An Infinite Family of Connected 1-Factorisations of Complete 3-Uniform Hypergraphs

Barbara Maenhaut[a] Jeremy Mitchell[a] Anna Puskás[b]

Submitted: Aug 4, 2023; Accepted: Nov 12, 2024; Published: Dec 17, 2024

Abstract

A connected 1-factorisation is a 1-factorisation of a hypergraph for which the union of each pair of distinct 1-factors is a connected hypergraph. A uniform 1-factorisation is a 1-factorisation of a hypergraph for which the union of each pair of distinct 1-factors is isomorphic to the same subhypergraph, and a uniform-connected 1-factorisation is a uniform 1-factorisation in which that subhypergraph is connected. Chen and Lu [Journal of Algebraic Combinatorics, 46(2) 475–497, 2017] describe a family of 1-factorisations of the complete 3-uniform hypergraph on $q+1$ vertices, where $q \equiv 2 \pmod 3$ is a prime power. In this paper, we show that their construction yields a connected 1-factorisation only when $q = 2, 5, 11$ or $q = 2^p$ for some odd prime p, and a uniform 1-factorisation only for $q = 2, 5, 8$ (each of these is a uniform-connected 1-factorisation).

Mathematics Subject Classifications: 05C70, 05C65, 05E18

1 Introduction

A *1-factor* of a graph G is a spanning 1-regular subgraph of G, and a *1-factorisation* of G is a collection of edge-disjoint 1-factors of G that partition the edge-set of G. It is natural to ask: under what conditions does the complete graph on n vertices (K_n) admit a 1-factorisation? It is clear that n must be even. By Kirkman's 1847 construction of 1-factorisations of K_n for all even integers $n \geqslant 2$ [8], this condition is sufficient.

Given a 1-factorisation of a graph G, a well-studied problem is to ask if the union of each pair of its 1-factors is isomorphic to the same subgraph H of G. Such a 1-factorisation is called a *uniform 1-factorisation* (U1F) of G and the subgraph H is called the *common graph*. Furthermore, a uniform 1-factorisation in which the common graph is a Hamilton cycle is called a *perfect 1-factorisation* (P1F). In the 1960's, Kotzig [9] posed a question

[a] School of Mathematics and Physics, The University of Queensland, Brisbane, Queensland, Australia (bmm@maths.uq.edu.au, jeremy.mitchell@uq.net.au).
[b] School of Mathematics & Statistics, University of Glasgow, Glasgow, Scotland (anna.puskas@glasgow.ac.uk).

which has become known as Kotzig's perfect 1-factorisation conjecture, namely that for each even integer n, the complete graph K_n admits a perfect 1-factorisation. Three infinite families of perfect 1-factorisations of complete graphs are known to exist, covering orders $n = p + 1$ and $n = 2p$ where p is an odd prime [3]. Perfect 1-factorisations of complete graphs are also known to exist for all even orders up to $n = 56$ and some other sporadic orders, however the conjecture remains open. For an updated overview of the problem, we recommend a survey by Rosa [1] and a paper on the number of non-isomorphic P1Fs of K_{16} by Gill and Wanless [7]. Recently, Davies, Maenhaut, and Mitchell [6], have generalised the notions of uniform and perfect 1-factorisations of graphs to the context of hypergraphs.

A *hypergraph* $\mathcal{H}$ consists of a non-empty vertex set $V(\mathcal{H})$ and an edge set $E(\mathcal{H})$ in which each element of $E(\mathcal{H})$ is a non-empty subset of the vertex set $V(\mathcal{H})$. The complete k-uniform hypergraph of order n, denoted K_n^k, is the hypergraph with n vertices, in which the edges are precisely all the k-subsets of the vertex set. A spanning 1-regular subhypergraph of a hypergraph is known as a *1-factor*. A partition of the edge set of a hypergraph $\mathcal{H}$ into 1-factors is called a *1-factorisation* of $\mathcal{H}$, and such a 1-factorisation having α 1-factors is often denoted by $\mathcal{F} = \{F_1, \dots, F_\alpha\}$. An obvious necessary condition for the existence of a 1-factorisation of the complete k-uniform hypergraph on n vertices is that $k|n$. Baranyai [2] showed that for $k \geqslant 3$, this condition is also sufficient.

A *path* between two vertices, x and y, of a hypergraph $\mathcal{H}$ is an alternating sequence of vertices and edges $[x = v_1, e_1, v_2, e_2, \dots, v_s, e_s, v_{s+1} = y]$, such that $v_1, v_2, \dots, v_{s+1}$ are distinct vertices of $\mathcal{H}$, and $e_1, e_2, \dots e_s$ are distinct edges of $\mathcal{H}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for $1 \leqslant i \leqslant s$. If every two vertices of a hypergraph $\mathcal{H}$ have a path between them, we say that $\mathcal{H}$ is *connected*. A *Berge cycle* in a hypergraph $\mathcal{H}$, is an alternating sequence of vertices and edges $(v_1, e_1, v_2, e_2, \dots, v_m, e_m)$, such that $[v_1, e_1, v_2, e_2, \dots, v_m]$ is a path in $\mathcal{H}$, $\{v_1, v_m\} \subseteq e_m$, and $e_m \in E(\mathcal{H}) \setminus \{e_1, e_2, \dots, e_{m-1}\}$. Note that each edge e_i may contain vertices other than v_i and v_{i+1} including vertices outside of $\{v_1, \dots, v_m\}$. A *Hamilton Berge cycle* in a hypergraph $\mathcal{H}$ is a Berge cycle in $\mathcal{H}$ for which $\{v_1, \dots, v_m\}$ is the vertex set of $\mathcal{H}$.

In [6], Davies et al. generalised uniform and perfect 1-factorisations of graphs to the context of hypergraphs in several different ways, leading to the following definitions. A *connected 1-factorisation* (C1F) is a 1-factorisation of a hypergraph for which the union of each pair of distinct 1-factors is a connected hypergraph. A *uniform 1-factorisation* (U1F) is a 1-factorisation of a hypergraph for which the union of each pair of 1-factors is isomorphic to the same subhypergraph, called the common hypergraph, and a *uniform-connected 1-factorisation* (UC1F) is a U1F in which the common hypergraph is connected. A *Hamilton-Berge 1-factorisation* (HB1F) is a 1-factorisation of a k-uniform hypergraph for which the union of each k-set of 1-factors has a Hamilton Berge cycle. In addition to showing some existence results of these generalisations, they also classified some known 1-factorisations as being C1Fs, HB1Fs, and U1Fs [6]. Of these known 1-factorisations, the infinite family of symmetric 1-factorisations of K_{q+1}^3 for $q \equiv 2 \pmod 3$ found by Chen and Lu [4] is the subject of this paper. We denote this 1-factorisation by $\mathcal{F}_q$, see §2.2. Davies et al. [6], showed that for $q = 2, 5, 8$ these 1-factorisations are C1Fs, U1Fs, UC1Fs, and HB1Fs, and that the 1-factorisations when $q = 11, 32$ are C1Fs and HB1Fs.

The main results of this paper determine when $\mathcal{F}_q$ is a C1F or a U1F.

Theorem 1. *$\mathcal{F}_q$ is a connected 1-factorisation if and only if $q \in \{2, 5, 11\}$ or $q = 2^p$ for some odd prime p.*

Theorem 2. *$\mathcal{F}_q$ is a uniform 1-factorisation if and only if $q \in \{2, 5, 8\}$, and in these cases it is a uniform-connected 1-factorisation.*

2 Preliminaries

Throughout this paper, q denotes a prime power that satisfies $q \equiv 2 \pmod 3$. Let $\mathbb{F} = \mathbb{F}_q$ be the finite field of order q, let $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$, and let $V = \mathbb{F} \cup \{\infty\}$ be the vertex set of the complete 3-uniform hypergraph of order $q + 1$, K^3_{q+1}. We will work with a particular 1-factorisation of K^3_{q+1}, originally given by Chen and Lu [4]. To describe this 1-factorisation, we first recall the action of the group $\mathrm{PSL}(2, q)$ on the projective line $\mathrm{PG}(1, q)$.

2.1 The action of $\mathbf{PSL(2, q)}$ on the projective line

Recall that $V = \mathbb{F} \cup \{\infty\}$ can be identified with the projective line $\mathrm{PG}(1, q)$. (In homogeneous coordinates, identify ∞ with $[1 : 0]^T$ and $x \in \mathbb{F}$ with $[x : 1]^T$.) The group $\mathrm{PSL}(2, q)$ is a quotient of $\mathrm{SL}(2, q)$:

$$\mathrm{PSL}(2,q) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathbb{F};\ \alpha\delta - \beta\gamma = 1 \right\} / \sim,$$

where two matrices are equivalent by $\sim$ if one is a scalar multiple of the other. The group $\mathrm{PSL}(2, q)$ acts faithfully on $\mathrm{PG}(1, q)$:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x + \beta y \\ \gamma x + \delta y \end{bmatrix}.$$

It is often more convenient to think of this action in terms of fractional linear transformations on $\mathbb{F} \cup \{\infty\}$. We may also write this as:

$$t_{\alpha,\beta,\gamma,\delta}(x) := \frac{\alpha x + \beta}{\gamma x + \delta}$$

for $x \in \mathbb{F} \cup \{\infty\}$, where $\frac{\alpha\infty+\beta}{\gamma\infty+\delta} = \frac{\alpha}{\gamma}$ for $\gamma \neq 0$, $\frac{\alpha\infty+\beta}{\delta} = \infty$ for $\alpha \neq 0$, and $\frac{\omega}{0} = \infty$ for $\omega \in \mathbb{F}^*$.

Recall that the action of $\mathrm{PSL}(2, q)$ on $\mathrm{PG}(1, q)$ is transitive. Note that we may think of $\mathrm{PSL}(2, q)$ as a subgroup of $\mathrm{PGL}(2, q) = \mathrm{GL}(2, q)/\sim$, and the action given above extends to one of $\mathrm{PGL}(2, q)$. Under this (extended) action, the stabiliser of the point ∞ is a subgroup. Indeed, it is the group of the invertible linear transformations of $\mathbb{F} \cup \{\infty\}$:

$$\mathrm{PGL}(2, q)_\infty = \{g_{\alpha,\beta} := t_{\alpha,\beta,0,1} \mid \alpha \in \mathbb{F}^*,\ \beta \in \mathbb{F}\}.$$

2.2 The construction of Chen and Lu

The following construction was given by Chen and Lu [4, Example 5.1, Lemma 5.2]. We present it here in a format convenient for our purposes. Let $f : V \to V$ be the map defined by

$$f(x) = \frac{1}{1-x} \text{ for } x \in V \setminus \{1, \infty\} \text{ and } f(1) = \infty \text{ and } f(\infty) = 0.$$

For any $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$ define $g_{\alpha,\beta} : V \to V$ as $g_{\alpha,\beta}(x) = \alpha x + \beta$ for $x \in \mathbb{F}$ and $g_{\alpha,\beta}(\infty) = \infty$. Set $m_{\alpha,\beta} := g_{\alpha,\beta} \circ f \circ g_{\alpha,\beta}^{-1}$. Here $g_{\alpha,\beta} \in \mathrm{PGL}(2, q)$ and the conjugation makes sense in this group, so that $m_{\alpha,\beta} \in \mathrm{PSL}(2, q)$. Then

$$m_{\alpha,\beta}(x) = \beta + \frac{\alpha^2}{\alpha + \beta - x} \text{ for } x \in \mathbb{F} \setminus \{\alpha + \beta\} \text{ and } m_{\alpha,\beta}(\alpha + \beta) = \infty,\ m_{\alpha,\beta}(\infty) = \beta.$$

Note that $m_{1,0} = f$ and $m_{\alpha,\beta}^2 = m_{\alpha,\beta}^{-1}$. In particular, $m_{\alpha,\beta} \in \mathrm{PSL}(2, q)$ has order 3 for any $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. It has no fixed points on $V = \mathrm{PG}(1, q)$. We also note that in the notation of [4], $f = t_{0,1,-1,1}$ and $m_{\alpha,\beta} = t_{-\beta,\alpha^2+\alpha\beta+\beta^2,-1,\alpha+\beta}$.

It follows that the sets

$$F_{\alpha,\beta} := \left\{\{x, m_{\alpha,\beta}(x), m_{\alpha,\beta}^{-1}(x)\} \mid x \in \mathbb{F} \cup \{\infty\}\right\}$$

are 1-factors of K_{q+1}^3 on V. Set

$$\mathcal{F}_q = \{F_{\alpha,\beta} \mid \alpha \in \mathbb{F}^*, \beta \in \mathbb{F}\}.$$

By [4, Lemma 5.2] $\mathcal{F}_q$ is a 1-factorisation of K_{q+1}^3 for $q \geqslant 5$ if $q \equiv 2 \pmod 3$. In fact for $q = 2$ it is the trivial 1-factorisation of K_3^3. Recall also that $\mathcal{F}_q$ has $\frac{q(q-1)}{2}$ 1-factors. In particular, each is represented by exactly two (α, β) pairs: $F_{\alpha,\beta} = F_{-\alpha,\alpha+\beta}$. Using the notation of [4], $F_{\eta^i,\beta}$ is $F_{i,\beta}$ in our notation, and their $\mathcal{PG}_{(q+1;3,\frac{q(q-1)}{2})}$ is our $\mathcal{F}_q$.

Remark 3. Observe that $f = m_{1,0}$ having no fixed points implies that no $x \in \mathbb{F}$ satisfies $x^2 - x + 1 = 0$.

In the language of the action of $\mathrm{PSL}(2, q)$ on $\mathrm{PG}(1, q)$, each 1-factor $F_{\alpha,\beta} \in \mathcal{F}_q$ is the set of orbits of $\langle m_{\alpha,\beta}\rangle$ on $V = \mathrm{PG}(1, q)$. The elements $m_{\alpha,\beta} \in \mathrm{PSL}(2, q)$ are exactly the conjugates of $f = m_{1,0}$ in $\mathrm{PGL}(2, q)$ by the stabiliser $\mathrm{PGL}(2, q)_\infty$. Each subgroup $\langle m_{\alpha,\beta}\rangle$ is isomorphic to the cyclic group of order 3.

3 Connected 1-Factorisations of $\mathcal{F}_q$

In this section we prove Theorem 1.1, noting that $\mathcal{F}_2, \mathcal{F}_5, \mathcal{F}_8$, $\mathcal{F}_{11}$ and $\mathcal{F}_{32}$ were shown to be C1Fs in [6]. The graph theoretic properties of the factorisation $\mathcal{F}_q$ can be rephrased in terms of the action of $\mathrm{PSL}(2, q)$ on $\mathrm{PG}(1, q)$. We explain this first. Note that whenever we refer to the action of a group on $\mathrm{PG}(1, q)$, we understand this to mean that the group is embedded into $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, and the action is the one described in §2.

Lemma 4. *The 1-factorisation $\mathcal{F}_q$ is a connected 1-factorisation if and only if the subgroup $H_{\alpha,\beta,\gamma,\delta} = \langle m_{\alpha,\beta}, m_{\gamma,\delta} \rangle$ acts transitively on $\mathrm{PG}(1,q)$ for each $(\alpha,\beta),(\gamma,\delta) \in \mathbb{F}_q^* \times \mathbb{F}_q$ such that $\langle m_{\alpha,\beta}\rangle \neq \langle m_{\gamma,\delta}\rangle$. Such a subgroup $H_{\alpha,\beta,\gamma,\delta}$ has at least four elements of order 3.*

Proof. As seen above, two vertices of V are in the same edge of $F_{\alpha,\beta}$ if and only if they are in the same orbit under the action of $\langle m_{\alpha,\beta}\rangle$ on $\mathrm{PG}(1,q)$. Similarly, the union of two 1-factors $F_{\alpha,\beta} \cup F_{\gamma,\delta}$ is connected if and only if the subgroup $H_{\alpha,\beta,\gamma,\delta}$ acts transitively on $\mathrm{PG}(1,q)$. Indeed, for $x, y \in V$ a path from x to y along edges from $F_{\alpha,\beta}$ and $F_{\gamma,\delta}$ corresponds to an element $w \in \mathrm{PSL}(2,q)$ such that $w(x) = y$ and $w = m_{\alpha,\beta}^{i_0} \circ m_{\gamma,\delta}^{i_1} \circ \cdots \circ m_{\gamma,\delta}^{i_\ell} \in H_{\alpha,\beta,\gamma,\delta}$, where $i_k \in \{-1,0,1\}$ for $0 \leqslant k \leqslant \ell$.

Furthermore $F_{\alpha,\beta} \neq F_{\gamma,\delta}$ exactly if $\langle m_{\alpha,\beta}\rangle \neq \langle m_{\gamma,\delta}\rangle$. In this case $m_{\alpha,\beta}, m_{\alpha,\beta}^{-1}, m_{\gamma,\delta}, m_{\gamma,\delta}^{-1}$ are four distinct elements of $H_{\alpha,\beta,\gamma,\delta}$ that each have order 3. □

According to Lemma 4, the subgroups of $\mathrm{PSL}(2,q)$ are relevant for deciding whether $\mathcal{F}_q$ is a C1F or not. These have been classified by Dickson [5, Chapter XII.]. We recall the following theorem, summarising the portion of the classification that is relevant to us.

Theorem 5. *[5, §260, pp285-286], see also [12, Theorem 6.25, 6.26] Let $q = p^r \equiv 2 \pmod 3$ where p is a prime. Then we have the following.*

(a) The group $\mathrm{PSL}(2,q)$ has exactly $\frac{q(q-1)}{2}$ cyclic subgroups of order 3.

(b) A subgroup isomorphic to $\mathrm{PSL}(2,p^s)$ exists in $\mathrm{PSL}(2,q)$ for any divisor s of r.

(c) If q is odd, then a subgroup isomorphic to A_4 exists in $\mathrm{PSL}(2,q)$.

(d) If q is even, i.e. $q = 2^r$ with r odd, then the subgroups of $\mathrm{PSL}(2,q)$ either have an order that is a divisor of q, are isomorphic to $\mathrm{PSL}(2,2^s)$ where s is a divisor of r, or are cyclic or dihedral.

We are now ready to show which q will result in $\mathcal{F}_q$ not being a C1F.

Lemma 6. *Let $q = p^r \equiv 2 \pmod 3$ where p is a prime. If either $q > 11$, or $p = 2$ and r is composite, then $\mathcal{F}_q$ is not a C1F.*

Proof. Since each 1-factor in $\mathcal{F}_q$ corresponds to a 3-element subgroup in $\mathrm{PSL}(2,q)$, part (a) of Theorem 5 implies that every subgroup of $\mathrm{PSL}(2,q)$ of order 3 corresponds to a 1-factor, i.e. is of the form $\langle m_{\alpha,\beta}\rangle$ for some $(\alpha,\beta) \in \mathbb{F}_q^* \times \mathbb{F}_q$. Lemma 4 then implies that $\mathcal{F}_q$ is not a C1F if $\mathrm{PSL}(2,q)$ has a subgroup H that has at least two subgroups of order 3, but does not act transitively on $\mathrm{PG}(1,q)$. Indeed the two distinct subgroups of order 3 would be $\langle m_{\alpha,\beta}\rangle$ and $\langle m_{\gamma,\delta}\rangle$ for some $(\alpha,\beta),(\gamma,\delta) \in \mathbb{F}_q^* \times \mathbb{F}_q$. Together they would generate a $H_{\alpha,\beta,\gamma,\delta} \leqslant H$ that does not act transitively on $\mathrm{PG}(1,q)$.

We now show that if $q > 11$ or $q = 2^r$ and r is composite, then $\mathrm{PSL}(2,q)$ has a subgroup H such that H has at least two distinct subgroups of order 3 but its order is not divisible by $q+1$, and hence it does not act transitively on $\mathrm{PG}(1,q)$. Consider first the case where q is odd and $q > 11$. Part (c) of Theorem 5 implies that the group $\mathrm{PSL}(2,q)$

has a subgroup H isomorphic to A_4. This H has (four) distinct subgroups of order 3, and has order 12, which is not a multiple of $q+1$. Therefore $\mathcal{F}_q$ is not a C1F. In the case that $q = 2^r$ where r is composite, let s be the smallest prime divisor of r. Part (b) of Theorem 5 implies that $\mathrm{PSL}(2, q)$ has a subgroup H isomorphic to $\mathrm{PSL}(2, 2^s)$. This subgroup H has $2^{s-1}(2^s - 1)$ distinct subgroups of order 3 by Theorem 5 (a). It also has order $|H| = (2^{2s} - 1)2^s$ [5, §239, p. 261]. Since $q = 2^r$, we know that $q + 1$ is coprime to 2^s. Furthermore, r is divisible by s, hence $2^{2s} - 1 < 2^r = q$, so $q + 1$ does not divide $2^{2s} - 1$. Thus $q + 1$ is not a divisor of $|H|$. This completes the proof. □

Remark 7. Note that while the proof of Dickson's classification of subgroups of $\mathrm{PSL}(2, q)$ where $q = p^r$ is subtle, one subgroup of the form $\mathrm{PSL}(2, p^s)$ for $s|r$ is easy to find. Indeed $\mathbb{F}_{p^s}$ is a subfield of $\mathbb{F}_q$ for any such s. Furthermore, if $p \geqslant 5$ is odd and $r > 2$, or if $p = 2$ and r is odd composite, then we can use such a subfield to show that $\mathcal{F}_q$ is not a C1F, by choosing appropriate α, β in the subfield, and considering the subgraph $F_{\alpha,\beta} \cup F_{1,0}$.

It remains to tackle the case where q is a power of 2 with prime exponent.

Lemma 8. *If $q = 2^r$ for some odd prime r, then $\mathcal{F}_q$ is a C1F.*

Proof. By Lemma 4 it suffices to show that no proper subgroup of $\mathrm{PSL}(2, 2^r)$ has at least four elements of order 3. This implies that for any $(\alpha, \beta), (\gamma, \delta) \in \mathbb{F}_q^* \times \mathbb{F}_q$ the group $H_{\alpha,\beta,\gamma,\delta}$ is the entire group $\mathrm{PSL}(2, 2^r)$ and thus acts transitively on $\mathrm{PG}(1, q)$. By Theorem 5 part (d), we have that for a proper subgroup H of $\mathrm{PSL}(2, q)$, one of the following holds: $|H|$ is a divisor of q, H is cyclic, dihedral, or $H \cong \mathrm{PSL}(2, 2) \cong S_3$ (the symmetric group on three letters). A proper subgroup H therefore has at most two elements of order 3. This completes the proof. □

The proof of Theorem 1 follows by combining Lemmas 6 and 8 with the knowledge from [6] that $\mathcal{F}_2, \mathcal{F}_5$, and $\mathcal{F}_{11}$ are C1Fs.

4 Uniform 1-Factorisations

In this section we prove Theorem 1.2, noting that $\mathcal{F}_5$ and $\mathcal{F}_8$ were shown to be U1Fs (and UC1Fs) in [6] and $\mathcal{F}_2$ is trivially a U1F.

For two distinct 1-factors F_1 and F_2 of a hypergraph, we say that a pair of vertices, $B = \{v_1, v_2\}$, is *repeated* in the pair F_1 and F_2 if $B \subseteq e$ for some edge $e \in F_1$ and $B \subseteq e'$ for some edge $e' \in F_2$. We call the number of repeated pairs in a pair of 1-factors the *pair overlap number*. If each pair of distinct 1-factors of a 1-factorisation have the same pair overlap number, we call that the *pair overlap number* of the 1-factorisation. Davies et al. [6] showed that if a U1F of K_n^3 exists then the pair overlap number of the 1-factorisation is 2. Thus in order to prove that $\mathcal{F}_q$ is not a U1F, we need only show that there exist two distinct 1-factors with pair overlap number not equal to 2.

Let $F_{1,0}$ and $F_{\alpha,\beta}$ be distinct 1-factors of $\mathcal{F}_q$, with corresponding functions f and $m_{\alpha,\beta}$ for $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Recall that this is the case if and only if $(\alpha, \beta) \notin \{(1, 0), (-1, 1)\}$ in $\mathbb{F}$. Observe that the pair overlap number of $F_{1,0}$ and $F_{\alpha,\beta}$ is

$$|\{x \in \mathbb{F} \cup \{\infty\} \ : \ f(x) = m_{\alpha,\beta}(x)\}| + |\{x \in \mathbb{F} \cup \{\infty\} \ : \ f^{-1}(x) = m_{\alpha,\beta}(x)\}|.$$

This means that every repeated pair corresponds to a solution to either $f(x) = m_{\alpha,\beta}(x)$ or $f^{-1}(x) = m_{\alpha,\beta}(x)$.

We will now consider the number of solutions for $f(x) = m_{\alpha,\beta}(x)$ with values of α and β that result in $F_{\alpha,\beta}$ being distinct from $F_{1,0}$. A solution to $f(x) = m_{\alpha,\beta}(x)$ gives us the equation

$$\frac{1}{1-x} = \beta + \frac{\alpha^2}{\alpha+\beta-x}.$$

We note that for $x = \infty$, $f(\infty) = 0$ and $m_{\alpha,\beta}(\infty) = \beta$ so there is at least one solution if $\beta = 0$, and only one if $\beta = 0$, $\alpha = -1$, and $F_{1,0} \neq F_{-1,0}$. Further, if $\beta = 0$, $\alpha \notin \{-1, 1\}$, then we also get the solution $x = \frac{\alpha}{1+\alpha}$. If $\alpha + \beta = 1$ then the only solutions are $x = 1$ and $x = -\alpha$. (Note $(\alpha, \beta) \neq (1, 0)$. We may have $\alpha = -1$.) We now consider the case where $\alpha + \beta \neq 1$. Then $f(x) = m_{\alpha,\beta}(x)$ implies $x \notin \{1, \alpha + \beta\}$, every solution is in $\mathbb{F}$ and $f(x) = m_{\alpha,\beta}(x)$ is equivalent to

$$0 = (\alpha^2 + \alpha\beta - \alpha + \beta^2 - \beta) - x(\alpha^2 + \alpha\beta + \beta^2 + \beta - 1) + \beta x^2.$$

We can also obtain the number of solutions for $f^{-1}(x) = m_{\alpha,\beta}(x)$ by noticing that for $x \in \mathbb{F} \cup \{\infty\}$ we have

$$|\{x : f^{-1}(x) = m_{\alpha,\beta}(x)\}| = |\{x : f(x) = m^{-1}_{\alpha,\beta}(x)\}| = |\{x : f(x) = m_{-\alpha,\alpha+\beta}(x)\}|.$$

To summarise the above we have the following.

Conditions	$\{x \in \mathbb{F} \cup \{\infty\} \mid f(x) = m_{\alpha,\beta}(x)\}$
$\beta = 0,\ \alpha = -1$	$\{\infty\}$
$\beta = 0,\ \alpha \notin \{-1, 1\}$	$\{\infty, \frac{\alpha}{1+\alpha}\}$
$\beta \neq 0,\ \alpha + \beta = 1$	$\{1, -\alpha\}$
$\beta \neq 0,\ \alpha + \beta \neq 1$, $((\alpha, \beta) \neq (-1, 1))$	$\{x \in \mathbb{F} \mid \beta x^2 - (\alpha^2 + \alpha\beta + \beta^2 + \beta - 1)x + (\alpha^2 + \alpha\beta + \beta^2 - \alpha - \beta) = 0\}$

Conditions	$\{x \in \mathbb{F} \cup \{\infty\} \mid f^{-1}(x) = m_{\alpha,\beta}(x)\}$
$\beta = 1,\ \alpha = 1$	$\{\infty\}$
$\beta = 1,\ \alpha \notin \{-1, 1\}$	$\{\infty, \frac{1}{1-\alpha}\}$
$\beta \neq 1,\ \alpha + \beta = 0$	$\{0, 1 - \alpha\}$
$\beta \neq 1,\ \alpha + \beta \neq 0$, $((\alpha, \beta) \neq (-1, 1))$	$\{x \in \mathbb{F} \mid (1 - \beta)x^2 + (\alpha^2 + \alpha\beta + \beta^2 - \alpha - \beta - 1)x + (\alpha + \beta) = 0\}$

We use the information in the tables to show that $\mathcal{F}_q$ is not a U1F if $q \notin \{2, 5, 8\}$. The cases of $5|q$ and $2|q$ are treated separately from that of other primes. We start with the case of primes greater than 5.

Lemma 9. *Let $q = p^\ell$ for some prime $p > 5$ and some integer $\ell \geqslant 1$ such that $q \equiv 2 \pmod 3$. Then $\mathcal{F}_q$ is not a U1F.*

Proof. Let $F_{1,0}$ and $F_{-1,0}$ be 1-factors of $\mathcal{F}_q$; we shall prove that the pair overlap number of this pair of 1-factors is not 2. $F_{1,0}$ and $F_{-1,0}$ are distinct, and from above we know that there is only one repeated pair corresponding to a solution to $f(x) = m_{-1,0}(x)$. Further, we know that $\{x \in \mathbb{F}_q \cup \{\infty\} \mid f^{-1}(x) = m_{-1,0}(x)\} = \{x \in \mathbb{F}_q \mid x^2 + x - 1 = 0\}$, and $x^2 + x - 1 = 0$ will have 2 solutions in $\mathbb{F}_q$ if 5 is a quadratic residue, and 0 if not. Thus the pair overlap number of this pair of 1-factors must be either 1 or 3, and thus $\mathcal{F}_q$ is not a U1F. □

Lemma 10. *Let $q = 5^\ell$ for some integer $\ell > 1$ such that $q \equiv 2 \pmod 3$. Then $\mathcal{F}_q$ is not a U1F.*

Proof. As in the proof of Lemma 9 we show that there is a choice of α, β such that the pair overlap number of $F_{1,0}$ and $F_{\alpha,\beta}$ is not 2. This implies that $\mathcal{F}_q$ is not a U1F. We shall show that for $\alpha \in \mathbb{F} \setminus \mathbb{F}_5$ the factor $F_{1,0}$ has a pair overlap number of 4 with at least one of $F_{\alpha,-\alpha}$, $F_{\alpha,1-\alpha}$ or $F_{\alpha^2,1-\alpha^2}$.

It follows from the tables above that if $\alpha \in \mathbb{F} \setminus \mathbb{F}_5$ and we set $\beta = -\alpha$ then $f^{-1}(x) = m_{\alpha,-\alpha}$ has the two distinct solutions 0 and $1-\alpha$. The solutions of $f(x) = m_{\alpha,-\alpha}$ are $x \in \mathbb{F}$ such that $\alpha x^2 + (\alpha^2 - \alpha - 1)x - \alpha^2 = 0$. The discriminant is $D_1 = (\alpha - 1)^2 \cdot (\alpha^2 - \alpha + 1)$. Therefore the pair overlap number between $F_{1,0}$ and $F_{\alpha,-\alpha}$ is 4 if $\alpha^2 - \alpha + 1$ is a square in $\mathbb{F}$.

Now set $\beta = 1 - \alpha$ in the tables above. If $\alpha \in \mathbb{F} \setminus \mathbb{F}_5$ then $f(x) = m_{\alpha,1-\alpha}$ has the two distinct solutions, 1 and $-\alpha$. The solutions of $f^{-1}(x) = m_{\alpha,1-\alpha}$ are $x \in \mathbb{F}$ such that $\alpha x^2 + (\alpha^2 - \alpha - 1)x + 1 = 0$. The discriminant is $D_2 = (\alpha + 1)^2 \cdot (\alpha^2 + \alpha + 1)$. Therefore the pair overlap number between $F_{1,0}$ and $F_{\alpha,1-\alpha}$ is 4 if $\alpha^2 + \alpha + 1$ is a square in $\mathbb{F}$.

Now take an $\alpha \in \mathbb{F} \setminus \mathbb{F}_5$. If $\alpha^2 - \alpha + 1 \in \mathbb{F}^2$ or $\alpha^2 + \alpha + 1 \in \mathbb{F}^2$ then the pair overlap number of $F_{1,0}$ with $F_{\alpha,-\alpha}$ or with $F_{\alpha,1-\alpha}$ is not 2 by the above paragraphs. Recall that $\mathbb{F}^*$ is a cyclic group, therefore the product of two non-squares is a square. Therefore if $\alpha^2 - \alpha + 1 \notin \mathbb{F}^2$ and $\alpha^2 + \alpha + 1 \notin \mathbb{F}^2$, then their product is a square: $(\alpha^2 - \alpha + 1)(\alpha^2 + \alpha + 1) = (\alpha^2)^2 + \alpha^2 + 1 \in \mathbb{F}^2$.

Observe that $q \equiv 2 \pmod 3$ implies that ℓ is odd. Therefore $\mathbb{F}$ does not contain the field of 25 elements. This implies that for $\alpha \in \mathbb{F} \setminus \mathbb{F}_5$ we have $\alpha^2 \in \mathbb{F} \setminus \mathbb{F}_5$. Thus using similar working to above, the pair overlap number of $F_{1,0}$ and $F_{\alpha^2,1-\alpha^2}$ is 4. □

We now turn our attention to the case where $q = 2^\ell$ for ℓ an odd integer, $\ell > 3$. We shall show that then $\mathcal{F}_q$ is not a U1F by proving that there is an $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$ such that the pair overlap number of $F_{1,0}$ with $F_{\alpha,1}$ or $F_{\alpha,0}$ is not 2. As in the case of odd characteristic, the proof involves considering the number of solutions of the equations $f(x) = m_{\alpha,\beta}(x)$ and $f^{-1}(x) = m_{\alpha,\beta}(x)$ in special cases. To do so we recall the following useful facts about the trace map.

Lemma 11. *Let ℓ be a positive integer and set $\mathbb{F} = \mathbb{F}_{2^\ell}$. The field extension $\mathbb{F}|\mathbb{F}_2$ is cyclic, its Galois group generated by the Frobenius automorphism $x \mapsto x^2$. The trace map: $\mathrm{Tr} = \mathrm{Tr}^{\mathbb{F}}_{\mathbb{F}_2} : \mathbb{F} \to \mathbb{F}_2$ given by*

$$\mathrm{Tr}(x) = \mathrm{Tr}^{\mathbb{F}_{2^\ell}}_{\mathbb{F}_2}(x) = \sum_{i=0}^{\ell-1} x^{2^i}$$

is an $\mathbb{F}_2-$*linear map. For any* $x \in \mathbb{F}$ *we have* $x^{2^\ell} = x$ *and thus* $\mathrm{Tr}(x) = \mathrm{Tr}(x^2)$ *for all* $x \in \mathbb{F}$. *For* $x \in \mathbb{F}$ *there exists an* $r \in \mathbb{F}$ *such that* $x = r^2 + r$ *if and only if* $\mathrm{Tr}(x) = 0$. *A quadratic equation* $x^2 + Lx + C$ *with* $L \neq 0$ *has two solutions in* $\mathbb{F}$ *if* $\mathrm{Tr}\left(\frac{C}{L^2}\right) = 0$, *and zero solutions otherwise. If* ℓ *is an odd integer then* $\mathrm{Tr}(1) = 1$.

Proofs of the facts collected in Lemma 11 can be found in many texts. For general facts about the trace map, see [10, §5]. The fact that $x = r^2+r$ has a solution if and only if $\mathrm{Tr}(x) = 0$ is the additive form of Hilbert's Theorem 90 [10, Theorem 6.3]. The statement about the number of roots of a quadratic equation follows from the Artin-Schreier theorem [10, Theorem 6.4] by a change of variables. See for example [11, Proposition 1].

To prove that $\mathcal{F}_{2^\ell}$ is not uniform for any $\ell > 3$, we will use the following lemma to allow us to find two 1-factors of $\mathcal{F}_{2^\ell}$, at least one of which will have pair overlap number 4 with the 1-factor $F_{1,0}$.

Lemma 12. *For every odd* $\ell > 3$, *there exists* $\gamma \in \mathbb{F}_{2^\ell} \setminus \mathbb{F}_2$ *such that* $\mathrm{Tr}(\gamma) = \mathrm{Tr}(\frac{1}{\gamma}) = 1$.

Proof. For a contradiction, suppose that if $\gamma \in \mathbb{F}_{2^\ell} \setminus \mathbb{F}_2$ and $\mathrm{Tr}(\gamma) = 1$, then $\mathrm{Tr}(\frac{1}{\gamma}) = 0$. Thus, $\mathrm{Ker}(\mathrm{Tr}) \setminus \{0\} = \left\{\frac{1}{\gamma} : \mathrm{Tr}(\gamma) = 1, \, \gamma \in \mathbb{F}_{2^\ell} \setminus \mathbb{F}_2\right\}$. It then follows that $\mathrm{Tr}(\gamma + \frac{1}{\gamma}) = 1$ for every $\gamma \in \mathbb{F}_{2^\ell} \setminus \mathbb{F}_2$, and hence the polynomial $x^{2^{\ell-1}}\mathrm{Tr}(x + \frac{1}{x}) + x^{2^{\ell-1}} = 0$ has $2^\ell - 2$ roots in $\mathbb{F}_{2^\ell}$. However, note that if x satisfies $\mathrm{Tr}(x + \frac{1}{x}) = 1$ we have:

$$x^{2^{\ell-1}} \cdot \mathrm{Tr}\left(x + \frac{1}{x}\right) = x^{2^{\ell-1}} \cdot \sum_{i=0}^{\ell-1} \left(x^{2^i} + x^{-2^i}\right) = x^{2^{\ell-1}}$$

$$\sum_{i=0}^{\ell-1} \left(x^{2^{\ell-1}+2^i} + x^{2^{\ell-1}-2^i}\right) = x^{2^{\ell-1}}$$

$$x + \sum_{i=0}^{\ell-2} \left(x^{2^{\ell-1}+2^i}\right) + x^{2^{\ell-1}} + \sum_{i=0}^{\ell-1} \left(x^{2^{\ell-1}-2^i}\right) = 0$$

where the last line follows from the fact that $x^{2^\ell} = x$ for every $x \in \mathbb{F}$ (see Lemma 11). The left-hand side here is a polynomial of degree $2^{l-1} + 2^{l-2}$ (with coefficients in $\mathbb{F}_2$). Therefore, it has at most $2^{l-1} + 2^{l-2}$ roots in the field $\mathbb{F}$. If $\ell > 3$ then $2^{\ell-1} + 2^{\ell-2} < 2^\ell - 2$, which is a contradiction with our earlier conclusion. Thus, there exists some $\gamma \in \mathbb{F}_{2^\ell} \setminus \mathbb{F}_2$ such that $\mathrm{Tr}(\gamma) = \mathrm{Tr}(\frac{1}{\gamma}) = 1$. ☐

With the existence of such a γ we can obtain the following result.

Lemma 13. $\mathcal{F}_{2^\ell}$ *is not a U1F for any odd* $\ell > 3$.

Proof. We show this by finding an α such that either the pair of 1-factors $F_{1,0}$ and $F_{\alpha,0}$ or the pair $F_{1,0}$ and $F_{\alpha+1,0}$ has pair overlap number 4. Recall the earlier discussion at the beginning of the section about the pair overlap number of the two 1-factors, $F_{1,0}$ and

$F_{\alpha,\beta}$. Specialising the corresponding tables to characteristic 2 and setting $\beta = 0$ we find the solutions as follows.

Conditions	$\{x \in \mathbb{F} \cup \{\infty\} \mid f(x) = m_{\alpha,\beta}(x)\}$	$\{x \in \mathbb{F} \cup \{\infty\} \mid f^{-1}(x) = m_{\alpha,\beta}(x)\}$
$\beta = 0$ $(\alpha \neq 1)$	$\{\infty, \frac{\alpha}{1+\alpha}\}$	$\{x \in \mathbb{F} \mid x^2 + (\alpha^2 + \alpha + 1)x + \alpha = 0\}$

By Lemma 12, there exists some $\gamma \in \mathbb{F}_{2^\ell} \setminus \mathbb{F}_2$ such that $\mathrm{Tr}(\gamma) = \mathrm{Tr}(\frac{1}{\gamma}) = 1$. As $Tr(\gamma) = 1$ it follows that $Tr(\gamma + 1) = 0$, thus there are 2 solutions to the equation $x^2 + x + 1 = \gamma$. Let α be one such solution (note that $\alpha \notin \{0, 1\}$). Therefore, $\mathrm{Tr}(\frac{1}{\gamma}) = \mathrm{Tr}(\frac{1}{\alpha^2+\alpha+1}) = 1$; furthermore,

$$\mathrm{Tr}\left(\frac{1}{(\alpha^2 + \alpha + 1)^2}\right) = \mathrm{Tr}\left(\frac{1}{\alpha^2 + \alpha + 1}\right) = 1.$$

We now consider the two possible values of $\mathrm{Tr}(\frac{\alpha}{(\alpha^2+\alpha+1)^2})$ separately. If $\mathrm{Tr}(\frac{\alpha}{(\alpha^2+\alpha+1)^2}) = 0$, then $x^2 + (\alpha^2 + \alpha + 1)x + \alpha = 0$ has two solutions, thus by the above table the 1-factors $F_{1,0}$ and $F_{\alpha,0}$ will have pair overlap 4. If $\mathrm{Tr}(\frac{\alpha}{(\alpha^2+\alpha+1)^2}) = 1$, then

$$\mathrm{Tr}\left(\frac{(\alpha + 1)}{((\alpha + 1)^2 + (\alpha + 1) + 1)^2}\right) = \mathrm{Tr}\left(\frac{\alpha}{(\alpha^2 + \alpha + 1)^2}\right) + \mathrm{Tr}\left(\frac{1}{(\alpha^2 + \alpha + 1)^2}\right) = 0.$$

Thus, the 1-factors $F_{1,0}$ and $F_{\alpha+1,0}$ will have pair overlap number 4.

Therefore, $\mathcal{F}_{2^\ell}$ will not be a U1F for all odd $\ell > 3$. □

The proof of Theorem 2 then follows from Lemmas 9, 10, 13, and the knowledge from [6] that $\mathcal{F}_2, \mathcal{F}_5, \mathcal{F}_8$ are both U1Fs and C1Fs.

5 Hamilton-Berge 1-Factorisations

A necessary condition for a 1-factorisation of K_n^k to be a Hamilton-Berge 1-factorisation is that the union of each k-set of 1-factors is connected.

We remark that the proof of Lemma 6 allows us to find three 1-factors whose union is disconnected. Indeed it suffices to find a subgroup H of $\mathrm{PSL}(2, q)$ such that H has at least 3 distinct subgroups of order 3, and H does not act transitively on $\mathrm{PG}(1, q)$. If q is odd and $q > 11$, then $H = A_4$, and if $q = 2^r$ with r odd composite, then $H = \mathrm{PSL}(2, 2^s)$ for $s \geqslant 3$ a divisor of r will satisfy this. From this it follows that if $q = p^r \equiv 2 \pmod 3$ for some prime p and $q > 11$ is odd, or if q is even for some (odd) composite r, then $\mathcal{F}_q$ cannot be an HB1F. Finally, it follows from Lemma 8 that if $q = 2^r$ for some odd prime r, then $\mathcal{F}_q$ satisfies the property that the union of each set of three distinct 1-factors is connected.

Thus $\mathcal{F}_q$ can only be an HB1F if $q \in \{2, 5, 11\}$ or $q = 2^p$ for some odd prime p. The 1-factorisations $\mathcal{F}_5, \mathcal{F}_8, \mathcal{F}_{11}$, and $\mathcal{F}_{32}$ were shown to be HB1Fs in [6], and the 1-factorisation $\mathcal{F}_2$ is trivially an HB1F. We have also shown computationally that $\mathcal{F}_{128}$ is an HB1F, which leads us to the following conjecture.

Conjecture 14. $\mathcal{F}_q$ is a Hamilton-Berge 1-factorisation if and only if $q \in \{2, 5, 11\}$ or $q = 2^p$ for some odd prime p.

Acknowledgements

The authors acknowledge the support of an Australian Government Research Training Program Scholarship, and the support of ARC grant DE200101802. The authors also thank the referees for their helpful comments and for their suggestions to streamline some proofs in Section 3.

References

[1] R. Alexander. Perfect 1-factorizations. *Mathematica Slovaca*, 69(3):479–496, 2019.

[2] Z. Baranyai. On the factorization of the complete uniform hypergraph. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I*, pages 91–108. Colloq. Math. Soc. János Bolyai, Vol. 10. 1975.

[3] D. Bryant, B. Maenhaut, and I. Wanless. New families of atomic Latin squares and perfect one-factorisations. *J. Combin. Theory Ser. A*, 113:608–624, 2006.

[4] H. Y. Chen and Z. P. Lu. Symmetric factorizations of the complete uniform hypergraph. *J. Algebraic Combin.*, 46(2):475–497, 2017.

[5] L. E. Dickson *Linear groups: With an exposition of the Galois field theory.* Vol. 6. BG Teubner, 1901.

[6] S. Davies, B. Maenhaut, and J. Mitchell. Perfect 1-factorisations of complete hypergraphs. *Australas. J. Combin.*, 85(1):35–48, 2022.

[7] M. J. Gill and I. M. Wanless. Perfect 1-factorisations of K_{16}. *Bulletin of the Australian Mathematical Society*, 101(2):177–185, 2020.

[8] T. P. Kirkman. On a problem in combinations. *Cambridge and Dublin Mathematical Journal*, 2:191–204, 1847.

[9] A. Kotzig. Problem 20. In *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)*, page 162. Publ. House Czech. Acad. Sci., Prague, 1964.

[10] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[11] K. Pommerening. Quadratic equations in finite fields of characteristic 2 (English Version). http://www.klauspommerening.de/MathMisc/QuGlChar2.pdf, 2012. [Online; accessed 18-May-2022].

[12] M. Suzuki. *Group theory. I*, volume 247 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, 1982. Translated from the Japanese by the author.