Cyclic Orderings of Paving Matroids

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Abstract

A matroid M of rank r is *cyclically orderable* if there is a cyclic permutation of the elements of M such that any r consecutive elements form a basis in M. An old conjecture of Kajitani, Miyano, and Ueno states that a matroid M is cyclically orderable if and only if for all $\emptyset \neq X \subseteq E(M)$, $\frac{|X|}{r(X)} \leqslant \frac{|E(M)|}{r(M)}$. In this paper, we verify this conjecture for all paving matroids.

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1 Introduction

A matroid M of rank r is **cyclically orderable** if there is a a cyclic permutation of the elements of M such that any r consecutive elements is a base.

For a matroid M and a subset $\emptyset \neq X \subseteq E(M)$, we define $\beta(X) := \frac{|X|}{r(X)}$, if $r(X) \neq 0$; otherwise, $\beta(X) := \infty$. Let $\gamma(M) = \max_{\emptyset \neq X \subseteq E(M)} \beta(X)$.

It turns out that the condition $\gamma(M) = \beta(E(M))$ is a necessary condition for a matroid M to be cyclically orderable. To see this, suppose $e_1 e_2 \cdots e_n$ is a cyclic ordering of a rank-r matroid M. Then for any nonempty subset $A \subseteq E(M)$, we have $r|A| = \sum_{i=1}^{n} |A \cap \{e_i, e_{i+1}, \ldots, e_{i+r}\}| \leqslant nr(A)$. The first equality follows from the fact that each element of A appears in exactly r sets $\{e_i, e_{i+1}, \ldots, e_{i+r}\}$ and the second inequality follows from the fact that $|A \cap \{e_i, e_{i+1}, \ldots, e_{i+r}\}| \leqslant r(A)$. Consequently, $\beta(A) \leqslant \beta(E(M))$ and hence $\gamma(M) = \beta(E(M))$. In light of this, the following conjecture of Kajitani, Miyano, and Ueno [6] seems natural:

Conjecture 1. A matroid M is cyclically orderable if and only if $\gamma(M) = \beta(E(M))$.

Despite having been around for decades, the above conjecture is only known to be true for a few special classes of matroids. In [2], the conjecture was shown to be true for sparse paving matroids. Perhaps the strongest result thus far can be found in [8] where it was shown that Conjecture 1 is true when $r(M)$ and $|E(M)|$ are relatively prime.

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2 Theorem (Van Den Heuvel and Thomasse)
Let M be a matroid for which $\gamma(M) = \beta(E(M))$. If $|E(M)|$ and $r(M)$ are relatively prime, then M has a cyclic ordering.

It follows from recent results in [1] on *split matroids*, a class which includes paving matroids, that the conjecture is true for paving matroids M where $|E(M)| \leqslant 2r(M)$. Coupled with Theorem 2, we can replace $2r(M)$ by $2r(M)+1$ in this bound since $|E(M)|$ and $r(M)$ are relatively prime when $|E(M)| = 2r(M)+1$. In this paper, we verify Conjecture 1 for all paving matroids.

Theorem 3. *Let M be a paving matroid where $\gamma(M) = \beta(E(M))$. Then M is cyclically orderable.*

For concepts, terminology, and notation pertaining to matroids, we shall follow Oxley [7] when possible. For a matroid M, $\mathcal{C}(M)$ will denote the set of all circuits of M.

For a finite set A and integer $k \leqslant |A|$, we let $\binom{A}{k}$ denote the set of all k-subsets of A. For a collection of subsets $\mathcal{A}$ and integer k we let $\binom{k}{\mathcal{A}}$ denote the set of all sets in $\mathcal{A}$ having cardinality k.

For a set A and elements $x_1, \dots, x_k$ we will often write, for convenience, $A + x_1 + x_2 + \cdots + x_k$ (resp. $A - x_1 - x_2 - \cdots - x_k$) in place of $A \cup \{x_1, \dots, x_k\}$ (resp, $A \backslash \{x_1, \dots, x_k\}$).

For a positive integer n, we let $[n]$ denote the set $\{1, \dots, n\}$.

1.1 Idea behind the proof

To prove the main theorem, we shall use induction on $|E(M)|$. To do this, we shall first remove a basis S from M so that the resulting matroid M' satisfies $\gamma(M') = \beta(E(M)-S)$. While generally such a basis S may not exist, we will show that such bases exist when $|E(M)| \geqslant 2r(M)+2$. Applying the inductive assumption, M' is cyclically orderable, with a cyclic ordering say $e_1 e_2 \cdots e_m$. We will show that for some $i \in [m]$ and some ordering of S, say $s_1 s_2 \cdots s_r$ (where $r = r(M)$), the ordering $e_1 \cdots e_i s_1 s_2 \cdots s_r e_{i+1} \cdots e_m$ is a cyclic ordering of M. To give a rough idea of how to prove this, we will illustrate the proof in the case where $r(M) = 3$.

Suppose $S = \{s_1, s_2, s_3\}$ is a basis of M where $\gamma(M \backslash S) = \beta(E(M)-S)$ and $r(M \backslash S) = 3$. Assume that $M' = M \backslash S$ has a cyclic ordering $e_1 e_2 \cdots e_m$. Suppose we try to insert the elements of S, in some order, between e_m and e_1, so as to achieve a cyclic ordering for M. Assume this is not possible. Then for every permutation π of $\{1,2,3\}$, $e_1 e_2 \cdots e_m s_{\pi(1)} s_{\pi(2)} s_{\pi(3)}$ is not a cyclic ordering of M. Thus for all permutations π of $\{1,2,3\}$, at least one of $\{e_{m-1}, e_m, s_{\pi(1)}\}$, $\{e_m, s_{\pi(1)}, s_{\pi(2)}\}$, $\{s_{\pi(2)}, s_{\pi(3)}, e_1\}$, or $\{s_{\pi(3)}, e_1, e_2\}$ is a circuit. As an exercise for the reader, one can now show that there exist distinct $i, j \in \{1,2,3\}$ such $\{s_i, e_{m-1}, e_m\}, \{s_j, e_1, e_2\}, S - s_i + e_m$, and $S - s_j + e_1$ are circuits. We may assume that $i = 1$ and $j = 2$. If instead, one were to assume that one could not insert the elements of S in some order between e_1 and e_2 so as to achieve a cyclic ordering of M, then as above, there exist distinct $i', j' \in \{1,2,3\}$, such that $\{s_{i'}, e_m, e_1\}, \{s_{j'}, e_2, e_3\}, S - s_{i'} + e_1$, and $S - s_{j'} + e_2$ are circuits. If $i' = 1$, then $\{s_1, e_{m-1}, e_m\}$ and $\{s_1, e_m, e_1\}$ are circuits. The circuit elimination axiom (together with the fact that M is a paving matroid) would then imply that $(\{s_1, e_{m-1}, e_m\} \cup \{s_1, e_m, e_1\}) - s_1 = \{e_{m-1}, e_m, e_1\}$ is a circuit, contradicting our assumption that $e_1 e_2 \cdots e_m$ is a cyclic ordering of M'. Also, if $i' = 2$, then $\{s_2, e_m, e_1\}$ and $\{s_2, e_1, e_2\}$ are circuits and hence by the circuit elimination axiom, $\{e_m, e_1, e_2\}$ is a circuit, a contradiction. Thus $i' \notin \{1, 2\}$ and hence $i' = 3$ and $\{e_m, e_1, s_3\}$ and $\{s_1, s_2, e_2\}$ are circuits. Given that $\{s_2, e_1, e_2\}$ is also a circuit, it follows that $\{e_1, e_2\} \subset \mathrm{cl}(\{s_1, s_2\})$. Now $j' \in \{1, 2\}$, and $\{s_{i'}, e_2, e_3\}$ is a circuit, implying that $e_3 \in \mathrm{cl}(\{s_1, s_2\})$. However, this is impossible since (by assumption) $\{s_1, s_2, s_3\}$ is a basis. Thus there must be some ordering of S so that when the elements of S are inserted (in this order) between e_m and e_1 or between e_1 and e_2, the resulting ordering is a cyclic ordering for M.

2 Removing a basis from a matroid

Let M be a paving matroid where $\gamma(M) = \beta(E(M))$. As a first step in the proof of Theorem 3, we wish to find a basis B of M where $\gamma(M \backslash B) = \beta(E(M) - B)$. Unfortunately, there are matroids where there is no such basis, as for example, the Fano plane. In this section, we will show that, despite this, such bases exist when $|E(M)| \geqslant 2r(M) + 2$.

The following is an elementary observation which we will refer to in a number of places.

Observation 4. *For a basis B in a matroid M and an element $x \in E(M) - B$, the set $B + x$ has a unique circuit which contains x.*

We will need the following strengthening of Edmonds' matroid partition theorem [3] given in [4]:

Theorem 5. *Let M be a matroid where $\gamma(M) = k + \varepsilon$, where $k \in \mathbb{N}$ and $0 \leqslant \varepsilon < 1$. Then $E(M)$ can be partitioned into $k+1$ independent sets with one set of size at most $\varepsilon r(M)$.*

We are now in a position to prove the main result of this section.

Proposition 6. *Let M be a paving matroid where $\gamma(M) = \beta(E(M))$, $|E(M)| \geqslant 2r(M) + 2$, and $r(M) \geqslant 3$. Then there is a basis B of M where $\gamma(M \backslash B) = \beta(E(M) - B)$ and $r(M \backslash B) = r(M)$.*

Proof. Let $\gamma(M) = k + \frac{\ell}{r(M)}$ where $0 \leqslant \ell < r(M)$ and $k \geqslant 2$. Then $|E(M)| = kr(M) + \ell$ and it follows by Theorem 5 that one can partition $E(M)$ into k independent sets $F_1, \ldots, F_k$ and one independent set F_{k+1} having at most ℓ elements. Since for all $i \in [k]$, $|F_i| \leqslant r(M)$ and $|F_{k+1}| \leqslant \ell$ it follows that $kr(M) + \ell = |E(M)| = \sum_{i=1}^{k} |F_i| + |F_{k+1}| \leqslant kr(M) + \ell$. Thus equality must hold in the inequality and as such, for all $i \in [k]$, $|F_i| = r(M)$ and $|F_{k+1}| = \ell$. Thus $F_1, \ldots, F_k$ are bases in M. Let $r = r(M)$. If $\ell = 0$, then $|E(M)| = kr \geqslant 3r$. In this case, we can take $B = F_k$ since for $M' = M \backslash F_k$, it is seen that $\gamma(M') = k - 1 = \beta(E(M'))$. Thus we may assume that $\ell > 0$.

Let $F_k = \{x_1, x_2, \ldots, x_r\}$. Suppose there exist distinct $i, j \in [r]$ for which $r((F_k - x_i) \cup F_{k+1}) = r((F_k - x_j) \cup F_{k+1}) = r - 1$. Let $x \in F_{k+1}$. Then $x + (F_k - x_i)$ and $x + (F_k - x_j)$ are (distinct) circuits, contradicting Observation 4. Thus there is at most one $i \in [r]$ for which $r((F_k - x_i) \cup F_{k+1}) = r - 1$. As such, we may assume that for $i = 1, \ldots, r-1$, $r((F_k - x_i) \cup F_{k+1}) = r$. Thus for $i = 1, \ldots, r-1$, there is a subset $A_i \subseteq F_k - x_i$ such that $B_i = A_i \cup F_{k+1}$ is a basis for M.

We shall show that the bases B_i, $i = 1, \ldots, r-1$ can be chosen so that for some $i \in [r-1]$, $B = B_i$ is a basis satisfying the proposition. Suppose that none of the bases B_i satisfy the proposition. Then for all $i \in [r-1]$, there is a subset $X_i \subseteq E(M) - B_i$ for which $\beta(X_i) > \beta(E(M) - B_i)$. Since $k > 1$, we have that $F_1 \subseteq E(M \backslash B_i)$ and hence $r(M \backslash B_i) = r$. Thus we have $\beta(E(M) - B_i) = k - 1 + \frac{\ell}{r}$. If $r(X_i) < r - 1$, then X_i is independent and hence $\beta(X_i) = 1 \leqslant \beta(E(M) - B_i)$. Thus $r(X_i) \geqslant r - 1$ and seeing as $\beta(X_i) > \beta(E(M) - B_i)$, we have $r(X_i) \leqslant r - 1$. Consequently, $r(X_i) = r - 1$ and $\beta(X_i) = \frac{|X_i|}{r-1} > k - 1 + \frac{\ell}{r}$. Since $r(X_i) = r - 1$, it follows that for $j = 1, \ldots, k-1$, $|X_i \cap F_j| \leqslant r - 1$. Consequently, $|X_i| \leqslant (k-1)(r-1) + \ell$. If $|X_i| < (k-1)(r-1) + \ell$, then $\beta(X_i) \leqslant k - 1 + \frac{\ell - 1}{r-1}$, implying that $\beta(X_i) \leqslant k - 1 + \frac{\ell}{r}$, contradicting our assumptions. Thus it follows that $|X_i| = (k-1)(r-1) + \ell$ and for all $i \in [r-1]$ and for all $j \in [k-1]$, $|X_i \cap F_j| = r - 1$, and $F_k - A_i \subset X_i$. Thus for all $i \in [r-1]$ and for all $j \in [k-1]$, $X_{ij} = X_i \cap F_j$ spans X_i. Since all circuits in M have size at least r, it follows that for all $j \in [k-1]$, and for all $x \in X_i - X_{ij}$, $X_{ij} + x$ is a circuit.

Suppose $k \geqslant 3$. Let $i, j \in [r-1]$ where i and j are distinct (noting that such i, j exists since $r \geqslant 3$). Since $r \geqslant 3$, there exists $x \in X_{i2} \cap X_{j2}$. We have that $x + X_{i1}$ and $x + X_{j1}$ are circuits. It follows by Observation 4 that $X_{i1} = X_{j1}$ and thus $\mathrm{cl}(X_i) = \mathrm{cl}(X_j)$. Let $X = \mathrm{cl}(X_i)$. Since $F_k - A_i \subset X_i$, $F_k - A_j \subset X_j$, $x_i \in F_k - A_i$ and $x_j \in F_k - A_j$, we have $\{x_i, x_j\} \subset X$. Since this applies to all $j \in [r-1] - i$, it follows that $F_k - x_r \subset X$. If $r((F_k - x_r) \cup F_{k+1}) = r$, then one could let x_r play the role of x_{r-1}, and it would follow that $x_r \in X$. This would imply that $F_k \subset X$, an impossibility (since $r(X) = r - 1$). Thus $r((F_k - x_r) \cup F_{k+1}) = r - 1$. Given that $F_k - x_r \subset X$, we have $F_{k+1} \subseteq \mathrm{cl}(F_k - x_r) \subset X$. Now it is seen that $\beta(X) = \frac{|X|}{r(X)} = \frac{k(r-1)+\ell}{r-1} = k + \frac{\ell}{r-1} > \gamma(M)$, a contradiction.

From the above, we have $k = 2$. Since $|E(M)| \geqslant 2r(M) + 2$, we have $\ell \geqslant 2$. Let $i \in [r-1]$.

Claim 7. *For all $j \in [r-1] - i$, one can choose B_j so that $X_{j1} = X_{i1}$.*

Proof. Let $j \in [r-1] - i$. Suppose there exists $x \in (F_2 - A_i) \cap (F_2 - A_j)$. Then $x \in X_i \cap X_j$ (since $F_2 - A_i \subset X_i$ and $F_2 - A_j \subset X_j$) and, given that $r(X_i) = r(X_j) = r - 1 = |X_{i1}| = |X_{j1}|$, it follows that $x + X_{i1}$ and $x + X_{j1}$ are circuits. It now follows by Observation 4 that $X_{i1} = X_{j1}$. Suppose instead that $(F_2 - A_i) \cap (F_2 - A_j) = \emptyset$. That is, $F_2 - A_i \subseteq A_j$ (and $F_2 - A_j \subseteq A_i$). Since $\ell \geqslant 2$, there exists $x_s \in F_2 - A_j - x_j$. Now $x_s + B_j$ contains a (unique) circuit C where $x_s \in C$. We claim that $C \cap (F_2 - A_i) \neq \emptyset$. To see this, we observe that $|A_j - (F_2 - A_i)| = r - 2\ell$. Thus

$$\begin{aligned} |C \cap (F_2 - A_i)| &= |C - x_s| - |C \cap ((A_j - (F_2 - A_i)) \cup F_3))| \\ &\geqslant |C| - 1 - ((r - 2\ell) + \ell) = |C| - 1 - r + \ell \geqslant \ell - 1 \geqslant 1. \end{aligned}$$

Let $x_t \in C \cap (F_2 - A_i)$. Observing that $B_j - x_t + x_s$ is also a basis, let $A'_j = A_j - x_t + x_s$ and $B'_j = B_j - x_t + x_s$. Then $B'_j = A'_j + F_3$ and moreover, $x_t \in (F_2 - A_i) \cap (F_2 - A'_j)$.

Now defining X_j as before, using B'_j in place of B_j, one obtains that $X_{i1} = X_{j1}$, as in the previous case. $\square$

By the above claim, we may assume that for all $j \in [r-1] - i$, the base B_j can be chosen so that $X_{i1} = X_{j1}$. Letting $X = \mathrm{cl}(X_i)$ and following similar reasoning as before, we have that $(F_2 - x_r) \cup F_3 \subset X$. Thus $\beta(X) = \frac{|X|}{r(X)} = \frac{2(r-1)+\ell}{r-1} = 2 + \frac{\ell}{r-1} > \gamma(M)$, a contradiction. It follows that for some $i \in [r-1]$, the proposition holds for $B = B_i$. $\square$

3 S-Pairs

In the second part of the proof of Theorem 3, we will need to establish the existence of certain circuits. More specifically, suppose S is a basis as described in Proposition 6 where we assume that $S = \{s_1, \ldots, s_r\}$. Suppose $e_1 e_2 \ldots e_m$ is cyclic ordering for $M' = M \backslash S$ and our aim is to extend this ordering to a cyclic ordering for M by inserting the elements of S, in some order, between e_m and e_1. Assuming this is not possible, it turns out (as in the case where $r(M) = 3$) that there must be certain circuits. For example, there are subsets $\{B_1, B_2\} \in \binom{S}{r-2}$ such that for all $s_i \in B_1$, $\{s_i, e_{m-r+2}, \ldots, e_m\} \in \mathcal{C}(M)$ and for all $s_i \in B_2, \{s_i, e_1, \ldots, e_{r-1}\} \in \mathcal{C}(M)$. The results in this section and its successor, lay the ground work to prove the existence of such circuits.

Let S be a finite, nonempty set. For $i = 1, 2$, let $\mathcal{S}_i \subseteq 2^S$. We call the pair $(\mathcal{S}_1, \mathcal{S}_2)$ an **S-pair** if it has the following properties.

(**S1**) For $i = 1, 2$, if $A, B \in \mathcal{S}_i$ where $|A| = |B| + 1$ and $B \subset A$, then $\binom{A}{|B|} \subseteq \mathcal{S}_i$.

(**S2**) For $i = 1, 2$, if $A, B \in \mathcal{S}_i$ where $|A| = |B|$ and $|A \cap B| = |A| - 1$, then $A \cup B \in \mathcal{S}_i$.

(**S3**) For $i = 1, 2$, $\binom{S}{1} \not\subseteq \mathcal{S}_i$ and $S \notin \mathcal{S}_i$.

(**S4**) For $k = 1, \ldots, |S| - 1$, if $\binom{S-x}{k} \subseteq \mathcal{S}_1$ for some $x \in S$, then $\binom{S-x}{|S|-k} \not\subseteq \mathcal{S}_2$.

In the next section, we shall need the following observations for an S-pair $(\mathcal{S}_1, \mathcal{S}_2)$ where $|S| = r$.

Observation 8. *Let $A \subseteq S$ where $\alpha = |A|$. Suppose that for some $i \in \{1, 2\}$ and some $j \in [\alpha]$, $\binom{A}{j} \subseteq \mathcal{S}_i$. Then for $k = j, \ldots, \alpha$, $\binom{A}{k} \subseteq \mathcal{S}_i$.*

Proof. We may assume that $j < \alpha$. Suppose that for some $k \in \{j, \ldots, \alpha - 1\}$, $\binom{A}{k} \subseteq \mathcal{S}_i$. Let $B \in \binom{A}{k+1}$. Let $\{b_1, b_2\} \subseteq B$ and for $s = 1, 2$, let $B_s = B - b_s$. By assumption, for $s = 1, 2$, $B_s \in \mathcal{S}_i$. It now follows by (**S2**) that $B = B_1 \cup B_2 \in \mathcal{S}_i$. Consequently, we have that $\binom{A}{k+1} \subseteq \mathcal{S}_i$. Arguing inductively, we see that for $k = j, \ldots, \alpha$, $\binom{A}{k} \subseteq \mathcal{S}_i$. $\square$

Observation 9. *Let $A \in \mathcal{S}_i$ where $\alpha = |A|$. Suppose that for some $j \in [\alpha - 1]$ and $x \in A$, we have $\binom{A-x}{j} \subseteq \mathcal{S}_i$. Then $\binom{A}{j} \subseteq \mathcal{S}_i$.*

Proof. Suppose first that $j = \alpha - 1$. Then $A' = A - x \in \mathcal{S}_i$. It follows by (**S1**) that $\binom{A}{\alpha-1} \subseteq \mathcal{S}_i$. Assume that $j < \alpha - 1$ and the assertion holds for $j+1$; that is, if $\binom{A-x}{j+1} \subseteq \mathcal{S}_i$, then $\binom{A}{j+1} \subseteq \mathcal{S}_i$. Suppose $\binom{A-x}{j} \subseteq \mathcal{S}_i$. Then by Observation 8, $\binom{A-x}{j+1} \subseteq \mathcal{S}_i$. Thus by assumption, $\binom{A}{j+1} \subseteq \mathcal{S}_i$. Let $B \in \binom{A}{j}$, where $x \in B$. Let $y \in A - B$ and let $B' = B - x + y$. Since $B' \in \binom{A-x}{j}$, it follows that $B' \in \mathcal{S}_1$. However, we also have that $B + y \in \mathcal{S}_i$. Thus it follows by (**S1**) that $B \in \mathcal{S}_i$. We now see that $\binom{A}{j} \subseteq \mathcal{S}_i$. The assertion now follows by induction. □

Observation 10. *Let $A \subseteq S$. Suppose for some $x \in A$, $i \in \{1, 2\}$, and $j \geqslant 2$, we have that $\{B \in \binom{A}{j} \mid x \in B\} \subseteq \mathcal{S}_i$. Then $\binom{A}{j} \subseteq \mathcal{S}_i$ and $A \in \mathcal{S}_i$.*

Proof. We may assume that $|A| \geqslant j + 1$. Let $B' \in \binom{A-x}{j}$. Let $\{y_1, y_2\} \subseteq B'$ and for $s = 1, 2$, let $B_s = B' - y_s + x$. By assumption, $\{B_1, B_2\} \subset \mathcal{S}_i$. It follows by (**S2**) that $B = B' + x = B_1 \cup B_2 \in \mathcal{S}_i$. Thus by (**S1**) we have that $\binom{B}{j} \subseteq \mathcal{S}_i$ and hence $B' \in \mathcal{S}_i$. It now follows that $\binom{A}{j} \subseteq \mathcal{S}_i$, and moreover, $A \in \mathcal{S}_i$ (by Observation 8). □

4 Order-consistent pairs

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of n elements and let $\mathcal{S}_1 \subseteq 2^S$ and $\mathcal{S}_2 \subseteq 2^S$. We say that the pair $(\mathcal{S}_1, \mathcal{S}_2)$ is **order-consistent** with respect to S if for any permutation π of $[n]$, there exists $i \in [n]$ for which either $\{s_{\pi(1)}, \cdots, s_{\pi(i)}\} \in \mathcal{S}_1$ or $\{s_{\pi(i)}, \ldots, s_{\pi(n)}\} \in \mathcal{S}_2$. Note that if $(\mathcal{S}_1, \mathcal{S}_2)$ is order-consistent, then $(\mathcal{S}_2, \mathcal{S}_1)$ is also order consistent. To see this, let π be a permutation of $[n]$ and let π' be the permutation which is the reverse of π; that is, for all $i \in [n]$, $\pi'(i) = \pi(n - i + 1)$. Since $(\mathcal{S}_1, \mathcal{S}_2)$ is order-consistent, there exists $i \in [n]$ such that either $\{s_{\pi'(1)}, \ldots, s_{\pi'(i)}\} \in \mathcal{S}_1$ or $\{s_{\pi'(i)}, \ldots, s_{\pi'(n)}\} \in \mathcal{S}_2$. Thus either $\{s_{\pi(n-i+1)}, \ldots, s_{\pi(n)}\} \in \mathcal{S}_1$ or $\{s_{\pi(1)}, \ldots, s_{\pi(n-i+1)}\} \in \mathcal{S}_2$. Given that this holds for all permutations π, it follows that $(\mathcal{S}_2, \mathcal{S}_1)$ is an order-consistent pair.

Let Π denote the set of all permutations of $[n]$ and let $\pi \in \Pi$. We say that a subset $A \in \mathcal{S}_1$ (resp. $B \in \mathcal{S}_2$) is π-**relevant** if there exists $i \in [n]$ such that $A = \{s_{\pi(1)}, \ldots, s_{\pi(i)}\}$ (resp. $B = \{s_{\pi(i)}, \ldots, s_{\pi(n)}\}$). Let $\Pi' \subseteq \Pi$ be a subset of permutations. We say that a subset $\mathcal{A} \subseteq \mathcal{S}_1$ (resp. $\mathcal{B} \subseteq \mathcal{S}_2$) is Π'-**relevant** if for all $A \in \mathcal{A}$ (resp. $B \in \mathcal{B}$), there exists $\pi \in \Pi'$ such that A (resp. B) is π-relevant. We say that $(\mathcal{A}, \mathcal{B})$ is order-consistent relative to Π' if for all $\pi \in \Pi'$, either there exists $A \in \mathcal{A}$ for which A is π-relevant, or there exists $B \in \mathcal{B}$ for which B is π-relevant. For $i \in [n]$, we let Π_i denote the set of permutations $\pi \in \Pi$ where $\pi(1) = i$. The following theorem will be instrumental in the proof of main theorem.

Theorem 11. *Let $S = \{s_1, \ldots, s_n\}$ be a set where $n \geqslant 3$ and let $(\mathcal{S}_1, \mathcal{S}_2)$ be an S-pair. Then $(\mathcal{S}_1, \mathcal{S}_2)$ is order-consistent if and only if there exists $(A_1, A_2) \in \binom{n-1}{\mathcal{S}_1} \times \binom{n-1}{\mathcal{S}_2}$, $A_1 \neq A_2$, and $\{B_1, B_2\} \subset \binom{S}{n-2}$ where for $i = 1, 2$, $B_i \cap A_i = B_1 \cap B_2 \in \binom{A_1 \cap A_2}{n-3}$ and $\binom{B_i}{1} \subset \mathcal{S}_i$.*

Proof. To prove sufficiency, suppose $A_i, B_i,\ i = 1, 2$ are as described in the theorem. Note that since $A_1 \neq A_2$, we have $A_1 \cup A_2 = S$. Also, since $B_1 \cap B_2 \subseteq A_1 \cap A_2$, we have $|B_1 \cap B_2| = n - 3 = |A_1 \cap A_2| - 1$. Now $B_1 \not\subseteq A_1$, for otherwise $|B_1 \cap B_2| = |A_1 \cap B_1| = |B_1| = n - 2$. Thus $B_1 \subseteq A_2$, and likewise, $B_2 \subseteq A_1$. For $i = 1, 2$, let $\mathcal{T}_i = \{A_i\} \cup \binom{B_i}{1}$. We need only show that $(\mathcal{T}_1, \mathcal{T}_2)$ is order-consistent. Suppose it is not. Clearly it is order-consistent relative to the set of permutations π for which $s_{\pi(1)} \in B_1$ or $s_{\pi(n)} \in B_2$. Let $\pi \in \Pi$ where $s_{\pi(1)} \notin B_1$ and $s_{\pi(n)} \notin B_2$. If $s_{\pi(1)} \notin A_2$, then $A_2 = \{s_{\pi(2)}, \dots, s_{\pi(n)}\}$ and A_2 is π-relevant. Thus $A_2 - B_1 = \{s_{\pi(1)}\} = (A_1 \cap A_2) - B_1$. By similar reasoning, we also have $A_1 - B_2 = \{s_{\pi(n)}\} = (A_1 \cap A_2) - B_1$. However, our assumptions imply that $(A_1 \cap A_2) - B_1 = (A_1 \cap A_2) - B_2$, and consequently, $s_{\pi(1)} = s_{\pi(r)}$. This yields a contradiction. It follows that $(\mathcal{T}_1, \mathcal{T}_2)$ is order-consistent.

To prove necessity, we shall use induction on n. It is a straightforward exercise to verify the assertion for $n = 3$. We shall assume that $n \geqslant 4$ and the assertion is valid to all values less than n. That is, if $|S| < n$, and $(\mathcal{S}_1, \mathcal{S}_2)$ is an S-pair which is order-consistent, then there exist sets $A_i, B_i,\ i = 1, 2$ as described in the theorem. Assume now that $S = \{s_1, \dots, s_n\}$ and $(\mathcal{S}_1, \mathcal{S}_2)$ is an S-pair which is order-consistent.

For all $k \in [n]$, let $S^k = S - s_k$ and let $\mathcal{S}_1^k = \{A - s_k \mid A \in \mathcal{S}_1 \text{ and } s_k \in A\}$ and $\mathcal{S}_2^k = \{A \in \mathcal{S}_2 \mid s_k \notin A\}$. We observe that properties (**S1**) and (**S2**) still hold for the pair $(\mathcal{S}_1^k, \mathcal{S}_2^k)$ whereas (**S3**) and (**S4**) may not.

(A) *For all $k \in [n]$, one of the following holds:*

*(**a1**)* $\{s_k\} \in \mathcal{S}_1$.

*(**a2**)* $S^k \in \mathcal{S}_2$.

*(**a3**)* $\binom{S^k}{1} \subseteq \mathcal{S}_2$.

*(**a4**) For some $D \in \binom{S^k}{n-2}$, and positive integers i, j where $i + j = n - 1$, $\binom{D}{i} \subseteq \mathcal{S}_1^k$ and $\binom{D}{j} \subseteq \mathcal{S}_2^k$.*

*(**a5**) There exist $(A_1^k, A_2^k) \in \binom{n-2}{\mathcal{S}_1^k} \times \binom{n-2}{\mathcal{S}_2^k}$, $A_1^k \neq A_2^k$, and $\{B_1^k, B_2^k\} \subseteq \binom{S^k}{n-3}$ where for $i = 1, 2$, $B_i^k \cap A_i^k = B_1^k \cap B_2^k \in \binom{A_1^k \cap A_2^k}{n-4}$ and $\binom{B_i^k}{1} \subseteq \mathcal{S}_i^k$.*

Proof. Let $k \in [n]$. Assume that none of (**a1**) - (**a4**) hold for k. We will show that (**a5**) must hold for k. Clearly $S^k \notin \mathcal{S}_1^k$, for otherwise this would mean that $S \in \mathcal{S}_1$ which is not allowed by (**S3**). We also have that $\binom{S^k}{1} \not\subseteq \mathcal{S}_1^k$. For if this was the case, then it would follow that for all $i \in [n] - k$, $\{s_i, s_k\} \in \mathcal{S}_1$. It would then follow by Observation 10 that $S \in \mathcal{S}_1$ violating (**S3**). Given that (**a2**) -(**a4**) do not hold, $(\mathcal{S}_1^k, \mathcal{S}_2^k)$ is seen to be an S^k-pair. Let $\pi \in \Pi_k$ and let $\pi' = \pi(2)\pi(3)\cdots\pi(n)$. Since $(\mathcal{S}_1, \mathcal{S}_2)$ is order-consistent, there exists $A \in \mathcal{S}_1$ or $B \in \mathcal{S}_2$ and $i \in [n]$ such that either $A = \{s_{\pi(1)}, \dots, s_{\pi(i)}\}$ or $B = \{s_{\pi(i)}, \dots, s_{\pi(n)}\}$. Given that (**a1**) and (**a2**) do not hold, it follows that in the former case, $i \geqslant 2$, $A' = \{s_{\pi(2)}, \dots, s_{\pi(i)}\} \in \mathcal{S}_1^k$ and hence A' is π'-relevant. In the latter case, $i \geqslant 3$ and $B' = \{s_{\pi(i)}, \dots, s_{\pi(n)}\} \in \mathcal{S}_2^k$ and B' is π'-relevant. Given that π was arbitrarily chosen from Π_k, we see that $(\mathcal{S}_1^k, \mathcal{S}_2^k)$ is order-consistent with respect to S^k. By the inductive assumption, there exist $(A_1^k, A_2^k) \in \binom{n-2}{\mathcal{S}_1^k} \times \binom{n-2}{\mathcal{S}_2^k}$, $A_1^k \neq A_2^k$, and $\{B_1^k, B_2^k\} \subset \binom{S^k}{n-3}$ where for $i = 1, 2$, $B_i^k \cap A_i^k = B_1^k \cap B_2^k \in \binom{A_1^k \cap A_2^k}{n-4}$ and $\binom{B_i^k}{1} \subset \mathcal{S}_i^k$. Thus (**a5**) holds for k. □

(B) *There is at most one integer k for which (**a2**) or (**a3**) holds.*

Proof. It suffices to prove that (**a2**) can hold for at most one integer k; if (**a3**) holds for some integer k, then it follows by Observation 8 that $S^k \in \mathcal{S}_2$, and hence (**a2**) holds for k. Suppose to the contrary that (**a2**) holds for distinct integers k and ℓ. Then $S^k \in \mathcal{S}_2$ and $S^\ell \in \mathcal{S}_2$. It then follows by (**S2**) that $S = S^k \cup S^\ell \in \mathcal{S}_2$. However, this violates (**S3**). Thus no two such integers can exist. □

(C) *Property (**a4**) holds for at most one integer k.*

Proof. Suppose (**a4**) holds for distinct integers k and ℓ. Then for some i, j, i', j' where $i + j = n - 1$, $i' + j' = n - 1$, and subsets $D \in \binom{S^k}{n-2}$ and $D' \in \binom{S^\ell}{n-2}$, we have $\binom{D}{i} \subseteq \mathcal{S}_1^k$, $\binom{D}{j} \subseteq \mathcal{S}_2^k$, $\binom{D'}{i'} \subseteq \mathcal{S}_1^\ell$, and $\binom{D'}{j'} \subseteq \mathcal{S}_2^\ell$. By Observation 10, we have that $F_1 = D + s_k \in \mathcal{S}_1$ and $F_2 = D' + s_\ell \in \mathcal{S}_1$. If $F_1 \neq F_2$, then by property (**S2**), $F_1 \cup F_2 = S \in \mathcal{S}_1$, violating (**S3**). Thus $F_1 = F_2 = S - s = S'$ for some $s \in S - s_k - s_\ell$ and $S' \in \mathcal{S}_1$.

Let $i^* = \max\{i, i'\}$ and $j^* = \min\{j, j'\}$. We claim that $\binom{S'}{i^*+1} \subseteq \mathcal{S}_1$ and $\binom{S'}{j^*} \subseteq \mathcal{S}_2$. To prove the first assertion, we first note that it is true when $i^* = n - 2$ since $S' \in \mathcal{S}_1$. We may assume that $i^* < n-2$. Then $i^* \leqslant n-3 = |D \cap D'| = |S' - s_k - s_\ell|$. Suppose first that $i^* = i$. Then by assumption, $\binom{D \cap D'}{i^*} \subset \mathcal{S}_1^k$. Thus for all $X \in \binom{S' - s_k - s_\ell}{i^*}$, $X + s_k \in \mathcal{S}_1$. It now follows by Observation 10 that $\binom{S' - s_\ell}{i^*+1} \subseteq \mathcal{S}_1$. Now Observation 9 implies that $\binom{S'}{i^*+1} \subseteq \mathcal{S}_1$. Suppose now that $i^* > i$. Then $i < n-3$ and it follows by assumption that $\binom{D \cap D'}{i} \subseteq \mathcal{S}_1^k$. It now follows by Observation 10 that $\binom{D'}{i+1} \subseteq \mathcal{S}_1$. Also, since $\binom{D}{i} \subseteq \mathcal{S}_1^k$, we have $\binom{D}{i+1} \subseteq \mathcal{S}_1$. Let $X \in \binom{S'}{i+1}$. If $X \subseteq D$ or $X \subseteq D'$, then $X \in \mathcal{S}_1$. Suppose neither occurs. Then $\{s_k, x_l\} \subseteq X$ and hence $X - s_k \in \binom{D}{i} \subseteq \mathcal{S}_1^k$. It follows that $X \in \mathcal{S}_1$. Consequently, $\binom{S'}{i+1} \subseteq \mathcal{S}_1$. Since $i + 1 \leqslant i^* + 1$, it follows by Observation 8 that $\binom{S'}{i^*+1} \subseteq \mathcal{S}_1$.

To prove that $\binom{S'}{j^*} \subseteq \mathcal{S}_2$, first suppose that $j^* = n - 2$. Then $j^* = j = j' = n - 2$. In this case, $D, D' \in \mathcal{S}_2$ and hence $S' = D \cup D' \in \mathcal{S}_2$ by (**S2**). It would then follow by (**S1**) that $\binom{S'}{n-2} \subseteq \mathcal{S}_2$. Thus we may assume that $j^* < n - 2$. We have that $\binom{D \cap D'}{j^*} \subseteq \mathcal{S}_2$. Given that $D \cap D' = S' - s_k - s_\ell$, it follows by Observation 9 that $\binom{S' - s_\ell}{j^*} \subseteq \mathcal{S}_2$ and this in turn implies that $\binom{S'}{j^*} \subseteq \mathcal{S}_2$.

Given that $i + j = i' + j' = n-1$, it follows that $i^* \leqslant n-1-j^*$, and hence $i^* + 1 + j^* \leqslant n$. By application of Observation 8, we have that $\binom{S'}{n-i^*-1} \subseteq \mathcal{S}_2$. However, we now have both $\binom{S'}{i^*+1} \subseteq \mathcal{S}_1$ and $\binom{S'}{n-i^*-1} \subseteq \mathcal{S}_2$, violating (**S4**). We conclude that (**a4**) can hold for at most one integer k. □

(D) *There exists $T \in \binom{S}{n-3}$ such that either $\binom{T}{1} \subseteq \mathcal{S}_1$ or $\binom{T}{1} \subseteq \mathcal{S}_2$.*

Proof. Assume that there is no subset $T \in \binom{S}{n-3}$ such that $\binom{T}{1} \subseteq \mathcal{S}_1$. Then there are at least three integers k for which (**a1**) does not hold. By (**B**) and (**C**), (**a2**) or (**a3**) holds for at most one integer k and (**a4**) holds for at most one integer k. Thus there exists $k \in [n]$ such that none of (**a1**) - (**a4**) hold. By (**A**), (**a5**) holds for k. Thus there exists $(A_1^k, A_2^k) \in \binom{n-2}{\mathcal{S}_1^k} \times \binom{n-2}{\mathcal{S}_2^k}$, $A_1^k \neq A_2^k$, and $\{B_1^k, B_2^k\} \subset \binom{S^k}{n-3}$ where for $i = 1, 2$, $B_i^k \cap A_i^k = B_1^k \cap B_2^k \in \binom{A_1^k \cap A_2^k}{n-4}$ and $\binom{B_i^k}{1} \subset \mathcal{S}_i^k$. Thus we see that $\binom{B_2^k}{1} \subseteq \mathcal{S}_2^k \subseteq \mathcal{S}_2$. This completes the proof. □

(E) *There exists* $T \in \binom{S}{n-2}$ *such that either* $\binom{T}{1} \subseteq \mathcal{S}_1$ *or* $\binom{T}{1} \subseteq \mathcal{S}_2$.

Proof. By (**D**), there exists $T \in \binom{S}{n-3}$ such that either $\binom{T}{1} \subseteq \mathcal{S}_1$ or $\binom{T}{1} \subseteq \mathcal{S}_2$. We claim that it suffices to prove the assertion when $\binom{T}{1} \subseteq \mathcal{S}_1$. For if instead $\binom{T}{1} \subseteq \mathcal{S}_2$, then redefine $\mathcal{S}_i^k$ so that for all $k \in [n]$, $\mathcal{S}_1^k = \{A \in \mathcal{S}_1 \mid s_k \notin A\}$ and $\mathcal{S}_2^k = \{A - s_k \mid A \in \mathcal{S}_2 \text{ and } s_k \in A\}$. Now it is seen that (**A**) - (**C**) still hold when in (**a1**) - (**a5**), we switch $\mathcal{S}_1$ with $\mathcal{S}_2$ and switch $\mathcal{S}_1^k$ with $\mathcal{S}_2^k$. Now one can use the same proof as in the case when $\binom{T}{1} \subseteq \mathcal{S}_1$.

By the above, we may assume that $\binom{T}{1} \subseteq \mathcal{S}_1$. Furthermore, we may assume that $T = \{s_1, \dots, s_{n-3}\}$. Next, we will show that either $\{s_i\} \in \mathcal{S}_1$ for some $i \in \{n-2, n-1, n\}$, or $\binom{S'}{1} \subseteq \mathcal{S}_2$ for some $S' \in \binom{S}{n-2}$. We may assume that (**a1**) and (**a3**) do not hold for all $k \in \{n-2, n-1, n\}$. Furthermore, by (**B**) and (**C**), (**a2**) holds for at most one integer $k \in \{n-2, n-1, n\}$ as does (**a4**). As such, we may assume that (**a2**) and (**a4**) do not hold for $k = n-2$. Thus by By (**A**), (**a5**) holds for $k = n-2$. Thus there exist $(A_1^{n-2}, A_2^{n-2}) \in \binom{n-2}{\mathcal{S}_1^{n-2}} \times \binom{n-2}{\mathcal{S}_2^{n-2}}$, $A_1^{n-2} \neq A_2^{n-2}$, and $\{B_1^{n-2}, B_2^{n-2}\} \subset \binom{S^{n-2}}{n-3}$ where for $i = 1, 2$, $B_i^{n-2} \cap A_i^{n-2} = B_1^{n-2} \cap B_2^{n-2} \in \binom{A_1^{n-2} \cap A_2^{n-2}}{n-4}$ and $\binom{B_i^{n-2}}{1} \subset \mathcal{S}_i^{n-2}$.

Suppose $s_i \in B_1^{n-2} \cap \{s_1, \dots, s_{n-3}\}$. By assumption, $\{s_i\} \in \mathcal{S}_1$. However, given that $s_i \in B_1^{n-2}$, we also have that $\{s_i\} \in \mathcal{S}_1^{n-2}$ and hence $\{s_i, s_{n-2}\} \in \mathcal{S}_1$. By (**S1**), $\{s_{n-2}\} \in \mathcal{S}_1$, a contradiction. Thus $B_1^{n-2} \cap \{s_1, \dots, s_{n-3}\} = \emptyset$ and hence $B_1^{n-2} \subseteq \{s_{n-1}, s_n\}$. Consequently, $n - 3 \leqslant 2$ and hence $n \leqslant 5$. To complete the proof, we need only consider two cases:

Case 1: $n = 5$.

We have $B_1^{n-2} = B_1^3 = \{s_4, s_5\}$. We may assume that $A_2^3 = \{s_1, s_4, s_5\}$, where $B_1^3 \cap B_2^3 = \{s_4\}$. Thus $A_1^3 = \{s_1, s_2, s_4\}$ and $B_2^3 = \{s_2, s_4\}$. Then $A_1^3 + s_3 = \{s_1, s_2, s_3, s_4\} \in \mathcal{S}_1$ and $A_2^3 \in \mathcal{S}_2$. Given that $\binom{B_2^3}{1} \subseteq \mathcal{S}_2$, we may assume that for all $i \in \{1, 3, 4\}$, $\{s_i\} \notin \mathcal{S}_2$. Since by assumption (**a1**) and (**a3**) do not hold for $k \in \{3, 4, 5\}$, it follows by (**A**) that for all $k \in \{4, 5\}$, one of (**a2**), (**a4**), or (**a5**) must hold.

Suppose (**a5**) holds for $k = 5$. Then arguing as above, we have that $B_1^5 = \{s_3, s_4\}$ and hence $A_1^5 = \{s_1, s_2, s_3\}$ or $A_2^5 = \{s_1, s_2, s_4\}$. Thus either $\{s_1, s_2, s_3, s_5\} \in \mathcal{S}_1$ or $\{s_1, s_2, s_4, s_5\} \in \mathcal{S}_1$. Given that $\{s_1, s_2, s_3, s_4\} \in \mathcal{S}_1$, it would follow by **(S2)** that $S \in \mathcal{S}_1$, contradicting **(S3)**. Thus (**a5**) does not hold for $k = 5$.

Suppose (**a4**) holds for $k = 5$. Then there exists a subset $D' \in \binom{S^5}{3}$ and integers i, j where $i + j = 4$ such that $\binom{D'}{i} \subseteq \mathcal{S}_1^5$ and $\binom{D'}{j} \subseteq \mathcal{S}_2^5$. Let $D = D' + s_5$. By Observation

10, it follows that $\binom{D}{i+1} \subseteq \mathcal{S}_1$ and $D \in \mathcal{S}_1$. Clearly $D \neq \{s_1, s_2, s_3, s_4\}$ and hence it follows by property (**S2**) that $D \cup \{s_1, s_2, s_3, s_4\} = S \in \mathcal{S}_1$, yielding a contradiction. Thus (**a2**) holds for $k = 5$ and hence $\{s_1, s_2, s_3, s_4\} \in \mathcal{S}_2$. By (**A**) and (**B**) it follows that either (**a4**) or (**a5**) holds for $k = 4$.

Suppose (**a5**) holds for $k = 4$. Arguing as before, we see that $B_1^4 = \{s_3, s_5\}$ and either $A_1^4 = \{s_1, s_2, s_3\}$ or $A_1^4 = \{s_1, s_2, s_5\}$. In the latter case, we have that $\{s_1, s_2, s_4, s_5\} \in \mathcal{S}_1$. It would then follow by (**S2**) that $\{s_1, s_2, s_3, s_4\} \cup \{s_1, s_2, s_4, s_5\} = S \in \mathcal{S}_1$, contradicting (**S3**). Thus we have that $A_1^4 = \{s_1, s_2, s_3\}$. It now follows that $\{s_3\} = A_1^4 \cap B_1^4 = B_1^4 \cap B_2^4$. Thus $s_3 \in B_2^4$, implying that $\{s_3\} \in \mathcal{S}_2$, contradicting our assumptions.

By the above, (**a4**) must hold for $k = 4$. Thus there exists a subset $D' \in \binom{S^4}{3}$ and integers i, j where $i + j = 4$ such that $\binom{D'}{i} \subseteq \mathcal{S}_1^4$ and $\binom{D'}{j} \subseteq \mathcal{S}_2^4$. It follows by Observation 10 that for $D = D' + s_4$, $\binom{D}{i+1} \subseteq \mathcal{S}_1$ and $D \in \mathcal{S}_1$. If $D \neq \{s_1, s_2, s_3, s_4\}$, then we would have $D \cup \{s_1, s_2, s_3, s_4\} = S \in \mathcal{S}_1$, contradicting (**S3**). Thus $D = \{s_1, s_2, s_3, s_4\}$ and consequently, $D' = \{s_1, s_2, s_3\}$. Given that $\binom{D'}{j} \subseteq \mathcal{S}_2$ and $\{s_3\} \notin \mathcal{S}_2$, it follows that $j \geqslant 2$.

Suppose $i = 1$. Then $\binom{D}{2} \subseteq \mathcal{S}_1$. Given that $\binom{B_1^3}{1} \subseteq \mathcal{S}_1^3$, it follows that $\{s_5\} \in \mathcal{S}_1^3$ and hence $\{s_3, s_5\} \in \mathcal{S}_1$. Thus we have $\{B \in \binom{S}{2} \mid s_3 \in B\} \subseteq \mathcal{S}_1$. It now follows by Observation 10 that $S \in \mathcal{S}_1$, contradicting (**S3**). Thus $i \geqslant 2$ and $i = j = 2$. We now have that $\binom{D}{3} \subseteq \mathcal{S}_1$. Given that $B_2^3 = \{s_2, s_4\} \in \mathcal{S}_2$ and $\binom{D'}{2} \subseteq \mathcal{S}_2$, it follows $\{B \in \binom{D}{2} \mid s_2 \in B\} \subseteq \mathcal{S}_2$. Thus by Observation 10, we have $\binom{D}{2} \subseteq \mathcal{S}_2$. However, we now have both $\binom{D}{3} \subseteq \mathcal{S}_1$ and $\binom{D}{2} \subseteq \mathcal{S}_2$, contradicting (**S4**). This completes the case $n = 5$.

Case 2: $n = 4$.

We may assume that $B_1^{n-2} = B_1^2 = \{s_4\}$, $A_1^2 = \{s_1, s_3\}$. There are two possible cases to consider for A_2^2 and B_2^2: either $A_2^2 = \{s_1, s_4\}$ and $B_2^2 = \{s_3\}$ or $A_2^2 = \{s_3, s_4\}$ and $B_2^2 = \{s_1\}$. We shall assume the former – the latter case can be handled similarly. We have that $A_1^2 = \{s_1, s_3\}$ and hence $A_1^2 + s_2 = \{s_1, s_2, s_3\} \in \mathcal{S}_1$ and $B_1^2 + s_2 = \{s_2, s_4\} \in \mathcal{S}_1$. We also have that $A_2^2 = \{s_1, s_4\} \in \mathcal{S}_2$ and $\{s_3\} \in \mathcal{S}_2$. We may assume that (**a1**) and (**a3**) do not hold for $k = 3$ or $k = 4$.

Suppose (**a5**) holds for $k = 3$. Then $B_1^3 = \{s_2\}$ or $B_1^3 = \{s_4\}$. In the former case, we have $A_1^3 = \{s_1, s_4\}$, and hence $A_1^3 + s_3 = \{s_1, s_3, s_4\} \in \mathcal{S}_1$. However, since $\{s_1, s_2, s_3\} \in \mathcal{S}_1$, it would follow that $\{s_1, s_3, s_4\} \cup \{s_1, s_2, s_3\} = S \in \mathcal{S}$, contradicting (**S3**). Thus $B_1^3 = \{s_4\}$ and $A_1^3 = \{s_1, s_2\}$. We have that $B_1^3 + s_3 = \{s_3, s_4\} \in \mathcal{S}_1$. However, given that $\{s_2, s_4\} \in \mathcal{S}_1$, it follows by (**S2**) that $\{s_3, s_4\} \cup \{s_2, s_4\} = \{s_2, s_3, s_4\} \in \mathcal{S}_1$. Again, since $\{s_1, s_2, s_3\} \in \mathcal{S}_1$, it follows that $\{s_2, s_3, s_4\} \cup \{s_1, s_2, s_3\} = S \in \mathcal{S}_1$, yielding a contradiction. We conclude that (**a5**) does not hold for $k = 3$. By similar arguments, one can also show that (**a5**) does not hold for $k = 4$ either.

Suppose (**a4**) holds for $k = 4$. Then there exists a subset $D' \in \binom{S^4}{2}$ and integers i, j where $i + j = 3$ such that $\binom{D'}{i} \subseteq \mathcal{S}_1^4$ and $\binom{D'}{j} \subseteq \mathcal{S}_2^4$. We have that $D = D' + s_4 \in \mathcal{S}_1$. Given that $\{s_1, s_2, s_3\} \in \mathcal{S}_1$, it follows by (**S2**) that $S = D \cup \{s_1, s_2, s_3\} \in \mathcal{S}_1$, a contradiction. Thus (**a4**) does not hold for $k = 4$ and hence (**a2**) holds for $k = 4$. Furthermore, since by (**B**), (**a2**) holds for at most one of $k = 3$ or $k = 4$, it must be the case that (**a4**) holds for

$k=3$. As such, there exists a subset $D' \in \binom{S^3}{2}$ and integers i,j where $i+j=3$ such that $\binom{D'}{i} \subseteq \mathcal{S}_1^3$ and $\binom{D'}{j} \subseteq \mathcal{S}_2^3$. We have that $D = D' + s_3 \in \mathcal{S}_1$. Given that $\{s_1, s_2, s_3\} \in \mathcal{S}_1$, if $D \neq \{s_1, s_2, s_3\}$, then by (**S2**), $S = D \cup \{s_1, s_2, s_3\} \in \mathcal{S}_1$, a contradiction. Thus we must have that $D = \{s_1, s_2, s_3\}$, and thus $D' = \{s_1, s_2\}$. If $j = 1$, then $\binom{D'}{1} \subseteq \mathcal{S}_2^3 \subseteq \mathcal{S}_2$. Given $|D'| = 2 = n-2$, the assertion holds in this case. Thus we may assume that $j=2$ and $i=1$. However, this means that $\{s_1\} \in \mathcal{S}_1^3$, implying that $\{s_1, s_3\} \in \mathcal{S}_1$. This in turn implies that $\{s_3\} \in \mathcal{S}_1$ (since $\{s_1\} \in \mathcal{S}_1$) yielding a contradiction. This completes the case for $n=4$. $\square$

By (**E**), there exists $i \in \{1,2\}$ and $T \in \binom{S}{n-2}$ for which $\binom{T}{1} \subseteq \mathcal{S}_i$. Using similar reasoning as before, it suffices to prove the case where $\binom{T}{1} \subseteq \mathcal{S}_1$ (see the first paragraph of the proof of (**E**)). Thus we may assume $\binom{T}{1} \subseteq \mathcal{S}_1$ and moreover, $T = \{s_1, \ldots, s_{n-2}\}$.

Suppose first that (**a1**) holds for $k = n-1$; that is, $\{s_{n-1}\} \in \mathcal{S}_1$. Then $\binom{S^n}{1} \subseteq \mathcal{S}_1$ and (by Observation 8), $S^n \in \mathcal{S}_1$. We shall show that (**a1**) - (**a5**) do not hold for $k=n$, violating (**A**). Clearly (**a1**) does not hold for $k=n$, for otherwise (**S3**) is violated. If (**a2**) or (**a3**) holds for $k=n$, then $S^n \in \mathcal{S}_2$. In this case, (**S4**) is violated. Suppose (**a4**) holds for $k=n$.

Then there exists $D' \in \binom{S^n}{n-2}$ and $1 \leqslant i \leqslant n-2$ where $\binom{D'}{i} \subseteq \mathcal{S}_1^n$, and $D = D' + s_n \in \mathcal{S}_1$. However, since $S^n \in \mathcal{S}_1$, it follows by (**S2**) that $D \cup S^n = S \in \mathcal{S}_1$, violating (**S3**). Thus (**a4**) does not hold for $k=n$. If (**a5**) holds for $k=n$, then there is a set $A_1^n \in \binom{n-2}{\mathcal{S}_1^n}$, implying that $D = A_1^n + s_n \in \mathcal{S}_1$. Again, we have $D \cup S^n = S \in \mathcal{S}_1$, a contradiction. This shows that (**a1**) - (**a5**) do not hold for $k=n$ (a contradiction) and hence (**a1**) can not hold for $k=n-1$. By similar arguments, one can also show that (**a1**) does not hold for $k=n$.

Suppose (**a2**) holds for $k=n-1$. Then $S^{n-1} = \{s_1, \ldots, s_{n-2}, s_n\} \in \mathcal{S}_2$. We will show that (**a4**) holds for $k=n$. By (**B**), neither (**a2**) nor (**a3**) holds for $k=n$. Suppose (**a5**) holds for $k=n$. Following a previous argument, we have that $\{s_1, \ldots, n-2\} \cap B_1^n = \emptyset$. Thus $B_1^n \subseteq \{s_{n-1}\}$ and $n \leqslant 4$. Given $n \geqslant 4$, it follows that $n=4$ and $B_1^4 = \{s_3\}$ and $A_1^4 = \{s_1, s_2\}$. Thus $S^3 = \{s_1, s_2, s_4\} \in \mathcal{S}_1$. Since for $i = 1,2$, $\{s_i\} \in \mathcal{S}_1$, it follows by Observation 9 that $\binom{S^3}{1} \subseteq \mathcal{S}_1$. However, this implies that $\{s_4\} \in \mathcal{S}_1$, a contradiction.

It follows from the above that, assuming (**a2**) holds for $k=n-1$, (**a4**) holds for $k=n$. Thus there exists $D' \in \binom{S^n}{n-2}$ and integers $i,j,\ i+j=n-1$, such that $\binom{D'}{i} \subseteq \mathcal{S}_1^n$ and $\binom{D'}{j} \subseteq \mathcal{S}_2^n$. Then $D = D' + s_n \in \mathcal{S}_1$. If $D' = \{s_1, \ldots, s_{n-2}\}$, then $D' \in \mathcal{S}_1$, (since $\binom{D'}{1} \subseteq \mathcal{S}_1$). It now follows by Observation 9 that $\binom{D}{1} \subseteq \mathcal{S}_1$. However, this implies that $\{s_n\} \in \mathcal{S}_1$, a contradiction. Thus $s_{n-1} \in D'$. We have $\binom{D'-s_{n-1}}{1} \subseteq \mathcal{S}_1$ and $D' - s_{n-1} \in \mathcal{S}_1$. Note that $D' \notin \mathcal{S}_1$; for otherwise, Observation 9 would imply that $\binom{D'}{1} \subseteq \mathcal{S}_1$, contradicting the fact that $\{s_{n-1}\} \notin \mathcal{S}_1$.

Suppose $i \leqslant n-3$. Then $\binom{D'-s_{n-1}}{i} \subseteq \mathcal{S}_1^n$. Thus for all $S' \in \binom{D'-s_{n-1}}{i}$, $S' \in \mathcal{S}_1$ and $S' + s_n \in \mathcal{S}_1$. It follows by (**S1**) that $\binom{S'+s_n}{i} \subseteq \mathcal{S}_1$. This in turn implies that $\binom{D'-s_{n-1}+s_n}{i} \subseteq \mathcal{S}_1$. By Observation 8, $D' - s_{n-1} + s_n \in \mathcal{S}_1$. However, we also have that $\{s_1, \ldots, s_{n-2}\} \in \mathcal{S}_1$ and thus $\{s_1, \ldots, s_{n-2}\} \cup (D' - s_{n-1} + s_n) = S^{n-1} \in \mathcal{S}_1$. Given that $S^{n-1} = D' - s_{n-1} + s_n + s_i$, for some $i \in [n-2]$, it follows by Observation 9 that $\binom{S^{n-1}}{i} \subseteq \mathcal{S}_1$. By Observation 8, we have $\binom{S^{n-1}}{i+1} \subseteq \mathcal{S}_1$. Since $\binom{D'}{j} \subseteq \mathcal{S}_2^n \subseteq \mathcal{S}_2$ and $S^{n-1} \in \mathcal{S}_2$ (since (**a2**) holds for $k = n-1$) and $S^{n-1} - s_i = D'$, for some $i \in [n-2]$, it follows by Observation 9 that $\binom{S^{n-1}}{j} \subseteq \mathcal{S}_2$. However, we have $\binom{S^{n-1}}{i+1} \subseteq \mathcal{S}_1$ and $\binom{S^{n-1}}{j} \subseteq \mathcal{S}_2$ and $i+1+j = n$, in violation of (**S4**).

From the above, we have $i = n-2$ and $j = 1$. Then $D' \in \mathcal{S}_1^n$ and $\binom{D'}{1} \subseteq \mathcal{S}_2$. Let $A_1 = D' + s_n$, $A_2 = S^{n-1}$, $B_1 = S - s_{n-1} - s_n$, and $B_2 = D'$. Then by the above, $(A_1, A_2) \in \binom{n-1}{\mathcal{S}_1} \times \binom{n-1}{\mathcal{S}_2}$ and $A_1 \neq A_2$. Furthermore, we have that for $i = 1, 2$, $\binom{B_i}{1} \subseteq \mathcal{S}_i$. We also see that $B_1 \cap B_2 = D' \cap \{s_1, \ldots, s_{n-2}\} = A_1 \cap B_1 = A_2 \cap B_2$. Thus in this case, the theorem is satisfied.

To finish the proof, we will show that no other options are possible. Suppose now that (**a2**) does not hold for $k = n-1$, and we may assume the same is true for $k = n$. Thus (**a3**) does not hold for $k = n-1$ or $k = n$.

Suppose (**a4**) holds for $k = n-1$. Then there exists $D' \in \binom{S^{n-1}}{n-2}$ and integers i, j, $i+j = n-1$, such that $\binom{D'}{i} \subseteq \mathcal{S}_1^{n-1}$ and $\binom{D'}{j} \subseteq \mathcal{S}_2^{n-1} \subseteq \mathcal{S}_2$. Then $D = D' + s_{n-1} \in \mathcal{S}_1$. As before, $D' \neq \{s_1, \ldots, s_{n-2}\}$. Thus $s_n \in D'$ and we may assume without loss of generality that $D' = \{s_1, \ldots, s_{n-3}, s_n\}$. By (**C**), (**a4**) does not hold for $k = n$. Thus (**a5**) holds for $k = n$ and there exist $(A_1^n, A_2^n) \in \binom{n-2}{\mathcal{S}_1^n} \times \binom{n-2}{\mathcal{S}_2^n}$, $A_1^n \neq A_2^n$, and $\{B_1^n, B_2^n\} \subseteq \binom{S^n}{n-3}$ where for $i = 1, 2$, $B_i^n \cap A_i^n = B_1^n \cap B_2^n \in \binom{A_1^n \cap A_2^n}{n-4}$ and $\binom{B_i^n}{1} \subseteq \mathcal{S}_i^n$. Arguing as before, we have $B_1^n \cap \{s_1, \ldots, s_{n-2}\} = \emptyset$. This in turn implies that $B_1^n = \{s_{n-1}\}$ and hence $n = 4$. Furthermore, we have that $A_1^n = A_1^4 = \{s_1, s_2\}$, implying that $\{s_1, s_2, s_4\} \in \mathcal{S}_1$. However, we also have that $D = \{s_1, s_3, s_4\} \in \mathcal{S}_1$. It follows by (**S2**) that $S = D \cup \{s_1, s_2, s_4\} \in \mathcal{S}_1$, violating (**S3**). Thus (**a4**) does not hold for $k = n-1$ and the same holds for $k = n$.

From the above, (**a5**) must hold for both $k = n-1$ and $k = n$. Using similar arguments as above, one can show that $n = 4$, $B_1^3 = \{s_4\}$, $A_1^3 = \{s_1, s_2\}$, $B_1^4 = \{s_3\}$, and $A_1^4 = \{s_1, s_2\}$. We have $A_1^3 + s_3 = \{s_1, s_2, s_3\} \in \mathcal{S}_1$ and $A_1^4 + s_4 = \{s_1, s_2, s_4\} \in \mathcal{S}_1$. It now follows by (**S2**) that $\{s_1, s_2, s_3\} \cup \{s_1, s_2, s_4\} = S \in \mathcal{S}_1$, contradicting (**S3**). This completes the proof of the theorem. □

5 Proof of Theorem 3

Let M be a paving matroid where $\gamma(M) = \beta(E(M))$ and $|E(M)| = n$.

5.1 The case $r(M) = 2$

Suppose $r(M) = 2$. We shall prove by induction on n that M is cyclically orderable. Theorem 3 is seen to be true when $n = 2$. Assume that it is true when $n = m-1 \geqslant 2$. We shall prove that it is also true for $n = m$. Assume that M is a paving matroid where $r(M) = 2$, $|E(M)| = m$ and $\gamma(M) = \beta(E(M)) = \frac{m}{2}$. For all elements $e \in E(M)$, let X_e denote the parallel class containing e and let $m(e) = |X_e|$. Then for all $e \in E(M)$, $\beta(X_e) = m(e) \leqslant \gamma(M) = \frac{m}{2}$. If there are elements $e \in E(M)$ for which $m(e) = \frac{m}{2}$, then choose f to be one such element. If no such elements exist, then let f be any element in M. Let $M' = M \backslash f$. Suppose there exists $X \subseteq E(M')$ for which $\beta(X) > \frac{m-1}{2} = \beta(E(M')$. Then clearly $r(X) = 1$. Thus $X \subseteq X_g$ for some $g \in E(M')$. Given that $m(g) \leqslant \frac{m}{2}$, it follows that $X = X_g$ and $m(g) = \frac{m}{2}$. By the choice of f, we also have $m(f) = \frac{m}{2}$. Then $E(M) = X_f \cup X_g$ and $E(M) = m = 2\ell$, for some integer ℓ. Now let $e_1e_2\cdots e_m$ be an ordering of $E(M)$ where for all i, $e_i \in X_f$, if i is odd, and $e_i \in X_g$, if i is even. This gives a cyclic ordering for M. Thus we may assume that $\gamma(M') = \beta(E(M')) = \frac{m-1}{2}$. By assumption, there is a cyclic ordering for M', say $e_1e_2\cdots e_{m-1}$. Since $m(f) \leqslant \frac{m}{2}$, there exists $i \in [m-1]$ such that $\{e_i, e_{i+1}\} \cap X_f = \emptyset$. Consequently, $e_1\cdots e_i f e_{i+1}\cdots e_{m-1}$ is seen to be a cyclic ordering for M. The proof now follows by induction.

5.2 The case where $|E(M)| \leqslant 2r(M) + 1$

Suppose $|E(M)| \leqslant 2r(M)+1$. As mentioned earlier, if $|E(M)| = 2r(M)+1$, then $|E(M)|$ and $r(M)$ are relatively prime and hence it follows by Theorem 2 that M has a cyclic ordering. Thus we may assume that $|E(M)| \leqslant 2r(M)$. It now follows by Theorem 5 that there are bases A and B for which $A \cup B = E(M)$.

The following is a well-known conjecture of Gabow [5].

12 Conjecture (Gabow)
Suppose that A and B are bases of a matroid N of rank r. Then there are orderings $a_1a_2\cdots a_r$ and $b_1b_2\cdots b_r$ of the elements of A and B, respectively, such that for $i = 1,\ldots,r-1$, $\{a_1,\ldots,a_i,b_{i+1},\ldots,b_r\}$ and $\{a_{i+1},\ldots,a_r,b_1\ldots,b_i\}$ are bases.

We observe that in the special case of Conjecture 12 where $E(N)$ is the union of two bases, the conjecture implies that N has a cyclic ordering. In [1], the authors verify, among other things, the above conjecture for *split matroids*, a class of matroids which includes all paving matroids. Given that the above conjecture is true for split matroids (and hence also paving matroids) and $E(M) = A \cup B$, it follows that M has a cyclic ordering.

5.3 The case where $|E(M)| \geqslant 2r(M) + 2$ and $r(M) \geqslant 3$.

In this section, we shall assume that $|E(M)| \geqslant 2r(M)+2$ and $r(M) \geqslant 3$. By Proposition 6, there exists a basis S of M for which $\gamma(M\backslash S) = \beta(E(M) - S)$ and $r(M\backslash S) = r(M)$. Let $r = r(M)$ and let $S = \{s_1,\ldots,s_r\}$. Let $M' = M\backslash S$ and let $m = |E(M')| = n - r$. By assumption, M' is cyclically orderable and we will assume that $e_1e_2\cdots e_m$ is a cyclic ordering. Our goal is to show that the cyclic ordering for M' can be extended to a cyclic ordering of M. To complete the proof of Theorem 3, we need only prove the following:

Proposition 13. *There exists $i \in [m]$ and a permutation π of $[r]$ such that $e_1e_2\cdots e_i s_{\pi(1)}s_{\pi(2)}\cdots s_{\pi(r)}e_{i+1}\cdots e_m$ is a cyclic ordering of M.*

Proof. Assume to the contrary that for all $i \in [m]$ and for all permutations π of $[r]$, $e_1e_2\cdots e_i s_{\pi(1)}s_{\pi(2)}\cdots s_{\pi(r)}e_{i+1}\cdots e_m$ is not a cyclic ordering of M. For all $j \in [m]$, we shall define a pair $(\mathcal{H}_1^j, \mathcal{H}_2^j)$, where for $i = 1, 2$, $\mathcal{H}_i^j \subseteq 2^S$. Let $x_1^j = e_{j-1},\ x_2^j = e_{j-2},\ldots,x_{r-1}^j =$

e_{j-r+1}, and let $y_1^j = e_j,\ y_2^j = e_{j+1}, \ldots, y_{r-1}^j = e_{j+r-2}$ where for all integers k, we define $e_k := e_\ell$ where

$$\ell := \begin{cases} k \; mod \; m & \text{if } k \; mod \; m \neq 0 \\ m & \text{otherwise.} \end{cases}$$

Let $X^j = \{x_1^j, \ldots, x_{r-1}^j\}$ and $Y^j = \{y_1^j, \ldots, y_{r-1}^j\}$.

Let π be a permutation of $[r]$. By assumption,
$e_1 \cdots e_{j-1} s_{\pi(1)} s_{\pi(2)} \cdots s_{\pi(r)} e_j \cdots e_m$ is not a cyclic ordering for M. Then there exists $i \in [r-1]$ such that either $\{x_1^j, \ldots, x_i^j\} \cup \{s_{\pi(1)}, \ldots, s_{\pi(r-i)}\}$ is dependent or $\{y_1^j, \ldots, y_i^j\} \cup \{s_{\pi(i+1)}, \ldots, s_{\pi(r)}\}$ is dependent. Since the smallest circuit has size r, this means that either $\{x_1^j, \ldots, x_i^j\} \cup \{s_{\pi(1)}, \ldots, s_{\pi(r-i)}\}$ or $\{y_1^j, \ldots, y_i^j\} \cup \{s_{\pi(i+1)}, \ldots, s_{\pi(r)}\}$ is a circuit. Let $\mathcal{C}_1^j$ be the set of all r-circuits which occur in the former case, and let $\mathcal{C}_2^j$ be the set of all r-circuits occurring in the latter case. That is, $\mathcal{C}_1^j$ is the set of all r-circuits C where for some $i \in [r-1]$, $\{x_1^j, \ldots, x_i^j\} \subset C \subset \{x_1^j, \ldots, x_i^j\} \cup S$, and $\mathcal{C}_2^j$ is set of all r-circuits C where for some $i \in [r-1]$, $\{y_1^j, \ldots, y_i^j\} \subset C \subseteq \{y_1^j, \ldots, y_i^j\} \cup S$. For $i = 1, 2$, let $\mathcal{H}_i^j = \{C \cap S \mid C \in \mathcal{C}_i^j\}$.

(A) *For all j, the pair $(\mathcal{H}_1^j, \mathcal{H}_2^j)$ is an S-pair which is order-consistent.*

Proof. It suffices to prove the assertion for $j = 1$. For convenience, we let $x_i = x_i^1$ $y_i = y_i^1,\ i = 1, \ldots, r-1$. Furthermore, we let $X = X^1,\ Y = Y^1$, $\mathcal{H}_1 = \mathcal{H}_1^1$, $\mathcal{H}_2 = \mathcal{H}_2^1$, $\mathcal{C}_1 = \mathcal{C}_1^1$, and $\mathcal{C}_2 = \mathcal{C}_2^1$. It follows from the definition of $(\mathcal{H}_1, \mathcal{H}_2)$ that it is order-consistent. We need only show that it is an S-pair. Suppose $A, B \in \mathcal{H}_1$ where $|A| = |B| + 1$ and $B \subset A$. Then for some $i \in [r-1]$, $C_1 = A \cup \{x_1, \ldots, x_i\} \in \mathcal{C}_1$ and $C_2 = B \cup \{x_1, \ldots, x_{i+1}\} \in \mathcal{C}_1$. Let $x \in B$. Then $x \in C_1 \cap C_2$ and hence by the circuit elimination axiom there is a circuit $C \subseteq (C_1 \cup C_2) - x = (A - x) \cup \{x_1, \ldots, x_{i+1}\}$. Thus $C = (A - x) \cup \{x_1, \ldots, x_{i+1}\}$ and hence $A - x \in \mathcal{H}_1$. Since this applies to any element $x \in B$, it follows that $\binom{A}{|B|} \subseteq \mathcal{H}_1$. The same arguments can be applied to $\mathcal{H}_2$. Thus (**S1**) holds.

To show that (**S2**) holds, suppose $A, B \in \mathcal{H}_1$ where $|A| = |B|$ and $|A \cap B| = |A| - 1$. There exists $i \in [r]$ such that $C_1 = \{x_1, \ldots, x_i\} \cup A \in \mathcal{C}_1$ and $C_2 = \{x_1, \ldots, x_i\} \cup B \in \mathcal{C}_1$. By the circuit elimination axiom, there exists a circuit $C \subseteq (C_1 \cup C_2) - x_i = (A \cup B) \cup \{x_1, \ldots, x_{i-1}\}$. Thus $C = (A \cup B) \cup \{x_1, \ldots, x_{i-1}\}$ is a circuit and hence $A \cup B \in \mathcal{H}_1$. The same reasoning applies if $A, B \in \mathcal{H}_2$. Thus (**S2**) holds.

To show that (**S3**) holds, suppose $\binom{S}{1} \subseteq \mathcal{H}_1$. Then for $i = 1, \ldots, r-1,\ C_i = X \cup \{s_i\}$ is a circuit, and consequently, $S \subseteq \mathrm{cl}(X)$. However, this is impossible since $|X| = r - 1 < r(S) = r$. Thus $\binom{S}{1} \not\subseteq \mathcal{H}_1$ and likewise, $\binom{S}{1} \not\subseteq \mathcal{H}_2$. Also, we clearly have that for $i = 1, 2$, $S \notin \mathcal{H}_i$ since S is a base of M. Thus (**S3**) holds.

Lastly, to show that (**S4**) holds, let $S' = S - s_r$. Suppose first that $\binom{S'}{r-1} \subseteq \mathcal{H}_1$ and $\binom{S'}{1} \subseteq \mathcal{H}_2$. Then $S' \in \mathcal{H}_1$ and hence $S' + x_1 \in \mathcal{C}_1$. Also, for all $i \in [r-1]$, $Y + s_i \in \mathcal{C}_2$. Thus $x_1 \in \mathrm{cl}(S')$ and $S' \subseteq \mathrm{cl}(Y)$. Given that S' is independent and $|S'| = |Y| = r - 1$, it follows that $\mathrm{cl}(S') = \mathrm{cl}(Y)$. However, this implies that $Y + x_1 = \{x_1, y_1, \ldots, y_{r-1}\} = \{e_m, e_1, \ldots, e_{r-1}\} \subseteq \mathrm{cl}(S')$, which contradicts the assumption that $\{e_m, e_1, \ldots, e_{r-1}\}$ is a basis of M.

Suppose now that for some $k \in [r-2]$, $\binom{S'}{k} \subseteq \mathcal{H}_1$ and $\binom{S'}{r-k} \subseteq \mathcal{H}_2$. We claim that $\{x_1, \ldots, x_{r-k}\} \cup \{y_1, \ldots, y_k\} \subseteq \mathrm{cl}(S')$. Following the proof of Observation 8, we have that for $j = k, \ldots, r-1$, $\binom{S'}{j} \subseteq \mathcal{H}_1$. In particular, $S' \in \mathcal{H}_1$, and hence $C_1 = S' + x_1 \in \mathcal{C}_1$. This implies that $x_1 \in \mathrm{cl}(S')$. However, seeing as $\binom{S'}{r-2} \subseteq \mathcal{H}_1$, we have that $C_2 = (S' - s_{r-1}) \cup \{x_1, x_2\} \in \mathcal{C}_1$. Given that $x_1 \in \mathrm{cl}(S')$, it follows that $x_2 \in \mathrm{cl}(S')$. Continuing, we see that $\{x_1, \ldots, x_{r-k}\} \subseteq \mathrm{cl}(S')$. By similar arguments, it can be shown that $\{y_1, \ldots, y_k\} \subseteq \mathrm{cl}(S')$. Thus proves our claim. It follows that $r(\{x_1, \ldots, x_{r-k}\} \cup \{y_1, \ldots, y_k\}) \leqslant r-1$. However, this is impossible since by assumption $\{x_1, \ldots, x_{r-k}\} \cup \{y_1, \ldots, y_k\}$ is a basis. Thus no such k exists. More generally, the same arguments can be applied to any $j \in [r]$ and $S' = S - s_j$. Thus (**S4**) holds. □

By (**A**), for all $j \in [m]$, $(\mathcal{H}_1^j, \mathcal{H}_2^j)$ is an S-pair which is order-consistent. Thus it follows by Theorem 11, that for all $j \in [m]$, there exists $(A_1^j, A_2^j) \in \binom{r-1}{\mathcal{H}_1^j} \times \binom{r-1}{\mathcal{H}_2^j}$, $A_1^j \neq A_2^j$, and $\{B_1^j, B_2^j\} \subseteq \binom{S}{r-2}$ where for $i = 1, 2$, $B_i^j \cap A_i^j = B_1^j \cap B_2^j \in \binom{A_1^j \cap A_2^j}{r-3}$ and $\binom{B_i^j}{1} \subseteq \mathcal{H}_i^j$.

Suppose $r > 4$. Given that $|B_1^1| = |B_1^2| = r-2$, it follows that there exists $s_i \in B_1^1 \cap B_1^2$. Then $\{s_i\} \in \mathcal{H}_1^1 \cap \mathcal{H}_1^2$ and consequently, $C_1 = \{s_i, e_{m-r+2}, \ldots, e_m\}$ and $C_2 = \{s_i, e_{m-r+3}, \ldots, e_m, e_1\}$ are distinct circuits in M. By the circuit elimination axiom, there exists a circuit $C \subseteq (C_1 \cup C_2) - s_i = \{e_{m-r+2}, \ldots, e_m, e_1\}$. However, this is impossible since by assumption, $\{e_{m-r+2}, \ldots, e_m, e_1\}$ is a basis. Therefore, $r \leqslant 4$.

Suppose $r = 3$. Without loss of generality, we may assume that $A_1^1 = \{s_1, s_2\}$, $B_1^1 = \{s_3\}$, $A_2^1 = \{s_2, s_3\}$, and $B_2^1 = \{s_1\}$. Then $\{s_3, e_m, e_{m-1}\}$ and $\{s_1, e_1, e_2\}$ are circuits. We have that $B_1^2 \neq \{s_3\}$ and $B_2^2 \neq \{s_1\}$; for if $B_1^2 = \{s_3\}$, then $B_1^1 = B_2^2 = \{s_3\}$ and it follows that $\{s_3, e_{m-1}, e_m\}$ and $\{s_3, e_1, e_m\}$ are circuits, implying that $\{e_{m-1}, e_m, e_1\}$ is a circuit – a contradiction. Similar reasoning applies if $B_2^2 = \{s_1\}$. Suppose that $B_1^2 = \{s_1\}$. Then $\{s_1, e_1, e_m\}$ is a circuit. However, seeing as $\{s_1, e_1, e_2\}$, is a circuit (since $B_2^1 = \{s_1\}$), it follows that $\{s_1, e_1, e_2\} \cup \{s_1, e_1, e_m\} - s_1 = \{e_m, e_1, e_2\}$ is a circuit, which is false since by assumption $\{e_m, e_1, e_2\}$ is a basis. Thus $B_1^2 \neq \{s_1\}$. Given that $B_1^2 \neq \{s_3\}$, it follows that $B_1^2 = \{s_2\}$ and $A_1^2 = \{s_1, s_3\}$. Since $B_2^2 \neq \{s_1\}$, it follows that $B_2^2 = \{s_3\}$ and $A_2^2 = \{s_1, s_2\}$. Since $A_1^1 = A_2^2 = \{s_1, s_2\}$, it follows that $\{s_1, s_2, e_m\}$ and $\{s_1, s_2, e_2\}$ are circuits. Furthermore, since $B_1^2 = \{s_2\}$, it follows that $\{s_2, e_1, e_m\}$ is a circuit. It is now seen that $\{e_m, e_1, e_2\} \subseteq \mathrm{cl}(\{s_1, s_2\})$, which contradicts the assumption that $\{e_m, e_1, e_2\}$ is a basis.

Lastly, suppose $r = 4$. Suppose $s_i \in B_1^1 \cap B_1^2$. Then $\{s_i, e_{m-2}, e_{m-1}, e_m\}$ and $\{s_i, e_{m-1}, e_m, e_1\}$ are circuits and hence $\{e_{m-2}, e_{m-1}, e_m, e_1\}$ is also a circuit, contradicting our assumptions. Thus $B_1^1 \cap B_1^2 = \emptyset$ and similarly, $B_2^1 \cap B_2^2 = \emptyset$. More generally, for all $i \in \{1, 2\}$ and $j \in [m]$, $B_i^j \cap B_i^{j+1} = \emptyset$. Since for all $i \in \{1, 2\}$, $|B_i^1| = |B_i^2| = 2$ it follows that for all $i \in \{1, 2\}$, $j \in [m]$, $B_i^j \cup B_i^{j+1} = S$. Without loss of generality, we may assume $B_1^1 = \{s_1, s_2\}$ and $B_1^2 = \{s_3, s_4\}$. Note that $B_1^1 = \{s_1, s_2\}$ means that $\{s_1, s_2\} \subset A_2^1$ and so $A_2^1 = \{s_1, s_2, s_3\}$ or $\{s_1, s_2, s_4\}$. Given that $B_1^1 \not\subseteq A_1^1 \cap A_2^1$, irregardless of whether A_2^1 is the former or latter we have that $A_1^1 = \{s_1, s_3, s_4\}$ or $\{s_2, s_3, s_4\}$. However, since the indexing of the elements of S is essentially arbitrary, one can assume that A_2^1 is any one of the first two choices and A_1^1 is any one of the latter two choices. Thus we may assume without loss of generality that $A_1^1 = \{s_2, s_3, s_4\}$ and $A_2^1 = \{s_1, s_2, s_4\}$. Since for all $i \in \{1,2\}$, $j \in [m]$, $B_i^j \cup B_i^{j+1} = S$, it follows that $B_1^1 = B_1^3 = \cdots = \{s_1, s_2\}$ and $B_1^2 = B_1^4 = \cdots = \{s_3, s_4\}$. In particular, m must be even. Corresponding, for $i = 1, 3, \ldots$, $A_1^i = \{s_1, s_3, s_4\}$ or $\{s_2, s_3, s_4\}$ and for $i = 2, 4, \ldots$ $A_2^i = \{s_1, s_2, s_3\}$ or $\{s_1, s_2, s_4\}$.

Given that $B_1^1 = \{s_1, s_2\}$ and $\binom{B_1^1}{1} \subseteq \mathcal{H}_1^1$, it follows that $\{s_1, e_{m-2}, e_{m-1}, e_m\}$ and $\{s_2, e_{m-2}, e_{m-1}, e_m\}$ are circuits.
Thus $\{s_1, s_2\} \subset \mathrm{cl}(\{e_{m-2}, e_{m-1}, e_m\})$. By the above, we have that $B_1^m = \{s_3, s_4\}$ and either $A_1^m = \{s_1, s_2, s_3\}$ or $A_1^m = \{s_1, s_2, s_4\}$. Suppose the former holds. Then $\{s_1, s_2, s_3, e_{m-1}\}$ is a circuit. Consequently, $s_3 \in \mathrm{cl}(\{e_{m-2}, e_{m-1}, e_m\})$. However, since $B_1^2 = \{s_3, s_4\} \in \mathcal{H}_1^2$, it follows that $\{s_3, e_{m-1}, e_m, e_1\}$ and $\{s_4, e_{m-1}, e_m, e_1\}$ are circuits. By the circuit elimination axiom, $\{s_3, s_4, e_{m-1}, e_m\}$ is a circuit and hence $s_4 \in \mathrm{cl}(\{e_{m-2}, e_{m-1}, e_m\})$. However, it now follows that $\{s_1, s_2, s_3, s_4\} \subset \mathrm{cl}(\{e_{m-2}, e_{m-1}, e_m\})$, yielding a contradiction. If instead, $A_1^m = \{s_1, s_2, s_4\}$, then similar arguments yield a contradiction. This concludes the case for $r = 4$. □

References

[1] K. Berczi and T. Schwarcz, *Exchange distance of basis pairs in split matroids*, SIAM J. on Discrete Math, **38** (1), 2024.

[2] J. Bonin, *Base Exchange Properties of Sparse Paving Matroids*, Adv. in Appl. Math., **50** (2013) no. 1, 6-15.

[3] J. Edmonds, *Minimum partition of a matroid into independent sets*, J. Res. Nat. Bur. Standards Sect. B **69 B**, (1965), 67-72.

[4] G. Fan, H. Jiang, P. Li, D. West, D. Yang, X. Zhi, *Extensions of matroid covering and packing*, European J. Combinatorics, Vol. 76 (2019), 117-122.

[5] H. Gabow, *Decomposing symmetric exchanges in matroid bases*, Mathematical Programming, **10 (1)** (1976), 271-276.

[6] Y. Kajitani, S. Ueno, H. Miyano, *Ordering of the elements of a matroid such that its consecutive w elements are independent*, Discrete Math. 72(1988), 187-194.

[7] J. Oxley, *Matroid Theory*, 2nd. ed., Oxford University Press, Oxford, 2011.

[8] J. Van Den Heuvel and S. Thomasse, *Cyclic orderings and cyclic arboricity of matroids*, Journal of Comb. Theory series B, Vol. 102, 3, (2012), 638-646.