# Circumference, minimum degree and clique number

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### Abstract

The circumference and the clique number of a graph is the length of a longest cycle and the largest order of a clique in it respectively. We show that the circumference of a 2-connected non-Hamiltonian graph $G$ is at least the sum of its clique number and minimum degree unless $G$ is one of two specific graphs.


**Mathematics Subject Classifications:** 05C88, 05C89

## 1 Introduction

The length of a longest cycle, the *circumference*, in a graph $G$ denoted by $c(G)$ is widely studied in extremal and structure graph theory. Let $\delta(G)$ be the minimum degree of $G$. In 1952, Dirac [2] showed that the circumference of a 2-connected $n$-vertex graph is at least $\min\{n, 2\delta(G)\}$. Since then, there are lots of results concerning the circumference of a graph (see [5] for a survey). The *clique number*, denoted by $\omega(G)$, is the maximum order of a clique in $G$. For integers $n \geqslant k > 2t$, let $H(n,k,t)$ be the graph with a vertex partition $A \cup B \cup C$ with sizes $t$, $n-k+t$ and $k-2t$ respectively, and whose edge set consists of all edges between $A$ and $B$ and in $A \cup C$. For $n = k-t-2+\ell(t-1)+2$, the graph $Z(n,k,t)$ denotes the $n$-vertex graph obtained from vertex-disjoint union of a clique $K_{k-t-2}$ and $\ell \geqslant 2$ copies of $K_{t-1}$ by adding two new vertices and joining them to each other and completely to all other vertices (See Fig. 1). Clearly, the minimum degree, the clique number and the circumference of $H(n,k,t)$ and $Z(n,k,t)$ are $t, k-t$ and $k-1$ respectively. We will establish the following theorem concerning the relation between $\delta(G)$, $\omega(G)$ and $c(G)$ in 2-connected graphs.

**Theorem 1.** *Let $G$ be a 2-connected $n$-vertex graph. Then $c(G) \geqslant \min\{n, \omega(G)+\delta(G)\}$ unless $G = H(n, \omega(G)+\delta(G), \delta(G))$ or $G = Z(n, \omega(G)+\delta(G), \delta(G))$.*

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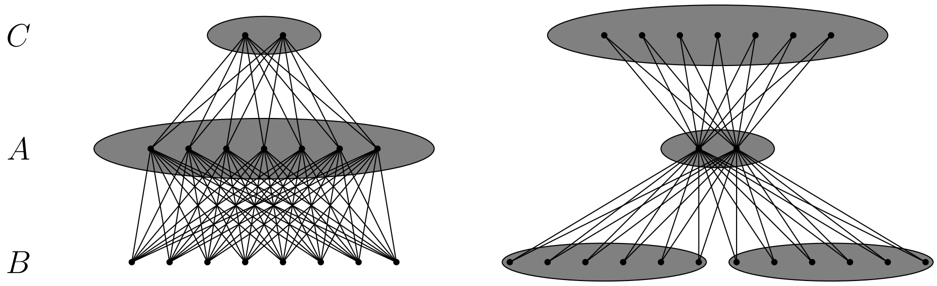


$H(17,16,7)$. $Z(19,16,7)$.

Figure 1. the vertices in each gray ellipse induce a complete graph

Circumference, minimum degree and clique number are important parameters of graphs. Hence it is meaningful to study the relation between them. Moreover, Theorem 1 can be applied to prove [8] a longstanding conjecture of Erdős, Simonovits and Sós [1] (determining the maximum number of edge colors in a complete graph such that there is no rainbow path of given length). In fact, Theorem 1 helps us to strengthen the following result of Füredi, Kostochka, Luo and Verstraëte [3, 4].

**Theorem 2** (Füredi, Kostochka, Luo and Verstraëte [3, 4])**.** *Let $G$ be an $n$-vertex 2-connected graph with $c(G) < k$. Let $\ell = \lfloor (k-1)/2 \rfloor$. Then $e(G) \leqslant \max\{e(H(n,k,\ell-1)), e(H(n,k,3))\}$ unless*

- *$k = 2\ell + 1$, $k \neq 7$, and $G \subseteq H(n,k,\ell)$;*
- *$k = 2\ell + 2$ or $k = 7$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most $\ell$;*[1] *or*
- *$G \subseteq H(n,k,2)$.*

Combining a stability result of the well-known Pósa lemma with Theorem 1, Ma and the author [6] proved the following theorem which strengthens Theorem 2 when $n$ is odd.[2]

**Theorem 3** (Ma and Yuan)**.** *Let $k = 2\ell+1 \geqslant 5$ be an odd integer and $n \geqslant k$. Let $G$ be an $n$-vertex 2-connected graph with $c(G) < k$. Then $e(G) < \max\{e(H(n,k,3)), e(H(n,k,\ell-1))\}$ unless*

- *$G$ is a subgraph of $H(n,k,2)$ or $H(n,k,\ell)$;*
- *$G = H(n,k,3)$ or $G = H(n,k,\ell-1)$;*
- *$G - A$ is a star forest for some $A \subseteq V(G)$ of size at most two for $k = 7$.*

Applying Theorem 3, the author confirms the above mentioned conjecture of Erdős, Simonovits and Sós.

[1] A *star forest* is a graph in which every component is a star.

[2] They also strengthened Theorem 2 when $n$ is even, but the result is quit complicate. So we do not list it here.

## 2 Notation and Pósa's Lemma

The general notation used in this paper is standard. Let $\overline{G}$ be the complement of $G$. Denote by $N_G(x)$ the set of neighbors of $x$ in $G$ and let $d_G(x)$ be the size of $N_G(x)$. Let $N_G[x] = N_G(x) \cup \{x\}$ and $d_G[x]$ be the size of $N_G[x]$. Let $P = x_1x_2 \cdots x_m$ be a path in $G$. For $x_i, x_j \in V(P)$, we use $x_iPx_j$ to denote the sub-path of $P$ between $x_i$ and $x_j$. For a vertex $x$ of $P$, denote $x^-$ and $x^+$ to be the immediate predecessor and successor of $x$ on $P$, respectively. For a subset $S$ of $V(P)$, let $S^+ = \{x^+ : x \in S\}$ and $S^- = \{x^- : x \in S\}$. We call $(i,j)$ a *crossing pair* of $P$ if $x_i \in N_P(x_m)$ and $x_j \in N_P(x_1)$ with $i < j$. A crossing pair $(i,j)$ is *minimal* in $P$ if $x_h \notin N_P(x_1) \cup N_P(x_m)$ for each $i < h < j$. We say a vertex $x \notin Y$ is *connected to* a vertex set $X \subseteq Y$ if there is a path starting from $x$, ending at $x' \in X$ and without containing any vertices of $Y \setminus \{x'\}$.

Our proof is based on the idea of the following well-known lemma of Pósa.

**Lemma 4** (Pósa [7]). *Let $G$ be an $n$-vertex 2-connected graph with a path $P = x_1 \dots x_k$. Then $c(G) \geqslant \min\{n, d_P(x_1) + d_P(x_k)\}$. Furthermore, if $P$ does not contain a crossing pair, then $c(G) \geqslant \min\{n, d_P(x_1) + d_P(x_k) + 1\}$.*

## 3 Proof of Theorem 1

Let $G$ be an edge-maximal counter-example. That is, $G$ is a 2-connected graph with $c(G) < \min\{n, \delta(G) + \omega(G)\}$, however for any edge $e \in E(\overline{G})$, $G + e$ will have a cycle of length at least $\min\{n, \delta(G) + \omega(G)\}$. If $\delta(G) \geqslant \omega(G)$, then Dirac's theorem shows that $c(G) \geqslant \min\{n, 2\delta(G)\} \geqslant \min\{n, \delta(G) + \omega(G)\}$, a contradiction. Thus we may assume that $\omega(G) > \delta(G)$. Let $\delta(G) = t$ and $\delta(G) + \omega(G) = k$. It is enough to show that $G = H(n,k,t)$ or $G = Z(n,k,t)$. If $t = 2$, then $\omega(G) = k - 2$. By $c(G) < k$, it is easy to see that $G = Z(n,k,2)$. Thus we may assume that $t \geqslant 3$.

Let $H$ be a complete subgraph of $G$ on $k-t$ vertices. Clearly, $G$ is not a complete graph, as otherwise $c(G) = n$ and we are done. Hence there is an edge $xy$ in $\overline{G}$ between $V(H)$ and $V(G) \setminus V(H)$, otherwise we have $\omega(G) > k-t$, contradicting $\delta(G) + \omega(G) = k$. By the maximality of $G$, if we adding $xy$ to $G$, then $G$ contains a cycle of length at least $\min\{n,k\}$, implying there is a path on at least $\min\{n,k\}$ vertices starting from $x$, and ending at $y$. Thus we can choose a maximal such path $P = x_1x_2Px_{m-1}x_m$ with $m \geqslant \min\{n,k\}$ starting from $x_1 \in V(H)$ and ending at $x_m \in V(G) \setminus V(H)$ ($N_H(x_1) \subseteq N_P(x_1)$, $N_G(x_m) = N_P(x_m)$ and it is possible that $(x_1, x_m) \neq (x, y)$.) Then we have $d_P(x_1) \geqslant d_H(x) = k - t - 1$ and $d_P(x_m) \geqslant \delta(G) \geqslant t$. Since $c(G) < \min\{n,k\}$, Lemma 4 implies that $P$ contains a minimal crossing pair $(i,j)$, $N_P[x_1] = N_H[x_1] = V(H)$, $d_P(x_m) = t$ and $n \geqslant k$. In fact, if $P$ does not contain a minimal crossing pair, then $c(G) \geqslant k \geqslant \min\{n,k\}$; if $N_P[x_1] \neq N_H[x_1]$ or $d_P(x_m) > t$, then $c(G) \geqslant (k - t - 1 + t) + 1 = k \geqslant \min\{n,k\}$ and if $n < k$, then $c(G) \geqslant k - t - 1 + t = k - 1 \geqslant \min\{n,k\}$. Hence we have $m \geqslant \min\{n,k\} \geqslant k$. Note that $N_P^-(x_1) \cap N_P[x_m] = \emptyset$ and $N_P^+(x_m) \cap N_P[x_1] = \emptyset$ (otherwise, there is a cycle of length at least $k$). The cycle $x_1Px_ix_mPx_jx_1$ of length $(k - t - 1) + (t + 1) - 1 = k - 1$ ($-1$ for the vertex $x_{j-1}$) implies for any minimal crossing pair $(i, j)$, we have

$$(N_P^-(x_1) \cup N_P[x_m]) \setminus \{x_{j-1}\} = V(x_1Px_i) \cup V(x_jPx_m) \text{ and } j - i = m - k + 2, \quad (1)$$

implying that each vertex in $V(x_1Px_i) \cup V(x_jPx_m)$ either belongs to $N_P[x_m]$ or belongs to $N_P^-(x_1)$.

Let $s_1 = \min\{h : x_h \in N_P(x_m)\}$ and $s_2 = \max\{h : x_h \in N_P(x_1)\}$. Consider the $m$-vertex paths $P^* = x_hPx_mx_{h-1}Px_1$ for $s_2 + 1 \leqslant h \leqslant m - 1$ (by $c(G) < k$, the possible neighbors of $x_h$ in $P^*$ are determined by the neighbors of $x_1$ in $P^*$). Since $c(G) < k$ and $\delta(G) \geqslant t$, the maximality of $m$ implies $N_P[x_h] = N_P[x_m]$ for $s_2 + 1 \leqslant h \leqslant m - 1$. Hence $x_m$ is not adjacent to any two consecutive vertices of $x_{s_1}Px_{s_2}$, otherwise $x_{\ell-1}Px_1x_{s_2}Px_\ell x_{s_2+1}Px_m$ is a cycle of length $k$ ($s_1+1 \leqslant \ell \leqslant s_2$), a contradiction. Similarly, $x_1$ is not adjacent to any two consecutive vertices of $x_{s_1}Px_{s_2}$ and $N_P[x_h] = N_P[x_1]$ for $2 \leqslant h \leqslant s_1 - 1$. From (1), we conclude that $C = N_P[x_1] \setminus V(x_1Px_{s_1-1}) = N_P[x_m] \setminus V(x_mPx_{s_2+1}) = \{x_{s_1}, x_{s_1+2}, \ldots, x_{s_2}\}$ where $s_1 \equiv s_2 \mod 2$ when $m = k$ (see Fig. 2); or $C = N_P[x_1] \setminus V(x_1Px_{s_1-1}) = N_P[x_m] \setminus V(x_mPx_{s_2+1}) = \{x_{s_1}, x_{s_2}\}$ with $s_2 - s_1 = m - k + 2$ when $m \geqslant k$.

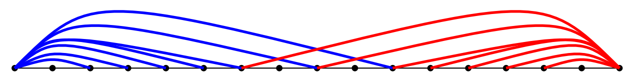

Figure 2. The neighors of $x_1$ and $x_m$ on $P$

Let $A = V(x_1Px_{s_1-1})$, $B = V(x_{s_2+1}Px_m)$ and $X = V(G) \setminus \{A \cup B \cup C\}$. Note that $G[A \cup C] = H$ and $G[B \cup C] = K_{t+1}$ are complete graphs (see Fig. 3).

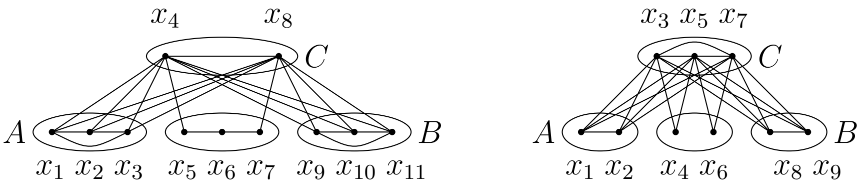

Figure 3. Subgraphs of $G$ with $k = 9$, $\delta(G) = 4$ and $\omega(G) = 5$

**Claim.** Each vertex of $X$ can only be connected to $C \subsetneq A \cup B \cup C$. Moreover, if $|C| \geqslant 3$, then $X$ is an independent set.

*Proof.* Let $x \in V(P) \cap X$. If $x$ is connected to $A \cup B \subsetneq A \cup B \cup C$, then in both cases ($|C| = 2$ and $|C| \geqslant 3$) we can easily find a cycle of length at least $k$, a contradiction. Now, let $x \in X \setminus V(P)$. Since $G$ is 2-connected, $x \in X$ is connected to $\{y_1, y_2\} \subseteq V(P)$ by two vertex-disjoint (except $x$) paths. If $y_1, y_2 \in A$ or $y_1, y_2 \in B$, then there is path on $m + 1$ vertices containing $V(P) \cup \{x\}$ starting from $A \subseteq V(H)$, ending at $B$, a contradiction to the maximality of $m$. If $y_1 \in A \cup B$ and $y_2 \in C$ or $y_1 \in A$ and $y_2 \in B$, then in both cases ($|C| = 2$ and $|C| \geqslant 3$) we can easily find a cycle of length at least $k$, a contradiction.

Now we prove the moreover part. Let $|C| \geqslant 3$. It is easy to see that for any two vertices $u, v \in V(P) \setminus (A \cup B \cup C)$ there is a path on $k$ vertices starting from $u$ and ending at $v$ (see Fig. 3). Hence $V(P) \setminus (A \cup B \cup C)$ is an independent set. Note that, in $G[V(P)]$, for any two vertices $u, v$ not both in $C$, there is a path on at least $k-1$ vertices starting from $u$ and ending at $v$ and for any two vertices $u, v$ in $C$, there is a path on $k-2$ vertices starting from $u$ and ending at $v$ (see Fig. 3). Thus by $G$ is 2-connected and $c(G) < k$, $X \setminus V(P)$ is an independent set and can be adjacent to $C$ of $V(P)$. □

If $|C| \geqslant 3$, then each vertex of $X$ can only be adjacent to $C$ in $G$ and $X$ is an independent set. Since $\delta(G) = t$, we must have $|C| = t$ and each vertex of $X$ is adjacent to all vertex of $C$. Thus it follows from the maximality of $G$ that $G = H(n, k, t)$.

Suppose that $|C| = 2$. Then $G[A]$ and $G[B]$ are cliques of sizes $k - t - 2$ and $t - 1$. Moreover, we have $\omega(G) - 2 = |A| \geqslant |B| = \delta(G) - 1$ by $\omega(G) > \delta(G)$. Recall that $G[A \cup C] = H$ and each vertex of $X$ is not adjacent to $B$. It is easy to see that the graph $G - B$ is still 2-connected with $\delta(G - B) \geqslant t$ (the vertices in $C \subseteq V(H)$ have degree at least $k - t - 1 \geqslant \delta(G)$ in $G - B$), $\omega(G - B) = k - t$ and $c(G - B) < k$.

Since $\delta(G - B) \geqslant t$ and each vertex of $X$ is not adjacent to $A \cup B$ by the claim, we have $|X| \geqslant t - 1$. If $|X| = t - 1$, then $G[X]$ is a complete graph on $t - 1$ vertices, whence $G = Z(k - t + 2(t - 1), k, t)$. Let $|X| \geqslant t$. Then $G - B$ has at least $k - t + t = k$ vertices.

We will show that $G - B$ is also an edge-maximal counter-example. Let $e = uv \in E(\overline{G - B})$. If $e$ is not incident with $A$, then $G + e$ contains a cycle of length at least $k$ containing no vertex of $B$ (by $|A| \geqslant |B|$), since any cycle containing $e$ ($e$ is not incident with $A$ and $B$) in $G + e$ cannot contain both of some vertex $a \in A$ and some vertex $b \in B$. Without loss of generality, let $u \in A$. We consider the path $\widetilde{P} = x_1 P x_{s_1} x_{s_2+1} P x_m x_{s_2} P x_{s_1+1}$. From the previous discussions, there exists a vertex set $B' = V(x_{s_2-1} P x_{s_1+1})$ such that $G[B' \cup C]$ is a complete graph on $t+1$ vertices. Moreover, each vertex of $B'$ is not adjacent to any vertex of $V(G) \setminus (B' \cup C)$. Since $G$ is 2-connected, by $|X| \geqslant t$, there is a vertex $z \in X \setminus B'$ which is connected to $C$ by two vertex-disjoint (except $z$) paths. If $v \in B'$, then $(G - B) + e$ contains a cycle of length at least $k$ containing $A \cup B' \cup C \cup \{z\}$. Let $v \notin B'$. If there is a cycle of length at least $k$ in $G + e$ ($e$ is not incident with $B$ and $B'$) contains vertices in $B$, then we can replace the vertices in $B$ with the vertices in $B'$ to get a cycle of length at least $k$ in $(G - B) + e$. Thus $G - B$ is also an edge-maximal counter-example.

Applying the above arguments to $G - B$, if $\delta(G - B) > t$, then repeat the previous proof, we have $c(G - B) \geqslant k$ by Lemma 4; if $G - B = H(n - t + 1, k, t)$ (the case $|C| = 3$ in the above arguments), then put back $B$, we can easily find a cycle of length $k$ in $G$ containing all vertices of $A$ and $B$ ($|B| = t - 1 \geqslant 2$), a contradiction; otherwise there is a copy of $K_{t-1}$ (the case $|C| = 2$ in the above arguments) joining only to $C$ such that after deleting it, the result graph $G^*$ on $n - 2(t - 1) \geqslant k$ vertices is still a 2-connected edge-maximal counter-example with $\delta(G^*) \geqslant t$, $\omega(G^*) = k - t$ and $c(G^*) < k$, whence $G = Z(n, k, t)$ by applying the above arguments repeatedly until $n - \ell(t - 1) = k - 1$ for some $\ell$. The proof of our theorem is complete. □


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