Coloring Hypergraphs from Random Lists

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Abstract

We introduce and study problems on coloring hypergraphs from randomly chosen color lists. Let $H = H(n)$ be an r-uniform hypergraph on n vertices where each vertex is assigned a list of colors of size k chosen independently and uniformly at random from all k-subsets of a set of colors of size $\sigma = \sigma(n)$. We prove a number of results concerning the probability of the existence of a list coloring of H from the random lists. One such result is that for fixed r, k and growing n, if $\sigma = \omega\left(n^{\frac{1}{k^2(r-1)}}\Delta^{\frac{1}{k(r-1)}}\right)$, and $H(n)$ has maximum degree $\Delta = O\left(n^{\frac{k-1}{k^2(k^2+k)r(r-1)}}\right)$, then with probability tending to 1, as $n \to \infty$, H has a proper coloring from the random lists. The bound on σ is tight. We also prove analogous results for hypergraphs of bounded maximum degree and complete hypergraphs where the size k of the random lists either is constant or an increasing function of n. In particular, for the complete r-uniform hypergraph K_n^r, if $\sigma = \lceil n/(r-1)\rceil$, then the property of being colorable from random lists of size k has a sharp threshold at $k(n) = \frac{\log n}{r-1}$.

Mathematics Subject Classifications: 05C15, 60C05

1 Introduction

A *proper coloring* of a graph or hypergraph H is an assignment of colors (usually positive integers) to the vertices of H so that no edge is monochromatic. As usual, by the *chromatic number* $\chi(H)$ of H we mean the number of colors needed for a proper coloring of H.

List coloring is a well-known generalization of ordinary vertex coloring which was introduced independently by Erdős et al [10] and Vizing [20] in the 70's, and has been studied extensively. Given a hypergraph H, assign to each vertex v of H a set $L(v)$ of colors. Such an assignment L is called a *list assignment* for H and the sets $L(v)$ are referred to as *lists* or *color lists*. If all lists have equal size k, then L is called a *k-list assignment*. If there is a proper coloring φ of H such that $\varphi(v) \in L(v)$ for all $v \in V(H)$, then H is *L-colorable* and φ is called an *L-coloring*. Furthermore, H is called *k-choosable* if it is L-colorable for every k-list assignment L.

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In this paper we consider the problem of coloring hypergraphs from randomly selected color lists. Assign lists of colors to the vertices of a hypergraph $H = H(n)$ with n vertices by choosing for each vertex v its list $L(v)$ independently and uniformly at random from all k-subsets of a color set $\mathcal{C} = \{1, 2, \ldots, \sigma\}$. Such a list assignment is called a *random* (k, σ)*-list assignment* for H, and we usually take k to be a fixed positive integer, unless otherwise stated. The question that we address in this paper is how large $\sigma = \sigma(n)$ should be in order to guarantee that with probability tending to 1 as $n \to \infty$ there is a proper coloring of the vertices of H with colors from the random list assignment. Alternatively, we can fix σ (as a function of n), and ask how large $k = k(n)$ should be to ensure that the random lists support a proper coloring with probability tending to 1. More generally, an event A_n occurs *with high probability* (abbreviated *whp*) if $\lim_{n\to\infty} \mathbb{P}[A_n] = 1$.

This problem of coloring from random lists was first studied by Krivelevich and Nachmias [16, 17] in the setting of ordinary graphs, and in a series of papers [4, 5, 6, 7] the present author generalized several results from [16, 17].

So far, problems on properly coloring from random lists have only been considered for graphs. Our aim in this paper is to introduce these kind of problems in the setting of hypergraphs. Indeed, we shall derive analogues for uniform hypergraphs of several results obtained earlier for graphs. We stress that in most cases the basic proof methods utilized in this paper are similar to the ones in papers on coloring graphs from random lists [6, 7]. Nevertheless, we need to generalize techniques from the setting of graphs to the setting of hypergraphs.

We consider r-uniform hypergraphs, where we assume r to be a fixed positive integer greater than 2. The *degree* of a vertex in a hypergraph is the number of edges it is contained in. As usual, we denote the maximum degree of a hypergraph H by $\Delta(H)$. As with ordinary graph coloring, the maximum degree is a natural invariant for bounding the chromatic number of a hypergraph, see e.g. [21] for some general background on hypergraph coloring as well as some recent results.

Let $H = H(n)$ be an r-uniform hypergraph on n vertices and maximum degree $\Delta(H)$. Our first result is that if H has bounded maximum degree, and L is a random (k, σ)-list assignment for H, then it is L-colorable provided that $\sigma = \omega\left(n^{\frac{1}{k^2(r-1)}}\right)$. We shall prove that this is tight in the sense that if $\sigma = o\left(n^{\frac{1}{k^2(r-1)}}\right)$, then there are examples of hypergraphs $H = H(n)$ that are not L-colorable whp.

For hypergraphs whose maximum degree grows with the number of vertices, we suggest the following, which generalizes a conjecture for graphs in [7], which was very recently proved by Hefetz and Krivelevich [18].

Conjecture 1. Let $H = H(n)$ be an r-uniform hypergraph on n vertices with maximum degree at most $\Delta = \Delta(n)$, $k \geqslant 2$ a fixed positive integer, and L a random (k, σ)-list assignment for H.

(i) If $\Delta = o\left(n^{\frac{1}{k^2-k}}\right)$ and $\sigma = \omega\left(n^{\frac{1}{k^2(r-1)}} \Delta^{\frac{1}{k(r-1)}}\right)$, then whp H is L-colorable.

(ii) There is a constant C, such that if $\Delta = \Omega\left(n^{\frac{1}{k^2-k}}\right)$ and $\sigma \geqslant C\Delta^{\frac{1}{r-1}}$, then whp H is

L-colorable.

Note that when Δ is bounded, then part (i) of the conjecture reduces to the result on bounded maximum degree.

Moreover, for the case $k = 1$, it is easily seen that for a r-uniform hypergraph $H = H(n)$ on n vertices with strictly increasing maximum degree $\Delta = \Delta(n)$, the "coarse threshold" for colorability from a random $(1, \sigma)$-list assignment trivially occurs at $\sigma = n^{1/(r-1)}\Delta^{1/(r-1)}$; thus if $\sigma = o(n^{1/(r-1)}\Delta^{1/(r-1)})$, then there are such hypergraphs that are whp not colorable from the random lists, while if $\sigma = \omega(n^{1/(r-1)}\Delta^{1/(r-1)})$, then such hypergraphs are whp colorable from the random lists.

We prove that part (i) of Conjecture 1 is true for a slightly more restrictive bound on the maximum degree, and also demonstrate that it is tight in the "coarse threshold sense". We also observe that a weaker version of part (ii) holds. Note that part (ii) would be best possible since a complete r-uniform hypergraph on n vertices has chromatic number $\lceil n/(r-1) \rceil$.

Next, we consider dense hypergraphs in Section 4 and prove some results concerning the colorability of complete r-uniform hypergraphs from randomly chosen color lists, and present a conjecture for complete multipartite hypergraphs.

Finally, in Section 5 we give some results on the problem of coloring hypergraphs from random lists of non-constant size. In particular we prove the following: let L be a random $(f(n), \lceil n/(r-1) \rceil)$-list assignment for the complete r-uniform hypergraph K_n^r, where $f(n)$ is some integer-valued function. Then for every $\varepsilon > 0$, K_n^r is L-colorable whp if $f(n) \geqslant (1+\varepsilon)\frac{\log n}{r-1}$. Indeed, this property has a sharp threshold at $f(n) = \frac{\log n}{r-1}$, which generalizes a result from [8]. This result can be interpreted as a result on so-called "palette sparsification" for complete hypergraphs (see e.g. [12] and references therein). Note further that the size $\lceil n/(r-1) \rceil$ on the color set is best possible, since it matches the chromatic number of K_n^r.

Throughout the paper, r is assumed to be a positive integer greater than 2. We use standard asymptotic notation and assumptions; n is large enough whenever necessary and we usually omit floor and ceiling signs to avoid cumbersome notation. e denotes the base of the natural logarithm and all our logarithms are indeed natural.

2 Hypergraphs of bounded maximum degree

We first consider the case of hypergraphs of bounded maximum degree. Let H be a hypergraph and L a list assignment for H. If H is not L-colorable, but removing any vertex (along with incident edges) from H yields an L-colorable hypergraph, then H is L-*critical*. Obviously, if L is a list assignment for a hypergraph H, and H is not L-colorable, then it contains a connected induced L-critical subhypergraph.

By a *path* in a hypergraph, we mean a sequence of distinct vertices $P = w_1 w_2 \dots w_k$, such that w_i and w_{i+1} are contained in an edge. (So the edges of a path are not necessarily distinct.) The *length* of such a path is the number of vertices contained in it minus one. Similarly, a *cycle* is a sequence of vertices $w_1 w_2 \dots w_k w_1$, where $w_1 w_2 \dots w_k$ is a path, and

w_1 and w_k are adjacent (i.e. contained in a common edge); its *length* is the number of vertices. The *girth* of a hypergraph is the length of its shortest cycle.

Suppose now that $H - w_1$ is L-colorable, where w_1 is some vertex of H. Given an L-coloring φ of $H - w_1$, a path $P = w_1 w_2 \dots w_t$ in H is called (φ, L)-*alternating* if there are colors $c_2, c_3, \dots, c_t$ such that $\varphi(w_i) = c_i$ and $c_i \in L(w_{i-1})$, $i = 2, \dots, t$. We allow such a path to have length 0 and thus only consist of w_1. The set of vertices which are adjacent to a vertex x in a hypergraph H is denoted by $N_H(x)$.

We shall need a structural lemma on list assignments of a hypergraph which do not support a proper coloring. The following is a hypergraph version of a lemma proved in [6]. Indeed, there are several natural ways of generalizing the lemma from [6] to the setting of r-uniform hypergraphs. It turns out that the following is sufficient for our purposes. For completeness, we provide a brief proof.

Lemma 2. *Let H be an r-uniform hypergraph and L a list assignment for H. If H is L-critical, then for any vertex $v_1 \in V(H)$, $H - v_1$ has an L-coloring φ that satisfies the following conditions:*

(i) All vertices in H lie on (φ, L)-alternating paths with origin at v_1.

(ii) For each color $c \in L(v_1)$, there are $r - 1$ vertices $w_1^c, \dots, w_{r-1}^c$, such that $\varphi(w_i^c) = c$ and $\{v_1, w_1^c, \dots, w_{r-1}^c\} \in E(H)$.

(iii) Define a rank function $R : V(H) \to \{0, 1, 2, \dots, |V(H)|\}$ on the vertices of H by setting $R(u) = j$ if a shortest (φ, L)-alternating path from v_1 to u has length j. Then for every vertex x of $H - v_1$ and every color $c \in L(x) \setminus \{\varphi(x)\}$, there is either
(a) a vertex $y \in N_H(x)$ colored $\varphi(y) = c$, or
(b) a vertex $z \in N_H(x)$ such that $c \in L(z)$ and $R(z) < R(x)$.

Proof. Let v_1 be any vertex of H. We shall prove that there is an L-coloring φ of $H - v_1$ satisfying (i)-(iii). Generally, for an L-coloring ψ of $H - v_1$, let W_i^ψ be the set of vertices w such that a shortest (ψ, L)-alternating path from v_1 to w has length i, d be the maximum integer for which there is an L-coloring γ of $H - v_1$ such that W_d^γ is non-empty, and set $W^\gamma = \cup_{i=0}^d W_i^\gamma$.

We define θ_1 to be the set of L-colorings ψ of $H - v_1$ with $|W_1^\psi|$ minimum, that is, θ_1 is the set of L-colorings ψ such that there is no L-coloring ψ' of $H - v_1$ with $|W_1^{\psi'}| < |W_1^\psi|$. Similarly, for each $i = 2, \dots, d$, let $\theta_i \subseteq \theta_{i-1}$ be the subset of θ_{i-1} consisting of L-colorings ψ with $|W_i^\psi|$ minimum, that is, θ_i is the set of colorings $\psi \in \theta_{i-1}$ such that there is no L-coloring $\psi' \in \theta_{i-1}$ satisfying $|W_i^{\psi'}| < |W_i^\psi|$. Since for each i, there is at least one such L-coloring, θ_i is non-empty for all $i = 1, \dots, d$. Let $\varphi \in \theta_d$ and denote by F the subhypergraph of H induced by W^φ.

It is straightforward to check that since H is L-critical and F contains all vertices on (φ, L)-alternating paths, $F = H$, so (i) holds. Moreover, (ii) clearly holds, since H is not L-colorable.

Consider now the rank function R defined above. We show that (a) or (b) of condition (iii) holds for every vertex u of $H - v_1$ and for every color of $L(u) \setminus \{\varphi(u)\}$. Suppose that

$x \in W_l^{\varphi}$, where $l \geqslant 1$, and that there is some color $c \in L(x)$ such that (iii) does not hold for x and c. Then no neighbor of x in H is colored c. Define a new coloring φ' of $H - v_1$ by setting

$$\varphi'(u) = \begin{cases} c & \text{if } u = x, \\ \varphi(u) & \text{if } u \notin \{x, v_1\}. \end{cases}$$

Then φ' is a proper L-coloring of $H - v_1$ and since (b) does not hold for x and c, $x \notin W_j^{\varphi'}$, for $j = 1, \dots, l-1$. Indeed,

$$W_j^{\varphi'} = W_j^{\varphi}, \text{ for } j = 1, \dots, l-1. \tag{1}$$

Moreover, $W_l^{\varphi'} = W_l^{\varphi} \setminus \{x\}$, which together with (1) contradict that $\varphi \in \theta_l$. Hence, (a) or (b) holds for every vertex u of $H - v_1$ and every color c in $L(u) \setminus \{\varphi(u)\}$. □

For a rank function R defined as in part (iii) of Lemma 2, we say that R is the rank function on $V(H)$ *induced by L and φ*.

Let F be a connected induced subhypergraph of a hypergraph H, v_1 a fixed vertex of F and $R : V(F) \to \{0, 1, \dots, |V(F)| - 1\}$ a rank function on the vertices of F. The triple (F, v_1, R) is *proper* if $R(v_1) = 0$, $R(u) > 0$ for each vertex $u \in V(F) \setminus \{v_1\}$, and if $R(u) = s$, then there is a vertex $x \in N_F(u)$ such that $R(x) = s - 1$. Note that if F, v_1 and R satisfy the conditions of Lemma 2 for some choice of L and φ, then (F, v_1, R) is proper. The next lemma gives an upper bound on the number of proper triples in an r-uniform hypergraph.

Lemma 3. *Let H be an r-uniform hypergraph on n vertices whose maximum degree is at most Δ. The number of proper triples (F, v_1, R), such that F is a connected, induced subhypergraph of H with m vertices does not exceed*

$$n \left(\Delta(r-1)\right)^{m-1} (m-1)!.$$

Proof. Suppose that (F, v_1, R) is a proper triple. Let G be the graph obtained from H by replacing every edge by an r-clique on the same vertices. By repeatedly removing edges from G we can construct a tree T with root v_1, such that if $u \in V(H)$ has rank $R(u) = t$ ($1 \leqslant t \leqslant m-1$), then u is adjacent to a vertex x in T with rank $R(x) = t - 1$, and v_1 is the unique vertex of rank $R(v_1) = 0$.

Moreover, given such a tree T in G with root v_1 and a rank function $R : V(T) \to \{0, 1, \dots, m-1\}$ satisfying these conditions, there is a uniquely determined proper triple (F, v_1, R) with $V(F) = V(T)$. Hence, the number of proper triples of H is bounded by the number of such trees in G. This latter quantity is bounded by

$$n(r-1)\Delta(2(r-1)\Delta)(3(r-1)\Delta)\dots((m-1)(r-1)\Delta) = n(r-1)^{m-1}\Delta^{m-1}(m-1)!,$$

because there are n ways of selecting v_1, then we have $(r-1)\Delta$ choices for a neighbor of v_1 as the next vertex v_2 of T; thereafter, there are at most $2((r-1)\Delta)$ ways of choosing an edge incident with v_1 or v_2 that connects one of these vertices with the next vertex of T, etc. □

Given a proper triple (F, v_1, R) of a hypergraph H and a list assignment L for H such that F is not L-colorable, we say that the triple (F, v_1, R) is L-*bad* (or just *bad*) if there is an L-coloring φ of $F - v_1$, such that F, v_1, R, L and φ satisfy conditions (i)-(iii) of Lemma 2 (with F in place of H). In particular, R is the rank function on $V(F)$ induced by L and φ. The next lemma gives an upper bound on the probability that a given proper triple (F, v_1, R) of H is L-bad.

Lemma 4. *Let L be a random (k, σ)-list assignment for an r-uniform hypergraph H with maximum degree at most Δ, where Δ is some positive integer. If (F, v_1, R) is a proper triple of H with $m = |V(F)|$, then*

$$\mathbb{P}[(F, v_1, R) \text{ is } L\text{-bad}] \leqslant \frac{\binom{\Delta}{k} k! \binom{k+\Delta(r-1)}{k-1}^{k(r-1)} \binom{\Delta k(r-1)}{k-1}^{m-k(r-1)-1}}{\sigma^{k(r-1)+1-m} \binom{\sigma}{k}^{m-1}}.$$

Proof. Suppose that (F, v_1, R) is a proper triple in H. To estimate the probability that (F, v_1, R) is bad, we first count how many choices for a proper coloring φ of $F - v_1$ we have. Then we count the number of list assignments L for F such that φ is an L-coloring of $F - v_1$ and the conditions (i)-(iii) of Lemma 2 holds.

There are at most $\binom{\sigma}{k}$ ways of choosing the color list $L(v_1)$. Since each color $c \in L(v_1)$ appears on at least $r-1$ vertices under φ, which together with v_1 are contained in a single edge, we have at most $\binom{\Delta}{k}$ ways of choosing these k edges, and then $(k-1)!$ ways of choosing the restriction of φ to these k edges containing v_1.

Thereafter, there are at most $\sigma^{m-k(r-1)-1}$ ways of choosing φ on all remaining vertices of F.

The rank function R induces a partition $\{U_0, U_1, \dots, U_d\}$ of $V(F)$, where U_i consists of the vertices w for which a shortest (φ, L)-alternating path from v_1 to w has length i. Consider a vertex w in U_1. Lemma 2 (iii) implies that we can include a color c in $L(w) \setminus \{\varphi(w)\}$ if there is a neighbor of w in F colored c or a vertex $u \in N_F(w)$, such that $c \in L(u)$ and $R(u) < R(w)$. It follows that we have at most $\binom{k+\Delta(r-1)}{k-1}$ choices for the rest of the colors of $L(w)$. Similarly, there are at most $\binom{\Delta k(r-1)}{k-1}$ ways of choosing the colors in $L(x) \setminus \{\varphi(x)\}$ for a vertex $x \in U_i$, $i \geqslant 2$, so the desired upper bound on the probability that (F, v_1, R) is L-bad follows. □

We can now prove the following theorem.

Theorem 5. *Let $H = H(n)$ be an r-uniform hypergraph on n vertices with bounded maximum degree, and L a random (k, σ)-list assingment for H. If $\sigma = \omega(n^{\frac{1}{k^2(r-1)}})$, then whp H is L-colorable.*

Given the above generalizations of lemmas from [6], the proof of this theorem is virtually identical to the proof of the main result in [6]. For completeness, we give the argument here as well.

Proof of Theorem 5. We will show that whp H has no connected induced L-critical subhypergraph. This will imply the theorem.

By Lemma 2, it suffices to prove that if $\sigma = \omega(n^{\frac{1}{k^2(r-1)}})$, then whp H does not contain an L-bad triple (F, v_1, R). We will use easy first moment calculations.

If (F, v_1, R) is L-bad, then F has at least $k(r-1)+1$ vertices. We first show that if (F, v_1, R) is bad, then whp F contains at most $(\Delta(r-1))^{k^2(r-1)}$ vertices. Consider a path P on ν vertices in H with origin at some vertex v. The probability that there is an L-coloring φ of $P - v$, such that P is (φ, L)-alternating is at most

$$\frac{\sigma^{\nu-1}\binom{\sigma-1}{k-1}^2\binom{\sigma-2}{k-2}^{\nu-2}}{\binom{\sigma}{k}^{\nu}} \leqslant \frac{k^{2\nu}}{\sigma^{\nu-1}},$$

Moreover, the number of distinct paths in H on ν vertices is at most $n(\Delta(r-1))^{\nu-1}$. Consequently, the expected number of paths P in H on at least $k^2(r-1)+1$ vertices, for which there is an L-coloring φ of $P - v$ such that P is (φ, L)-alternating is at most

$$\sum_{\nu=k^2(r-1)+1}^{n} \frac{n(\Delta(r-1))^{\nu-1}k^{2\nu}}{\sigma^{\nu-1}} = o(1),$$

since $\sigma(n) = \omega(n^{1/(k^2(r-1))})$. Hence, whp there is no list coloring φ of a subhypergraph of H with a (φ, L)-alternating path of length at least $k^2(r-1)+1$.

Now, by Lemma 2, if F is a subhypergraph of H that is in a bad triple (F, v_1, R), then there is an L-coloring φ of $F - v_1$ such that all vertices of F lie on (φ, L)-alternating paths with origin at v_1. Since whp the maximum length of such a path in H is at most $k^2(r-1)$, the maximum number of vertices in a subhypergraph of H that is in a bad triple is whp at most

$$1 + \Delta(r-1) + (\Delta(r-1))^2 + \cdots + (\Delta(r-1))^{k^2(r-1)-1} \leqslant (\Delta(r-1))^{k^2(r-1)}.$$

Let X_m be a random variable counting the number of bad triples (F, v_1, R) in H on m vertices and set $X = \sum_{m=k(r-1)+1}^{n} X_m$. By Lemmas 3 and 4, we have that

$$\begin{aligned}
\mathbb{P}[H \text{ contains a bad triple}] \leqslant& \mathbb{E}[X] \\
\leqslant& \sum_{m=k(r-1)+1}^{(\Delta(r-1))^{k^2(r-1)}} n(\Delta(r-1))^{m-1}(m-1)! \\
&\times \frac{\binom{\Delta}{k}k!\binom{k+\Delta(r-1)}{k-1}^{k(r-1)}\binom{\Delta k(r-1)}{k-1}^{m-k(r-1)-1}}{\sigma^{k(r-1)+1-m}\binom{\sigma}{k}^{m-1}} + o(1) \\
\leqslant& \sum_{m=k(r-1)+1}^{(\Delta(r-1))^{k^2(r-1)}} n(m-1)^{m-1}\frac{(\Delta k(r-1))^{km}}{\sigma^{(k-1)m+k(r-2)+1}} + o(1) \\
=& O\Big(\frac{n}{\sigma^{k(r-2)+1}}\Big) \sum_{m=k(r-1)+1}^{(\Delta(r-1))^{k^2(r-1)}} \Big(\frac{k^k(r-1)^k\Delta^k m}{\sigma^{k-1}}\Big)^m
\end{aligned}$$

$$
\begin{aligned}
&= O\Big(\frac{n}{\sigma^{k^2(r-1)}}\Big) \sum_{m=0}^{\infty} \Big(\frac{k^k(\Delta(r-1))^{k+k^2(r-1)}}{\sigma^{k-1}}\Big)^m \\
&= o(1),
\end{aligned}
$$

provided that $\sigma(n) = \omega(n^{\frac{1}{(r-1)k^2}})$. $\square$

Next, we show that Theorem 5 is best possible in the "coarse threshold sense". We will show that for positive integers k and $r \geqslant 2$, and $n \geqslant (r-1)k+1$ (large enough), there is an r-uniform hypergraph $H = H(n)$ with n vertices and bounded maximum degree such that if $\sigma(n) = o\left(n^{\frac{1}{k^2(r-1)}}\right)$ and L is a random (k,σ)-list assignment for H, then whp H is not L-colorable.

In fact, we can take H to be a disjoint union of cliques of suitable size: we define H to be a hypergraph on n vertices consisting of $\left\lfloor \frac{n}{k(r-1)+1} \right\rfloor$ $(k(r-1)+1)$-cliques and some isolated vertices, where k and r are fixed positive integers. Now, if L is a random (k,σ)-list assignment for H, then H is L-colorable if and only if there is no clique in H where all vertices have identical lists. The probability that there is no such clique in H is

$$
\left(1 - \binom{\sigma}{k}^{-k(r-1)}\right)^{\left\lfloor \frac{n}{k(r-1)+1} \right\rfloor} = O\left(\exp\left(-\frac{n}{\sigma^{k^2(r-1)}(k(r-1)+1)}\right)\right),
$$

which tends to 0 as $n \to \infty$, given that $\sigma(n) = o\left(n^{\frac{1}{k^2(r-1)}}\right)$. We conclude that Theorem 5 is best possible.

Informally speaking, for an r-uniform hypergraph the "threshold" for list colorability from a random (k,σ)-list assignment coincides with the threshold for disappearance of $(k(r-1)+1)$-cliques where all vertices get identical lists. Thus, as expected, it is possible to prove a better bound on σ which ensures list colorability for hypergraphs of larger girth. For the case of graphs, some results in this direction are obtained in [6]. However, as pointed out therein, it seems difficult to obtain tight results here, since this would require determining the smallest non-k-choosable graph/r-uniform hypergraph with given girth, which, to the best of our knowledge, is a non-trivial open question. Consequently, we choose not to pursue this line of investigation here.

3 Hypergraphs with increasing maximum degree

In this section we consider hypergrahs $H = H(n)$ whose maximum degree increases with the number of vertices. We first prove the following theorem, which proves a slightly weaker version of part (i) of Conjecture 1.

Theorem 6. *Let $H = H(n)$ be an r-uniform hypergraph on n vertices with maximum degree at most $\Delta = \Delta(n)$, k a fixed positive integer, and L a random (k,σ)-list assignment for H. If $\sigma = \omega\left(n^{\frac{1}{k^2(r-1)}}\Delta^{\frac{1}{k(r-1)}}\right)$ and $\Delta = O\left(n^{\frac{k-1}{k^2(k^2+k)r(r-1)}}\right)$, then whp H is L-colorable.*

The proof of this theorem is similar to the proof of Theorem 5, so we only sketch the argument.

Proof of Theorem 6. (sketch). It suffices to show that H contains no connected induced L-critical subhypergraph. We first consider the case when F has exactly $k(r-1)+1$ vertices. If F is not L-colorable and $|V(F)| = k(r-1)+1$, then F is a $(k(r-1)+1)$-clique where all vertices have identical lists. By a result in [15], the number of $(k(r-1)+1)$-cliques in an r-uniform hypergraph with maximum degree Δ is at most

$$n\frac{\binom{(r-1)\Delta^{\frac{1}{r-1}}}{k(r-1)+1}}{(r-1)\Delta^{\frac{1}{r-1}}},$$

so the expected number of $(k(r-1)+1)$-cliques in H where the vertices get identical lists is at most

$$O\left(n\Delta^k\right)\binom{\sigma}{k}^{-k(r-1)}, \tag{2}$$

which tends to 0 as $n \to \infty$, because $\sigma(n) = \omega\left(n^{\frac{1}{k^2(r-1)}}\Delta^{\frac{1}{k(r-1)}}\right)$. Hence, whp there is no bad proper triple (F, v_1, R) in H satisfying that $|V(F)| = k(r-1)+1$.

Let us now consider the case when $|V(F)| \geqslant k(r-1)+2$. As in the proof of Theorem 5, we shall prove that whp H contains no bad triple.

Proceeding as in that proof, one may first prove that whp there is no L-coloring φ of H such that there is a (φ, L)-alternating path of length at least $k^2(r-1)+k(r-1)+1$, and consequently, whp a bad proper triple in H contains at most

$$(\Delta(r-1))^{(k^2+k)(r-1)}$$

vertices.

Continuing along the lines of the proof of Theorem 5, it therefore suffices to verify that the sum

$$\begin{aligned}&\sum_{m=k(r-1)+2}^{(\Delta(r-1))^{(k^2+k)(r-1)}} n(m-1)^{m-1}\frac{(\Delta k(r-1))^{km}}{\sigma^{(k-1)m+k(r-2)+1}}\\ &= O\Big(\frac{n\Delta^{k^2(r-1)+2k}}{\sigma^{k^2(r-1)+k-1}}\Big)\sum_{m=0}^{\infty}\Big(\frac{k^k(\Delta(r-1))^{k+(k^2+k)(r-1)}}{\sigma^{k-1}}\Big)^m\end{aligned}$$

tends to 0 as $n \to \infty$, provided that $\Delta = O\left(n^{\frac{k-1}{k^2(k^2+k)r(r-1)}}\right)$ and $\sigma = \omega\left(n^{\frac{1}{k^2(r-1)}}\Delta^{\frac{1}{k(r-1)}}\right)$. $\square$

Next, let us prove that Theorem 6 is tight, that is, the bound on σ in part (i) of Conjecture 1 is best possible in the "coarse threshold sense". We will show that for positive integers $k \geqslant 2$ and $n \geqslant k+1$ (large enough), and each increasing integer-valued function $\Delta = O(n^{\frac{1}{k^2-k}})$, there is an r-uniform hypergraph $H = H(n, \Delta)$ with n vertices and maximum degree approximately Δ such that if $\sigma(n) = \Theta\left(w_n^{-1} n^{\frac{1}{k^2(r-1)}} \Delta^{\frac{1}{k(r-1)}}\right)$, where $w_n \to \infty$ arbitrarily slowly, and L is a random (k, σ)-list assignment for H, then whp H is not L-colorable.

We set $n_\Delta = \left\lfloor \frac{n}{\Delta^{\frac{1}{r-1}}} \right\rfloor$, and let H be an r-uniform graph on n vertices which is the disjoint union of n_Δ complete hypergraphs, each of which has $\Delta^{\frac{1}{r-1}}$ vertices, and possibly some isolated vertices. Let $J_1, \ldots, J_{n_\Delta}$ be the non-trivial components of H and let L be a random (k, σ)-list assignment for H. As we shall see, whp there is at least one $(k(r-1)+1)$-clique in H where all vertices have identical lists, which means that whp H is not L-colorable.

Let X be a random variable counting the number of $(k(r-1)+1)$-cliques in H where all vertices have identical lists. Then

$$\mathbb{E}[X] = \left\lfloor \frac{n}{\Delta^{\frac{1}{r-1}}} \right\rfloor \binom{\Delta^{\frac{1}{r-1}}}{k(r-1)+1} \binom{\sigma}{k}^{-k(r+1)} = \Theta\left(n \Delta^k \sigma^{-k^2(r-1)}\right). \tag{3}$$

To prove that $\mathbb{P}[X > 0] = 1 - o(1)$ we use the second moment method with the inequality due to Chebyshev in the following form:

$$\mathbb{P}[Y = 0] \leqslant \frac{\mathrm{Var}[Y]}{\mathbb{E}[Y]^2}, \tag{4}$$

valid for all non-negative random variables Y. Since X is a sum of indicator random variables, we can use the following approach from [2].

Let $X = X_1 + \cdots + X_d$, where each X_i is the indicator random variable for the event that the vertices of a $(k(r-1)+1)$-clique in H get identical lists. Let A_i be the event corresponding to X_i, that is, $X_i = 1$ if A_i occurs and $X_i = 0$ otherwise. For indices i, j we write $i \sim j$ if $i \neq j$ and the events A_i, A_j are not independent. Set

$$\Pi = \sum_{i \sim j} \mathbb{P}[A_i \wedge A_j].$$

When $i \sim j$, we have

$$\mathrm{Cov}[X_i, X_j] = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] \leqslant \mathbb{E}[X_i X_j] = \mathbb{P}[A_i \wedge A_j]$$

and when $i \neq j$ and not $i \sim j$ then $\mathrm{Cov}[X_i, X_j] = 0$. Thus

$$\mathrm{Var}[X] \leqslant \mathbb{E}[X] + \Pi$$

and the following proposition follows from (4).

Claim 7. *If* $\mathbb{E}[X] \to \infty$ *and* $\Pi = o(\mathbb{E}[X]^2)$, *then* $\mathbb{P}[X > 0] = 1 - o(1)$.

It is clear from (3) that $\mathbb{E}[X] \to \infty$ if $\sigma(n) = \Theta\left(w_n^{-1} n^{\frac{1}{k^2(r-1)}} \Delta^{\frac{1}{k(r-1)}}\right)$. Let us show that the second criterion of Claim 7 is satisfied. If $i \sim j$, then clearly A_i and A_j are events for different intersecting $(k(r-1)+1)$-cliques, which are in the same component of H, to have identical lists on all vertices. So all vertices in these two cliques have identical lists.

Now, since H has n_Δ components and two distinct copies of the $(k(r-1)+1)$-clique have at most $k(r-1)$ vertices in common we have

$$\Pi = O\left(n_\Delta\right) \sum_{l=1}^{k(r-1)} \binom{\Delta^{1/(r-1)}}{k(r-1)+1+l} \binom{\sigma}{k}^{-(k(r-1)+l)} = O\left(n \Delta^{k+2/(r-1)} \sigma^{-k^2(r-1)-k}\right),$$

since this sum is dominated by its first term, and thus $\Pi = o(\mathbb{E}[X]^2)$ as required. We conclude that Theorem 6 is best possible in the "coarse threshold sense".

Unfortunately, the dense case of Conjecture 1 seems difficult. Using our proof method we can show that part (ii) of the conjecture holds for somewhat cruder bounds on σ. We choose to present only one such result, but remark that it is possible to prove slighly stronger results by bounding Δ from above.

Proposition 8. *Let $H = H(n)$ be an r-uniform hypergraph on n vertices and L a random (k, σ)-list assignment for H. If $\sigma(n) = \omega(n^{\frac{1}{k(r-1)}} \Delta^{\frac{1}{r-1}})$, then whp H is L-colorable.*

Proof. (sketch). As before, we need to show that H has no induced L-critical subhypergraph. By Lemma 2 it suffices to show that whp H contains no vertex v with $k(r-1)$ distinct neighbors $w_1, \ldots, w_{k(r-1)}$, such that $L(v) = \{c_1, \ldots, c_k\}$, and

$$c_i \in \{L(w_{(i-1)(r-1)+1}), \ldots, L(w_{i(r-1)})\}, \text{ and}$$

$$\{v, w_{(i-1)(r-1)+1}, \ldots, w_{i(r-1)}\} \in E(H),\ i = 1, \ldots, k.$$

The expected number of such vertices in H does not exceed

$$n \binom{\Delta}{k} \frac{k! \binom{\sigma-1}{k-1}^{k(r-1)}}{\binom{\sigma}{k}^{k(r-1)}},$$

which tends to 0 as $n \to \infty$. $\square$

We also remark that since the threshold in Theorem 6, is due to the appearance of $(k(r-1)+1)$-cliques where all vertices get identical lists, it is possible to prove better bounds on σ which ensure list colorability whp for hypergraphs of larger girth. As in the preceding section, we do not prove any such explicit bounds here, but refer the interested reader to [6, 7] for corresponding results for graphs.

4 Dense hypergraphs

In this section we consider coloring dense hypergraphs from randomly chosen lists. Recall that K_n^r denotes the complete r-uniform hypergraph on n vertices, and consider a random $(1,\sigma)$-list assignment for K_n^r. The expected number of monochromatic edges is $\binom{n}{r}\frac{\sigma}{\sigma^r}$, which tends to zero as $n \to \infty$, provided that $\sigma = \omega(n^{r/(r-1)})$. If, on the other hand, $\sigma = o(n^{r/(r-1)})$, then straightforward second moment calculations show that whp there is a monochromatic edge in K_n^r. Thus, we have the following.

Proposition 9. *For the complete r-uniform hypergraph K_n^r, the property of being colorable from a random $(1,\sigma)$-list assignment has a coarse threshold at $\sigma = n^{r/(r-1)}$.*

Next, consider a random (k,σ)-list assignment for K_n^r where $k \geqslant 2$. Recall that the chromatic number of K_n^r is $\lceil n/(r-1)\rceil$ since any r-subset of vertices must get at least two distinct colors. We shall prove that having just slightly more colors available implies that whp K_n^r is colorable from a random 2-list assignment chosen from this set of colors. The following result is thus best possible up to the multiplicative constant e.

Theorem 10. *For every $\varepsilon > 0$, if $\sigma(n) \geqslant (1+\varepsilon)\frac{en}{r-1}$ and L is a random $(2,\sigma)$-list assignment for K_n^r, then it is L-colorable whp.*

To prove this theorem, we need the following lemma.

Lemma 11. *If L is a list assignment for K_n^r for which there is no L-coloring, then there is a set of vertices S of size $(r-1)s+1$, $s \geqslant 1$, such that*

$$|\cup_{v\in S} L(v)| = s.$$

The proof of this lemma is similar to standard textbook proofs of Halls' theorem for matchings in bipartite graphs, so we just briefly sketch the proof.

Proof. If K_n^r is not L-colorable, then it contains a minimal vertex-induced subhypergraph with this property. Let T be the vertex set of this subhypergraph.

We consider a bipartite graph B with parts T and $C = \cup_{v\in T} L(v)$, where a vertex of v is adjacent to a color c if and only if $c \in L(v)$. Since T is the vertex set of a minimal vertex-induced subhypergraph that is not L-colorable, there is an L-coloring φ of the subhypergraph of K_n^r induced by $T - \{v\}$, for some vertex $v \in T$. We define a spanning subgraph F of B by including an edge $e = uc$ in F if and only if $\varphi(u) = c$.

Now, consider the set of vertices A that are reachable by F-alternating paths in B (edges alternately in F and not in F) with origin at v. Then every vertex of $A \cap C$ has degree $r-1$ in F, since otherwise we may exchange edges along an F-alternating path with origin at v and then color T accordingly to obtain a coloring of T where no edge is monochromatic.

Furthermore, every vertex in $A \cap T$ except v has degree 1 in F. Hence, $S = A \cap T$ is the required set. $\square$

Proof of Theorem 10. We use Lemma 11 for proving the theorem. Let L be a random $(2, \sigma)$-list assignment for K_n^r. We shall prove that if $\sigma(n) \geqslant (1+\varepsilon)\frac{en}{r-1}$, then whp K_n^r contains no set as in Lemma 11, and consequently, K_n^r is L-colorable. We shall refer to such a set simply as *bad*.

Now, there are $\binom{n}{(r-1)s+1}$ choices for a set S of size $(r-1)s+1$ from $V(K_n^r)$, and $\binom{\sigma}{s}$ choices for a set of s colors. Thus, the expected number of bad sets of size $(r-1)s+1$ is at most

$$\binom{n}{(r-1)s+1}\binom{\sigma}{s}\frac{\binom{s}{2}^{(r-1)s+1}}{\binom{\sigma}{2}^{(r-1)s+1}} \leqslant \left(\frac{ne}{(r-1)s+1}\right)^{(r-1)s+1}\left(\frac{\sigma e}{s}\right)^{s}\left(\frac{s(s-1)}{\sigma(\sigma-1)}\right)^{(r-1)s+1}.$$

Thus the total number of expected bad sets is at most

$$\sum_{s=2}^{\frac{n-1}{r-1}} e^{rs+1}\left(\frac{\sigma}{s}\right)^{s}\left(\frac{n(s-1)}{(r-1)\sigma(\sigma-1)}\right)^{(r-1)s+1}$$

$$= O(1)\sum_{s=2}^{\frac{n-1}{r-1}}\left(e^{r}\left(\frac{s}{\sigma}\right)^{r-2}\left(\frac{n}{(r-1)(\sigma-1)}\right)^{r-1}\right)^{s}$$

which tends to 0, provided that $\sigma(n) \geqslant (1+\varepsilon)\frac{en}{r-1}$. □

We conclude this section by considering complete multipartite hypergraphs. Denote by $K_{t\times n}^r$ the complete r-uniform t-partite hypergraph where each part has n vertices, where t is a positive integer and edges have at most one vertex from each part.

Proposition 12. *Let L be a random (k, σ)-list assignment for $K_{t\times n}^r$.*

(i) If $k = 1$, then the property of being L-colorable has a coarse threshold at $\sigma(n) = n^{r/(r-1)}$.

(ii) If $k = 2$, and $\sigma(n) \geqslant 2(1+\varepsilon)(t-r+1)n$, then $K_{t\times n}^r$ is L-colorable whp.

The proof of part (i) is similar to the proof of the corresponding result for K_n^r, so we omit it.

Part (ii) can be deduced from the result in [4] that for the complete s-partite graph $K_{s\times n}$ with n vertices in each part, the property of being colorable from a random $(2, \sigma)$-list assigment has a sharp threshold at $\sigma = 2(s-1)n$. Indeed, from $K_{t\times n}^r$ we can form a complete multipartite graph G with $t-r+2$ parts by simply removing $r-2$ parts from $K_{t\times n}^r$ and replacing every edge in $K_{t\times n}^r$ with a 2-subset of elements contained in G. Now, any proper coloring of G also yields a proper coloring of $K_{t\times n}^r$, so the result follows.

We do not believe that part (ii) of Proposition 12 is best possible. Rather, we would like to suggest the following.

Conjecture 13. Let L be a random $(2, \sigma)$-list assignment for $K_{t\times n}^r$. Then the property of $K_{t\times n}^r$ of being L-colorable has a sharp threshold at $\sigma = 2\frac{t-1}{r-1}n$.

Finally, let us note that the result in [4] for complete multipartite graphs in fact also holds for growing $s = s(n)$, so Proposition 12 holds for growing $t = t(n)$ as well.

5 Coloring from random lists of non-constant size

The problem of coloring graphs from random lists of non-constant size was, to the best of our knowledge, first considered in [3, 8], where some results for complete and complete bipartite graphs were obtained. More recently, it was proved independently by Keevash [14] and Jain and Pham [11] that for the complete bipartite graph $K_{n,n}$, the property of being edge-colorable from a random (k, n)-list assignment (for the edges of $K_{n,n}$) has a threshold at $k = \log n$ (thereby improving on several previous results [3, 8, 13, 19]), which had been conjectured independently by several authors. It would be interesting to consider similar questions on edge coloring complete r-partite r-uniform hypergraphs, although this is beyond the scope of this paper. We suggest the following:

Conjecture 14. Let $H = H(n)$ be a complete r-uniform r-partite hypergraph on $r \times n$ vertices. There is a constant $C > 0$, such that if L is a random $(C \log n, n)$-list assignment for the edges of H, then H is L-colorable whp.

Let us also note that similar questions have recently been studied in the context of "palette sparsification", see [12] and references therein. Earlier on, in [8] it was proved that for the complete graph K_n the property of being colorable from a random $(f(n), n)$-list assignment has a sharp threshold at $f(n) = \log n$, which can be interpreted as a palette sparsification result for K_n (generalized in [12]). Here we shall prove a similar result for hypergraphs by fixing the number of colors to be $n_r = \lceil n/(r-1) \rceil$ and ask for the minimum size $f(n)$ of lists chosen independently and uniformly at random from $[n_r] = \{1, \ldots, n_r\}$ so that whp there is a proper coloring of K_n^r from the random lists. Indeed, this property has a sharp threshold at $f(n) = \frac{\log n}{r-1}$, which generalizes the result from [8].

Theorem 15. *Let L be a random $(f(n), \lceil n/(r-1) \rceil)$-list assignment for K_n^r, where $f(n)$ is some integer-valued function. Then for every $\varepsilon > 0$, K_n^r is L-colorable whp if $f(n) \geqslant (1+\varepsilon)\frac{\log n}{r-1}$, and not L-colorable whp if $f(n) \leqslant (1-\varepsilon)\frac{\log n}{r-1}$.*

Proof. We shall use Lemma 11 to prove the theorem, and thus show that whp there is no bad set if $f(n) \geqslant (1+\varepsilon)\log n/(r-1)$ and our color set has size $\sigma(n) = n_r = \lceil n/(r-1) \rceil$, where *bad* is defined as in the proof of Theorem 10.

Let us first prove that if $f(n) = c \log n/(r-1)$ and $c < 1$, then whp there is no list coloring of K_n^r. Indeed, in this case there is whp a color that does not appear in any of the lists; hence, there cannot be a list coloring of K_n^r since it has chromatic number n_r.

Now, the probability that a given color does not appear in any list is

$$\frac{\binom{n_r-1}{f(n)}^n}{\binom{n_r}{f(n)}^n} \sim \left(\frac{n_r - f(n)}{n_r}\right)^n \sim \exp(-c \log n).$$

Hence, the expected number $\mathbb{E}[X]$ of such colors, counted by the random variable X, is $n_r e^{-c \log n} \to \infty$, if $c < 1$. A straightforward second moment argument now shows that $\mathbb{P}[X = 0] \to 0$ as $n \to \infty$.

Next, we consider the case when $c > 1$. Let us first note that if there is a bad set, then either such a set has size at most $\lceil n/2 \rceil$ or there is a subset S of the color set $[n_r]$ of size $s \leqslant \lceil n_r/2 \rceil$ such that these colors only appear in lists of t vertices in K_n^r, where $(r-1)(s-1) \leqslant t \leqslant (r-1)s - 1$. Indeed, this is easily seen by considering a bipartite graph B with parts $V(K_n^r)$ and $[n_r]$ as in the proof of Lemma 11. Moreover, such a subset S of $[n_r]$ of minimal size in B also satisfies the following:

(i) every vertex of S except one, denoted by u, has $r-1$ unique neighbors in $N_B(S)$, where $N_B(S)$ is the set of neighbors of S in B; u has r' neighbors in $N_B(S)$, where $0 \leqslant r' \leqslant r-2$;

(ii) every vertex of S has at least one additional neighbor in $N_B(S)$ except the $r-1$ neighbors mentioned in (i).

Part (i) follows from the fact that S is minimal, and (ii) follows from the fact that every vertex in $[n_r]$ whp has degree at least r in B. This latter statement is due to the fact that every vertex degree in $V(B) \cap [n_r]$ is whp greater than A, for any constant A, when $c > 1$, which is easily shown using first moment calculations as above.

We call such a subset S of $[n_r]$ of size at most $\lceil n_r/2 \rceil$, satisfying (i) and (ii), a *color-bad* set. Apparently, whp there is no color-bad set of size 1 if $c > 1$. Let us prove that whp there is no color-bad set of greater size if $c > 1$.

Let W_s be a random variable counting the number of such color-bad sets S of size s where the set of neighbors of S in B has size t, for some t such that $(r-1)(s-1) \leqslant t \leqslant (r-1)s - 1$.

Now, there are $\binom{n_r}{s}$ ways of choosing such a set S, $\binom{n}{t}$ ways of choosing $N_B(S)$, thereafter $t!$ ways of choosing a subgraph $B[S \cup N_B(S)]$ satisfying (i), and then t^s ways of choosing the edges due to the condition (ii). Next, we have at most

$$\binom{n_r - 1}{f(n) - 1}^{t-s} \binom{n_r - 2}{f(n) - 2}^{s}$$

ways of choosing the remaining neighbors of the vertices in $N_B(S)$, and $\binom{n_r - s}{f(n)}^{n-t}$ choices for the neighbors of the vertices in $V(K_n^r) \setminus N_B(S)$ in B. Thus, we deduce that

$$\begin{aligned}
\mathbb{E}[W_s] &\leqslant \frac{\binom{n_r}{s}\binom{n}{t} t! t^s \binom{n_r-1}{f(n)-1}^{t-s} \binom{n_r-2}{f(n)-2}^{s} \binom{n_r-s}{f(n)}^{n-t}}{\binom{n_r}{f(n)}^{n}} \\
&\leqslant \frac{(n_r)_s}{s!} \frac{(n)_t}{t!} t! t^s \left(\frac{(n_r-1)_{f(n)-1} f(n)!}{(n_r)_{f(n)} (f(n)-1)!} \right)^{t-s} \left(\frac{(n_r-2)_{f(n)-2} f(n)!}{(n_r)_{f(n)} (f(n)-2)!} \right)^{s} \left(\frac{(n_r-s)_{f(n)}}{(n_r)_{f(n)}} \right)^{n-t} \\
&\leqslant \left(\frac{(n_r)_s (n)_t (f(n))^{t+s}}{n_r^{t-s} n_r^s (n_r-1)^s} \right) \frac{t^s}{s!} \exp\left(-\frac{s}{n_r} f(n)(n-t)\right) \\
&\leqslant \left(\frac{A(c \log n)^r}{n^{c/2}} \right)^{s},
\end{aligned}$$

since $s \leqslant \lceil n_r/2 \rceil$ and $t \leqslant (r-1)s-1$, and where A is an absolute constant. Hence,

$$\sum_{s=2}^{\lceil n_r/2 \rceil} \mathbb{E}[W_s] = o(1),$$

given that $c > 1$.

Next, let us prove that whp there is no bad set in K_n^r. Let Y_s be a random variable counting the number of bad sets of size $(r-1)s+1$ in K_n^r. To complete the proof, we need to show that

$$\sum_{s=f(n)}^{\lceil n_r/2 \rceil} \mathbb{E}[Y_s] = o(1).$$

Now, there are $\binom{n}{(r-1)s+1}$ choices for a set S of size $(r-1)s+1$ from $V(K_n^r)$, and $\binom{n_r}{s}$ choices for a set of s colors from $[n_r]$. Thus, the expected number of bad sets of size $(r-1)s+1$ is at most

$$\binom{n}{(r-1)s+1}\binom{n_r}{s}\frac{\binom{s}{f(n)}^{(r-1)s+1}}{\binom{n_r}{f(n)}^{(r-1)s+1}} \leqslant \left(\frac{ne}{(r-1)s+1}\right)^{(r-1)s+1}\left(\frac{n_r e}{s}\right)^{s}\left(\frac{s}{n_r}\right)^{(s(r-1)+1)f(n)},$$

so

$$\sum_{s=f(n)}^{\lceil n_r/2 \rceil} \mathbb{E}[Y_s] \leqslant \sum_{s=f(n)}^{\lceil n_r/2 \rceil}\left(\frac{ne}{(r-1)s+1}\right)^{(r-1)s+1}\left(\frac{n_r e}{s}\right)^{s}\left(\frac{s}{n_r}\right)^{(s(r-1)+1)f(n)}.$$

It is readily verified that this sum tends to 0 as $n \to \infty$, e.g. by separately considering the cases when s is at most n/w_n, were $w_n \to \infty$ arbitrarily slowly, and when s is greater than n/w_n. This completes the proof of Proposition 15. □

It would be interesting to investigate similar questions on palette sparsification for general r-uniform hypergraphs. In light of the classic result by Erdős and Lovasz [9], that the (list-)chromatic number of an r-uniform hypergraph H is at most

$$\left\lceil \frac{r-1}{r-2}\left((r-2)\Delta(H)\right)^{1/(r-1)} \right\rceil$$

we suggest the following problem.

Problem 16. Let $H = H(n)$ be an r-uniform hypergraph with maximum degree Δ, and C a positive constant. What is the smallest $k = k(n, C, r)$ so that H whp is list colorable from a random $(k, C\Delta^{1/(r-1)})$-list assignment?

To the best of our knowledge, this problem is completely open, and in light of the above theorem for complete hypergraphs, it is tempting to conjecture that $k = \Theta((\log n/(r-1))$ for small C (which would be a generalization of recent results on palette sparsificiation, see e.g. [12]), and perhaps that k could be as small as $\Theta\left((\log n)^{1/r}\right)$ if C is sufficiently large, which would be a generalization of a result by Alon and Assadi [1].

Finally, let us note that it is possible to prove versions of some of the above results for random list assignments of non-constant size. Indeed, we can easily obtain a version of Theorem 6 for lists of non-constant size. For an r-uniform hypergraph $H = H(n)$ on n vertices with maximum degree at most Δ, and L a random (k, σ)-list assignment for H with $k = \omega(1)$, we have the following.

Proposition 17. *Suppose that $k = O(\log^{1/4} n)$ and $\Delta = O\left(n^{\frac{k-1}{k^2(k^2+k)r(r-1)}}\right)$. For any $\varepsilon > 0$, H is whp L-colorable if one of the following holds:*

(i) $k = o(\log \Delta)$ *and* $\sigma(n) \geqslant (1+\varepsilon)n^{\frac{1}{k^2(r-1)}}\Delta^{\frac{1}{k(r-1)}}k$, *or*

(ii) $k = C\log\Delta$ *and* $\sigma(n) \geqslant (1+\varepsilon)n^{\frac{1}{k^2(r-1)}}e^{\frac{1}{C(r-1)}}k$, *or*

(iii) $k = \omega(\log\Delta)$ *and* $\sigma(n) \geqslant (1+\varepsilon)n^{\frac{1}{k^2(r-1)}}k$.

The proof of the above theorem is along the same lines as the proof of Theorem 6. Since k is relatively small compared to σ and n, the same arguments and calculations as in the proof of Theorem 6 yield the required conclusions.

Note further that the example after Thereom 6, shows that the bound on σ in the above theorem is best possible up to the multiplicative factor k, provided that $k = o(\Delta^{\frac{1}{2(r-1)}})$.

For lists of greater size we have the following analogue of Proposition 8 which is valid for all Δ:

Proposition 18. *Let H and L be as above.*

(i) If $k = o(\log n)$, then for any $\epsilon > 0$, if $\sigma(n) \geqslant (1+\epsilon)n^{\frac{1}{k(r-1)}}\Delta^{\frac{1}{r-1}}k$, then whp H is L-colorable.

(ii) For every constant $C > 0$, there is a constant $A > 0$ such that if $k \geqslant A\log n$ and $\sigma \geqslant C\Delta^{1/(r-1)}\log n$, then whp H is L-colorable.

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