Directed graphs without rainbow stars

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Abstract

In a rainbow version of the classical Turán problem one considers multiple graphs on a common vertex set, thinking of each graph as edges in a distinct color, and wants to determine the minimum number of edges in each color, or their sum, which guarantees existence of a rainbow copy (having each edge from a different graph) of a given graph. Here, we find an optimal solution for this problem, both for the minimum and the sum, for any directed star and any number of colors.

Mathematics Subject Classifications: 05C20, 05C35

1 Introduction

One of the central topics in extremal graph theory, known as the Turán problem, is to determine the maximum number of edges of a graph on n vertices that does not contain a copy of a given graph F as a subgraph. Equivalently, the minimum number of edges that forces the existence of F as a subgraph. Research on this topic and its various generalizations, provides a deep understanding of the relationship between various global properties and local structures of graphs.

Recently, a rainbow version of this problem has been intensively studied. In this variant, for an integer $c \geqslant 1$ we consider a collection of c graphs $\mathcal{G} = (G_1, \dots, G_c)$ on a common vertex set and say that a graph F is a *rainbow subgraph* of $\mathcal{G}$ (or $\mathcal{G}$ *contains* F) if there exists an injective function $\varphi : E(F) \to [c]$ such that for each $e \in E(F)$ it holds $e \in G_{\varphi(e)}$. In other words, we think of each graph in $\mathcal{G}$ as edges in a distinct color and $\mathcal{G}$ as a c-edge-colored multigraph with each color spanning a simple graph. We want to force the existence of a rainbow copy of F in $\mathcal{G}$ by having a large number of edges in each graph. Typically, by bounding the value of $\min_{1 \leqslant i \leqslant c} e(G_i)$ [1, 2], i.e., the number of edges

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in each graph, or the sum $\sum_{1\leqslant i\leqslant c} e(G_i)$ [4, 10], but other measures are also considered, in particular the product $\prod_{1\leqslant i\leqslant c} e(G_i)$ [7, 8], or more general functions of the number of edges [5, 9].

Such a rainbow Turán problem was also considered for directed graphs in [3], where the optimal solution (up to a lower order error term) for $\min_{1\leqslant i\leqslant c} e(G_i)$ and $\sum_{1\leqslant i\leqslant c} e(G_i)$ was provided, for any number of colors, when a directed or transitive rainbow triangle is forbidden. Here, we continue this line of research for directed graphs. As in the undirected setting, a natural next case to consider is the problem for forbidden trees, especially since a solution for trees gives lower bounds for any denser forbidden structures. It was shown in [9] that among all trees the least restrictive is to forbid stars, i.e., a solution for stars gives an upper bound for any tree, which motivates the study of stars. In contrast to the undirected setting, the rainbow Turán problem for directed stars is not so straightforward. In particular, as shown in Theorem 4, there are different and quite unintuitive extremal constructions.

Let $S_{p,q}$ be the orientation of a star on $p+q+1$ vertices with *center* vertex of indegree p and outdegree q. Forbidding a rainbow $S_{p,q}$ in a collection of graphs $\mathcal{G} = (G_1, \ldots, G_c)$ is analogous to forbidding a rainbow $S_{q,p}$ in the collection of graphs obtained by changing the orientation of every edge in each graph from $\mathcal{G}$. Thus it is enough to consider this rainbow Turán problem for $S_{p,q}$ only when $p \leqslant q$. As this problem is trivial when the number of colors c is less than the number of edges in the forbidden rainbow graph, we consider only $c \geqslant p+q$.

In Section 2 we consider a star $S_{0,q}$ as the forbidden rainbow graph and prove, for every $n > c \geqslant q \geqslant 1$, exact bounds for $\sum_{i=1}^{c} e(G_i)$ (Theorem 1) and $\min_{1\leqslant i\leqslant c} e(G_i)$ (Theorem 2). In Section 3 we consider $S_{p,q}$ as the forbidden rainbow graph for any $q \geqslant p \geqslant 1$ and prove bounds for $\sum_{i=1}^{c} e(G_i)$ (Theorem 3) and $\min_{1\leqslant i\leqslant c} e(G_i)$ (Theorem 4), which are tight up to a lower order error term. Additionally, in Section 4 we provide exact bounds for any $c \geqslant 2$ and $n \geqslant 3$ when a rainbow $S_{1,1}$, i.e., directed path of length 2, is forbidden, for both the sum (Theorem 5) and the minimum (Theorem 6) of the number of edges.

2 Rainbow directed $S_{0,q}$

In this section the forbidden rainbow graph is a directed star with all edges oriented away from the center. As noted earlier, the problem is the same if we forbid a directed star with all edges oriented to the center. The following theorem provides the optimal bound for the sum of the number of edges in all graphs.

Theorem 1. *For integers $n > c \geqslant q \geqslant 1$, every collection of directed graphs $G_1, \ldots, G_c$ on a common set of n vertices containing no rainbow $S_{0,q}$ satisfies*

$$\sum_{i=1}^{c} e(G_i) \leqslant (q-1)(n^2 - n).$$

Moreover, this bound is sharp.

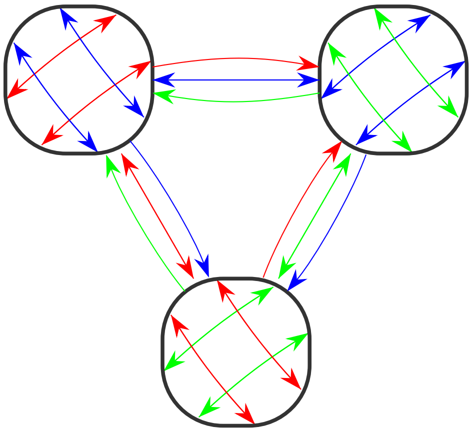

Figure 1: The optimal construction for 3 colors and forbidden rainbow $S_{0,3}$.

Proof. The sharpness of the bound follows from taking a collection of $q-1$ complete graphs, but there are more extremal constructions. We split the vertex set into disjoint subsets and *assign* to each of them a different set of $q-1$ colors. This way each vertex has $q-1$ colors assigned. Then, for any two vertices u, v and color $i \in [c]$, we add edge uv to G_i if color i is assigned to u. Clearly this construction does not contain a rainbow $S_{0,q}$ as no vertex has positive outdegree in q graphs, and each vertex has the sum of outdegrees over all graphs equal to $(q-1)(n-1)$ giving the total number of edges equal to $(q-1)(n^2-n)$. An example construction of this type is shown in Figure 1.

To prove the upper bound, consider a collection of graphs $G_1, G_2, \dots, G_c$ on a common set V of n vertices that does not contain a rainbow $S_{0,q}$. Let $v \in V$ be an arbitrary vertex. We will show that the total number of edges outgoing from v is bounded by $(q-1)(n-1)$. Let H be an auxiliary bipartite graph between the vertices $V \setminus \{v\}$ and colors in $[c]$ created by connecting vertex $u \in V \setminus \{v\}$ and color $i \in [c]$ if $vu \in E(G_i)$. The number of edges in H is equal to the number of outgoing edges from v that we want to bound. Note that the existence of a matching of size q in H means that there exists a rainbow $S_{0,q}$ with center v, which is not possible. Therefore, the maximum matching in H is of size at most $q-1$. Recall that König's theorem states that the maximum matching of H is at least as large as the minimum vertex cover. Hence the minimum vertex cover is also of size at most $q-1$. As $n-1 \geqslant c$, this implies that the maximum number of edges in H is bounded by $(q-1)(n-1)$, as desired. By summing it over all vertices we obtain $\sum_{i=1}^{c} e(G_i) \leqslant (q-1)(n^2-n)$. □

Note that the bound $n > c$ in Theorem 1 is indeed needed as demonstrated by the following collections of graphs. If $c \geqslant n \geqslant q$, for each vertex v, add edges in each color from v to $q-1$ other arbitrary vertices. This way

$$\sum_{i=1}^{c} e(G_i) = (q-1)cn > (q-1)(n^2-n)$$

and there is no rainbow $S_{0,q}$. While if $n \leqslant q$, then a collection of complete directed graphs does not contain rainbow $S_{0,q}$ and satisfies

$$\sum_{i=1}^{c} e(G_i) = c(n^2 - n) > (q-1)(n^2 - n).$$

The same constructions also provide an exact bound for $\min_{1 \leqslant i \leqslant c} e(G_i)$ for any $n \leqslant c$.

Theorem 1 implies that for integers $n > c \geqslant q \geqslant 1$, every collection of directed graphs $G_1, \dots, G_c$ on a common set of n vertices containing no rainbow $S_{0,q}$ satisfies

$$\min_{1 \leqslant i \leqslant c} e(G_i) \leqslant \frac{q-1}{c}\left(n^2 - n\right).$$

Moreover, if $n(q-1)$ is divisible by c, then one can make a construction as detailed in the beginning of the proof of Theorem 1, in which the number of edges in each graph is the same (see Figure 1). This means that for such n the above bound is sharp. We can actually extend this to obtain the optimal bound for every $n > c$.

Theorem 2. *For integers $n > c \geqslant q \geqslant 1$, every collection of directed graphs $G_1, \dots, G_c$ on a common set of n vertices containing no rainbow $S_{0,q}$ satisfies*

$$\min_{1 \leqslant i \leqslant c} e(G_i) \leqslant \left\lfloor \frac{n(q-1)}{c} \right\rfloor (n-1) + r,$$

where r is the remainder of $n(q-1)$ when divided by c. Moreover, this bound is sharp.

Proof. We proceed similarly as in the proof of Theorem 1. Consider a collection of graphs $G_1, G_2, \dots, G_c$ on a common set V of n vertices that does not contain a rainbow $S_{0,q}$. For any vertex $v \in V$ we consider an auxiliary bipartite graph H between the vertices $V \setminus \{v\}$ and colors in $[c]$ created by connecting vertex $u \in V \setminus \{v\}$ and color $i \in [c]$ if $vu \in E(G_i)$. Since the existence of a matching of size q in H means that there exists a forbidden rainbow $S_{0,q}$ centered in v, the maximum matching in H is of size $q-1$. From König's theorem the minimum vertex cover is also of size at most $q-1$. Let a_v be the number of vertices in the minimum vertex cover that are in the part of H related with the colors. Similarly, let b_v denote the number of vertices in the same minimum vertex cover that are in the other part of H, related with the vertices $V \setminus \{v\}$. In particular, in a_v colors there are at most $n-1$ edges outgoing from v and in all other colors there are at most b_v edges outgoing from v. Let $a = \sum_{v \in V} a_v$ and $b = \sum_{v \in V} b_v$. Since $a_v + b_v \leqslant q-1$, we have $a + b \leqslant n(q-1)$.

There exists a color i which appears at most $\left\lfloor \frac{a}{c} \right\rfloor$ times in the minimum vertex covers, which gives that

$$e(G_i) \leqslant \left\lfloor \frac{a}{c} \right\rfloor (n-1) + b \leqslant \left\lfloor \frac{n(q-1) - b}{c} \right\rfloor (n-1) + b.$$

Note that increasing b by 1 either increases the above bound by 1 or decreases it by $n-2$, and the decrease happens after at most $c-1$ increases. Since $n > c$, the maximum value of the bound is achieved in the last moment before the first decrease, which occurs for b equal to the remainder of $n(q-1)$ when divided by c, as desired.

The sharpness of the bound follows from a modification of a construction described in the proof of Theorem 1. We enumerate the vertex set by consecutive integers from 1 to n and assign to every vertex $j \in [n]$ all colors from $(j-1)(q-1)$ to $j(q-1)-1$ considered modulo c. This way each vertex has $q-1$ colors assigned and in total we have $n(q-1)$ assignments. Now, remove the last r assignments. Since r is the remainder of $n(q-1)$ when divided by c, after the removal each color was assigned the same number of times. For every $i \in [c]$, add to G_i an edge from each vertex with color i assigned to any other vertex. This gives $\left\lfloor \frac{n(q-1)}{c} \right\rfloor (n-1)$ edges in each graph. Note that every vertex v having x assignments removed has positive outdegree in exactly $q-1-x$ graphs, so we may still add edges from v to arbitrary x vertices and avoid creating a rainbow $S_{0,q}$. This addition increases the number of edges in each graph by the total number of removed assignments, which is equal to r. Altogether we obtain the required number of edges in each graph. □

3 General rainbow directed star

In this section, for integers $p, q, c \geqslant 1$, we consider a collection of c directed graphs $G_1, \dots, G_c$ on a common vertex set and forbid rainbow star $S_{p,q}$, which is a star with the center of indegree p and outdegree q. Since the problem is trivial if $c < p+q$ and symmetric with respect to p and q, it is enough to consider only $c \geqslant p+q$ and $p \leqslant q$. The following theorem provides the optimal bound for the total number of edges in all graphs.

Theorem 3. *For integers $q \geqslant p \geqslant 1$, $c \geqslant p+q$ and n, every collection of directed graphs $G_1, \dots, G_c$ on a common set of n vertices containing no rainbow $S_{p,q}$ satisfies*

$$\sum_{i=1}^{c} e(G_i) \leqslant \begin{cases} (p+q-1)n^2 + o(n^2) & \text{if} \quad c \leqslant p+2q-1+2\sqrt{pq}, \\ \left(\frac{(c-p+1)^2}{4(c-q+1)} + p - 1\right) n^2 + o(n^2) & \text{if} \quad c \geqslant p+2q-1+2\sqrt{pq}. \end{cases}$$

Moreover, the above bounds are tight up to a lower order error term.

Proof. The claimed bound is tight in the first case when the collection consists of exactly $p+q-1$ complete directed graphs and all other graphs are empty. This construction obviously has no rainbow $S_{p,q}$. In the second case, as the first $p-1$ graphs we take complete directed graphs, while for the remaining graphs we split the vertex set into two disjoint sets A of size $\frac{c-p+1}{2(c-q+1)}n$ and C of size $\frac{c+p-2q+1}{2(c-q+1)}n$. For $i \in [q-1] \setminus [p-1]$, in graph G_i we add all edges inside A and from C to A. While for $i \in [c] \setminus [q-1]$, in graph G_i we add all edges from C to A. Note that vertices in A have nonzero outdegree only in $q-1$ graphs, while vertices in C have nonzero indegree in $p-1$ graphs, so none of them can be the center of a rainbow $S_{p,q}$. The total number of edges in all graphs in such construction is equal to

$$(p-1)n^2 + |A|^2(q-p) + |A||C|(c-p+1) + o(n^2) = \left(\frac{(c-p+1)^2}{4(c-q+1)} + p - 1\right) n^2 + o(n^2),$$

as desired.

To prove the upper bound, consider a collection of directed graphs $G_1, \dots, G_c$ on a common set V of n vertices containing no rainbow $S_{p,q}$. Note that a vertex in V may have nonzero indegree in p graphs and nonzero outdegree in different q graphs, for example if a pair of vertices is connected by edges in all the graphs in both directions. Or more generally, if a vertex is the center of a noninjective homomorphic image of a rainbow $S_{p,q}$. Recall that a colored graph removal lemma, implied by the Szemerédi Regularity Lemma (see [6] or Theorem 4.3 in [9]), states that for any graph F if a collection of graphs on n vertices has $o(n^{|V(F)|})$ homomorphic images of a rainbow F, then one can delete $o(n^2)$ edges in total to remove all homomorphic images of a rainbow F. Therefore, since the collection $G_1, \dots, G_c$ contains no rainbow $S_{p,q}$, it contains $o(n^{p+q+1})$ homomorphic images of a rainbow $S_{p,q}$ and we can remove all of them by deleting $o(n^2)$ edges in total. Thus, we may assume that no vertex in V has nonzero indegree in p graphs and nonzero outdegree in q different graphs.

We split the vertex set V into three disjoint sets. Let B be the set of vertices incident to edges in at most $p+q-1$ graphs, A be the set of vertices in $V \setminus B$ that have nonzero outdegree in at most $q-1$ graphs, and C be the set of vertices in $V \setminus B$ having nonzero indegree in at most $p-1$ graphs. Let us denote $\alpha = |A|$, $\beta = |B|$ and $\gamma = |C|$.

Note that any two vertices in A may be connected by at most $2(q-1)$ edges ($q-1$ in each direction), vertices in B by at most $2(p+q-1)$ edges, while vertices in C by at most $2(p-1)$ edges. Additionally, between any vertices in A and B we have at most $p+2q-2$ edges ($q-1$ from A to B and $p+q-1$ from B to A), between vertices in B and C there are at most $2p+q-2$ edges, while between vertices in A and C we have at most $c+p-1$ edges. This gives an upper bound for the total number of edges in all the graphs of

$$
\begin{aligned}
&(q-1)\alpha^2 + (p+q-1)\beta^2 + (p-1)\gamma^2 + (p+2q-2)\alpha\beta + (2p+q-2)\beta\gamma + (c+p-1)\alpha\gamma \\
&= \left(\frac{(c-p+1)^2}{4(c-q+1)} + p - 1\right)(\alpha+\gamma)^2 - (c-q+1)\left(\frac{c-p+1}{2(c-q+1)}(\alpha+\gamma) - \alpha\right)^2 \\
&\quad + (p+q-1)\beta^2 + (p+2q-2)\alpha\beta + (2p+q-2)\beta\gamma \\
&\leqslant \left(\frac{(c-p+1)^2}{4(c-q+1)} + p - 1\right)(\alpha+\gamma)^2 + (p+q-1)\beta^2 + (p+2q-2)\beta(\alpha+\gamma) \\
&= \left(\frac{(c-p+1)^2}{4(c-q+1)} + p - 1\right)(n-\beta)^2 + (p+q-1)\beta^2 + (p+2q-2)\beta(n-\beta).
\end{aligned}
$$

The obtained bound is a quadratic function of $\beta \in [0, n]$ with the coefficient of β^2 equal to $\frac{(c+p-2q+1)^2}{4(c-q+1)}$, so it is convex and its maximum is reached for $\beta = 0$ or $\beta = n$. It is easy to verify that for $c \leqslant p + 2q - 1 + 2\sqrt{pq}$ the maximum occurs when $\beta = n$, while for $c \geqslant p + 2q - 1 + 2\sqrt{pq}$ it occurs when $\beta = 0$, which gives the desired bounds. □

The bound in Theorem 3 is not realizable for $p \geqslant 2$ in any collection of directed graphs each having the same number of edges. This is due to the fact that in order to have exactly $c+p-1$ edges between the vertices in A and C, all edges from A to C must be in the same $p-1$ colors, and so in other colors we have fewer edges. Thus, for $p \geqslant 2$ the optimal bound for $\min_{1\leqslant i\leqslant c} e(G_i)$ is different. On the other hand, when $p=1$ the second bound in Theorem 3 can be obtained in such a collection of directed graphs, so this theorem implies the optimal bound for $\min_{1\leqslant i\leqslant c} e(G_i)$ for $c \geqslant 2q+2\sqrt{q}$. It occurs that the same construction gives the optimal bound already for $c \geqslant \max\{q+\sqrt{q}, 2q-2\}$, but for smaller values of c there are different optimal constructions.

The theorem below gives the optimal bound for $\min_{1\leqslant i\leqslant c} e(G_i)$ for any integers p and q.

Theorem 4. *For integers $q \geqslant p \geqslant 1$, $c \geqslant p+q$ and n, every collection of directed graphs $G_1, \dots, G_c$ on a common set of n vertices containing no rainbow $S_{p,q}$ satisfies the following. The values*

$$t_1 = 2p+q-1, \qquad t_2 = \begin{cases} \frac{(q-1)(p+q-1)}{q-p-1} & \text{if} \quad q \geqslant p+2, \\ \infty & \text{if} \quad q \leqslant p+1, \end{cases}$$

$$t_3 = p+q-1+\sqrt{pq}, \qquad t_4 = q-1+\sqrt{(q-1)(q-p)}$$

satisfy $t_1 \leqslant t_2$, $t_1 \leqslant t_3$, and either $t_2 \leqslant t_3 \leqslant t_4$ or $t_4 \leqslant t_3 \leqslant t_2$.
If $t_2 \leqslant t_3 \leqslant t_4$, then

$$\min_{1\leqslant i\leqslant c} e(G_i) \leqslant \begin{cases} \frac{(p+q-1)^2}{c^2} n^2 + o(n^2) & \text{if} \quad c \leqslant t_1, \\ \frac{(c-q+1)^2(p+q-1)^2}{4c^2 p(c-p-q+1)} n^2 + o(n^2) & \text{if} \quad t_1 \leqslant c \leqslant t_2, \\ \frac{q-1}{c} n^2 + o(n^2) & \text{if} \quad t_2 \leqslant c \leqslant t_4, \\ \frac{(c^2-(p-1)(q-1))^2}{4c^2(c-p+1)(c-q+1)} n^2 + o(n^2) & \text{if} \quad c \geqslant t_4. \end{cases}$$

While if $t_4 \leqslant t_3 \leqslant t_2$, then

$$\min_{1\leqslant i\leqslant c} e(G_i) \leqslant \begin{cases} \frac{(p+q-1)^2}{c^2} n^2 + o(n^2) & \text{if} \quad c \leqslant t_1, \\ \frac{(c-q+1)^2(p+q-1)^2}{4c^2 p(c-p-q+1)} n^2 + o(n^2) & \text{if} \quad t_1 \leqslant c \leqslant t_3, \\ \frac{(c^2-(p-1)(q-1))^2}{4c^2(c-p+1)(c-q+1)} n^2 + o(n^2) & \text{if} \quad c \geqslant t_3. \end{cases}$$

Moreover, the above bounds are tight up to a lower order error term.

Proof. First, we prove the inequalities between thresholds t_i for $i \in [4]$. Note that $t_3 \geqslant t_1$ since $q \geqslant p$. If $q \leqslant p+1$, then clearly $t_4 \leqslant t_3 \leqslant t_2$ and $t_1 \leqslant t_2$. While for $q \geqslant p+2$ we have

$$t_2 = \frac{(q-1)(p+q-1)}{q-p-1} = \frac{(q-p-1+p)(q-p-1+2p)}{q-p-1} = q+2p-1+\frac{2p^2}{q-p-1} \geqslant t_1.$$

One can also verify that for $q \geqslant p+2$ each of the inequalities $t_2 \leqslant t_3$ and $t_3 \leqslant t_4$ is equivalent to the inequality $q(q-p-1)^2 \geqslant p(p+q-1)^2$, so either $t_2 \leqslant t_3 \leqslant t_4$ or $t_4 \leqslant t_3 \leqslant t_2$, as desired.

Consider a collection of directed graphs $G_1, \dots, G_c$ on a common set V of n vertices containing no rainbow $S_{p,q}$. Similarly as in the proof of Theorem 3 we use a colored graph removal lemma to remove all homomorphic images of a rainbow $S_{p,q}$ by deleting $o(n^2)$ edges in total. Thus, we may assume that no vertex in V has nonzero indegree in p graphs and nonzero outdegree in q different graphs.

We split the vertex set V into disjoint sets. Let B be the set of vertices incident to edges in at most $p+q-1$ graphs, A be the set of vertices in $V \setminus B$ that have nonzero outdegree in at most $q-1$ graphs, and C be the set of vertices in $V \setminus B$ having nonzero indegree in at most $p-1$ graphs. Additionally, for each $i \in [c]$, let $A_i \subset A$ be the set of vertices in A that have nonzero outdegree in G_i, similarly $C_i \subset C$ be the set of vertices in C that have nonzero indegree in G_i, while $B_i \subset B$ be the set of vertices in B incident to edges in G_i. For every $i \in [c]$, we denote $\alpha_i = |A_i|$, $\beta_i = |B_i|$, $\gamma_i = |C_i|$, $\alpha = |A|$, $\beta = |B|$ and $\gamma = |C|$.

Observe that for every $i \in [c]$,

$$e(G_i) \leqslant (\alpha_i + \beta_i + \gamma)(\alpha + \beta_i + \gamma_i), \tag{1}$$

because edges of G_i can go only from vertices in $A_i \cup B_i \cup C$ to vertices in $A \cup B_i \cup C_i$.

For integers $x, y \geqslant 1$, by averaging over all colors, there is $j \in [c]$ such that

$$y\alpha + x\alpha_j + (x+y)\beta_j + x\gamma + y\gamma_j \leqslant \frac{1}{c}\sum_{i=1}^{c} y\alpha + x\alpha_i + (x+y)\beta_i + x\gamma + y\gamma_i.$$

Since

$$\sum_{i=1}^{c} \alpha_i \leqslant (q-1)\alpha, \qquad \sum_{i=1}^{c} \beta_i \leqslant (p+q-1)\beta \qquad \text{and} \qquad \sum_{i=1}^{c} \gamma_i \leqslant (p-1)\gamma,$$

from (1) we obtain

$$\begin{aligned} e(G_j) &\leqslant \frac{1}{xy}(x\alpha_j + x\beta_j + x\gamma)(y\alpha + y\beta_j + y\gamma_j) \\ &\leqslant \frac{1}{4xy}\big(y\alpha + x\alpha_j + (x+y)\beta_j + x\gamma + y\gamma_j\big)^2 \\ &\leqslant \frac{1}{4c^2xy}\big((yc + x(q-1))\alpha + (x+y)(p+q-1)\beta + (xc + y(p-1))\gamma\big)^2. \end{aligned} \tag{2}$$

In particular, for $x = 1$ and $y = 1$, since $p \leqslant q$ and $\alpha + \gamma = n - \beta$, this gives

$$\begin{aligned} e(G_j) &\leqslant \frac{1}{4c^2}\big((c+q-1)\alpha + 2(p+q-1)\beta + (c+p-1)\gamma\big)^2 \\ &\leqslant \frac{1}{4c^2}\big((c+q-1)(\alpha+\gamma) + 2(p+q-1)\beta\big)^2 \\ &= \frac{1}{4c^2}\big((c+q-1)n + (2p+q-1-c)\beta\big)^2. \end{aligned}$$

If $c \leqslant t_1$, then the above expression is maximized at $\beta = n$ and we obtain

$$e(G_j) \leqslant \frac{(p+q-1)^2}{c^2} n^2,$$

which gives the first bound in both cases of the theorem. This bound is achieved if $A = \emptyset$, $C = \emptyset$ and B is divided into $\binom{c}{p+q-1}$ equal-sized sets, one for each subset of $p+q-1$ colors assigned to the set, with edges in G_i, for $i \in [c]$, between any vertices from sets having color i assigned. This is depicted in Figure 2 for $p = 1$, $q = 2$ and $c = 3$.

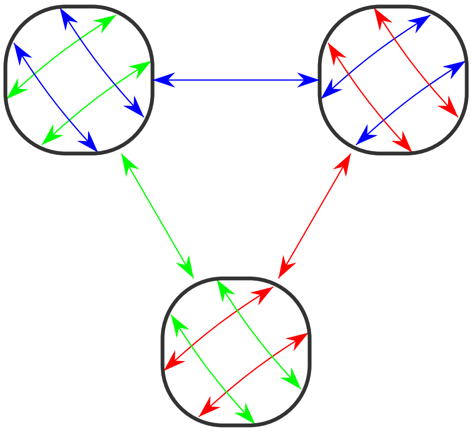

Figure 2: The optimal construction for 3 colors and a forbidden rainbow $S_{1,2}$.

Consider now $c \geqslant t_1$ and take $x = c - p - q + 1$ and $y = p$. This gives

$$yc + x(q-1) = (x+y)(p+q-1) = (c-q+1)(p+q-1),$$

so from (2) we obtain

$$\begin{aligned} e(G_j) &\leqslant \frac{1}{4c^2 p(c-p-q+1)} \big((c-q+1)(p+q-1)(\alpha+\beta) + ((c-p-q+1)c + p(p-1))\gamma\big)^2 \\ &= \frac{1}{4c^2 p(c-p-q+1)} \big((c-q+1)(p+q-1)n - (pq - (c-p-q+1)^2)\gamma\big)^2. \end{aligned}$$

If $c \leqslant t_3$, then the above expression is maximized at $\gamma = 0$ and gives

$$e(G_j) \leqslant \frac{(c-q+1)^2(p+q-1)^2}{4c^2 p(c-p-q+1)} n^2.$$

This bound is achieved when $C = \emptyset$, the set B is of size $\frac{(q-1)(p+q-1)-c(q-p-1)}{2p(c-p-q+1)} n$ and is divided into $\binom{c}{p+q-1}$ equal-sized sets, one for each subset of $p+q-1$ colors assigned to the set, while set A is of size $\frac{(p+q-1)(c-2p-q+1)}{2p(c-p-q+1)} n$ and is divided into $\binom{c}{q-1}$ equal-sized sets, one for each subset of $q-1$ colors assigned to the set. We put edges in G_i, for $i \in [c]$, between any vertex from a set in $A \cup B$ having color i assigned to any other vertex in A or a vertex in B having color i assigned. This construction is possible only if the mentioned sizes of sets A and B are non-negative, which occurs when $c \geqslant t_1$ and $c \leqslant t_2$, and so it proves the second bound in both cases of the theorem. Note that if $q \leqslant p+1$, then the size of B is always positive, which explains why t_2 is defined in this way. This construction is illustrated in Figure 3 for $p=1$, $q=2$ and $c=3$, despite the fact that in this case the optimal size of B is equal to 0.

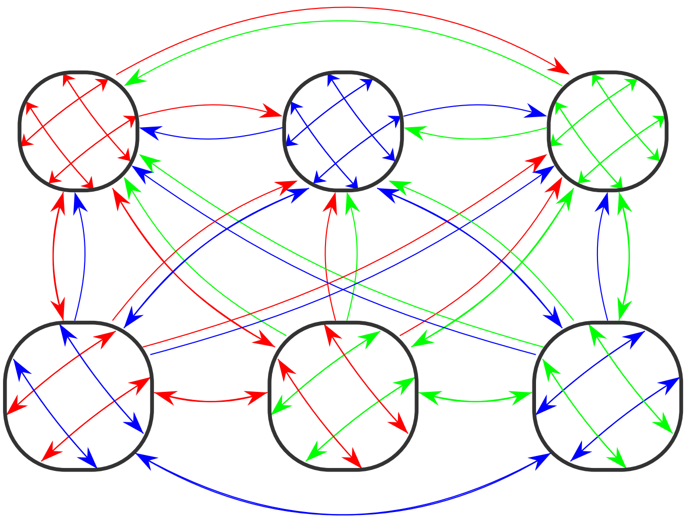

Figure 3: A construction for 3 colors and a forbidden rainbow $S_{1,2}$ with nonempty sets A and B.

In order to prove the third bound in the first case of the theorem, consider c satisfying $t_2 \leqslant c \leqslant t_4$ and take $x=c$ and $y=q-1$. From (2) we get

$$e(G_j) \leqslant \frac{1}{4c^3(q-1)}\big(2c(q-1)\alpha+(c+q-1)(p+q-1)\beta+(c^2+(p-1)(q-1))\gamma\big)^2.$$

The assumption $c \leqslant t_4$ implies that $c^2+(p-1)(q-1) \leqslant 2c(q-1)$, so

$$\begin{aligned} e(G_j) &\leqslant \frac{1}{4c^3(q-1)}\big(2c(q-1)(\alpha+\gamma)+(c+q-1)(p+q-1)\beta\big)^2 \\ &= \frac{1}{4c^3(q-1)}\big(2c(q-1)n-(c(q-p-1)-(q-1)(p+q-1))\beta\big)^2. \end{aligned}$$

Since $c \geqslant t_2$, the expression above is maximized at $\beta=0$ and gives

$$e(G_j) \leqslant \frac{q-1}{c}n^2,$$

which proves the third bound in the first case of the theorem. This is the same bound as when forbidding a rainbow $S_{0,q}$, so it is achieved when $B=C=\emptyset$, while set A is divided into $\binom{c}{q-1}$ equal-sized sets, one for each subset of $q-1$ colors assigned to the set, and we

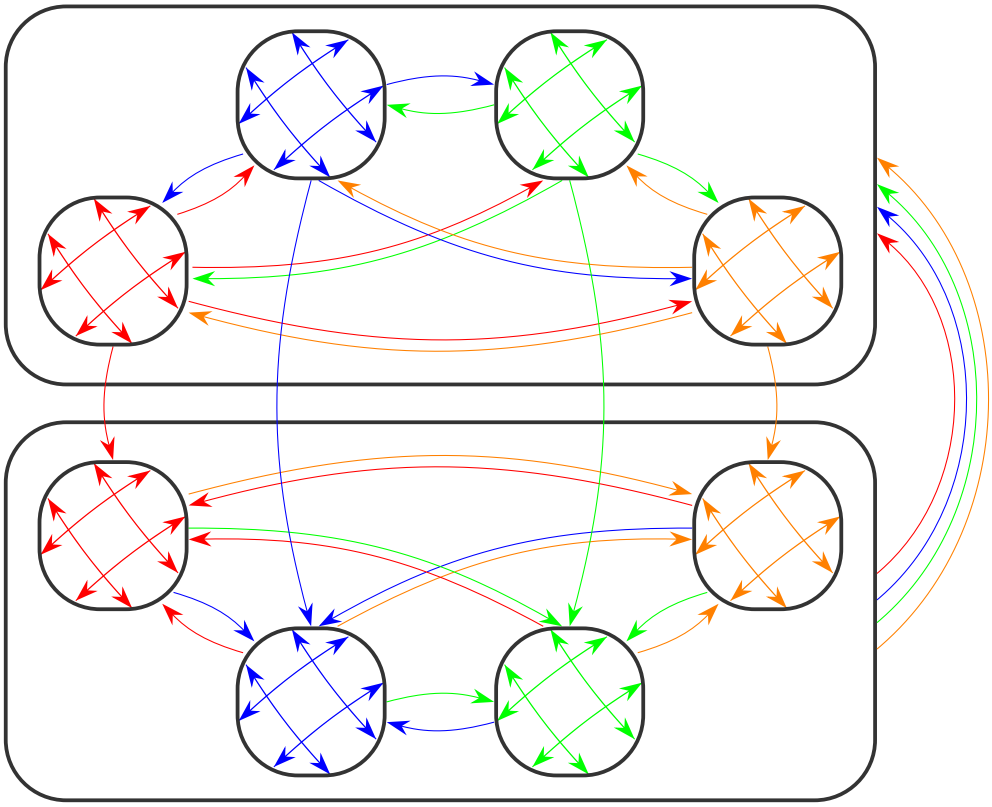

Figure 4: A construction for 4 colors and a forbidden rainbow $S_{2,2}$ with nonempty sets A and C.

put edges in G_i, for $i \in [c]$, between any vertex from a set having color i assigned to any other vertex. This is depicted for $q = 3$ and $c = 3$ in Figure 1.

Finally, consider the last remaining bound, where we have $c \geqslant t_3$ and $c \geqslant t_4$. From (2) for $x = c - p + 1$ and $y = c - q + 1$ we obtain

$$\begin{aligned} e(G_j) &\leqslant \frac{1}{4c^2(c-p+1)(c-q+1)}\big((c^2-(p-1)(q-1))(\alpha+\gamma) + (2c-p-q+2)(p+q-1)\beta\big)^2 \\ &= \frac{1}{4c^2(c-p+1)(c-q+1)}\big((c^2-(p-1)(q-1))n - ((c-p-q+1)^2 - pq)\beta\big)^2. \end{aligned}$$

Since $c \geqslant t_3$, the above expression is maximized at $\beta = 0$ and gives

$$e(G_j) \leqslant \frac{(c^2-(p-1)(q-1))^2}{4c^2(c-p+1)(c-q+1)}.$$

This is achieved when $B = \emptyset$, the set A is of size $\frac{(c-p+1)^2+(p-1)(q-p)}{2(c-p+1)(c-q+1)}n$ and is divided into $\binom{c}{q-1}$ equal-sized sets, one for each subset of $q-1$ colors assigned to the set, similarly, the set C is of size $\frac{(c-q+1)^2-(q-1)(q-p)}{2(c-p+1)(c-q+1)}n$ and is divided into $\binom{c}{p-1}$ equal-sized sets, one for each subset of $p-1$ colors assigned to the set. We put edges in G_i, for $i \in [c]$, from any vertex in A having color i assigned and any vertex in C to any vertex in A and any vertex in C having color i assigned. This is depicted in Figure 4 for $c = 4$ and a forbidden $S_{2,2}$ just for illustration, despite the fact that in this case it is not the extremal construction. The sizes of sets A and C are non-negative if $c \geqslant t_4$, so it gives the last bound in both cases of the theorem. $\square$

4 Rainbow directed $S_{1,1}$

In this section we consider $c \geqslant 2$ directed graphs $G_1, G_2, \ldots, G_c$ on a common vertex set and forbid a rainbow star $S_{1,1}$, which is a directed path of length 2. The following theorem provides the optimal bound for the sum of the number of edges in all graphs.

Theorem 5. *For integers $c \geqslant 2$ and $n \geqslant 3$, every collection of directed graphs $G_1, \ldots, G_c$ on a common set of n vertices containing no rainbow $S_{1,1}$ satisfies*

$$\sum_{i=1}^{c} e(G_i) \leqslant \begin{cases} n^2 - n & \text{if } c \leqslant 3, \\ c\left\lfloor \frac{n^2}{4} \right\rfloor & \text{if } c \geqslant 4. \end{cases}$$

Moreover, the above bounds are sharp.

Proof. The bound for $c \leqslant 3$ is obtained when G_1 is a complete directed graph and all G_i for $i \in [c] \setminus \{1\}$ are empty graphs. While for $c \geqslant 4$ the bound is achieved when each graph is the same balanced complete bipartite graph with all edges oriented in the same direction. This is depicted in Figure 5.

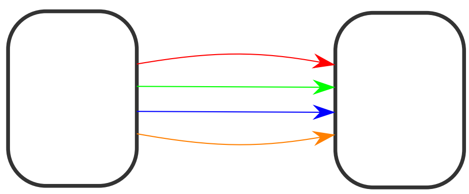

Figure 5: The optimal construction for at least 4 colors and a forbidden rainbow $S_{1,1}$.

We prove the upper bound by induction on n. For $n = 3$ and $n = 4$ it is easy to verify that the bound holds. Consider then a collection of directed graphs $G_1, \ldots, G_c$ on a common set V of $n \geqslant 5$ vertices containing no rainbow $S_{1,1}$ and assume that the bound holds for all collections on a smaller number of vertices.

For a subset $U \subset V$ and a vertex $v \in V$ by $e(U, v)$ we denote the total number of edges in all graphs $G_1, \ldots, G_c$ between the set U and the vertex v. If U consists of only one vertex u, we write $e(u, v)$ instead of $e(\{u\}, v)$ for brevity.

Assume first that there are two vertices u and v such that there exists an edge uv in at least two colors. To avoid rainbow $S_{1,1}$, for each vertex $x \in V \setminus \{u, v\}$, there are no edges xu nor vx, and if $ux \in E(G_i)$ and $xv \in E(G_j)$, then $i = j$. This means that

$e(\{u,v\},x) \leqslant c$. If there is no edge vu in any color, then $e(u,v) \leqslant c$ and together with the inductive assumption on $V \setminus \{u,v\}$ for $c \leqslant 3$ we obtain

$$\sum_{i=1}^{c} e(G_i) \leqslant c + c(n-2) + (n-2)^2 - (n-2) = n^2 - n - (3-c)(n-1) - (n-3) \leqslant n^2 - n,$$

while for $c \geqslant 4$ we have

$$\sum_{i=1}^{c} e(G_i) \leqslant c + c(n-2) + c\left\lfloor \frac{(n-2)^2}{4} \right\rfloor = c\left\lfloor \frac{n^2}{4} \right\rfloor.$$

On the other hand, if there is an edge vu in some color, then between $\{u,v\}$ and x we can have edges only in that color, so $e(\{u,v\},x) \leqslant 2$. Hence, for $c \leqslant 3$ we obtain

$$\sum_{i=1}^{c} e(G_i) \leqslant 2c + 2(n-2) + (n-2)^2 - (n-2) = n^2 - n - 2(n-4) - 2(3-c) \leqslant n^2 - n,$$

while for $c \geqslant 4$ we have

$$\sum_{i=1}^{c} e(G_i) \leqslant 2c + 2(n-2) + c\left\lfloor \frac{(n-2)^2}{4} \right\rfloor = c\left\lfloor \frac{n^2}{4} \right\rfloor - (c-4)(n-3) - 2(n-4) \leqslant c\left\lfloor \frac{n^2}{4} \right\rfloor.$$

Therefore, we are left with the case when between any pair of vertices there are at most two edges (at most one edge in each direction). But then $\sum_{i=1}^{c} e(G_i) \leqslant 2\binom{n}{2} = n^2 - n$, which gives the correct bound for $c \leqslant 3$ and is smaller than $c\left\lfloor \frac{n^2}{4} \right\rfloor$ for $c \geqslant 4$, as desired. ☐

Since the extremal construction for $c \geqslant 4$ in Theorem 5 has the same number of edges in each graph, this theorem immediately implies the optimal bound for $\min_{1 \leqslant i \leqslant c} e(G_i)$ if $c \geqslant 4$. We will show that for $c \leqslant 3$ the same bound holds.

Theorem 6. *For any integers $c \geqslant 2$ and $n \geqslant 4$, every collection of directed graphs $G_1, \ldots, G_c$ on a common set of n vertices containing no rainbow $S_{1,1}$ satisfies*

$$\min_{1 \leqslant i \leqslant c} e(G_i) \leqslant \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Moreover, this bound is sharp.

Proof. The bound is achieved when each graph is the same balanced complete bipartite graph with all edges oriented in the same direction (see Figure 5).

Note that it is enough to consider $c = 2$, because if the theorem is true for 2 colors, then it also holds for any larger number of colors. We proceed by induction on n. For $n = 4$ it is easy to verify that the theorem holds (note that for $n = 3$ it is not true as one can have a directed triangle oriented clockwise in G_1 and anticlockwise in G_2). Consider then two directed graphs G_1, G_2 on a common set V of $n \geqslant 5$ vertices containing no rainbow $S_{1,1}$ and assume that the theorem holds for all pairs of graphs on a smaller number of vertices.

Assume first that there exists a vertex $v \in V$ incident to at most 2 edges in each of the graphs. Then, from the induction assumption on $V \setminus \{v\}$ we obtain

$$\min_{1 \leqslant i \leqslant 2} e(G_i) \leqslant 2 + \left\lfloor \frac{(n-1)^2}{4} \right\rfloor = \left\lfloor \frac{n^2 - 2n + 9}{4} \right\rfloor \leqslant \left\lfloor \frac{n^2}{4} \right\rfloor .$$

Therefore, we may assume that no such vertex exists in V. In particular, every vertex has outdegree or indegree at least 2 in at least one of the graphs.

We split the vertex set V into disjoint sets based on colors and directions of incident edges, similarly as in Section 3. Let α and γ be the number of vertices that have outdegree 0, respectively indegree 0, in both graphs. For $i \in [2]$, let β_i be the number of remaining vertices that are incident only to edges in G_i. The remaining vertices in V can be divided into 4 sets depending whether outdegree or indegree is large and in which graph. For $i \in [2]$, let δ_i^+ be the number of vertices having outdegree at least 2 in G_i and δ_i^- the number of vertices having indegree at least 2 in G_i. Let M be the set of pairs of vertices u, v such that $uv \in E(G_1)$ and $vu \in E(G_2)$. Note that every vertex appears in at most one pair in M as otherwise it has no other incident edges which contradicts the condition proven in the previous paragraph. Moreover, all $\delta_1^+ + \delta_1^- + \delta_2^+ + \delta_2^-$ vertices must appear in M. Thus, we can set $m = |M| = \frac{1}{2}(\delta_1^+ + \delta_1^- + \delta_2^+ + \delta_2^-)$.

Note that all edges in G_1, except those in M, are going from $\gamma + \beta_1 + \delta_1^+$ vertices and to $\alpha + \beta_1 + \delta_1^-$ vertices. From symmetry, we may assume without loss of generality that

$$\beta_1 + \frac{1}{2}\delta_1^+ + \frac{1}{2}\delta_1^- \leqslant \beta_2 + \frac{1}{2}\delta_2^+ + \frac{1}{2}\delta_2^-,$$

we obtain

$$\begin{aligned}
e(G_1) &\leqslant (\gamma + \beta_1 + \delta_1^+)(\alpha + \beta_1 + \delta_1^-) + m \\
&\leqslant \left\lfloor \frac{(\gamma + \beta_1 + \delta_1^+ + \alpha + \beta_1 + \delta_1^-)^2}{4} \right\rfloor + m \\
&\leqslant \left\lfloor \frac{(\gamma + \alpha + \beta_1 + \beta_2 + \frac{1}{2}\delta_1^+ + \frac{1}{2}\delta_1^- + \frac{1}{2}\delta_2^+ + \frac{1}{2}\delta_2^-)^2}{4} \right\rfloor + m \\
&= \left\lfloor \frac{(n-m)^2}{4} \right\rfloor + m \\
&= \left\lfloor \frac{n^2 - m(2n - m - 4)}{4} \right\rfloor \\
&\leqslant \left\lfloor \frac{n^2}{4} \right\rfloor,
\end{aligned}$$

because $m \leqslant \frac{1}{2}n$ and $n \geqslant 4$, which concludes the proof. □

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References

[1] R. Aharoni, M. DeVos, S. González Hermosillo de la Maza, A. Montejano, R. Šámal, A rainbow version of Mantel's Theorem, *Adv. Combin.* (2020), 2020:2.

[2] S. Babiński, A. Grzesik, Graphs without a rainbow path of length 3, *SIAM Journal on Discrete Mathematics* 38 (1) (2024), 629–644.

[3] S. Babiński, A. Grzesik, M. Prorok, Directed graphs without rainbow triangles. arXiv:2308.01461, 2023.

[4] D. Chakraborti, J. Kim, H. Lee, H. Liu, J. Seo, On a rainbow extremal problem for color-critical graph, *Random Structures Algorithms* 64(2) (2024), 460–489.

[5] V. Falgas-Ravry, K. Markström, E. Räty, Rainbow Variations on a Theme by Mantel: Extremal Problems for Gallai Colouring Templates, *Combinatorica* (2024).

[6] J. Fox, A new proof of the graph removal lemma, *Ann. of Math.* 174 (2011), 561–579.

[7] P. Frankl, Graphs without rainbow triangles. arXiv:2203.07768, 2022.

[8] P. Frankl, E. Győri, Z. He, Z. Lv, N. Salia, C. Tompkins, K. Varga, X. Zhu, Extremal Results for Graphs Avoiding a Rainbow Subgraph, *Electron. J. Combin.* 31(1) (2024), #P1.28.

[9] S. Im, J. Kim, H. Lee, H. Seo, On rainbow Turán Densities of Trees. arXiv:2312.15956, 2023.

[10] P. Keevash, M. Saks, B. Sudakov, J. Verstraëte, Multicolour Turán problems, *Adv. Appl. Math.* 33(2) (2004), 238–262.