Piercing Independent Sets in Graphs without Large Induced Matching

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Abstract

Given a graph G, denoted by $h(G)$ the smallest size of a subset of $V(G)$ which intersects every maximum independent set of G. We prove that any graph G without induced matching of size t satisfies $h(G) \leqslant \omega(G)^{3t-3+o(1)}$. This resolves a conjecture of Hajebi, Li and Spirkl [Hitting all maximum stable sets in P_5-free graphs. J. Combin. Theory Ser. B, 165:142-163, 2024].

Mathematics Subject Classifications: 05C69, 05D15

1 Introduction

For a graph G, let $h(G)$ denote the smallest size of a subset of $V(G)$ intersecting every maximum independent set of G. Bollobás, Erdős and Tuza [3, 7] conjectured that any graph G with linear (in $|G|$) independence number must have sublinear $h(G)$. This easy-to-state conjecture is surprisingly difficult and remains wide open. It is then natural to consider special families of graphs. For example, Alon [1] suggested to study 3-colorable graphs. Motivated by this and some similarities to another well-studied problem of χ-boundedness, Hajebi, Li and Spirkl [8] recently considered the problem of bounding $h(G)$ by the clique number $\omega(G)$. Indeed, they observed that (1) given G, $h(H) \leqslant \omega(H)$ for every induced $H \subseteq G$ if and only if G is perfect; (2) there are graphs G with arbitrarily large $h(G)$ and girth; and (3) if for any graph G in a hereditary family $\mathcal{G}$ (i.e. closed under taking induced subgraphs), $h(G)$ can be bounded by a polynomial function of $\omega(G)$, then $\mathcal{G}$ satisfies the Erdős-Hajnal conjecture [5], yet another central conjecture in extremal and structural graph theory, that every graph in $\mathcal{G}$ has polynomial size clique or independent set. We refer

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interested readers to the survey of Scott and Seymour [14] for more about χ-boundedness and the recent work [2, 12, 13] for more on the Erdős-Hajnal conjecture.

Hajebi, Li and Spirkl [8] proved that for graphs G without induced P_5, $h(G) \leqslant f(\omega(G))$ for some function f. They raised the following conjecture that a similar phenomenon holds for graphs without large induced matching, and furthermore one can take a polynomial binding function f. An *induced matching* of size t consists of vertex set $\{a_i, b_i\}_{i\in[t]}$ and edge set $\{a_ib_i\}_{i\in[t]}$.

Conjecture 1 ([8])**.** Let G be a graph without induced matching of size t, then $h(G) \leqslant \omega(G)^{O_t(1)}$.

Hajebi, Li and Spirkl [8] resolved the first case $t = 2$. Quoting them, '*it is not even known (strikingly enough)*' whether $h(G)$ can be bounded by any function of $\omega(G)$ for $t = 3$.

In this short note, we prove Theorem 1. Our proof is inspired by the recent work in [10].

Theorem 2. *Let G be a graph without induced matching of size t, then we have $h(G) \leqslant 10t^t\omega(G)^{3t-3}\log\omega(G)$.*

2 The proof

For a set system $\mathcal{F} \subseteq 2^V$, the *transversal number* of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is the minimum size of subset $T \subseteq V$ such that $T \cap F \neq \emptyset$ for every set $F \in \mathcal{F}$. The *fractional transversal number* of $\mathcal{F}$, denoted by $\tau^*(\mathcal{F})$, is the minimum of $\sum_{v\in V} g(v)$, taken over all functions $g\colon V \to [0,1]$ such that $\sum_{v\in F} g(v) \geqslant 1$ for every $F \in \mathcal{F}$. Note that by definition, $\tau^*(\mathcal{F}) \leqslant \tau(\mathcal{F})$. The *Vapnik-Chervonenkis dimension* (VC-dimension for short) of $\mathcal{F}$ is the maximum cardinality of a subset $S \subseteq V$ such that for every $S' \subseteq S$ there exists $F \in \mathcal{F}$ with $F \cap S = S'$. The well-known ε-net theorem of Haussler and Welzl [9] (also see the book of Matoušek [11] and [4]) provides an inverse relation for set systems with bounded VC-dimension.

Theorem 3 ([9])**.** *Let $\mathcal{F}$ be a set system with VC-dimension d, then we have $\tau(\mathcal{F}) \leqslant 2d\tau^*(\mathcal{F})\log(11\tau^*(\mathcal{F}))$.*

Now, let G be an n-vertex graph with no induced matching of size t. Consider the set system $\mathcal{F} \subseteq 2^{V(G)}$ consisting of all maximum independent sets of G. By definition, $h(G) = \tau(\mathcal{F})$. Considering the constant function $g \equiv 1/\alpha(G)$ over $V(G)$, we see that

$$\tau^*(\mathcal{F}) \leqslant n/\alpha(G) \leqslant \chi(G) \leqslant \omega(G)^{2t-2},$$

where the last inequality is a classical result of Wagon [15] on χ-boundedness for graphs without large induced matching. It remains to bound the VC-dimension of $\mathcal{F}$.

Suppose the VC-dimension of $\mathcal{F}$ is d and $S = \{v_1, v_2, \dots, v_d\}$ is a set such that for any subset $S' \subseteq S$, there is a maximum independent set $I_{S'} \in \mathcal{F}$ with $I_{S'} \cap S = S'$. In particular, the set S itself is an independent set by considering I_S. For each $i \in [d]$, as $v_i \notin I_{S\setminus\{v_i\}}$, and $I_{S\setminus\{v_i\}}$ is a maximum independent set, there exists some vertex $u_i \in I_{S\setminus\{v_i\}}$ such that $v_i u_i \in E(G)$ and $v_j u_i \notin E(G)$ for any $j \neq i$. Since S is an independent set, we must have $u_i \notin S$. Moreover, $u_1, \ldots, u_d$ are all distinct. Indeed, if $u_j = u_i$ for some $j \neq i$, then $v_j u_j = v_j u_i \notin E(G)$, a contradiction. Thus, $\{v_i, u_i\}_{i\in[d]}$ forms a matching of size d, and $v_1, v_2, \ldots, v_d$ form an independent set. Then, as G has no induced matching of size t, $G[\{u_1, \ldots, u_d\}]$ has no independent set of size t, and so by off-diagonal Ramsey [6], $d < R(\omega(G)+1, t) \leqslant \binom{\omega(G)+t-1}{t-1}$. Thus by Theorem 3, we have

$$h(G) \leqslant 2\binom{\omega(G)+t-1}{t-1}\omega(G)^{2t-2}\log(11\omega(G)^{2t-2}) \leqslant 10t^t\omega(G)^{3t-3}\log\omega(G),$$

completing the proof.

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