, \ldots, v_i\}]$ for every $i \in [n]$. Let $k := \left\lfloor \frac{3\Delta}{2} \right\rfloor + 2$, and let $\mathscr{C}(H)$ denote the set of odd k-colourings of each induced subgraph $H \subseteq G$. We construct an odd k-colouring of G greedily by constructing a sequence of partial colourings $(\sigma_i)_{i\in[n]}$ that satisfies the following induction hypothesis.

$$\sigma_i \in \mathscr{C}(H_i) \text{ and } |\mathcal{U}_{\sigma_i}(u)| \geqslant 1 \text{ for every vertex } u \in N_G(V(H_i)). \tag{IH}$$

For the base case, we may begin with the empty colouring σ_0. Let us now assume that we have constructed σ_i that satisfies (IH). In order to maintain (IH), we forbid that $\sigma_{i+1}(v_{i+1})$ is one of $\{\sigma_i(u) : u \in N_{H_i}(v_{i+1})\} \cup \{w_{\sigma_i}(u) : u \in N_G(v_{i+1}) \text{ and } u \text{ is } \sigma_i\text{-critical}\}$. If at most $k-1$ colours are forbidden for v_{i+1}, then there remains at least one possible choice for $\sigma_{i+1}(v_{i+1})$, and letting $\sigma_{i+1}(u) = \sigma_i(u)$ for every $u \in V(H_i)$ we have that σ_{i+1} satisfies (IH).

We may now assume that all k colours are forbidden for v_{i+1}. Let $X \subseteq N_G(v_{i+1})$ be the set of neighbours of v_{i+1} that forbid exactly one colour for v_{i+1}, and $Y \subseteq N_G(v_{i+1}) \setminus X$ be the set of neighbours of v_{i+1} that forbid exactly two colours for v_{i+1} (so $Y \subseteq V(H_i)$ and every vertex $y \in Y$ is σ_i-critical). We claim that there is a vertex $y \in Y$ such that $\sigma_i(y)$ is forbidden only by y for v_{i+1}. Indeed, otherwise the number of forbidden colours for v_{i+1} would be at most $|X| + \frac{3}{2}|Y| \leqslant \frac{3}{2}\Delta < k$, a contradiction. We also claim that $\sigma_i(y)$ is not a witness colour of v_{i+1} in σ_i. Indeed, otherwise every colour of $N(v_{i+1}) \setminus \{y\}$ appears at least twice, hence v_{i+1} has at most $\left\lfloor \frac{\Delta-1}{2} \right\rfloor + 1$ adjacent colours in σ_i. Since there are at most Δ witness colours in $N_G(v_{i+1})$, there are at most $\left\lfloor \frac{3\Delta+1}{2} \right\rfloor \leqslant k-1$ forbidden colours for v_{i+1}, a contradiction. Let us set $\sigma_{i+1}(v_{i+1}) \coloneqq \sigma_i(y)$, and $\sigma_{i+1}(u) \coloneqq \sigma_i(u)$ for every vertex $u \in V(H_i) \setminus \{y\}$.

There remains to define $\sigma_{i+1}(y)$. Since $|\mathcal{U}_{\sigma_i}(y)| = 1$, it means that every colour in $N_G(y) \cap V(H_{i+1})$ appears at least twice in σ_{i+1} except $w_{\sigma_i}(y)$ and $\sigma_{i+1}(v_{i+1}) = \sigma_i(y)$. So y has at most $\lfloor \frac{\Delta-2}{2} \rfloor + 2 = \lfloor \frac{\Delta}{2} \rfloor + 1$ adjacent colours in σ_{i+1}. Since $N_G(y)$ contains at most Δ witness colours in σ_{i+1}, there are at most $\lfloor \frac{3\Delta}{2} \rfloor + 1 = k-1$ forbidden colours for y in σ_{i+1}, and so there remains at least one possible choice for $\sigma_{i+1}(y)$. This ends the proof of the induction.

We conclude that σ_n is an odd k-colouring of G, which proves that $\chi_o(G) \leqslant k$, as desired. $\square$

4 An asymptotic version of the Odd Colouring Conjecture

The section is devoted to the proof of Theorem 3.

4.1 Set-up of the required terminology

Let G be a connected graph of maximum degree Δ. Given a vertex-colouring $\sigma \colon V(H) \to [k]$ of some induced subgraph H of G, for each $v \in V(H)$ a colour x is said to be an *odd colour* of v if x is an odd colour of $N_H(v)$ in σ. Let $w_\sigma(v)$ denote the unique odd colour of v in σ if such a colour exists; otherwise $w_\sigma(v)$ is undefined. When it exists, we say that $w_\sigma(v)$ is the *witness colour* of v in σ.

Let $k > \Delta$ be some integer. Let V^- be the subset of $V(G)$ consisting of all vertices of degree less than $k/2$, and $V^+ \coloneqq V(G) \setminus V^-$ be the set of vertices of degree at least $k/2$. We denote $G^+ \coloneqq G[V^+]$. For every $X \subseteq V^+$, we say that a proper partial k-colouring $\sigma \colon X \to [k]$ of G^+ is *admissible* if every vertex $v \in V^-$ having $N_G(v) \subseteq X$ has an odd colour in σ. Finally, we let V^{++} be the set of vertices $v \in V^+$ having $N_G(v) \subseteq V^+$.

4.2 Colouring vertices of large degree.

Let $\boldsymbol{\sigma} \colon V^+ \to [k]$ be a uniformly random admissible colouring of G^+. For every $v \in V^{++}$, we let $B_{\boldsymbol{\sigma}}(v)$ be the random event that $N_G(v)$ has no odd colour in $\boldsymbol{\sigma}$. The goal of this subsection will be to show that, with non-zero probability, no event $B_{\boldsymbol{\sigma}}(v)$ occurs.

Lemma 16. *If $\Delta \geqslant 49$ and $k \geqslant \Delta + 4(\ln\Delta + \ln\ln\Delta + 3)$, then there exists an admissible colouring $\sigma\colon V^+ \to [k]$ of G^+ such that every vertex $v \in V^{++}$ has an odd colour in σ.*

Proof of Theorem 16. Fix $k \coloneqq \Delta + \eta$, for some integer $\eta \geqslant 1$ whose precise value will be determined later in the proof, and let $\boldsymbol{\sigma}$ be a uniformly random admissible k-colouring of G^+. Such a colouring exists, since each vertex v has at most Δ constraints (at most $\deg_{V^+}(v)$ constraints because of the adjacent colours, and at most $\deg_{V^-}(v)$ constraints because of the adjacent witness colours). In particular, we have $|L_{\boldsymbol{\sigma}}(v)| \geqslant \eta$ for every $v \in V^+$. We want to show that, with non-zero probability, no bad event $B_{\boldsymbol{\sigma}}(v)$ occurs for $v \in V^{++}$.

Let $m \leqslant k/2$ be some integer whose explicit value will be determined later in the proof. For every $v \in V^{++}$, we pick an arbitrary subset $M(v) \subseteq N(v)$ of size m. Then we let $\Gamma(v) \coloneqq N[M(v)]$. For a vertex $u \in V^{++}$, the outcome of $B_{\boldsymbol{\sigma}}(u)$ is entirely determined by the colours assigned to vertices in $N(u)$. So if we fix the realisation of $\boldsymbol{\sigma}$ outside of $M(v)$, we in particular fix the outcomes of all events $B_{\boldsymbol{\sigma}}(u)$ such that $M(v) \cap N[u] = \emptyset$. This holds for every $u \notin \Gamma(v)$. We wish to apply Lemma 12 to those bad events, with that definition of $\Gamma(v)$. To that end, let Σ_0 be the set of possible realisations of $\boldsymbol{\sigma}_{|V^+ \setminus M(v)}$ such that no event $B_{\boldsymbol{\sigma}}(u)$ occurs for $u \notin \Gamma(v)$. For every $Z \subseteq V^{++} \setminus \Gamma(v)$, we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(v) \,\middle|\, \bigcap_{u\in Z} \overline{B_{\boldsymbol{\sigma}}(u)}\right] \leqslant \sup_{\sigma_0 \in \Sigma_0} \mathbb{P}\left[B_{\boldsymbol{\sigma}}(v) \mid \boldsymbol{\sigma}_{|V^+\setminus M(v)} = \sigma_0\right] \leqslant \sqrt{2}\left(\frac{2m}{e\eta}\right)^{m/2},$$

by Lemma 15 applied to the graph G^+ with $\mathscr{C}(G^+)$ being the set of admissible k-colourings of G^+.

Let us fix $\eta \coloneqq 2m$, so that this is at most $\sqrt{2}e^{-m/2}$. Since $v \in V^{++}$, we know that $\deg_{G^+}(v) \geqslant k/2 \geqslant \eta/2 \geqslant m$, so this lets us pick any value for m. To apply Lemma 12, we need an upper bound of $\frac{1}{em\Delta}$ for that probability, which holds precisely when $m \geqslant -2W_{-1}\left(-\frac{1}{2\sqrt{2}e\Delta}\right)$. We may therefore pick $m \coloneqq \left\lceil -2W_{-1}\left(-\frac{1}{2\sqrt{2}e\Delta}\right)\right\rceil$; a careful analysis of that value yields that $2m \leqslant \lceil 4(\ln\Delta + \ln\ln\Delta + 3)\rceil$ when $\Delta \geqslant 49$. ☐

4.3 Colouring vertices of small degree.

Proof of Theorem 3. Since the bound on $\chi_o(G)$ that we want to prove is weaker than that of Corollary 4 if $\Delta \leqslant 65$, we may assume that $\Delta \geqslant 66$.

By Theorem 16, G^+ has an admissible k-colouring $\sigma\colon V^+ \to [k]$ with $k = \Delta + \lceil 4(\ln\Delta + \ln\ln\Delta + 3)\rceil$, such that every $v \in V^{++}$ has an odd colour.

It remains to colour the vertices of V^-. Let us denote $V^- = \{v_1, \dots, v_t\}$ with $t = |V^-|$. For $i \in [t]$ let H_i be the subgraph of G induced by $V^+ \cup \{v_1, \dots, v_i\}$. Let $H_0 = G^+$ and $\sigma_0 = \sigma$. Let us inductively construct a sequence of partial colouring σ_i on H_i. We assign $\sigma_i(v) = \sigma_{i-1}(v)$ for $v \in V(H_{i-1})$, and assign a colour to v_i which avoids the colours and witness colours of all vertices of $N_G(v_i)$ with respect to σ_{i-1}. Each time we colour $v \in V^-$, each neighbour u of v yields at most 2 forbidden colours (its colour $\sigma(u)$, and its witness colour if it exists), so there are less than k forbidden colours for v.

We claim that σ_t is an odd colouring of G. Clearly σ_t is a proper colouring. On the other hand, every vertex $v \in V(G) \setminus V^{++}$ has at least one neighbour in V^-. When we colour its last neighbour u with respect to σ_i, since the witness colour of v is forbidden, the assignment of u yields an odd colour to v. This ensures that the sequence of colouring terminates, and ends the proof of Theorem 3. □

5 Odd colourings of hypergraphs with constrained edge sizes

We recall that, given a graph-hypergraph pair $(G, \mathcal{H})$, an *odd k-colouring of* $(G, \mathcal{H})$ is a mapping $\sigma\colon V \to [k]$ that is both a proper colouring of G and an odd colouring of $\mathcal{H}$. Given a set $S \subseteq V(\mathcal{H})$, we let $\mathcal{H}[S] = (S, \{e \cap S : e \in E(\mathcal{H})\})$ be the sub-hypergraph of $\mathcal{H}$ induced by S (note that this definition allows the possibility that $\mathcal{H}[S]$ contains $\emptyset$ as an edge).

5.1 A bound in terms of the chromatic number for quasi-regular graphs

The proof of Theorem 3 relies on the probabilistic method by analysing the behaviour of a uniformly random admissible colouring of a given graph G. It turns out that we have exploited the randomness of only a subset of the vertices of G: a subset of m neighbours of each vertex $v \in V^{++}$, where $m = \Theta(\ln \Delta(G))$. If G has a large minimum degree, we may restrict the random choices to a small subset of vertices that should suffice to have an odd colour in every neighbourhood, and colour the other vertices with an optimal proper colouring.

Theorem 17. *Let $(G, \mathcal{H})$ be a graph-hypergraph pair. Assume that $\Delta(\mathcal{H}) \geqslant 49$, and fix $\eta \coloneqq \lceil 4(\ln \Delta(\mathcal{H}) + \ln \ln \Delta(\mathcal{H}) + 3) \rceil$. For every subset of vertices $S \subseteq V(\mathcal{H})$, if $\epsilon(\mathcal{H}[S]) \geqslant \eta/2$, then $(G, \mathcal{H})$ has an odd k-colouring, where*

$$k \leqslant \chi(G \setminus S) + \Delta(G[S]) + \eta.$$

Proof. Let $k_0 \coloneqq \chi(G \setminus S)$ and let σ_0 be a proper k_0-colouring of $G \setminus S$. Let $k \coloneqq k_0 + \Delta(G[S]) + \eta$, and let $\boldsymbol{\sigma}$ be a uniformly random proper k-colouring of G that satisfies $\boldsymbol{\sigma}_{|G\setminus S} = \sigma_0$. For every $e \in E(\mathcal{H})$, we let $B_{\boldsymbol{\sigma}}(e)$ be the random event that e has no colour appearing at odd times in $\boldsymbol{\sigma}$. Let us show that, with non-zero probability, no event $B_{\boldsymbol{\sigma}}(e)$ occurs. Let $m \leqslant \eta/2$ be an integer, and for every edge $e \in E(\mathcal{H})$ let $M(e) = \{u_1, u_2, \ldots, u_m\} \subseteq e \cap S$ be a subset of m vertices in e. Let us recolour the vertices in $M(e)$ in turn with a uniformly random available colour. Each time we recolour u_i, the neighbours of u_i in S forbid at most $\deg_S(u_i) \leqslant \Delta(G[S])$ colours, and the neighbours of u_i not in S forbid at most $k_0 = \chi(G \setminus S)$ colours (these colours are fixed by σ_0). So there are at least η available colours for u_i. In particular, we have $|L_{\boldsymbol{\sigma}}(v)| \geqslant \eta$ for each $v \in V(G)$.

We apply Lemma 12 with $\Gamma(e) \coloneqq \{e' \in E(\mathcal{H}) : e' \cap M(e) \neq \emptyset\}$ for every edge $e \in E(\mathcal{H})$, and obtain that, with non-zero probability, none of the events $B_{\boldsymbol{\sigma}}(e)$ occurs. The size of $\Gamma(e)$ is at most $m\Delta(\mathcal{H})$. Let Σ_1 be the set of possible realisations of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$ such that no event $B_{\boldsymbol{\sigma}}(e')$ occurs for $e' \notin \Gamma(e)$. For every $Z \subseteq E(\mathcal{H}) \setminus \Gamma(e)$, we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \,\middle|\, \bigcap_{e' \in Z} \overline{B_{\boldsymbol{\sigma}}(e')}\right] \leqslant \sup_{\sigma_1 \in \Sigma_1} \mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \mid \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma_1\right]$$
$$\leqslant \sqrt{2}\left(\frac{2m}{e\eta}\right)^{m/2} = \sqrt{2}\, e^{-m/2},$$

by Lemma 15 applied to the graph G with $\mathscr{C}(G)$ being the set of proper k-colourings of G. As in the proof of Lemma 16, by fixing $m \coloneqq \left\lceil -2W_{-1}\left(-\frac{1}{2\sqrt{2}e\Delta(\mathcal{H})}\right)\right\rceil \leqslant \eta/2$, the above probability is at most $\frac{1}{e\cdot m\Delta(\mathcal{H})}$. This proves the existence of a proper k-colouring σ of G such that every vertex has an odd colour in σ, as desired. □

By taking $S = V(G)$, Theorem 17 has the following result as a corollary.

Corollary 18. *Let $(G, \mathcal{H})$ be a graph-hypergraph pair. Assume that $\Delta(\mathcal{H}) \geqslant 49$, and fix $\eta \coloneqq \lceil 4(\ln \Delta(\mathcal{H}) + \ln\ln \Delta(\mathcal{H}) + 3)\rceil$. If $\epsilon(\mathcal{H}) \geqslant \eta/2$, then there exists an odd $(\Delta(G) + \eta)$-colouring of $(G, \mathcal{H})$.*

We next show how to find a set S satisfying the hypothesis of Theorem 17 such that $\Delta(G[S])$ is as small as possible.

Lemma 19. *Let $(G, \mathcal{H})$ be a graph-hypergraph pair. Let $\Delta \coloneqq \Delta(G) + \Delta(\mathcal{H})$, and let us assume that the minimum edge size in $\mathcal{H}$ is $\epsilon(\mathcal{H}) \geqslant 12 \ln \Delta$. Let $r \coloneqq \min\{\epsilon(\mathcal{H}), \Delta(G)\}$. Then for every m satisfying $4 \ln \Delta \leqslant m \leqslant r/3$, if Δ is large enough, there is a subset $S \subseteq V(G)$ such that $\epsilon(\mathcal{H}[S]) \geqslant m$, and $\Delta(G[S]) \leqslant \frac{\Delta(G)}{r}(m + \sqrt{60\, m \ln \Delta})$.*

Proof. Let us for short denote $D \coloneqq (m + \sqrt{60\, m \ln \Delta})\, \frac{\Delta(G)}{r}$. Let $\mathbf{S}$ be a random subset of $V(G)$ obtained by taking each vertex independently uniformly at random with probability $p = \frac{m + \sqrt{11\, m \ln \Delta}}{r}$. We note that, since $4 \ln \Delta \leqslant m \leqslant r/3$ by assumption, we have $p < 1$ and so this probability is well-defined. The result follows if there exists a realization S such that every $e \in E(\mathcal{H})$ has $|e \cap S| \geqslant m$ and every $v \in V(G)$ satisfies $\deg_S(v) \leqslant D$. Let $V(G)$ be ordered arbitrarily. For every edge $e \in E(\mathcal{H})$, we let $\widetilde{e}$ consist of the first r vertices (with respect to that order) of e. For every edge $e \in E(\mathcal{H})$, let $\mathbf{X}_e \coloneqq |\widetilde{e} \cap \mathbf{S}|$ be the random variable that counts the number of vertices of $\widetilde{e}$ in $\mathbf{S}$, and we write $\mu_e \coloneqq \mathbb{E}\left[\mathbf{X}_e\right] = m + \sqrt{11 m \ln \Delta}$. For every vertex $v \in V(G)$, we let $\mathbf{X}_v \coloneqq |N_G(v) \cap \mathbf{S}|$ be the random variable that counts the number of neighbours of v in $\mathbf{S}$, and we write $\mu_v \coloneqq \mathbb{E}\left[\mathbf{X}_v\right] = \frac{m + \sqrt{11 m \ln \Delta}}{r} \cdot \deg_G(v) \leqslant \frac{\Delta(G)}{r}(m + \sqrt{11 m \ln \Delta})$. For every edge $e \in E(\mathcal{H})$, let B'_e be the random event that $\mathbf{X}_e < m$, and for every vertex $v \in V(G)$, let B_v be the random event that $\mathbf{X}_v > D$.

Observe that, for every pair of vertices (u, v) such that $N(u) \cap N(v) = \emptyset$, the random events B_u and B_v are independent. So the bad event B_v is dependent with at most $\Delta(G)^2$ bad events B_u, and at most $\Delta(G)\Delta(\mathcal{H})$ bad events B'_e. So the dependency-degree of B_v is at most $\Delta(G)\Delta \leqslant \Delta^2$. Moreover, for every pair of edges (e, f), such that $\widetilde{e} \cap \widetilde{f} = \emptyset$, the random events B'_e and B'_f are independent. So the bad event B'_e is dependent with at most $r\Delta(\mathcal{H})$ bad events B'_f, and at most $r\Delta(G)$ bad events B_v. So the dependency-degree of B'_e is at most $r\Delta \leqslant \Delta^2$. Hence we may apply the LLL in order to show that, with non-zero probability, no event B_v or B'_e occurs. Regarding that the maximum degree of the dependency-graph of those random events is at most Δ^2, it suffices to prove that

$$\mathbb{P}\left[\mathbf{X}_e < m\right] \leqslant \frac{1}{e\Delta^2} \quad \text{and} \quad \mathbb{P}\left[\mathbf{X}_v > D\right] \leqslant \frac{1}{e\Delta^2}.$$

We do so by applying Chernoff bounds on the random variable $\mathbf{X}_e$ of expectancy μ_e:

$$\begin{aligned}
\mathbb{P}\left[\mathbf{X}_e < m\right] &\leqslant \exp\left(-\frac{(\mu_e - m)^2}{2\mu_e}\right) \\
&= \exp\left(-\frac{11m\ln\Delta}{2m + 2\sqrt{11m\ln\Delta}}\right) = \exp\left(-\frac{11\ln\Delta}{2 + 2\sqrt{\frac{11\ln\Delta}{m}}}\right) \\
&\leqslant \Delta^{-\frac{11}{2+\sqrt{11}}} \leqslant \frac{1}{e\Delta^2} \qquad \text{if } \Delta \text{ is large enough.}
\end{aligned}$$

We do the same with the random variable $\mathbf{X}_v$ of expectancy μ_v:

$$\begin{aligned}
\mathbb{P}\left[\mathbf{X}_v > D\right] &= \exp\left(-\frac{(D - \mu_v)^2}{2D}\right) \\
&\leqslant \exp\left(-\frac{(\sqrt{60} - \sqrt{11})^2 m\ln\Delta}{2m + 2\sqrt{60m\ln\Delta}} \cdot \frac{\Delta(G)}{r}\right) \\
&\leqslant \exp\left(-\frac{(\sqrt{60} - \sqrt{11})^2 \ln\Delta}{2 + 2\sqrt{\frac{60\ln\Delta}{m}}} \cdot \frac{\Delta(G)}{r}\right) \\
&\leqslant \Delta^{-\frac{(\sqrt{60}-\sqrt{11})^2}{2+2\sqrt{15}}} \leqslant \frac{1}{e\Delta^2} \qquad \text{if } \Delta \text{ is large enough.}
\end{aligned}$$

By Lemma 12, with non-zero probability, $\mathbf{S}$ satisfies the conclusion of Lemma 19. □

We note that it is possible to drop the condition that Δ is large enough in the statement of Lemma 19 if we set $D := \frac{\Delta(G)}{r}(m + \sqrt{30m(1 + 2\ln\Delta)})$ and $p := \frac{1}{r}(m + \sqrt{5.5m(1 + 2\ln\Delta)})$ instead, and assume that $m \geqslant 2 + 4\ln\Delta$ and $\epsilon(\mathcal{H}) \geqslant 6 + 12\ln\Delta$.

We may combine Theorem 17 and Lemma 19 in order to obtain that, for a quasi-regular graph G (that is, a graph G where the ratio $\Delta(G)/\delta(G)$ is bounded), the odd chromatic number of G is not too far from its chromatic number.

Corollary 20. *Let G be a graph of maximum degree Δ large enough, and minimum degree $\delta \geqslant 12\ln(2\Delta)$. Then*

$$\chi_o(G) \leqslant \chi(G) + \lceil 4(\ln\Delta + \ln\ln\Delta + 3)\rceil + \frac{20\Delta\ln(2\Delta)}{\delta} = \chi(G) + O\left(\frac{\Delta\ln\Delta}{\delta}\right) \quad \textit{as } \Delta \to \infty.$$

Note that the minimum degree condition in Corollary 20 can be dropped. Indeed, if it is not fulfilled, then the upper bound on $\chi_o(G)$ is larger than that given by Theorem 4.

5.2 Graphs of small chromatic number

Given a graph-hypergraph pair $(G, \mathcal{H})$, if we can find a set S that satisfies the hypothesis of Theorem 17 such that $\chi(G[S])$ is much smaller than $\Delta(G[S])$, we can use another approach to obtain a better bound.

Theorem 21. *Let $(G, \mathcal{H})$ be a graph-hypergraph pair. Assume that $\Delta(\mathcal{H}) \geqslant 49$, and fix $\eta \coloneqq \lceil 4(\ln \Delta(\mathcal{H}) + \ln\ln \Delta(\mathcal{H}) + 3) \rceil$. For every subset of vertices $S \subseteq V(\mathcal{H})$, if $\epsilon(\mathcal{H}[S]) \geqslant \eta/2$, then $(G, \mathcal{H})$ has an odd k-colouring, where*

$$k \leqslant \chi(G \setminus S) + \eta\, \chi(G[S]).$$

Proof. Let $G_0 \coloneqq G \setminus S$ and $G_1 \coloneqq G[S]$. For each $i \in \{0, 1\}$, we write $k_i \coloneqq \chi(G_i)$, and we let σ_i be a proper k_i-colouring of G_i.

We define a random proper k-colouring $\boldsymbol{\sigma}$ of G as follows, where $k = k_0 + \eta k_1$. For every $v \in S$, draw some random value $\mathbf{x}_v$ uniformly at random from $[\eta]$, and let $\boldsymbol{\sigma}(v) \coloneqq (\sigma_1(v), \mathbf{x}_v)$. For every $v \notin S$, let $\boldsymbol{\sigma}(v) \coloneqq \sigma_0(v)$. Let us order the vertices in $V(G)$ arbitrarily. For every $e \in E(\mathcal{H})$, we let $B_{\boldsymbol{\sigma}}(e)$ be the random (bad) event that e has no odd colour. Let $m \coloneqq \eta/2$, and let $M(e)$ contain the smallest m vertices of $e \cap S$. Let σ be a possible realisation of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$. By construction, for every $v \in S$, there are η choices in $L_{\boldsymbol{\sigma}}(v)$. Hence we may apply Lemma 15 and obtain that

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \;\middle|\; \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma\right] \leqslant \sqrt{2}\left(\frac{2m}{e\eta}\right)^{m/2} = \sqrt{2}\, e^{-m/2}.$$

As explained in the proof of Theorem 3, this is at most $\frac{1}{em\Delta(\mathcal{H})}$. For an edge $e' \in E(\mathcal{H})$, the outcome of $B_{\boldsymbol{\sigma}}(e')$ is entirely determined by the realisation of $\boldsymbol{\sigma}_{|e'}$. So if we fix the realisation of $\boldsymbol{\sigma}$ outside of $M(e)$, we in particular fix the outcomes of all events $B_{\boldsymbol{\sigma}}(e')$ such that $M(e) \cap e' = \emptyset$. So we set $\Gamma(e) \coloneqq \{e' : e' \cap M(e) \neq \emptyset\}$, and observe that these sets have size at most $m\Delta(\mathcal{H})$. We may now apply Lemma 12 to the bad events $(B_{\boldsymbol{\sigma}}(e))$, with that definition of $\Gamma(e)$, and conclude that with positive probability, no event $B_{\boldsymbol{\sigma}}(e)$ occurs. So there is a realisation of $\boldsymbol{\sigma}$ that is an odd k-colouring of $(G, \mathcal{H})$. This concludes the proof. □

Corollary 22. *Let G be a graph of maximum degree $\Delta \geqslant 49$ and minimum degree at least $\eta/2$, where $\eta \coloneqq \lceil 4(\ln \Delta + \ln\ln \Delta + 3) \rceil$. Then*

$$\chi_o(G) \leqslant \eta\, \chi(G),$$

and this is tight up to a multiplicative constant for a family of graphs of increasing chromatic numbers.

Proof. The upper bound on k is a direct consequence of Theorem 21 where $\mathcal{H}$ is the neighbourhood-hypergraph of G, and $S = V(G)$. Let us prove the tightness of the bound.

Fix an integer $k_0 \geqslant 2$, and let $n_0 \geqslant k_0$ be an even integer. Let G_0 be a complete k_0-partite graph, with parts $X_1, \ldots, X_{k_0}$ all of size $2n_0$. For every $i \in [k_0]$, and for every

$S \in \binom{X_i}{n_0}$, we add a vertex with neighbourhood S in G_0. Let G be the obtained graph; let us show that $\chi_o(G) \geqslant k_0(n_0+1)$. We write $k := \chi_o(G)$, and let σ be an odd k-colouring of G. First observe that we must have $\sigma(X_i) \cap \sigma(X_j) = \emptyset$ for every $i \neq j$, otherwise we would find a monochromatic edge in σ. So it suffices to show that $|\sigma(X_i)| \geqslant n_0 + 1$ for every $i \in [k_0]$. Let us assume for the sake of contradiction that $|\sigma(X_i)| \leqslant n_0$. For every odd colour of X_i, we remove one vertex with that colour from X_i. We are left with at least n_0 vertices. We now remove monochromatic pairs of vertices from X_i until exactly n_0 vertices remain. We obtain a set S with no odd colour, and by construction there is a vertex in $V(G)$ such that $N(v) = S$. So v has no odd colour in σ, a contradiction.

The maximum degree of G is $\frac{1}{2}\binom{2n_0}{n_0} + (k_0-1)\cdot 2n_0 < 4^{n_0}$ when n_0 is large enough, and $\chi(G) = \chi(G_0) = k_0$. So we have

$$\chi_o(G) > \chi(G)\log_4 \Delta(G),$$

while the minimum degree of G is $n_0 \geqslant \log_4 \Delta(G)$.

Note that the minimum degree of G is smaller than what is required by a factor $4\ln 2$. If we want to meet the required lower bound for the minimum degree of G, we may let the size of each X_i be $\frac{7}{6}n_0$ instead of $2n_0$. We can now prove that more than $\frac{n_0}{6}$ colours must appear on each part X_i. We still have $\chi(G) = k_0$ and $\delta(G) = n_0$, while

$$\begin{aligned}\Delta(G) &= (k_0-1)\cdot\frac{7}{6}n_0 + \binom{7n_0/6}{n_0}\\ &= (k_0-1)\cdot\frac{7}{6}n_0 + \binom{7n_0/6}{n_0/6}\\ &\leqslant (k_0-1)\cdot\frac{7}{6}n_0 + (7e)^{n_0/6}.\end{aligned}$$

Since $6/\ln(7e) > 2.03$, when n_0 is large enough we have $\delta(G) = n_0 \geqslant 2(\ln\Delta(G) + \ln\ln\Delta(G) + 3)$, as required. On the other hand, we have $\chi_o(G) > \frac{1}{6}k_0 n_0$, so we are only a factor 12 away from the upper bound guaranteed in that regime[1]. □

6 The effect of multiplying the constraints

We may now wonder what happens when we seek for an odd colouring of a given graph-hypergraph pair $(G,\mathcal{H})$, such that every edge $e \in E(\mathcal{H})$ has *many* odd colours. By a greedy algorithm, we can immediately get $\chi_o^h(G) \leqslant (h+1)\Delta(G) + 1$ for any graph G and integer h, or more generally that $\chi_o^h(G,\mathcal{H}) \leqslant h\Delta(\mathcal{H}) + \Delta(G) + 1$ given a graph-hypergraph pair $(G,\mathcal{H})$. We will show that we can ensure much better upper bounds with an additional reasonable minimum edge size condition in $\mathcal{H}$. We will rely on an extension of Lemma 15 to h-odd colourings, that we present hereafter.

[1] With a more refined estimate of the binomial coefficient, one can replace $7/6$ with $53/45$, and conclude that we are only a factor $45/4$ away from best possible.

Lemma 23. *Let $(G, \mathcal{H})$ be a graph-hypergraph pair, and $\mathscr{C}(G)$ a set of colourings of G. Let $\boldsymbol{\sigma}$ be drawn uniformly at random from $\mathscr{C}(G)$, and assume that there exists an integer τ such that we deterministically have $|L_{\boldsymbol{\sigma}}(v)| \geqslant \tau$ for every integer $v \in V(G)$. Let $h \geqslant 1$ and $t \geqslant 0$ be integers that satisfy $m := h - 1 + t \leqslant \epsilon(\mathcal{H})$, and $B_{\boldsymbol{\sigma}}(e)$ be the bad event that e has less than h odd colours in $\boldsymbol{\sigma}$. Then, for every subset $M(e) \subseteq e$ of size m, and for every possible realisation σ_0 of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$, we have*

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \;\middle|\; \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma_0\right] \leqslant \sqrt{2}\binom{m}{t}\left(\frac{2t}{e\tau}\right)^{t/2},$$

for every $m \leqslant \epsilon(\mathcal{H})$.

Proof. Let $e \in E(\mathcal{H})$, and let $M(e) = \{u_1, \ldots, u_m\}$ be a fixed subset of m vertices in e. Let σ_0 be a possible realisation of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$. Let $\boldsymbol{\sigma}_0$ be drawn uniformly at random from the extensions of σ_0 to $\mathscr{C}(G)$. For every $1 \leqslant i \leqslant m$, we let $\boldsymbol{\sigma}_i \in \mathscr{C}(G)$ be obtained from $\boldsymbol{\sigma}_{i-1}$ by resampling the colour of u_i uniformly at random from $L_{\boldsymbol{\sigma}_{i-1}}(u_i)$. For every $i \leqslant m$, let $\mathbf{S}_i$ be the number of odd colours of e in $\boldsymbol{\sigma}_i$. For every $i \geqslant 1$, we have $\mathbf{S}_i = \mathbf{S}_{i-1} - 1$ if $\boldsymbol{\sigma}_i(u_i)$ is one of the $\mathbf{S}_{i-1}$ odd colours of e in $\boldsymbol{\sigma}_{i-1}$; since there are at least τ choices for $\boldsymbol{\sigma}_i(u_i)$ this happens with probability at most $\frac{h-1}{\tau}$ if $\mathbf{S}_{i-1} = h - 1$. Otherwise, we have $\mathbf{S}_i = \mathbf{S}_{i-1} + 1$. So the sequence $(\mathbf{S}_i)_{i\leqslant m}$ satisfies the hypotheses of Lemma 14, which yields

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}_m}(e)\right] = \mathbb{P}\left[\mathbf{S}_m \leqslant h-1\right] \leqslant \sqrt{2}\binom{m}{h-1}\left(\frac{2(m-h+1)}{e\tau}\right)^{\frac{m-h+1}{2}} = \sqrt{2}\binom{m}{t}\left(\frac{2t}{e\tau}\right)^{t/2}.$$

Since we resample the colours uniformly at random, the random colourings $(\boldsymbol{\sigma}_i)_{i\leqslant m}$ are identically distributed. Therefore, if $\boldsymbol{\sigma}$ is drawn uniformly at random from $\mathscr{C}(G)$, we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \;\middle|\; \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma_0\right] = \mathbb{P}\left[B_{\boldsymbol{\sigma}_0}(e)\right] = \mathbb{P}\left[B_{\boldsymbol{\sigma}_m}(e)\right],$$

and the conclusion follows. □

Theorem 24. *Let $(G, \mathcal{H})$ be a graph-hypergraph pair with $\Delta(\mathcal{H}) \geqslant 49$, and let h be a given integer. If there exists a subset of vertices $S \subseteq V(\mathcal{H})$ such that $\min\{h-1, \epsilon(\mathcal{H}[S])-h+1\} \geqslant 2(\ln\Delta(\mathcal{H}) + \ln\ln\Delta(\mathcal{H}) + 3)$, then $(G, \mathcal{H})$ has an h-odd k-colouring, where*

$$k \leqslant \begin{cases} \chi(G \setminus S) + \Delta(G[S]) + 32(h-1) & \text{if } \epsilon(\mathcal{H}[S]) \geqslant 2(h-1); \\ \chi(G \setminus S) + \Delta(G[S]) + 2e^2\,\frac{\epsilon(\mathcal{H}[S])^{2+1/\ln\Delta(\mathcal{H})}}{\epsilon(\mathcal{H}[S])-h+1} & \text{otherwise.} \end{cases}$$

Proof. Let $k_0 := \chi(G \setminus S)$ and let σ_0 be a proper k_0-colouring of $G \setminus S$. Fix $k := k_0 + \Delta(G[S]) + \eta$, for some integer $\eta \geqslant 1$ whose precise value will be determined later in the proof, and let $\boldsymbol{\sigma}$ be a uniformly random proper k-colouring of G that satisfies $\boldsymbol{\sigma}_{|G\setminus S} = \sigma_0$. For every edge $e \in E(\mathcal{H})$, we let $B_{\boldsymbol{\sigma}}(e)$ be the (bad) random event that e contains less than h odd colours in $\boldsymbol{\sigma}$. Let us show that, with non-zero probability, no event $B_{\boldsymbol{\sigma}}(e)$ occurs.

Let us write $t := \min\{\epsilon(\mathcal{H}[S]) - h + 1, h - 1\}$ and $m := h - 1 + t$. Let $e \in E(\mathcal{H})$ and $M(e) = \{v_1, v_2, \ldots, v_m\} \subseteq e \cap S$ be a subset of m vertices in e. Let us recolour the vertices in $M(e)$ in turn with a uniformly random available colour. Each time we recolour v_i, the neighbours of v_i in S forbid at most $\deg_S(v_i) \leqslant \Delta(G[S])$ colours, and the neighbours of v_i not in S forbid at most $k_0 = \chi(G \setminus S)$ colours (these colours are fixed by σ_0). So there are at least η available colours for v_i. In particular, we have $|L_{\boldsymbol{\sigma}}(v)| \geqslant \eta$ for each $v \in M(e)$.

We apply Lemma 12 with $\Gamma(e) \coloneqq \{e' \in E(\mathcal{H}) : e' \cap M(e) \neq \emptyset\}$ for every edge $e \in E(\mathcal{H})$, and obtain that, with non-zero probability, none of the events $B_{\boldsymbol{\sigma}}(e)$ occurs. The size of $\Gamma(e)$ is at most $m\Delta(\mathcal{H})$. Let Σ_0 be the set of possible realisations of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$ such that no event $B_{\boldsymbol{\sigma}}(e')$ occurs for $e' \notin \Gamma(e)$. For every $Z \subseteq E(\mathcal{H}) \setminus \Gamma(e)$, we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \;\middle|\; \bigcap_{e' \in Z} \overline{B_{\boldsymbol{\sigma}}(e')}\right] \leqslant \sup_{\sigma_0 \in \Sigma_0} \mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \mid \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma_0\right] \leqslant \sqrt{2}\binom{m}{t}\left(\frac{2t}{e\eta}\right)^{\frac{t}{2}},$$

by Lemma 23 applied to the graph G with $\mathscr{C}(G)$ being the set of proper k-colourings of G.

Let us fix $\eta \coloneqq 2t\,(\frac{m}{t} \times \binom{m}{t})^{2/t}$, so that the above probability is at most $\sqrt{2}\frac{t}{m}e^{-t/2}$. We can apply Lemma 12 if this is at most $\frac{1}{em\Delta(\mathcal{H})}$, which is equivalent to

$$\sqrt{2}e^{-t/2} \leqslant \frac{1}{et\Delta(\mathcal{H})}.$$

As in the proof of Lemma 16, this holds whenever $t \geqslant \left\lceil -2W_{-1}\left(-\frac{1}{2\sqrt{2}e\Delta(\mathcal{H})}\right)\right\rceil$. Since by hypothesis we have $\Delta(\mathcal{H}) \geqslant 49$ and $t \geqslant \lceil 2(\ln\Delta(\mathcal{H}) + \ln\ln\Delta(\mathcal{H}) + 3)\rceil$, this inequality is verified. If $t = h-1$ then

$$\eta = 2(h-1)\left(2\binom{2h-2}{h-1}\right)^{\frac{2}{h-1}} \leqslant 2(h-1)\left(2^{2h-2}\right)^{\frac{2}{h-1}} = 32(h-1).$$

Otherwise, we have $m = \epsilon(\mathcal{H}[S]) > 2t$, and

$$\eta = 2t\left(\frac{m}{t}\binom{m}{t}\right)^{\frac{2}{t}} \leqslant 2t\left(\frac{m}{t}\right)^{\frac{2}{t}}\left(\frac{me}{t}\right)^{2} \leqslant 2e^2\,\frac{m^{2+\frac{1}{\ln\Delta(\mathcal{H})}}}{m-h+1}.$$

This proves the existence of a proper k-colouring σ of $(G,\mathcal{H})$ such that every edge has at least h odd colours in σ, as desired. ☐

Remark 25. The second bound of Theorem 24 is at most $\chi(G\setminus S)+\Delta(G[S])+2e^3\,\frac{\delta(G[S])^2}{\delta(G[S])-h+1}$ when $\mathcal{H}$ is the neighbourhood-hypergraph of G.

The following corollary can be derived from Theorem 24 by setting $S \coloneqq V(G)$.

Corollary 26. *Let $(G,\mathcal{H})$ be a graph-hypergraph pair with $\Delta(\mathcal{H}) \geqslant 49$, and let h be a given integer. Let us assume that* $\min\{h-1, \epsilon(\mathcal{H})-h+1\} \geqslant 2(\ln\Delta(\mathcal{H})+\ln\ln\Delta(\mathcal{H})+3)$. *If $\epsilon(\mathcal{H}) \geqslant 2(h-1)$, then there exists an h-odd $(\Delta(G)+32(h-1))$-colouring of $(G,\mathcal{H})$; Otherwise, $(G,\mathcal{H})$ admits an h-odd $(\Delta(G)+2e^2\,\frac{\epsilon(\mathcal{H})^{2+1/\ln\Delta(\mathcal{H})}}{\epsilon(\mathcal{H})-h+1})$-colouring.*

We also extend Theorem 21 to h-odd colourings.

Theorem 27. *Let $(G, \mathcal{H})$ be a graph-hypergraph pair with $\Delta(\mathcal{H}) \geqslant 49$, and let h be a given integer. If there exists a subset of vertices $S \subseteq V(\mathcal{H})$ such that $\min\{h-1, \epsilon(\mathcal{H}[S])-h+1\} \geqslant 2(\ln \Delta(\mathcal{H}) + \ln\ln \Delta(\mathcal{H}) + 3)$, then $(G, \mathcal{H})$ has an h-odd k-colouring, where*

$$k \leqslant \begin{cases} \chi(G \setminus S) + 32(h-1)\,\chi(G[S]) & \text{if } \epsilon(\mathcal{H}[S]) \geqslant 2(h-1); \\ \chi(G \setminus S) + 2e^2\,\frac{\epsilon(\mathcal{H}[S])^{2+1/\ln \Delta(\mathcal{H})}}{\epsilon(\mathcal{H}[S])-h+1}\,\chi(G[S]) & \text{otherwise.} \end{cases}$$

Proof. Let $G_0 \coloneqq G \setminus S$ and $G_1 \coloneqq G[S]$. For each $i \in \{0,1\}$, we write $k_i \coloneqq \chi(G_i)$, and we let σ_i be a proper k_i-colouring of G_i.

We define a random proper k-colouring $\boldsymbol{\sigma}$ of G as follows, where $k = k_0 + \eta k_1$ and η is some positive integer whose precise value will be determined later in the proof. For every $v \in S$, draw some random value $\mathbf{x}_v$ uniformly at random from $[\eta]$, and let $\boldsymbol{\sigma}(v) \coloneqq (\sigma_1(v), \mathbf{x}_v)$. For every $v \notin S$, let $\boldsymbol{\sigma}(v) \coloneqq \sigma_0(v)$. Let us order the vertices in $V(G)$ arbitrarily. For every $e \in E(\mathcal{H})$, we let $B_{\boldsymbol{\sigma}}(e)$ be the random (bad) event that e contains less than h odd colours in $\boldsymbol{\sigma}$. Let us write $t \coloneqq \min\{\epsilon(\mathcal{H}[S]) - h + 1, h-1\}$, and let us fix $m \coloneqq h - 1 + t$. We let $M(e)$ contain the smallest m vertices of $e \cap S$. Let σ be a possible realisation of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$. By construction, for every $v \in S$, there are η choices in $L_{\boldsymbol{\sigma}}(v)$.

For an edge $e' \in E(\mathcal{H})$, the outcome of $B_{\boldsymbol{\sigma}}(e')$ is entirely determined by the realisation of $\boldsymbol{\sigma}_{|e'}$. So if we fix the realisation of $\boldsymbol{\sigma}$ outside of $M(e)$, we in particular fix the outcomes of all events $B_{\boldsymbol{\sigma}}(e')$ such that $M(e) \cap e' = \emptyset$. So we set $\Gamma(e) \coloneqq \{e' : e' \cap M(e) \neq \emptyset\}$, and observe that these sets have size at most $m\Delta(\mathcal{H})$. Let Σ_0 be the set of possible realisations of $\boldsymbol{\sigma}_{|V(G)\setminus M(e)}$ such that no event $B_{\boldsymbol{\sigma}}(e')$ occurs for $e' \notin \Gamma(e)$. Hence we may apply Lemma 23 and obtain that for every $Z \subseteq E(\mathcal{H}) \setminus \Gamma(e)$, we have

$$\mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \,\middle|\, \bigcap_{e' \in Z} \overline{B_{\boldsymbol{\sigma}}(e')}\right] \leqslant \sup_{\sigma_0 \in \Sigma_0} \mathbb{P}\left[B_{\boldsymbol{\sigma}}(e) \mid \boldsymbol{\sigma}_{|V(G)\setminus M(e)} = \sigma_0\right] \leqslant \sqrt{2}\binom{m}{t}\left(\frac{2t}{e\eta}\right)^{\frac{t}{2}}.$$

As explained in the proof of Theorem 24, this is at most $\frac{1}{em\Delta(\mathcal{H})}$ by fixing $\eta \coloneqq 2t\,(\frac{m}{t} \times \binom{m}{t})^{2/t}$ when $\Delta(\mathcal{H}) \geqslant 49$. We then have $\eta \leqslant 32(h-1)$ if $\epsilon(\mathcal{H}[S]) \geqslant 2(h-1)$; otherwise $\eta \leqslant 2e^2\,\frac{\epsilon(\mathcal{H}[S])^{2+1/\ln \Delta(\mathcal{H})}}{\epsilon(\mathcal{H}[S])-h+1}$. We may now apply Lemma 12 to the bad events $(B_{\boldsymbol{\sigma}}(e))$, with that definition of $\Gamma(e)$, and conclude that with positive probability, no event $B_{\boldsymbol{\sigma}}(e)$ occurs. So there is a realisation of $\boldsymbol{\sigma}$ that is an h-odd k-colouring of $(G, \mathcal{H})$. This concludes the proof. □

6.1 Constructions

In the current section, we have derived upper bound for $\chi_o^h(G, \mathcal{H})$ given an integer h and a graph-hypergraph pair $(G, \mathcal{H})$ that satisfies that $\epsilon(\mathcal{H}) - h$ is sufficiently large. We now show that, if no such restriction holds, then the bound obtained by a greedy colouring may be close to best possible.

We propose a construction that relies on the existence of Steiner 2-designs, which was proven in [23]. Given an integer q, a Steiner 2-design on $[q]$ is a collection of sets of uniform size k (that we call *blocks*), such that every pair of vertices from $[q]$ is contained in exactly one set. We denote it $S(2,k;q)$. Among its many properties, it must contain exactly $\binom{q}{2}/\binom{k}{2}$ blocks, and each vertex is contained in exactly $\frac{q-1}{k-1}$ blocks.

Proposition 28. *For every integer $h \geqslant 1$, there exists a family of graphs G of increasing maximum degree Δ and of minimum degree $h+1$ such that*

$$\chi_o^h(G) \geqslant h\Delta + 1.$$

Proof. By [23, Theorem 2.1], we may find an arbitrarily large integer q such that a Steiner 2-design $S(2,h+1;q)$ exists. We let $H = (X,Y,E)$ be its (bipartite) incidence graph. We have $|X| = q$, $|Y| = \binom{q}{2}/\binom{h+1}{2}$, every vertex $y \in Y$ has degree $h+1$, every vertex $x \in X$ has degree $\frac{q-1}{h}$, and every pair of vertices $x, x' \in X$ is contained in the neighbourhood of some vertex $y \in Y$. In particular, $\delta(H) = h+1$ and $\Delta(H) = \frac{q-1}{h}$.

Let σ be an h-odd k-colouring of H, where $k = \chi_o^h(H)$. So every neighbourhood of size $h+1$ must be rainbow in σ (i.e. contains no pair of vertices with the same colour), otherwise the number of odd colours in this neighbourhood is at most $h-1$. Since every pair of vertices from X is contained in a neighbourhood of size $h+1$, we infer that X is rainbow, and so $k \geqslant |X| = q = h\Delta(H) + 1$. □

One interesting special case of Theorem 24 is the following.

Corollary 29. *Let G be a Δ-regular graph, let $\lceil 2(\ln\Delta + \ln\ln\Delta + 3)\rceil \leqslant t \leqslant \Delta$ be a given integer, and let $h \coloneqq \Delta + 1 - t$. Then*

$$\chi_o^h(G) = O\left(\frac{\Delta^2}{t}\right).$$

We now show that Corollary 29 is tight up to a multiplicative constant.

Proposition 30. *For every even integer $\Delta \geqslant 2$ and $1 \leqslant t \leqslant \Delta$, there is a Δ-regular graph G such that, letting $h \coloneqq \Delta + 1 - t$, one has*

$$\chi_o^h(G) > \frac{1}{2}\frac{\Delta^2}{t+1}.$$

Proof. Let $n \coloneqq \frac{\Delta}{2} + 1$, and let $G \coloneqq L(K_{n,n})$ be the line-graph of a complete bipartite graph. Let $k \coloneqq \chi_o^h(G)$, and let σ be a h-odd k-colouring of G.

Let $v \in V(G)$. The neighbourhood of v can be covered with two cliques, so each colour in $N(v)$ has at most 2 occurrences. Let $s_\sigma(v)$ denote the number of colours with 2 occurrences in $N(v)$; we must have $s_\sigma(v) \leqslant (t-1)/2$. So

$$S \coloneqq \sum_{v \in V(G)} s_\sigma(v) \leqslant \frac{t-1}{2} n^2. \tag{2}$$

Let $(M_1, \ldots, M_k)$ be the colour classes of σ. If M_i has size m_i, then it has a contribution of $m_i(m_i-1)$ to S. Indeed, M_i is a matching of size m_i in $K_{n,n}$, and there are $m_i(m_i-1)$

edges incident to 2 edges from M_i in $K_{n,n}$; each of them corresponds to a vertex in G with a monochromatic pair of colour i in its neighbourhood. So by convexity we have

$$S = \sum_{i=1}^{k} m_i(m_i - 1) \geqslant k \cdot \frac{n^2}{k}\left(\frac{n^2}{k} - 1\right) = n^2\left(\frac{n^2}{k} - 1\right).$$

Combining this with (2), we obtain

$$k \geqslant \frac{2n^2}{t+1} > \frac{1}{2}\frac{\Delta^2}{t+1}.$$ □

We finish this section with the following consequence of Theorem 27.

Corollary 31. *Let G be a graph of maximum degree $\Delta \geqslant 49$, and let $h \geqslant 2(\ln\Delta + \ln\ln\Delta + 3)$ be a given integer. If the minimum degree of G is at least $2h$, then*

$$\chi_o^h(G) \leqslant 32(h-1)\,\chi(G),$$

and this is tight up to a multiplicative constant for a family of graphs of increasing chromatic numbers.

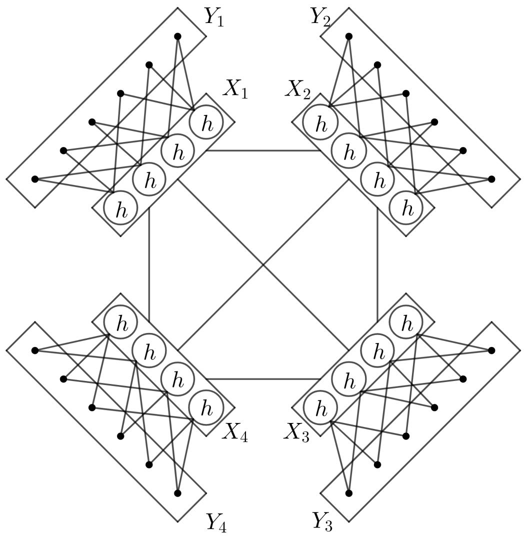

Figure 1: The construction in Corollary 31 when $k_0 = n_0 = 4$. A line between two sets stands for a complete bipartite graph.

Proof. The upper bound can be derived from Theorem 27 by letting $\mathcal{H}$ be the neighbourhood-hypergraph of G, and setting $S := V(G)$. To prove the tightness of the bound, we show that given any $\varepsilon > 0$, there is a graph G such that

$$\chi_o^h(G) \geqslant (2-\varepsilon)h\,\chi(G),$$

where h and $\chi(G)$ are arbitrary integers.

Fix an integer $k_0 \geqslant 2$, and let $n_0 \geqslant k_0$ be an even integer. Let G_0 be a complete k_0-partite graph, with parts $X_1, \dots, X_{k_0}$ all of size hn_0. For every $i \in [k_0]$, we partition X_i into n_0 h-sets, and for every pair $\{S_1, S_2\}$ of these h-sets in X_i we add a vertex with neighbourhood $S_1 \cup S_2$ in G_0; we denote Y_i the set of $\binom{n_0}{2}$ vertices thus added to G_0. Let G be the obtained graph (see Fig. 1); let us show that

$$\chi_o^h(G) \geqslant \left(2 - \frac{2}{n_0}\right) hk_0.$$

We write $k \coloneqq \chi_o^h(G)$, and let σ be an h-odd k-colouring of G. First observe that we must have $\sigma(X_i) \cap \sigma(X_j) = \emptyset$ for every $i \neq j$, otherwise we would find a monochromatic edge in σ. So it suffices to show that $|\sigma(X_i)| \geqslant \left(2 - \frac{2}{n_0}\right) h$ for every $i \in [k_0]$. On the one hand, each vertex $v \in Y_i$ has at least h odd colours. On the other hand, for each X_i and each colour c in σ, let $0 \leqslant p \leqslant n_0$ be the number of h-sets in X_i where c is an odd colour. Clearly, c is an odd colour of $p(n_0 - p) \leqslant \frac{n_0^2}{4}$ vertices in Y_i. So we need at least $\frac{4\binom{n_0}{2}}{n_0^2} h = \left(2 - \frac{2}{n_0}\right) h$ colours to make every vertex have h odd colours in $\sigma(X_i)$.

We also have $\chi(G) = \chi(G_0) = k_0$, so this concludes the proof. □

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