Finite s-geodesic-transitive digraphs

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Abstract

This paper initiates the investigation of the family of (G, s)-geodesic-transitive digraphs with $s \geqslant 2$. We first give a global analysis by providing a reduction result. Let Γ be such a digraph and let N be a normal subgroup of G maximal with respect to having at least 3 orbits. Then the quotient digraph Γ_N is $(G/N, s')$-geodesic-transitive where $s' = \min\{s, \mathrm{diam}(\Gamma_N)\}$, G/N is either quasiprimitive or bi-quasiprimitive on $V(\Gamma_N)$, and Γ_N is either directed or an undirected complete graph. Moreover, it is further shown that if Γ is not $(G, 2)$-arc-transitive, then G/N is quasiprimitive on $V(\Gamma_N)$.

On the other hand, we also consider the case that the normal subgroup N of G has one orbit on the vertex set. We show that if N is regular on $V(\Gamma)$, then Γ is a circuit, and particularly each (G, s)-geodesic-transitive normal Cayley digraph with $s \geqslant 2$, is a circuit.

Finally, we investigate $(G, 2)$-geodesic-transitive digraphs with either valency at most 5 or diameter at most 2. Let Γ be a $(G, 2)$-geodesic-transitive digraph. It is proved that: if Γ has valency at most 5, then Γ is $(G, 2)$-arc-transitive; if Γ has diameter 2, then Γ is a balanced incomplete block design with the Hadamard parameters.

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1 Introduction

A finite *digraph* (short for *directed graph*) Γ is a pair $(V(\Gamma), Arc(\Gamma))$ consisting of a finite set $V(\Gamma)$ of vertices and a subset $Arc(\Gamma)$ of the Cartesian product $V(\Gamma) \times V(\Gamma)$. Each element (u, v) of $Arc(\Gamma)$ is a *directed edge* of Γ from the vertex u to the vertex v, and sometimes (u, v) is written as $u \to v$, where $\to$ is an antisymmetric irreflexive relation on

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$V(\Gamma)$. An *arc* of Γ is an ordered pair of adjacent vertices. Hence for two vertices u and v, $u \to v$ is equivalent to that (u, v) is an arc. For each vertex $v \in V(\Gamma)$, we use $\Gamma^-(v) = \{u \in V | u \to v\}$ to denote the set of *in-neighbours* of v and use $\Gamma^+(v) = \{u \in V | v \to u\}$ to denote the set of *out-neighbours* of v. We say that a digraph Γ is k-regular if both the set $\Gamma^-(v)$ and the set $\Gamma^+(v)$ have size k for all $v \in V(\Gamma)$, and Γ is *regular* if it is k-regular for some positive integer k. The integer k is called the *valency* of a k-regular digraph.

For a non-negative integer s, an s-arc in a digraph (or graph) Γ is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices with $v_i \to v_{i+1}$ for each $i = 0, \ldots, s-1$ (we need to assume further that $v_i \neq v_{i+2}$ if Γ is a graph). This digraph (or graph) is said to be (G, s)-*arc-transitive* if its automorphism group G is transitive on all the s-arcs. The family of s-arc-transitive undirected graphs has been studied intensively, beginning with the seminal result of Tutte [25, 26], and particularly it was shown in 1981 by Weiss [29] that finite undirected graphs of valency at least 3 can only be s-arc-transitive for $s \leqslant 7$. For related results refer [1, 11, 15, 19, 23, 28]. In contrast with the situation for undirected graphs, there exist infinitely many classes of s-arc-transitive digraphs with unbounded s other than directed cycles. Constructions for such classes of digraphs were initiated by Praeger [22] in 1989 and have stimulated a lot of research. A few years later, Conder, Lorimer and Praeger [4] showed that for every integer $k \geqslant 2$ and every integer $s \geqslant 1$ there are infinitely many finite k-regular (G, s)-arc-transitive digraphs with G quasiprimitive on the vertex set. Giudici, Li and Xia [12] in 2017 solved the long-standing existence problem of vertex-primitive 2-arc-transitive digraphs by constructing an infinite class of such digraphs. One year later, Giudici and Xia [13] investigated vertex-quasiprimitive 2-arc-transitive digraphs, and reduced the problem of vertex-primitive 2-arc-transitive digraphs to almost simple groups. Their work includes a complete classification of vertex-quasiprimitive 2-arc-transitive digraphs where the action on vertices has quasiprimitive type SD or CD. Given integers k and m, Morgan, Potočnik and Verret [20] constructed a G-arc-transitive graph of valency k and an L-arc-transitive oriented digraph of out-valency k such that G and L both admit blocks of imprimitivity of size m. For more work see [3, 9, 18, 27].

An s-arc $(v_0, v_1, \ldots, v_s)$ in a digraph Γ is called an s-*geodesic* if the distance from v_0 to v_s is s. A digraph (or graph) is said to be (G, s)-*geodesic-transitive* if its automorphism subgroup G is transitive on the set of i-geodesics for each $i \leqslant s$. The simplest of s-geodesic-transitive digraphs are directed cycles.

By definition, each s-geodesic of Γ is an s-arc, but the converse is not true, for instance a 2-arc (u, v, w) of Γ satisfying $u \to w$ is not a 2-geodesic. Thus the family of s-arc-transitive digraphs is contained in the family of s-geodesic-transitive digraphs.

The possible local structures of s-geodesic-transitive undirected graphs for $s \geqslant 2$ were characterized by Devillers, Li, Praeger and the author [5], it was proved that for a vertex u, either $[\Gamma(u)] \cong m\mathrm{K}_r$ for some integers $m \geqslant 2, r \geqslant 1$, or $[\Gamma(u)]$ is a connected graph of diameter 2. Also the families of 2-geodesic-transitive graphs of valency 4 and of prime valency have been determined in [6] and [7], respectively. The 2-geodesic-transitive undirected graphs have also been extensively studied in the literature, see for example, [10, 14, 16].

In this paper, we initiate the study of finite (G, s)-geodesic-transitive digraphs for

$s \geqslant 2$. Our first theorem gives a global analysis for such digraphs, and it provides a reduction result.

Theorem 1. *Let Γ be a connected (G, s)-geodesic-transitive digraph for some $s \geqslant 2$, and let N be a normal subgroup of G maximal with respect to having at least 3 orbits. Then Γ_N is connected $(G/N, s')$-geodesic-transitive where $s' = \min\{s, \mathrm{diam}(\Gamma_N)\}$, and G/N is either quasiprimitive or bi-quasiprimitive on $V(\Gamma_N)$. Moreover, Γ_N is either a digraph or an undirected complete graph.*

Theorem 1 directly leads to the following corollary which shows that for $(G, 2)$-geodesic-transitive that is not $(G, 2)$-arc-transitive digraphs, if G has a normal subgroup N maximal with respect to having at least 2 orbits, then G/N is not bi-quasiprimitive on $V(\Gamma_N)$.

Corollary 2. *Let Γ be a connected $(G, 2)$-geodesic-transitive that is not $(G, 2)$-arc-transitive digraph. Let N be a normal subgroup of G maximal with respect to having at least 2 orbits. Then G/N is quasiprimitive on $V(\Gamma_N)$, and Γ_N is either a connected $(G/N, 2)$-geodesic-transitive digraph or a connected G/N-arc-transitive undirected complete graph.*

By Theorem 1, for $s \geqslant 2$ each connected (G, s)-geodesic-transitive digraph has a connected $(G/N, s')$-geodesic-transitive quotient digraph corresponding to a normal subgroup N of G such that G/N acts quasiprimitively or bi-quasiprimitively on the vertex set of the quotient digraph. Thus a preliminary step in determining all s-geodesic-transitive digraphs may be the determination of the base ones. We achieved this in Proposition 10, in the case where G is soluble. It is shown that Γ is a circuit with r vertices where $r = 4$ or r is a prime. Our next theorem is another contribution to the determination of connected (G, s)-geodesic-transitive vertex quasiprimitive digraphs.

For integer $s \geqslant 2$ and a (G, s)-geodesic-transitive digraph Γ, the second theorem shows that if G has a nontrivial regular normal subgroup, then Γ is known.

Theorem 3. *Let Γ be a (G, s)-geodesic-transitive digraph for some $s \geqslant 2$, and let N be a nontrivial normal subgroup of G. Suppose that N is regular on $V(\Gamma)$. Then Γ is a circuit. In particular, each (G, s)-geodesic-transitive G-normal Cayley digraph with $s \geqslant 2$, is a circuit.*

We give a remark of Theorem 3.

Remark 4. Let Γ be a connected $(G, 2)$-geodesic-transitive but not $(G, 2)$-arc-transitive digraph. If G is quasiprimitive on $V(\Gamma_N)$ of type HA, HS, HC or TW, then G has a normal subgroup that acts regularly on the vertex set, and by Theorem 3, Γ is a circuit.

Our third theorem investigates $(G, 2)$-geodesic-transitive digraphs with either valency at most 5 or diameter at most 2.

Theorem 5. *Let Γ be a G-arc-transitive digraph. Then the following statements hold.*

(i) If Γ has valency at most 5, then Γ is $(G,2)$-geodesic-transitive if and only if Γ is $(G,2)$-arc-transitive.

(ii) If Γ is $(G,2)$-geodesic-transitive of diameter 2, then Γ is a balanced incomplete block design with the Hadamard parameters.

2 Preliminaries

In this section, we will give some definitions about groups and digraphs that will be used in the paper. For the group theoretic terminology not defined here we refer the reader to [2, 8, 30].

All digraphs in this paper are finite and simple. For a digraph Γ, we use $V(\Gamma)$ and $Arc(\Gamma)$ to denote its vertex set and arc set, respectively.

An *automorphism* of a digraph Γ is a permutation π of $V(\Gamma)$ which has the property that $u \to v$ if and only if $u^\pi \to v^\pi$. The set of all automorphisms of Γ, with the operation of composition forms a group which is called the *automorphism group* of Γ, and denoted by $\mathrm{Aut}(\Gamma)$.

A digraph Γ is called *G-vertex-transitive* or *G-arc-transitive* if there is some $G \leqslant \mathrm{Aut}(\Gamma)$ such that G acts transitively on its vertex set or arc set, respectively. It is obvious that each G-arc-transitive digraph is G-vertex-transitive and each G-vertex-transitive digraph is regular.

The number of arcs traversed in the shortest directed path from u to v is called the *distance* in Γ from u to v, and is denoted by $d_\Gamma(u,v)$. The maximum value of the distance function in Γ is called the *diameter* of Γ, and denoted by $\mathrm{diam}(\Gamma)$. Define $\Gamma_i^+(u) = \{v \in V(\Gamma) | d_\Gamma(u,v) = i\}$ for $i \geqslant 1$. In particular $\Gamma_1^+(u) = \Gamma^+(u)$.

A subdigraph X of a digraph Γ is an *induced subdigraph* if (u,v) is an arc of X if and only if (u,v) is an arc in Γ. When $U \subseteq V(\Gamma)$, we denote by $[U]$ the subdigraph of Γ induced by U. Let Σ be a digraph. For a positive integer m, the digraph consisting of m vertex disjoint copies of Σ is denoted by $m\Sigma$.

A connected digraph Γ is called *strongly connected* if, for all $u, v \in V(\Gamma)$, there is a t-arc $(u = u_0, u_1, \ldots, u_t = v)$ for some positive integer t. Let Γ be a finite regular digraph. By [21, Lemma 2], if the underlying undirected graph of Γ (with $\{u,v\}$ an edge if either (u,v) or (v,u) is an arc of Γ) is connected, then Γ is strongly connected.

The *girth* or *directed girth* of a digraph is the minimum length of a closed path with at least three vertices.

For integers $r \geqslant 3$, an r-arc $(w_0, w_1, \ldots, w_r)$ with distinct vertices is called a *circuit* of length r if $w_r = w_0$. A shortest circuit is called a *minimal circuit.* Thus the girth of Γ is the length of a minimal circuit.

A transitive permutation group $G \leqslant \mathrm{Sym}(\Omega)$ is said to be *regular* on Ω, if for any $\omega \in \Omega$, the stabilizer $G_\omega = 1$.

For a non-empty subset S of a group H the *Cayley digraph* $\Gamma = \mathrm{Cay}(H,S)$ is defined to be the digraph with vertex set H and with arc set $Arc(\Gamma) = \{(h, xh) | h \in H, x \in S\}$. The arc set $Arc(\Gamma)$ is anti-symmetric provided that $S \cap S^{-1}$ is empty, where $S^{-1} = \{x^{-1} | x \in S\}$.

In addition, Γ is regular of valency $|S|$, and Γ is connected whenever S is a generating set for H. Also, Γ admits as an automorphism group the semidirect product $H : \mathrm{Aut}(H, S)$, where H acts by right translation and the set stabilizer $\mathrm{Aut}(H, S) = \{\alpha \in \mathrm{Aut}(H) | S^\alpha = S\}$ of S in the automorphism group of H acts by conjugation; in particular, $\mathrm{Aut}(H, S)$ acts transitively on S by conjugation if and only if $\mathrm{Cay}(H, S)$ is $(H : \mathrm{Aut}(H, S), 1)$-arc transitive. Moreover, $\mathrm{Cay}(H, S)$ is said to be *G-normal* if H is a normal subgroup of G.

Let Γ be a connected (G, s)-geodesic-transitive digraph where $s \geqslant 1$, and let N be a normal subgroup of G with more than two orbits in $V(\Gamma)$. Then the *quotient digraph* Γ_N is defined as the digraph with vertices the N-orbits in $V(\Gamma)$ and with (A, B) an arc, where A and B are N-orbits, if and only if, for some $a \in A$ and $b \in B$, (a, b) is an arc of Γ.

A transitive permutation group $G \leqslant \mathrm{Sym}(\Omega)$ is said to be *quasiprimitive*, if every non-trivial normal subgroup of G is transitive on Ω, while G is said to be *bi-quasiprimitive* if every non-trivial normal subgroup of G has at most two orbits on Ω and there exists one which has exactly two orbits on Ω. Quasiprimitivity is a generalization of primitivity as every normal subgroup of a primitive group is transitive, but there exist quasiprimitive groups which are not primitive. Praeger [23] generalized the O'Nan-Scott Theorem for primitive groups to quasiprimitive groups and showed that a finite quasiprimitive group is one of eight distinct types: Holomorph Affine (HA), Almost Simple (AS), Twisted Wreath product (TW), Product Action (PA), Simple Diagonal (SD), Holomorph Simple (HS), Holomorph Compound (HC) and Compound Diagonal (CD). For more information about quasiprimitive and bi-quasiprimitive permutation groups, refer to [24].

Lemma 6. *Let Γ be a connected G-arc-transitive digraph of valency $k \geqslant 2$. Let (u, v) be an arc of Γ. Then the following statements hold.*

(1) $\Gamma^+(u) \neq \{v\} \cup (\Gamma^+(u) \cap \Gamma^+(v))$.

(2) $\Gamma^+(u) \cap \Gamma^+(v) = \emptyset$ *if and only if each* 2*-arc of* Γ *is a* 2*-geodesic.*

Proof. (1) Suppose that $\Gamma^+(u) = \{v\} \cup (\Gamma^+(u) \cap \Gamma^+(v))$. Then since Γ is G-arc-transitive, the vertex stabilizer G_u acts transitively on $\Gamma^+(u)$, and so $[\Gamma^+(u)]$ is a digraph with out-valency $k - 1$. Therefore, for any two vertices $x, y \in \Gamma^+(u)$, we have two arcs (x, y) and (y, x). Again using the G-arc-transitive property of Γ, we know that Γ must be an undirected graph, which contradicts that Γ is a digraph. Thus $\Gamma^+(u) \neq \{v\} \cup (\Gamma^+(u) \cap \Gamma^+(v))$.

(2) It is obvious. □

3 Reduction

Let Γ be a connected (G, s)-geodesic-transitive digraph for some $s \geqslant 2$. In this section we study the nature of intransitive normal subgroups N of G. The first lemma shows that each orbit of N is arc-less.

Lemma 7. *Let Γ be a connected (G,s)-geodesic-transitive digraph for some $s \geqslant 1$, and let N be a nontrivial intransitive normal subgroup of G. Then there is no N-orbit contains any arc of Γ.*

Proof. Suppose that there exists one N-orbit B_0 that contains an arc (u,v) of Γ. Then by [8, Theorem 1.6A], B_0 is a block of the G-action on $V(\Gamma)$. The subgroup N is not transitive on $V(\Gamma)$ leading to that it has at least two orbits. Since Γ is connected and N is transitive on B_0, it follows that $\Gamma^+(u)$ intersects nontrivially with some other N-orbit, say B_1, and set $v' \in \Gamma^+(u) \cap B_1$. Then (u,v') is an arc. Since Γ is a (G,s)-geodesic-transitive digraph for some $s \geqslant 1$, G_u has an element that can maps v to v', which is impossible, as B_0 is a block of G and G_u fixes B_0 setwise. Thus there is no N-orbit contains any arc of Γ. □

Lemma 8. *Let Γ be a (G,s)-geodesic-transitive digraph for some $s \geqslant 2$, and let N be a nontrivial normal subgroup of G. If N has 2 orbits on $V(\Gamma)$, then Γ is $(G,2)$-arc-transitive and bipartite.*

Proof. Suppose that N has 2 orbits on $V(\Gamma)$, say Δ_1 and Δ_2. Then by Lemma 7, neither Δ_1 nor Δ_2 contains any arc of Γ, and so Γ is a bipartite digraph.

Let (u,v) be an arc of Γ where $u \in \Delta_1$ and $v \in \Delta_2$. Then since Γ is a bipartite digraph, it has no cycles of length 3 in the underlying undirected graph, and so $\Gamma^+(u) \cap \Gamma^+(v) = \emptyset$. Hence by Lemma 6 each 2-arc of Γ is a 2-geodesic. We conclude that Γ is $(G,2)$-arc-transitive. □

Let Γ_N be the quotient digraph with vertices the N-orbits in $V(\Gamma)$ and with (A,B) an arc, where A and B are N-orbits, if and only if, for some $a \in A$ and $b \in B$, (a,b) is an arc of Γ.

Now we show that each normal subgroup N of G with at least 3 orbits on vertices corresponds to a connected $(G/N,s')$-geodesic-transitive quotient digraph Γ_N.

Lemma 9. *Let Γ be a connected (G,s)-geodesic-transitive digraph for some $s \geqslant 2$, and let N be a normal subgroup of G with at least 3 orbits. Then Γ_N is either directed or an undirected complete graph.*

Proof. Suppose that Γ_N is not directed. Then Γ_N has two arcs (B_1,B_2) and (B_2,B_1) where B_1,B_2 are N-orbits. By the definition of Γ_N, there exist $u,u' \in B_1$ and $v,v' \in B_2$ such that (v,u) and (u',v') are two arcs of Γ. Since N is transitive on each orbit, we can assume that $u = u'$. Then (v,u,v') is a 2-arc of Γ. By Lemma 7, neither B_1 nor B_2 contains an arc of Γ. Thus (v,v') and (v',v) are not arcs and hence (v,u,v') is a 2-geodesic of Γ.

Since Γ is $(G,2)$-geodesic-transitive, it follows that $G_{v,u}$ is transitive on $\Gamma_2^+(v) \cap \Gamma^+(u)$. The stabilizer $G_{v,u}$ setwise fixes B_1 and B_2, and so $\Gamma_2^+(v) \cap \Gamma^+(u) \subseteq B_2$.

Let (B_1,B_3) be an arc of Γ_N where $B_3(\neq B_2)$ is an N-orbit. Then (B_2,B_1,B_3) is a 2-arc of Γ_N, and there exist $u'' \in B_1$ and $w \in B_3$ such that (u'',w) is an arc of Γ. The group N is transitive on each orbit leading to that we can assume $u = u''$. Hence (v,u,w) is a 2-arc. Due to $\Gamma_2^+(v) \cap \Gamma^+(u) \subseteq B_2$ and $B_3 \neq B_2$, we know that (v,w) must be an arc of

Γ, and so (B_2, B_3) is an arc of Γ_N. As a consequence $\Gamma_N^+(B_1) = \{B_2\} \cup (\Gamma_N^+(B_1) \cap \Gamma_N^+(B_2))$ and $\Gamma_N^+(B_2) = \{B_1\} \cup (\Gamma_N^+(B_1) \cap \Gamma_N^+(B_2))$, and so $(\Gamma_N)_2^+(B_1) \cap \Gamma_N^+(B_2) = \emptyset$. Since Γ_N is G/N-arc-transitive, it follows that Γ_N has diameter 1 and it is an undirected complete graph. □

We are ready to prove Theorem 1.

Proof of Theorem 1. Since Γ is a connected digraph, it is easy to see that the quotient digraph Γ_N is also connected. Since N is a normal subgroup of G maximal with respect to having at least 3 orbits, it follows that all normal subgroups of G/N are transitive or have two orbits on $V(\Gamma_N)$. Thus G/N is quasiprimitive or bi-quasiprimitive on $V(\Gamma_N)$. Moreover, by Lemma 9, Γ_N is either directed or an undirected complete graph.

Let $(B_0, B_1, B_2, \ldots, B_t)$ and $(C_0, C_1, C_2, \ldots, C_t)$ be two t-geodesics of Γ_N where $t \leqslant s' = \min\{s, \mathrm{diam}(\Gamma_N)\}$.

Then by the definition of Γ_N, there exist $x_i \in B_i$ and $x'_{i+1} \in B_{i+1}$ such that (x_i, x'_{i+1}) is an arc of Γ. Since N is transitive on each B_i, we have $x_j \in B_j$ such that $(x_0, x_1, x_2, \ldots, x_t)$ is a t-geodesic of Γ. Similarly, there exist $y_i \in C_i$ such that $(y_0, y_1, y_2, \ldots, y_t)$ is a t-geodesic of Γ. As $t \leqslant s' \leqslant s$ and Γ is (G, s)-geodesic-transitive, the group G has an element g such that $(x_0, x_1, x_2, \ldots, x_t)^g = (y_0, y_1, y_2, \ldots, y_t)$, and g induces the element gN of G/N, such that $(B_0, B_1, B_2, \ldots, B_t)^{gN} = (C_0, C_1, C_2, \ldots, C_t)$. Thus Γ_N is $(G/N, s')$-geodesic-transitive. We conclude the proof. □

Theorem 1 directly leads to Corollary 2 which is a reduction result on $(G, 2)$-geodesic-transitive but not $(G, 2)$-arc-transitive digraphs.

Proof of Corollary 2. If N has exactly 2 orbits on $V(\Gamma)$, then by Lemma 8, Γ is $(G, 2)$-geodesic-transitive implying that Γ is $(G, 2)$-arc-transitive, contradicting our assumption. Thus N has at least 3 orbits on $V(\Gamma)$. It follows from Theorem 1 that G/N is quasiprimitive or bi-quasiprimitive on $V(\Gamma_N)$, and Γ_N is connected $(G/N, s')$-geodesic-transitive where $s' = \min\{2, \mathrm{diam}(\Gamma_N)\}$. Moreover, as each arc-transitive digraph has diameter at least 2, we know that Γ_N is either a connected $(G/N, 2)$-geodesic-transitive digraph or a connected G/N-arc-transitive undirected complete graph

Assume that G/N acts bi-quasiprimitively on $V(\Gamma_N)$. Then G/N has a nontrivial normal subgroup which has exactly 2 orbits on $V(\Gamma_N)$, say Δ'_1 and Δ'_2. Applying Lemma 7, there are no arcs of Γ_N in Δ'_1 or Δ'_2 and Γ_N is a bipartite digraph. It follows that neither Δ'_1 nor Δ'_2 contains an arc of Γ, and so Γ is a bipartite digraph. We conclude that each 2-arc of Γ is a 2-geodesic. Therefore, Γ is $(G, 2)$-arc-transitive, a contradiction. Hence G/N is not bi-quasiprimitive on $V(\Gamma_N)$. □

Now we prove Theorem 3 to show that: for an integer $s \geqslant 2$ and a (G, s)-geodesic-transitive digraph Γ, if G has a nontrivial regular normal subgroup, then Γ is a circuit.

Proof of Theorem 3. Let (u, v) be an arc. Then $\Gamma^+(u) \neq \Gamma^+(v)$. Suppose that $\Gamma^+(u) \cap \Gamma^+(v) = \emptyset$. Then each 2-arc of Γ is a 2-geodesic. Since Γ is (G, s)-geodesic-transitive for some $s \geqslant 2$, it follows that Γ is $(G, 2)$-arc-transitive, and by [22, Theorem 3.1], Γ is a directed cycle.

In the remainder, we consider the case that $\Gamma^+(u) \cap \Gamma^+(v) \neq \emptyset$. Assume that Γ is not a directed cycle. Then the valency m of Γ is at least 2.

Since N is regular on $V(\Gamma)$, we can identify $V(\Gamma)$ with N so that $\Gamma = \mathrm{Cay}(N, S)$ where S is a subset of $N \setminus \{1_N\}$ and $|S| = m \geqslant 2$. Moreover, N acts by right multiplication and, for $x \in N$ and $i \geqslant 1_N$, we denote by $\Gamma_i^+(x)$ the set of vertices at distance i from x.

Since N is a normal subgroup of G, it follows that $G_{1_N} = \mathrm{Aut}(N, S)$, and by the (G, s)-geodesic-transitivity with $s \geqslant 2$, $\mathrm{Aut}(N, S)$ is transitive on both $S = \Gamma^+(1_N)$ and $\Gamma_2^+(1_N)$. Thus all elements of S have the same order, and all elements of $\Gamma_2^+(1_N)$ have the same order.

Let $x \in S$. Assume that $\langle x\rangle \setminus \{1_N\} \nsubseteq S$. Let i be the smallest positive integer such that $x^i \notin S$. Note that $x^i \neq 1_N$ and due to $\langle x\rangle \setminus \{1_N\} \nsubseteq S$, we know that $x^i = x \times x^{i-1} \in \Gamma_2^+(1_N) \cap \Gamma^+(x)$. As Γ is $(G, 2)$-geodesic-transitive, $G_{1_N,x}$ acts transitively on $\Gamma_2^+(1_N) \cap \Gamma^+(x)$. Since $G_{1_N,x} \leqslant G_{1_N} \leqslant \mathrm{Aut}(N, S)$ and $G_{1_N,x}$ fixes x, it follows that $G_{1_N,x}$ fixes x^i. Hence $\Gamma_2^+(1_N) \cap \Gamma^+(x) = \{x^i\}$, and so $\Gamma^+(x) = (\Gamma^+(x) \cap S) \cup \{x^i\}$. It indicates that $\Gamma^+(1_N) = (\Gamma^+(x) \cap S) \cup \{x\}$, contradicting Lemma 6 (1). Thus $\langle x\rangle \setminus \{1_N\} \subseteq S$, and so $x, x^{-1} \in S$. This fact forces that both $(1, x)$ and $(x, 1)$ are arcs of Γ, which is a contradiction.

Therefore, Γ has valency 1 and it is a directed cycle.

Finally, if $\Gamma = \mathrm{Cay}(T, S)$ is a (G, s)-geodesic-transitive normal Cayley digraph for some $s \geqslant 2$, then T is a nontrivial normal subgroup of G and it acts regularly on $V(\Gamma)$, by the previous argument, Γ is a directed cycle. The proof is completed. □

Let Γ be a connected $(G, 2)$-geodesic-transitive but not $(G, 2)$-arc-transitive digraph. If G is quasiprimitive on $V(\Gamma_N)$ of type HA, HS, HC or TW, then G has a normal subgraph that acts regularly on the vertex set, and by Theorem 3, Γ is a directed cycle.

By Theorem 1, for $s \geqslant 2$ each connected (G, s)-geodesic-transitive digraph has a connected $(G/N, s')$-geodesic-transitive quotient digraph corresponding to a normal subgroup N of G such that G/N acts quasiprimitively or bi-quasiprimitively on the vertex set of the quotient digraph. Thus a preliminary step in determining all s-geodesic-transitive digraphs may be the determination of the base one. We shall achieve this in the following proposition in the case where G is soluble.

Proposition 10. *Let Γ be a connected $(G, 2)$-geodesic-transitive digraph such that G is soluble. Suppose that G acts quasiprimitively or bi-quasiprimitively on $V(\Gamma)$. Then Γ is a circuit. Moreover, if G acts primitively on $V(\Gamma)$, then Γ is a circuit with prime number of vertices.*

Proof. Since G is soluble, it follows that G has a non-trivial abelian characteristic subgroup, and say N. As G acts quasiprimitively or bi-quasiprimitively on $V(\Gamma)$, N has at most two orbits in the vertex set.

If N has one orbit in $V(\Gamma)$, then as it is an abelian group, N acts regularly on $V(\Gamma)$, and by Theorem 3, Γ is a circuit.

Assume now that N has exactly two orbits in $V(\Gamma)$. Applying Lemma 8, each 2-arc of Γ is a 2-geodesic. Thus the $(G, 2)$-geodesic-transitive property of Γ indicates that Γ is a $(G,2)$-arc-transitive digraph. Since N is semiregular on $V(\Gamma)$, it follows from Theorem 3.3 of [22] that Γ is circuit.

Moreover, if G acts primitively in $V(\Gamma)$, then by the previous argument, Γ is a circuit. Since G has only trivial blocks in $V(\Gamma)$, it follows that $|V(\Gamma)|$ must be a prime. This concludes the proof. □

4 Two-geodesic-transitive digraphs of small valency

In this section, we investigate the relationship between $(G,2)$-geodesic-transitive property and $(G,2)$-arc-transitive property in small valency digraphs.

Lemma 6 leads directly to the following result about valency 2 digraphs.

Lemma 11. *Let Γ be a digraph of valency 2. If Γ is $(G,2)$-geodesic-transitive, then Γ is $(G,2)$-arc-transitive.*

Proof. Suppose that Γ is $(G,2)$-geodesic-transitive. Let (u,v) be an arc. Then since Γ has valency 2 and applying Lemma 6 (1), we have $|\Gamma^+(u)\cap\Gamma^+(v)|\neq 1$, it follows that $\Gamma^+(u)\cap\Gamma^+(v)=\emptyset$. Thus by Lemma 6 (2), each 2-arc of Γ is a 2-geodesic, and so Γ is $(G,2)$-arc-transitive. □

Lemma 12. *Let Γ be a G-arc-transitive digraph of valency r. Then $|\Gamma^+(u)\cap\Gamma^+(v)|\leqslant\lfloor\frac{r}{2}\rfloor$ for each arc (u,v).*

Proof. Suppose to the contrary that $|\Gamma^+(u)\cap\Gamma^+(v)|=t>\lfloor\frac{r}{2}\rfloor$ for some arc (u,v). Then since Γ is G-arc-transitive, it follows that $[\Gamma^+(u)]$ is a vertex-transitive digraph of valency t.

Let $\Delta_1=\Gamma^+(u)\cap\Gamma^+(v)$ and $\Delta_2=\Gamma^+(u)\setminus(\Gamma^+(v)\cup\{v\})$. Then $\Delta_1\cap\Delta_2=\emptyset$ and $\Gamma^+(u)=\{v\}\cup\Delta_1\cup\Delta_2$. Moreover, $r=|\Gamma^+(u)|=1+|\Delta_1|+|\Delta_2|$. Since $|\Delta_1|=t>\lfloor\frac{r}{2}\rfloor$, it follows that $|\Delta_2|=r-1-t<r-1-\lfloor\frac{r}{2}\rfloor<t$.

Note that $|\Gamma^-(v)\cap\Gamma^+(u)|=t>\lfloor\frac{r}{2}\rfloor$. Hence we can find at least one vertex v' of Δ_1 such that $v'\in\Gamma^-(v)\cap\Gamma^+(u)$, and so both (v,v') and (v',v) are arcs, this forces that $[\Gamma^+(u)]$ is not a digraph, which is a contradiction. □

Lemma 13. *Let Γ be a $(G,2)$-geodesic-transitive digraph of valency $r\geqslant 3$. Let (u,v) be an arc. Then $|\Gamma^+(u)\cap\Gamma^+(v)|\neq 1$.*

Proof. Suppose to the contrary that $|\Gamma^+(u)\cap\Gamma^+(v)|=1$. Then as Γ is G-arc-transitive, $[\Gamma^+(u)]$ is vertex-transitive of valency 1, and is therefore isomorphic to $k\Sigma$ where $k\geqslant 1$, and Σ is a connected circuit with $|V(\Sigma)|\geqslant 3$.

We assume that $r=ke$ where $e=|V(\Sigma)|\geqslant 3$. Then $[\Gamma^+(u)]$ is the union of k disjoint Σ. Moreover, the G-vertex-transitive property of Γ indicates that for each vertex x, $[\Gamma^+(x)]$ is the union of k disjoint Σ.

Set $\Gamma^+(u)=\{v=v_{11},v_{12},\dots,v_{1e},v_{21},v_{22},\dots,v_{2e},\dots,v_{k1},v_{k2},\dots,v_{ke}\}$ and assume $v_{11}\to v_{12}\to\cdots\to v_{1e}\to v_{11}$. Then $[v_{11},v_{12},\dots,v_{1e}]\cong\Sigma$ is a connected circuit. Since $[\Gamma^+(u)]\cong k\Sigma$, it follows that $G_{u,v_{11}}=G_{u,v_{12}}=G_{u,v_{13}}=\cdots=G_{u,v_{1e}}$.

As $|\Gamma^+(u)\cap\Gamma^+(v_{11})|=1$, we have $|\Gamma_2^+(u)\cap\Gamma^+(v_{11})|=r-1=ke-1=(k-1)e+(e-1)$. Set $\Gamma_2^+(u)\cap\Gamma^+(v_{11})=\{w_{11},w_{12},\ldots,w_{1(e-1)},w_{21},w_{22},\ldots,w_{2e},\ldots,w_{k1},w_{k2},\ldots,w_{ke}\}$. Then $\Gamma^+(v_{11})=\{v_{12},w_{11},w_{12},\ldots,w_{1(e-1)},w_{21},w_{22},\ldots,w_{2e},\ldots,w_{k1},w_{k2},\ldots,w_{ke}\}$. Note that $[\Gamma^+(v_{11})]\cong k\Sigma$. Without loss of generality, we can assume that $v_{12}\to w_{11}\to w_{12}\to w_{13}\to\ldots\to w_{1(e-1)}\to v_{12}$. Since Γ is $(G,2)$-geodesic-transitive, it follows that $G_{u,v_{11}}$ is transitive on $\Gamma_2^+(u)\cap\Gamma^+(v_{11})$. Further, applying the facts that $G_{u,v_{11}}=G_{u,v_{12}}$ and $w_{1(e-1)}\to v_{12}$, we would have $w_{11}\to v_{12}$, contradicts that $v_{12}\to w_{11}$.

Thus $|\Gamma^+(u)\cap\Gamma^+(v)|\neq 1$. □

For G-arc-transitive digraphs of valency 3, we have the following lemma.

Lemma 14. *Let Γ be a $(G,2)$-geodesic-transitive digraph of valency 3. Then Γ is $(G,2)$-arc-transitive.*

Proof. Let (u,v) be an arc of Γ. Since Γ is $(G,2)$-geodesic-transitive of valency 3, it follows that $|\Gamma^+(u)\cap\Gamma^+(v)|=0,1,2$. Furthermore, by Lemmas 12 and 13, we have $|\Gamma^+(u)\cap\Gamma^+(v)|\neq 1,2$.

Thus $|\Gamma^+(u)\cap\Gamma^+(v)|=0$, and so $\Gamma^+(u)\cap\Gamma^+(v)=\emptyset$. Therefore each 2-arc of Γ is a 2-geodesic, and so Γ is $(G,2)$-arc-transitive. □

Lemma 15. *Let Γ be a $(G,2)$-geodesic-transitive digraph of valency 4. Then Γ is $(G,2)$-arc-transitive.*

Proof. Suppose that Γ is $(G,2)$-geodesic-transitive. For each arc (u,v), since Γ has valency 4, it follows that $|\Gamma^+(u)\cap\Gamma^+(v)|\leqslant 3$, that is, $|\Gamma^+(u)\cap\Gamma^+(v)|=0,1,2,3$. Moreover, by Lemmas 12 and 13, $|\Gamma^+(u)\cap\Gamma^+(v)|\neq 1,3$.

Suppose that $|\Gamma^+(u)\cap\Gamma^+(v)|=2$. Set $\Gamma^+(u)=\{v=v_1,v_2,v_3,v_4\}$. We can assume $v_1\to v_2$ and $v_1\to v_3$. And so $v_2\nrightarrow v_1$ and $v_3\nrightarrow v_1$. Since Γ is G-arc-transitive, it follows that G_u acts transitively on $\Gamma^+(u)$, and so $|\Gamma^+(u)\cap\Gamma^+(v_2)|=2$ and $|\Gamma^+(u)\cap\Gamma^+(v_3)|=2$. Hence we must have $v_2\to v_3$, $v_2\to v_4$, and $v_3\to v_2$, $v_3\to v_4$, which is a contradiction. Thus $|\Gamma^+(u)\cap\Gamma^+(v)|\neq 2$.

Therefore we must have $\Gamma^+(u)\cap\Gamma^+(v)=0$. It implies that each 2-arc is a 2-geodesic, and so Γ is $(G,2)$-arc-transitive. □

Lemma 16. *Let Γ be a $(G,2)$-geodesic-transitive digraph of valency 5. Then $|\Gamma^+(u)\cap\Gamma^+(v)|\neq 2$ for any arc (u,v).*

Proof. Let (u,v) be an arc. Suppose that $|\Gamma^+(u)\cap\Gamma^+(v)|=2$. Since Γ is G-arc-transitive, G_u is transitive on $\Gamma^+(u)$, and so $[\Gamma^+(u)]$ is a vertex-transitive digraph with valency 2 and 5 vertices. Moreover, for each vertex $x\in\Gamma^+(u)$, the set $\Gamma^+(u)\cap\Gamma^+(x)$ contains precisely 2 vertices.

Set $\Gamma^+(u)=\{v=v_1,v_2,v_3,v_4,v_5\}$ and assume

$$v_1\to v_2,\ v_1\to v_3.$$

Suppose that $v_2 \nrightarrow v_3$ and $v_3 \nrightarrow v_2$. Then $v_2 \to v_4$, $v_2 \to v_5$ and $v_3 \to v_4$, $v_3 \to v_5$. As a consequence, $v_4 \nrightarrow v_2$, $v_4 \nrightarrow v_3$ and $v_5 \nrightarrow v_2$, $v_5 \nrightarrow v_3$. Hence $v_4 \to v_1$ and $v_5 \to v_1$. Since $|\Gamma^+(u) \cap \Gamma^+(v_4)| = 2$ and $v_4 \nrightarrow v_2$, $v_4 \nrightarrow v_3$, it follows that $v_4 \to v_5$, and so $v_5 \nrightarrow v_4$. By the previous, $v_5 \nrightarrow v_2$ and $v_5 \nrightarrow v_3$, we must have $\Gamma^+(u) \cap \Gamma^+(v_5) = \{v_1\}$, which contradicts that $|\Gamma^+(u) \cap \Gamma^+(v_5)| = 2$.

Thus either $v_2 \to v_3$ or $v_3 \to v_2$. Without loss of generality, assume that

$$v_2 \to v_3.$$

Then $v_3 \nrightarrow v_2$, and as a result $v_3 \to v_4$ and $v_3 \to v_5$. Furthermore, $v_2 \to$ one of v_4, v_5. Assume that $v_2 \to v_4$. Then $v_4 \nrightarrow v_2$ and $v_4 \nrightarrow v_3$. Hence $v_4 \to v_1$ and $v_4 \to v_5$.

Since $\Gamma^+(u) \cap \Gamma^+(v_1) = \{v_2, v_3\}$ and since $v_2 \to v_3$, $v_3 \nrightarrow v_2$, it follows that G_{u,v_1} also fixes v_2 and v_3 pointwise, and so $G_{u,v_1} = G_{u,v_2} = G_{u,v_3}$. Due to $v_4 \to v_5$, and by a similar argument, we would have $G_{u,v_1} = G_{u,v_4} = G_{u,v_5}$. Thus

$$G_{u,v_1} = G_{u,v_2} = G_{u,v_3} = G_{u,v_4} = G_{u,v_5}. \qquad (*)$$

It leads to the fact $G_{u,v_1}^{\Gamma^+(u)} = 1$, that is, $G_u^{\Gamma^+(u)}$ acts regularly on $\Gamma^+(u)$. Let K be the kernel of the G_u-action on $\Gamma^+(u)$. Then $G_u^{\Gamma^+(u)} \cong G_u/K$.

Since Γ is $(G,2)$-geodesic-transitive, it follows that G_{u,v_1} is transitive on $\Gamma_2^+(u) \cap \Gamma^+(v_1)$. Due to $G_u^{\Gamma^+(u)}$ acts regularly on $\Gamma^+(u)$ and $G_u^{\Gamma^+(u)} \cong G_u/K$, the group K is transitive on $\Gamma_2^+(u) \cap \Gamma^+(v_1)$. However, each element $k \in K$ fixes vertices v_1, v_2 and v_3, and so $k \in G_{v_1,v_2,v_3}$. For the reason that $v_2, v_3 \in \Gamma^+(v_1)$, applying the G-arc-transitive property of Γ and $(*)$, we know that k acts trivially on $\Gamma^+(v_1)$, and as a consequence k acts trivially on $\Gamma_2^+(u) \cap \Gamma^+(v_1)$, which is a contradiction. Therefore Γ is not $(G,2)$-geodesic-transitive. □

Lemma 17. *Every $(G,2)$-geodesic-transitive digraph of valency 5 is also $(G,2)$-arc-transitive.*

Proof. Let Γ be a $(G,2)$-geodesic-transitive digraph of valency 5 and let (u,v) be an arc. Then $|\Gamma^+(u) \cap \Gamma^+(v)| \leqslant 4$, that is, $|\Gamma^+(u) \cap \Gamma^+(v)| = 0, 1, 2, 3, 4$. Moreover, by Lemmas 12, 13 and 16, $|\Gamma^+(u) \cap \Gamma^+(v)| \neq 1, 2, 3, 4$.

Thus we must have $\Gamma^+(u) \cap \Gamma^+(v) = 0$. It leads to that each 2-arc is a 2-geodesic, and so Γ is $(G,2)$-arc-transitive. □

Now we prove Theorem 5.

Proof of Theorem 5. Let Γ be a G-arc-transitive digraph of valency $r \geqslant 2$. Since each 2-geodesic of Γ is a 2-arc, it follows that if Γ is $(G,2)$-arc-transitive, then it must be $(G,2)$-geodesic-transitive. Assume conversely that Γ is $(G,2)$-geodesic-transitive. If Γ has valency r where $2 \leqslant r \leqslant 5$, then by Lemmas 11, 14, 15, 17, Γ is $(G,2)$-arc-transitive, and (i) holds.

Now let Γ be a $(G,2)$-geodesic-transitive digraph of diameter 2. Then Γ is G-distance-transitive. By [17, Theorem 4.2], Γ is known, and it is a balanced incomplete block design with the Hadamard parameters, and (ii) holds. □

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