$(\vec{P}_6,$ triangle)-free oriented graphs have bounded dichromatic number

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Abstract

The dichromatic number of an oriented graph is the minimum size of a partition of its vertices into acyclic induced subdigraphs. We prove that oriented graphs with no induced directed path on six vertices and no triangle have bounded dichromatic number. This is one (small) step towards the general conjecture asserting that for every oriented tree T and every integer k, any oriented graph that does not contain an induced copy of T nor a clique of size k has dichromatic number at most some function of k and T.

Mathematics Subject Classifications: 05C15, 05C20

1 Introduction

In this paper, we only consider *graphs* or *directed graphs* (*digraphs* in short) with no loops, no parallel edges or arcs, and no anti-parallel arcs. In particular our digraphs contain no cycle of length 2. Note that this class of digraphs is usually referred to as *oriented graphs*, but we will use the term digraphs in this paper for the sake of brevity.

Given an undirected graph G, we denote by $\omega(G)$ the size of a maximum clique of G and by $\chi(G)$ its chromatic number. A class of graphs $\mathcal{C}$ is *χ-bounded* if there exists a function f, such that every graph G in $\mathcal{C}$ satisfies $\chi(G) \leqslant f(\omega(G))$.

Given a graph (resp. a digraph) H, we denote by $Forb_{ind}(H)$ the class of graphs (resp. digraphs) that do not contain H as an induced subgraph (resp. induced subdigraph). A celebrated and still wide open question in the area of graph colouring is the following conjecture of Gyárfás [11] and Sumner [19] (see [17] for a survey on χ-boundedness).

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Conjecture 1 (Gyárfás-Sumner)**.** For any forest F, $Forb_{ind}(F)$ is χ-bounded.

In this paper, we study an analogue of this conjecture for digraphs. For a digraph D we denote by $\omega(D)$ the clique number of the underlying graph of D and by $\vec{\chi}(D)$ its *dichromatic number*, that is the minimum integer k such that the set of vertices of D can be partitioned into k acyclic subdigraphs. A class of digraphs $\mathcal{C}$ is $\vec{\chi}$-bounded if there exists a function f such that every digraph D in $\mathcal{C}$ satisfies $\vec{\chi}(D) \leqslant f(\omega(D))$.

Conjecture 2 (Aboulker, Charbit, Naserasr [4])**.** For any oriented forest $\vec{F}$, $Forb_{ind}(\vec{F})$ is $\vec{\chi}$-bounded.

It is enough to prove it for oriented trees (the proof is the same as for the undirected case, and can be found in [18], Proposition 1.6). An *oriented star* is an oriented tree with at most one non-leaf vertex. Chudnovsky, Scott and Seymour [8] proved it for oriented stars as well as for two of the four possible orientations of the path on 4 vertices: $\rightarrow\leftarrow\leftarrow$ and $\leftarrow\rightarrow\rightarrow$ (they actually prove that for any integer k and any oriented graph $\vec{H}$ where $\vec{H}$ is either an oriented star or $\rightarrow\leftarrow\leftarrow$, or $\leftarrow\rightarrow\rightarrow$, any digraph in $Forb_{ind}(\vec{H})$ with clique number at most k can be partitioned into a bounded number of stable sets, which is clearly stronger). Cook, Masařík, Pilipczuk, Reinald and Souza [9] proved it for the two other orientations of the paths on 4 vertices: $\rightarrow\rightarrow\rightarrow$ and $\rightarrow\leftarrow\rightarrow$. Nothing more is known.

Proving the conjecture for directed paths is already a very challenging case. In this paper, we go a step further in this direction by proving the following, where $\vec{P_6}$ denotes the directed path on 6 vertices.

Theorem 3. *For every $D \in Forb_{ind}(\vec{P_6})$ with $\omega(D) \leqslant 2$, $\vec{\chi}(D) \leqslant 382$.*

Note that we did not try to optimise the bound.

Context and Related Works It has been a central question in graph theory over the past 40 years to understand what substructures are forced by large chromatic number. Or equivalently, which forbidden substructures imply bounded chromatic number. The notion of χ-boundedness deals with this question.

Similarly, the notion of $\vec{\chi}$-boundedness deals with the analogous question for digraphs and dichromatic number: a class $\mathcal{C}$ is $\vec{\chi}$-bounded if for every k there exists a value ϕ_k such that any digraph in $\mathcal{C}$ with dichromatic number larger than ϕ_k must contain some orientation of a clique (or *tournament*) on k vertices. It turns out that the acyclic tournament on k vertices (denoted by $\overrightarrow{TT_k}$) is sufficient to characterise this notion: indeed every tournament on 2^k vertices contains $\overrightarrow{TT_k}$, and therefore a class $\mathcal{C}$ is $\vec{\chi}$-bounded if and only if for every k there exists a value c_k such that any digraph in $\mathcal{C}$ of dichromatic number at least c_k contains a $\overrightarrow{TT_k}$.

More generally, given a class of digraphs $\mathcal{C}$, a digraph H is a *hero in* $\mathcal{C}$ if there is a constant c_H such that digraphs of $\mathcal{C}$ that do not contain H as an induced subdigraph have dichromatic number at most c_H. The discussion above is that a class $\mathcal{C}$ is $\vec{\chi}$-bounded if and only if for every integer k, $\overrightarrow{TT_k}$ is a hero in $\mathcal{C}$, and Conjecture 2 can be rephrased as : *for every oriented forest $\vec{F}$, for every integer k, $\overrightarrow{TT_k}$ is a hero in $Forb_{ind}(\vec{F})$.* Additionally, a result of [13] implies that if H is *not* an oriented forest, then no digraph is a hero in $Forb_{ind}(H)$ except for K_1 (the digraph on one vertex) and $\overrightarrow{TT}_2$.

In a seminal paper, Berger et al. [5] give a simple inductive characterisation of heroes in the class of tournaments (these contain, of course, $\overrightarrow{TT}_k$, but many more). Note that if a class of digraphs $\mathcal{C}$ contains all tournaments, then a hero in $\mathcal{C}$ is, in particular, a hero in tournaments, but a hero in tournaments does not need to be a hero in $\mathcal{C}$. Every class considered in the following contains all tournaments.

Let $\overline{K_k}$ be the digraph on k vertices with no arc and observe that the class of tournaments is the same as $Forb_{ind}(\overline{K_2})$. Harutyunyan et al. [12] extended the above result of Berger at al. [5] by proving that, for every $k \geqslant 3$, heroes in $Forb_{ind}(\overline{K_k})$ are the same as heroes in tournaments.

Following these works, a systematic study of heroes in classes of digraphs of the form $Forb_{ind}(\overrightarrow{F})$ where $\overrightarrow{F}$ is an oriented forest has been initiated in [4]. In particular, it is proved that if $\overrightarrow{F}$ is not a disjoint union of oriented stars, then the only possible heroes in $Forb_{ind}(\overrightarrow{F})$ are the transitive tournaments. On the other hand, it was conjectured in [4] that if $\overrightarrow{S}$ is a disjoint union of oriented stars, then heroes in $Forb_{ind}(\overrightarrow{S})$ are the same as heroes in tournaments, but this turned out to be false. In the paragraph below, we give a quick overview of the results on this particular question.

The result of Chudnovsky et al. [8] mentioned earlier implies that transitive tournaments are heroes in $Forb_{ind}(\overrightarrow{S})$ for any disjoint union of oriented stars $\overrightarrow{S}$. In [2], it is proved that heroes in $Forb_{ind}(\overrightarrow{P_3})$ are the same as heroes in tournaments. Denote by $K_1 + \overrightarrow{TT}_2$ the disjoint union of K_1 and $\overrightarrow{TT}_2$ and observe that $Forb_{ind}(K_1 + \overrightarrow{TT}_2)$ is the class of oriented complete multipartite graphs. Heroes in $Forb_{ind}(K_1 + \overrightarrow{TT}_2)$ have been investigated in [2] where it is proved that they form a strict super class of transitive tournaments, and a strict subclass of heroes in tournaments (which disproved the aforementioned conjecture of [4]). Finally, heroes in $Forb_{ind}(\overrightarrow{K}_{1,2})$ (where $\overrightarrow{K}_{1,2}$ denotes the oriented star on three vertices with a vertex of out-degree 2, digraphs in this class are called *locally-out tournaments*) were studied in [1] and [18] (they are still conjectured to be the same as heroes in tournaments).

A digraph is *t-chordal* if all its induced directed cycles have length t. Surprisingly, for every $t \geqslant 3$, the class of t-chordal digraphs has been proved [6] to not be $\vec{\chi}$-bounded. Note that t-chordal digraphs are defined by forbidding an infinite number of digraphs, contrary to results mentioned above.

An *oriented chordal graph* is an orientation of a chordal graph. This is again a class of digraphs with a distinct flavour, obtained by taking all possible orientations of a class of (undirected) graphs. Heroes in oriented chordal graphs have been fully characterised in [3].

2 Definitions

Let D be a digraph. We denote by $V(D)$ its set of vertices and by $A(D)$ its set of arcs. For $X \subset V(D)$ we define $N^+(X) = \{y \in V(D) \setminus X, \exists x \in X \text{ such that } xy \in A(D)\}$ and

$N^-(X) = \{y \in V(D) \setminus X, \exists x \in X$ such that $yx \in A(D)\}$. A *subdigraph* of D is a digraph obtained from D by removing some arcs and some vertices (with all arcs incident to these vertices). If only vertices are removed, it is an *induced subdigraph.* For a given set of vertices $X \subseteq V(D)$, we denote by $D[X]$ the induced subdigraph obtained by removing $V(D) \setminus X$. Given a set of digraphs $\mathcal{H}$, we say that a digraph D is *$\mathcal{H}$-free* if it contains no induced subdigraph isomorphic to some member of $\mathcal{H}$. We denote by $Forb_{ind}(\mathcal{H})$ the class of $\mathcal{H}$-free digraphs. We say that D is *triangle-free* if $\omega(D) \leqslant 2$. Given a digraph H we say that D does not *contain* (or *has no*) H if D does not contain H as a (not necessarily induced) subdigraph.

We write $x \to y$ when $xy \in A(D)$. A *trail* of a digraph D is a sequence of vertices $x_1x_2 \dots x_p$ such that $x_ix_{i+1} \in A(D)$ for each $i < p$ and each arc is used once (but vertices can be used several times). It is *closed* if $x_1 = x_p$ and its *length* is its number of arcs. We say *odd closed trail* for a closed trail of odd length. A trail (resp. closed trail) in which vertices are pairwise distinct is called a *directed path* (resp. *directed cycle*). The directed path of length $k-1$ is denoted by $\overrightarrow{P}_k$.

A *k-dicolouring* of D is a partition of $V(D)$ into k sets $V_1, \dots, V_k$ such that $D[V_i]$ is acyclic for every $i = 1, \dots, k$. The *dichromatic number* of D, denoted by $\overrightarrow{\chi}(D)$ and introduced by Neumann-Lara [15] is the minimum integer k such that D admits a k-dicolouring. We will sometimes extend $\overrightarrow{\chi}$ to subsets of vertices, using $\overrightarrow{\chi}(X)$ to mean $\overrightarrow{\chi}(D[X])$ where $X \subseteq V(D)$. For a set $\mathcal{C}$ of digraphs we write $\overrightarrow{\chi}(\mathcal{C})$ to denote the maximum of $\overrightarrow{\chi}(D)$ over all elements D in $\mathcal{C}$, and write $\overrightarrow{\chi}(\mathcal{C}) = \infty$ if this is not bounded.

3 Preliminaries

A set of vertices X is *dipolar* if for every $x \in X$, $N^+(x) \subseteq X$ or $N^-(x) \subseteq X$. This notion was first introduced in [4] under the name "nice set" and has been renamed "dipolar set" in [9]. The main tool using dipolar sets is the following lemma. We include its proof because it is short and enlightening for people unfamiliar with the dichromatic number.

Lemma 4 (Lemma 17 in [4])**.** *Let $\mathcal{C}$ be a class of digraphs closed under taking induced subdigraph. Suppose that there exists a constant c such that each digraph $D \in \mathcal{C}$ has a dipolar set S such that $\overrightarrow{\chi}(S) \leqslant c$. Then $\overrightarrow{\chi}(\mathcal{C}) \leqslant 2c$.*

Proof. Let $D \in \mathcal{C}$ be a minimal counter example, that is: $\overrightarrow{\chi}(D) = 2c + 1$ and for every proper subdigraph H of D, $\overrightarrow{\chi}(H) \leqslant 2c$. By the hypothesis, D admits a dipolar set S, such that $\overrightarrow{\chi}(S) \leqslant c$. Set $S^+ = \{x \in S \mid N^-(x) \subseteq S\}$ and $S^- = \{x \in S \mid N^+(x) \subseteq S\}$. By definition of a dipolar set, $S = S^+ \cup S^-$.

The key observation is that any directed cycle that intersects S and $V(D) \setminus S$ intersects both S^+ and S^-. Hence, by minimality of D, we can dicolour $V(D) \setminus S$ with $2c$ colors. We can then extend this dicoloring to D by using colours $1, \dots, c$ for S^+ and $c+1, \dots, 2c$ for $S^- \setminus S^+$. □

The strategy to prove our result is to show that every digraph in our class has a dipolar set with dichromatic number at most 191 and then apply Lemma 4. The next two results give simple techniques to bound the dichromatic number of a digraph, they will be extensively used to prove that the dichromatic number of some dipolar set is bounded. The first one is well known and is a special case of a much stronger result proved in [7]. We give the proof anyway for sake completeness.

Proposition 5. *If a digraph D does not contain odd directed cycles as subdigraphs, then $\vec{\chi}(D) \leqslant 2$.*

Proof. Let D be a digraph with no odd directed cycle and since the dichromatic number of a digraph is the maximum of the dichromatic number of its strong components, we can assume without loss of generality that D is strongly connected. In that case, we prove that the underlying graph G of D is in fact bipartite. Assume by contradiction G contains an odd cycle $C = c_1 \to c_2 \to \ldots \to c_{2k+1} \to c_1$. For $i = 1, \ldots, 2k+1$, let P_i be a shortest directed path from c_i to c_{i+1} (indices being taken modulo $2k+1$). Observe that either $P_i = c_i c_{i+1}$, or $c_{i+1}c_i \in A(D)$, in which case P_i has odd length, for otherwise $P_i \cup \{c_{i+1}c_i\}$ is an odd directed cycle. Hence the union of the P_i for $i = 1 \ldots 2k+1$ forms a closed odd trail, which contains an odd directed cycle, a contradiction. □

The next result is the dichromatic version of the celebrated Gallai [10], Hasse [14], Roy [16] and Vitaver [20] Theorem asserting that the chromatic number is upper-bounded by the largest size of a directed path. In a nutshell: the dichromatic number is upper-bounded by the largest size of a directed path of some feedback arc set.

Proposition 6. *Let D be a digraph. Given a total ordering of the vertices of D, we say that an arc xy is forward if x precedes y in this ordering, and backward otherwise. The two following propositions are equivalent*

- $\vec{\chi}(D) \leqslant k$
- *There exists an ordering of the vertices of D such that there exists no directed path on $k+1$ vertices consisting only of backward arcs.*

Proof. One direction is easy : if $\vec{\chi}(D) \leqslant k$ then there exists a partition $(C_1, C_2, \ldots C_k)$ of $V(D)$ with C_i inducing an acyclic digraph. We construct an order on $V(D)$ by putting all vertices of C_i before all vertices of C_{i+1} for each i and within each class we use a topological sort. It is clear that in the resulting order, there can be no path on more than k vertices where all arcs go backward since a backward arcs goes from one class to a previous one.

For the converse direction, assume that D has an ordering on its vertices such that there exists no directed path on $k+1$ vertices consisting only of backward arcs and let us prove that D is k-dicolourable. For every $x \in V(D)$, define $f(x)$ to be the maximum number of vertices in a path consisting only of backward arcs and ending in x. By definition $1 \leqslant f(x) \leqslant k$. Define $C_i = f^{-1}(i)$ and let us prove that C_i does not contain any backward arc. Assume by contradiction xy is such an arc. Then there exists a path on i vertices ending in x consisting only of backward arcs, which implies that $f(y) \geqslant i+1$, contradiction. So each C_i induces an acyclic digraph, and thus $\vec{\chi}(D) \leqslant k$. □

The last lemma of this section is used to find induced directed paths.

Lemma 7. *Let D be a triangle-free digraph, C a (not necessarily induced) odd directed cycle of D and $d \in N(C)$. Then there exists consecutive vertices $a \to b \to c$ of C such that*

- *either $d \to a \to b \to c$ is an induced $\overrightarrow{P}_4$,*
- *or $a \to b \to c \to d$ is an induced $\overrightarrow{P}_4$,*
- *or $d \to a \to b \to c \to d$ is a C_4 (in particular, $d \in N^+(C) \cap N^-(C)$).*

Proof. Assume $a \in N^-(C)$. Let us denote by $x_1, \dots, x_{2k+1}$ the vertices of C (i.e. $\forall i \leqslant 2k$, $x_i x_{i+1} \in A(D)$ and $x_{2k+1}x_1 \in A(D)$). Assume without loss of generality that $ax_1 \in A(D)$. Let $1 \leqslant p \leqslant k$ be the maximum integer such that $ax_{2p+1} \in A(D)$. Since the digraph is triangle-free, $ax_{2k+1} \notin A(D)$, so $p \leqslant k$. It is straightforward to see that $b = x_{2p+1}$, $c = x_{2p+2}$, $d = x_{2p+3}$ satisfies either the first or third item of the lemma. By reversing the arcs of the digraph, the same proof works if $a \in N^+(C)$. □

We will often use this lemma the following way : if $a \in N^-(C) \setminus N^+(C)$ (resp. $a \in N^+(C) \setminus N^-(C)$), then the first (resp. the second) output holds.

4 Proof of Theorem 3

For a subset X of vertices, we define recursively the sets $N_k^+(X)$, $N_k^-(X)$ and $N_k(X)$ by $N_0^+(X) = N_0^-(X) = N_0(X) = X$, and for $k \geqslant 1$:

$$\begin{aligned} N_k^+(X) &= N^+(N_{k-1}^+(X)) \setminus \bigcup_{i<k} N_i(X) \\ N_k^-(X) &= N^-(N_{k-1}^-(X)) \setminus \bigcup_{i<k} N_i(X) \\ N_k(X) &= N_k^+(X) \cup N_k^-(X) \end{aligned}$$

In other words, $N_k^+(X)$ is the set of vertices y such that there exists a vertex $x \in X$ with a directed path from y to x of length k, but there is no directed path of length at most $k-1$ from y to x nor from x to y. Note that there can be a path of length at most $k-1$ linking x and y in the underlying graph, that is, a path containing $\to\leftarrow$ or $\leftarrow\to$ at some point.

We gather in the following claim several straightforward facts that we will use in the proof.

Claim 8. *For any $X \subset V$, the following hold*

1. *$N_1^+(X) = N^+(X)$, $N_1^-(X) = N^-(X)$ and $N_1(X) = N^+(X) \cup N^-(X)$*
2. *There are no arcs between X and $N_k(X)$ for $k > 1$.*

3. If $x \in N_{k-1}(X)$, then either $N^+(x) \subseteq \bigcup_{i \leqslant k} N_i(X)$ or $N^-(x) \subseteq \bigcup_{i \leqslant k} N_i(X)$.

4. If $v \in N_k^+(X)$ (respectively $N_k^-(X)$), there exists a directed path $v_0 v_1 \dots v_k$ (respectively $v_k v_{k-1} \dots v_0$) such that $v_k = v$ and $v_i \in N_i^+(X)$ for every $i \geqslant 0$.

Items 1, 2 and 3 follow from the definition and item 4 is easy to prove by induction on k.

Let now D be a triangle-free digraph in $Forb_{ind}(\overrightarrow{P_6})$. Let C be a (not necessarily induced) odd directed cycle of D of minimum length (we may assume it exists by Proposition 5). During the proof, for simplicity, we write C for $V(C)$, $D[C]$ for $D[V(C)]$ and $N_k(C)$ for $N_k(V(C))$.

We are going to prove that the set

$$S = C \cup N(C) \cup N_2(C) \cup N_3(C)$$

is dipolar and has dichromatic number at most 191, which implies Theorem 3 by Lemma 4.

Claim 9. *S is dipolar. Moreover, $\vec{\chi}(N_3(C)) \leqslant 2$.*

Proof. To prove that S is dipolar, we need to prove that for every vertex x in S, either $N^+(x)$ or $N^-(x)$ is contained in S. Note that by Claim 8 item 3, this is trivial if $x \in C \cup N_1(C) \cup N_2(C)$.

Assume now that $x \in N_3^+(C)$ and let us prove that $N^+(x) \subseteq N_1(C) \cup N_2(C)$, which will imply both parts of the claim, since this proves that $N_3^+(C)$ is an independent set.

By Claim 8 item 4, there exists a directed path $v_0 \to v_1 \to v_2 \to v_3$, where $v_3 = x$ and $v_i \in N_i^+(C)$. If $v_1 \in N^+(C) \setminus N^-(C)$, then, by Lemma 7, there exists $a, b, c \in C$ such that $abcv_1$ is an induced $\overrightarrow{P_4}$. Since there is no arc between C and $N_2(C) \cup N_3(C)$ (by Claim 8 item 2) and D is triangle-free, $a \to b \to c \to v_1 \to v_2 \to v_3$ is an induced $\overrightarrow{P_6}$, a contradiction.

So we can assume $v_1 \in N^+(C) \cap N^-(C)$. Consider $y \in N^+(x)$, and let us prove that $y \in N_1(C) \cup N_2(C)$. Let t be the in-neighbour of v_0 in C and observe that $t \to v_0 \to v_1 \to v_2 \to v_3 \to y$ is a $\overrightarrow{P_6}$ and the only way for it not to be induced (because of (Claim 8 item 2)) is that y is adjacent with one of $\{t, v_0, v_1\}$. If y is adjacent with t or v_0, then $y \in N_1(C)$. If y is adjacent with v_1, and since $v_1 \in N^+(C) \cap N^-(C)$, we get that $y \in N_2(C)$. We thus have proven that $y \in N_1(C) \cup N_2(C)$. Similarly, if $x \in N_3^-(C)$, then $N^-(x) \subseteq N_1(C) \cup N_2(C)$, which concludes the proof of this claim. □

Claim 10. $\vec{\chi}(D[C]) \leqslant 3$.

Proof. By minimality of C, removing a vertex from $D[C]$ yields a digraph with no odd directed cycle, which thus has dichromatic number at most 2 by Proposition 5. □

Set $C = x_1 x_2 \dots x_{2k+1} x_1$.

Claim 11. $\vec{\chi}(N^+(C) \setminus N^-(C)) \leqslant 4$ *and* $\vec{\chi}(N^-(C) \setminus N^+(C)) \leqslant 4$.

Proof. Let us prove that $\vec{\chi}(N^+(C) \setminus N^-(C)) \leqslant 4$. We first prove that $N^+(x_1) \cup N^+(x_2)$ intersects all odd directed cycles of $N^+(C) \setminus N^-(C)$. Suppose that it is not the case, and let C' be such an odd directed cycle. Let $i \geqslant 3$ be minimum such that x_i has an out-neighbour in C' (so that $x_1, \dots, x_{i-1}$ don't). Since $C' \subset N^+(C) \setminus N^-(C)$, x_i does not have an in-neighbour in C', so by Lemma 7 applied to C', there are 3 consecutive vertices a, b, c of C', such that $x_i \to a \to b \to c$ is an induced $\vec{P}_4$. By the choice of i, we then have that $x_{i-2} \to x_{i-1} \to x_i \to a \to b \to c$ is an induced $\vec{P}_6$, a contradiction. Now, $N^+(C) \setminus N^-(C)$ can be partitioned into two stable sets and a digraph with no odd directed cycle, and thus be 4-dicoloured. By directional duality, $\vec{\chi}(N^-(C) \setminus N^+(C)) \leqslant 4$. ☐

Claim 12. *$\vec{\chi}(N_2^+(C) \setminus N_2^-(C)) \leqslant 2$ and $\vec{\chi}(N_2^-(C) \setminus N_2^+(C)) \leqslant 2$.*

Proof. We prove that $\vec{\chi}(N_2^+(C) \setminus N_2^-(C)) \leqslant 2$. Assume by contradiction this is not the case, so that by Proposition 5 we get an odd directed cycle C' in $N_2^+(C) \setminus N_2^-(C)$. Let u be a vertex in $N^+(C) \cap N^-(C')$, which is non empty by definition of $N_2^+(C)$.

If $u \in N^+(C) \setminus N^-(C)$, then by Lemma 7, there exist $a, b, c \in C$ such that $a \to b \to c \to u$ is an induced $\vec{P}_4$, which along with a vertex $v \in N^+(u) \cap V(C')$ and the out-neighbour of v in $V(C')$ forms an induced $\vec{P}_6$, a contradiction (remember that by Claim 8 Item 2, there is no arc between C and C').

Thus $u \in N^+(C) \cap N^-(C)$ and since $V(C')$ is disjoint from $N_2^-(C)$, u has no in-neighbour in $V(C')$. Hence, by Lemma 7 applied on C', there exist $a, b, c \in V(C')$ such that $u \to a \to b \to c$ is an induced $\vec{P}_4$, which along with any $v \in N^-(u) \cap C$ and the in-neighbour of v in C forms an induced $\vec{P}_6$, a contradiction. ☐

Claim 13. *$\vec{\chi}(N^+(C) \cap N^-(C)) \leqslant 30$. Moreover, if for every $x \in C$, both $N_2^+(x)$ and $N_2^-(x)$ are stable sets, then $\vec{\chi}(N_2^+(C) \cap N_2^-(C)) \leqslant 30$.*

Proof. The same proof works for the two assertions of the claim. Let $\ell \in \{1, 2\}$ and observe that, by hypotheses (triangle-free for $\ell = 1$, or the assumption of the second sentence for $\ell = 2$), for every $x \in C$, both $N^{\ell+}(x)$ and $N^{\ell-}(x)$ are stable sets. In particular, the result holds if C has length 5, so we may assume assume it has length at least 7.

Let $X = (N^{\ell+}(C) \cap N^{\ell-}(C)) \setminus N^{\ell}(\{x_1, \dots, x_6\})$. It is enough to prove that $\vec{\chi}(X) \leqslant 30 - 12 = 18$.

For each vertex $v \in X$, choose (arbitrarily) a vertex x_i (resp. x_j) in C such that there is a directed path of length l from v to x_i (resp. from x_j to v). Set $out(v) = i$ and $in(v) = j$ so that we define two functions out and in from X to $\{1, \dots, 2k+1\}$.

In the case where $\ell = 2$, let p_v^+ (resp. p_v^-) be a vertex such that $v \to p_v^+ \to x_{out(v)}$ (resp. $x_{in(v)} \to p_v^- \to v$). In the rest of the proof, $v \to p_v^+ \to x_{out(v)}$ is understood as $v \to x_{out(v)}$ in the case where $\ell = 1$.

For $i \in [0, 5]$, let $X_i = \{v \in X \mid out(v) = i \mod 6\}$ and then define $X_{i,\geqslant} = \{v \in X_i \mid out(v) \geqslant in(v)\}$ and $X_{i,<} = \{v \in X_i \mid out(v) < in(v)\}$. It is enough to prove that $\vec{\chi}(X_{i,\geqslant}) \leqslant 2$ and $\vec{\chi}(X_{i,<}) \leqslant 1$ for $i = 0, \dots, 5$.

So now i is fixed and we define a total order $\prec$ on X_i the following way: we say first that $u \prec v$ when $out(u) < out(v)$ and then extend arbitrarily this partial ordering to a total ordering of X_i.

We first prove that $\overrightarrow{\chi}(X_{i,\geqslant}) \leqslant 2$ using Proposition 6 applied to the reversal of $\prec$ defined above. Suppose then by contradiction that there exist $a, b, c \in X_{i,\geqslant}$ such that $a \prec b \prec c$ and $ab, bc \in A(D)$. Since $N^{\ell-}(x)$ is a stable set for every $x \in C$, $out(a) \neq out(b)$ and $out(b) \neq out(c)$ and thus

$$out(c) \geqslant 6 + out(b) \geqslant 12 + out(a) \geqslant 12 + in(a)$$

If $in(a)$ has the same parity as $out(a)$ (and thus as $out(b)$ and $out(c)$), then $x_1 \to x_2 \to \cdots \to x_{in(a)} \to p_a^- \to a \to b \to c \to p_c^+ \to x_{out(c)} \to \cdots \to x_{2k+1} \to x_1$ is an odd closed trail (it does not need to be a directed cycle because $p_a^- = p_c^+$ is possible) and otherwise, $x_1 \to x_2 \to \cdots \to x_{in(a)} \to p_a^- \to a \to b \to p_b^+ \to x_{out(b)} \to \cdots \to x_{2k+1} \to x_1$ is an odd directed cycle. In both cases we get an odd directed trail that has strictly less vertices than C, and since an odd closed trail contains an odd directed cycle, we get our contradiction. Thus $\overrightarrow{\chi}(X_{i,\geqslant}) \leqslant 2$.

We now prove that $\overrightarrow{\chi}(X_{i,<}) \leqslant 1$. Suppose that there exist $a, b \in X_{i,<}$ such that $b \prec a$ and $ab \in A(D)$. Thus $out(b) + 6 \leqslant out(a) < in(a)$. If $out(a)$ and $in(a)$ do not have the same parity, then $x_{out(a)} \to x_{out(a)+1} \to \cdots \to x_{in(a)} \to p_a^+ \to a \to p_a^- \to x_{out(a)}$ is an odd closed trail. Otherwise $out(a)$ and thus $out(b)$ have the same parity as $in(a)$, and then $x_{out(b)} \to \cdots \to x_{in(a)} \to p_a^- \to a \to b \to p_b^+ \to x_{out(b)}$ is an odd directed cycle. In both cases it has strictly less vertices than C, a contradiction. Thus $\overrightarrow{\chi}(X_{i,<}) \leqslant 1$ by Proposition 6. □

Let $\overrightarrow{C}_{3,2}$ be the digraph with vertices u, v_1, v_2, w_1, w_2 and arcs $uv_1, v_1v_2, v_2w_2, uw_1, w_1w_2$ (so $\overrightarrow{C}_{3,2}$ is the digraph consisting of two directed paths from u to w_2, one being $u \to v_1 \to v_2 \to w_2$, and the other $u \to w_1 \to w_2$). Observe that if $G \in Forb_{ind}(\overrightarrow{C}_{3,2})$, then for every $x \in V(G)$, $N_2^+(x)$ and $N_2^-(x)$ are stable sets. Hence, by the previous claims, we get that for every triangle-free digraphs $G \in Forb_{ind}(\{\overrightarrow{P}_6, \overrightarrow{C}_{3,2}\})$, the set $Q \cup N(Q) \cup N_2(Q) \cup N_3(Q)$, where Q is an odd directed cycle of G of minimum length, is dipolar and has dichromatic number at most $3+4+4+2+2+2+30+30 = 77$. Hence, by Lemma 4 we get that:

Claim 14. *Triangle-free digraphs in $Forb_{ind}(\{\overrightarrow{P}_6, \overrightarrow{C}_{3,2}\})$ have dichromatic number at most 144.*

We are now able to prove the last bit of the proof.

Claim 15. $\overrightarrow{\chi}(N_2^+(C) \cap N_2^-(C)) \leqslant 144$.

Proof. By Claim 14, we may assume that $N_2^+(C) \cap N_2^-(C)$ contains $\overrightarrow{C}_{3,2}$ as an induced subdigraph. Thus there exists $u, v_1, v_2, w_1, w_2 \in N_2^+(C) \cap N_2^-(C)$ such that uv_1, uw_1, v_1v_2, w_1w_2, v_2w_2 are arcs of D. Moreover, there exists $r, s \in C$, and $t \in N^+(C)$ such that $rs, st, tu \in A(D)$. Now, since $r \to s \to t \to u \to v_1 \to v_2$ is not induced, t and v_2 are adjacent, and since $r \to s \to t \to u \to w_1 \to w_2$ is not induced, t and w_2 are adjacent. Hence t, v_2, w_2 forms a triangle, a contradiction. □

Altogether, we get that $\overrightarrow{\chi}(S) \leqslant 3+4+4+30+2+2+144+2 = 191$, and thus $\overrightarrow{\chi}(D) \leqslant 382$.

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