Kromatic Quasisymmetric Functions

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Abstract

We provide a construction for the kromatic symmetric function $\overline{X}_G$ of a graph introduced by Crew, Pechenik, and Spirkl using combinatorial (linearly compact) Hopf algebras. As an application, we show that $\overline{X}_G$ has a positive expansion into multifundamental quasisymmetric functions. We also study two related quasisymmetric q-analogues of $\overline{X}_G$, which are K-theoretic generalizations of the quasisymmetric chromatic function of Shareshian and Wachs. We classify exactly when one of these analogues is symmetric. For the other, we derive a positive expansion into symmetric Grothendieck functions when G is the incomparability graph of a natural unit interval order.

Mathematics Subject Classifications: 05C15, 05E05, 16T30

1 Introduction

The purpose of this note is to re-examine the algebraic origins of the *kromatic symmetric function* of a graph that was recently introduced by Crew, Pechenik, and Spirkl [4], and to study two quasisymmetric analogues of this power series.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{P} = \{1, 2, 3, \dots\}$, and $[n] = \{i \in \mathbb{P} : i \leqslant n\}$ for $n \in \mathbb{N}$, so that $[0] = \varnothing$. All graphs are undirected by default, and are assumed to be simple with a finite set of vertices. We do not distinguish between isomorphic graphs.

If G is any graph then we write $V(G)$ for its set of vertices and $E(G)$ for its set of edges. A *proper coloring* of G is a map $\kappa : V(G) \to \mathbb{P}$ with $\kappa(u) \neq \kappa(v)$ for all $\{u, v\} \in E(G)$. For maps $\kappa : V \to \mathbb{P}$ let $x^\kappa = \prod_{i \in V} x_{\kappa(i)}$ where $x_1, x_2, \dots$ are commuting variables.

Definition 1 (Stanley [20]). The *chromatic symmetric function* of G is the power series $X_G := \sum_\kappa x^\kappa$ where the sum is over all proper colorings κ of G.

Example 2. If $G = K_n$ is the *complete graph* with $V(G) = [n]$ then X_G is $n!$ times the *elementary symmetric function* $e_n := \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$.

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A poset is $(3+1)$-*free* if it does not contain a 3-element chain $a < b < c$ whose elements are all incomparable to some fourth element d. The *incomparability graph* of a poset is the graph whose vertices are the set of elements in the poset and whose edges are the unordered pairs $\{x, y\}$ with $x \not\leq y$ and $y \not\leq x$. The *Stanley–Stembridge conjecture* [21] asserts that if G is the incomparability graph of a $(3+1)$-free poset then X_G has a positive expansion into elementary symmetric functions. This conjecture has several refinements and generalizations [9, 18], and has been resolved in a number of interesting special cases [5, 10, 11]. Hikita [12] has recently announced a general proof of the conjecture.

Let G be an *ordered graph*, that is, a graph with a total order $<$ on its vertex set $V(G)$. An *ascent* (respectively, *descent*) of a map $\kappa : V(G) \to \mathbb{P}$ is an edge $\{u, v\} \in E(G)$ with $u < v$ and $\kappa(u) < \kappa(v)$ (respectively, $\kappa(u) > \kappa(v)$). Let $\mathrm{asc}_G(\kappa)$ and $\mathrm{des}_G(\kappa)$ be the number of ascents and descents of κ. Shareshian and Wachs [18] introduced the following q-analogue of X_G:

Definition 3 ([18]). The *chromatic quasisymmetric function* of an ordered graph G is $X_G(q) = \sum_\kappa q^{\mathrm{asc}_G(\kappa)} x^\kappa$ where the sum is over all proper colorings κ of G.

Example 4. One has $X_{K_n}(q) = [n]_q! e_n$ for the q-numbers $[i]_q := \frac{1-q^i}{1-q}$ and $[n]_q! := \prod_{i=1}^n [i]_q$.

Let $\mathsf{Set}(\mathbb{P})$ be the set of finite nonempty subsets of positive integers. For a map $\kappa : V \to \mathsf{Set}(\mathbb{P})$ define $x^\kappa = \prod_{i \in V} \prod_{j \in \kappa(i)} x_j$. A *proper set-valued coloring* is a map $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$ with $\kappa(u) \cap \kappa(v) = \varnothing$ for all $\{u, v\} \in E(G)$. Using set-valued colorings in Definition 1 leads to a "K-theoretic" analogue of X_G:

Definition 5 (Crew, Pechenik, and Spirkl [4]). The *kromatic symmetric function* of a graph G is the power series $\overline{X}_G = \sum_\kappa x^\kappa \in \mathbb{Z}[\![x_1, x_2, \ldots]\!]$ where the sum is over all proper set-valued colorings of G.

Example 6. If $G = K_n$ then $\overline{X}_G = n! \sum_{r=n}^\infty \left\{{r \atop n}\right\} e_r$ where $\left\{{r \atop n}\right\}$ is the Stirling number of the second kind.

Remark 7. Given $\alpha : V \to \mathbb{P}$, let $\mathsf{Clan}_\alpha(V)$ be the set of pairs (v, i) with $v \in V$ and $i \in [\alpha(v)]$. If G is a graph and $\alpha : V(G) \to \mathbb{P}$ is any map, then the *α-clan graph* $\mathsf{Clan}_\alpha(G)$ has vertex set $\mathsf{Clan}_\alpha(V(G))$ and edges $\{(v, i), (w, j)\}$ whenever $\{v, w\} \in E(G)$ or both $v = w$ and $i \neq j$. As observed in [4], one has $\overline{X}_G = \sum_{\alpha : V(G) \to \mathbb{P}} \frac{1}{\alpha!} X_{\mathsf{Clan}_\alpha(G)}$ where $\alpha! := \prod_v \alpha(v)!$. Many properties of X_G extend to $\overline{X}_G$ via this identity, but some interesting features of $\overline{X}_G$ cannot be explained by this formula alone.

Our main results in Section 3 provide a natural construction for $\overline{X}_G$ using the theory of *combinatorial Hopf algebras*. This approach requires some care, as $\overline{X}_G$ does not have bounded degree. We explain things precisely in terms of *linearly compact Hopf algebras* after reviewing a similar, simpler construction of X_G in Section 2, following [1].

As an application of our approach, we show that $\overline{X}_G$ has a positive expansion into *multifundamental quasisymmetric functions*. We also study two related q-analogues of $\overline{X}_G$, which give K-theoretic generalizations of $X_G(q)$. We classify exactly when one of these analogues is symmetric. For the other, we extend a theorem of Crew, Pechenik,

and Spirkl [4] (which also generalizes a theorem of Shareshian and Wachs [18]) to derive a positive expansion into symmetric Grothendieck functions when G is the incomparability graph of a natural unit interval order. These results are contained in Section 4.

Acknowledgments

2 Background

Let $\mathbb{K}$ be an integral domain; in most results, one can assume this is $\mathbb{Z}$.

2.1 Hopf algebras

Write $\otimes = \otimes_{\mathbb{K}}$ for the tensor product over $\mathbb{K}$. A $\mathbb{K}$-*algebra* is a $\mathbb{K}$-module A with $\mathbb{K}$-linear product $\nabla : A \otimes A \to A$ and unit $\iota : \mathbb{K} \to A$ maps. Dually, a $\mathbb{K}$-*coalgebra* is a $\mathbb{K}$-module A with $\mathbb{K}$-linear coproduct $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to \mathbb{K}$ maps. The (co)product and (co)unit maps must satisfy several associativity axioms; see [8, §1].

A $\mathbb{K}$-module A that is both a $\mathbb{K}$-algebra and a $\mathbb{K}$-coalgebra is a $\mathbb{K}$-*bialgebra* if the coproduct and counit maps are algebra morphisms. A bialgebra with a direct sum decomposition $A = \bigoplus_{n \in \mathbb{N}} A_n$ is *graded* if its (co)product and (co)unit are graded maps. A bialgebra $A = \bigoplus_{n \in \mathbb{N}} A_0$ is *connected* if there is an isomorphism $A_0 \cong \mathbb{K}$ and the unit and counit are obtained by composing this isomorphism with the usual inclusion and projection maps $A_0 \hookrightarrow A$ and $A \twoheadrightarrow A_0$.

Let $\mathrm{End}(A)$ denote the set of $\mathbb{K}$-linear maps $A \to A$. This set is a $\mathbb{K}$-algebra for the product $f * g := \nabla \circ (f \otimes g) \circ \Delta$. The unit of this *convolution algebra* is the composition $\iota \circ \epsilon$ of the unit and counit of A. A bialgebra A is a *Hopf algebra* if the identity map $\mathrm{id} : A \to A$ has a two-sided inverse $\mathsf{S} : A \to A$ in $\mathrm{End}(A)$. When it exists, we call S the *antipode* of A.

We mention a few common examples of Hopf algebras.

Example 8. Let Graphs_n be the free $\mathbb{K}$-module spanned by all isomorphism classes of undirected graphs with n vertices, and set $\mathsf{Graphs} = \bigoplus_{n \in \mathbb{N}} \mathsf{Graphs}_n$. One views Graphs as a connected, graded Hopf algebra with product $\nabla(G \otimes H) = G \sqcup H$ and coproduct $\Delta(G) = \sum_{S \sqcup T = V(G)} G|_S \otimes G|_T$ for graphs G and H, where $\sqcup$ means disjoint union and $G|_S$ is the induced subgraph of G on S.

A *lower set* in a directed acyclic graph $D = (V, E)$ is a set $S \subseteq V$ such that if a directed path connects $v \in V$ to $s \in S$ then $v \in S$. An *upper set* in D is a set $S \subseteq V$ such that if a directed path connects $s \in S$ to $v \in V$ then $v \in S$. Finally, an *antichain* in D is a set of vertices S such that no directed path connects any $s \in S$ to any other $t \in S$.

Example 9. Let DAGs_n for $n \in \mathbb{N}$ be the free $\mathbb{K}$-module spanned by all isomorphism classes of directed acyclic graphs with n vertices, and set $\mathsf{DAGs} = \bigoplus_{n \in \mathbb{N}} \mathsf{DAGs}_n$. One views DAGs as a connected, graded Hopf algebra with product $\nabla(C \otimes D) = C \sqcup D$ and coproduct $\Delta(D) = \sum D|_S \otimes D|_T$ for directed acyclic graphs graphs C and D, where the sum is over all disjoint unions $S \sqcup T = V(D)$ with S a lower set and T an upper set.

A *labeled poset* is a pair (D, γ) consisting of a directed acyclic graph D and an injective map $\gamma : V(D) \to \mathbb{Z}$. We consider $(D, \gamma) = (D', \gamma')$ if there is an isomorphism $D \xrightarrow{\sim} D'$, written $v \mapsto v'$, such that $\gamma(u) - \gamma(v)$ and $\gamma'(u') - \gamma'(v')$ have the same sign for all edges $u \to v \in E(D)$.

If (D_1, γ_1) and (D_2, γ_2) are labeled posets then let $\gamma_1 \sqcup \gamma_2 : V(D_1 \sqcup D_2) \to \mathbb{Z}$ be any injective map such that $(\gamma_1 \sqcup \gamma_2)(u) - (\gamma_1 \sqcup \gamma_2)(v)$ has the same sign as $\gamma_i(u) - \gamma_i(v)$ for all $i \in \{1, 2\}$ and $u, v \in V(D_i)$.

Example 10. Let $\mathsf{LPosets}_n$ be the free $\mathbb{K}$-module spanned by all labeled poset with n vertices, and set $\mathsf{LPosets} = \bigoplus_{n \in \mathbb{N}} \mathsf{LPosets}_n$. This is a connected, graded Hopf algebra with product $\nabla((D_1, \gamma_1) \otimes (D_2, \gamma_2)) = (D_1 \sqcup D_2, \gamma_1 \sqcup \gamma_2)$ and coproduct $\Delta((D, \gamma)) = \sum (D|_S, \gamma|_S) \otimes (D|_T, \gamma|_T)$ where the sum is over all disjoint decompositions $S \sqcup T = V(D)$ with S a lower set and T an upper set.

A *(strict) composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is a finite sequence of positive integers, called its *parts*. We say that α is a composition of $|\alpha| := \sum_i \alpha_i \in \mathbb{N}$.

Example 11. Fix a composition α and let $x_1, x_2, \ldots$ be a countable sequence of commuting variables. The *monomial quasisymmetric function* of α is the power series $M_\alpha = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_l} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_l}^{\alpha_l}$. Let $\mathsf{QSym} = \mathbb{K}\text{-span}\{M_\alpha : \alpha \text{ any composition}\}$ be the ring of quasisymmetric functions of bounded degree. This ring is a graded connected Hopf algebras for the coproduct $\Delta(M_\alpha) = \sum_{\alpha = \alpha' \alpha''} M_{\alpha'} \otimes M_{\alpha''}$ where $\alpha' \alpha''$ denotes concatenation of compositions, and the counit that acts on power series by setting $x_1 = x_2 = \cdots = 0$.

A *partition* is a composition sorted into decreasing order. We sometimes write $\lambda = 1^{m_1} 2^{m_2} \cdots$ to denote the partition with exactly m_i parts equal to i. We also let $\ell(\lambda) = m_1 + m_2 + \ldots$ denote the number of parts of λ.

Example 12. The *elementary symmetric function* of a partition λ is the product $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ where $e_n := M_{1^n}$. These power series are a basis for the Hopf subalgebra $\mathsf{Sym} \subset \mathsf{QSym}$ of symmetric functions of bounded degree.

2.2 Combinatorial Hopf algebras

Following [1], a *combinatorial Hopf algebra* (H, ζ) is a graded, connected Hopf algebra H of finite graded dimension with an algebra homomorphism $\zeta : H \to \mathbb{K}$.

Example 13. The pair $(\mathsf{QSym}, \zeta_\mathsf{Q})$ is an example of a combinatorial Hopf algebra, where $\zeta_\mathsf{Q} : \mathsf{QSym} \to \mathbb{K}$ is the map $\zeta_\mathsf{Q}(f) = f(1, 0, 0, \ldots)$, which sends $M_{(n)} \mapsto 1$ and $M_\alpha \mapsto 0$ for all compositions α with at least two parts.

For a graph G define $\zeta_{\mathsf{Graphs}}(G) = 0^{|E(G)|}$ where $0^0 := 1$. For a directed acyclic graph D likewise set $\zeta_{\mathsf{DAGs}}(D) = 0^{|E(D)|}$ for each directed acyclic graph D. These formulas extend to linear maps $\mathsf{Graphs} \to \mathbb{K}$ and $\mathsf{DAGs} \to \mathbb{K}$. Finally let $\zeta_{\mathsf{LPosets}} : \mathsf{LPosets} \to \mathbb{K}$ be the linear map with $\zeta_{\mathsf{LPosets}}((D, \gamma)) = 1$ if $\gamma(u) < \gamma(v)$ for all edges $u \to v \in E(D)$ and with $\zeta_{\mathsf{LPosets}}((D, \gamma)) = 0$ otherwise.

Example 14. The pairs $(\mathsf{Graphs}, \zeta_{\mathsf{Graphs}})$, $(\mathsf{DAGs}, \zeta_{\mathsf{DAGs}})$, and $(\mathsf{LPosets}, \zeta_{\mathsf{LPosets}})$ are all combinatorial Hopf algebras.

A morphism of combinatorial Hopf algebras $\Psi : (H, \zeta) \to (H', \zeta')$ is a graded Hopf algebra morphism $\Psi : H \to H'$ with $\zeta = \zeta' \circ \Psi$. Results in [1] show that there exists a unique morphism (with an explicit formula) from any combinatorial Hopf algebra to $(\mathsf{QSym}, \zeta_{\mathsf{Q}})$. Moreover, the image of this morphism is contained in the Hopf subalgebra $\mathsf{Sym} \subset \mathsf{QSym}$ if H is cocommutative. Of greatest relevance here [1, Ex. 4.5], the unique morphism $(\mathsf{Graphs}, \zeta_{\mathsf{Graphs}}) \to (\mathsf{QSym}, \zeta_{\mathsf{Q}})$ assigns each graph G to its chromatic symmetric function X_G.

We might as well also describe the unique morphisms of combinatorial Hopf algebras from $(\mathsf{DAGs}, \zeta_{\mathsf{DAGs}})$ and $(\mathsf{LPosets}, \zeta_{\mathsf{LPosets}})$ to $(\mathsf{QSym}, \zeta_{\mathsf{Q}})$. This requires a little more notation. A *P-partition* [19] of a labeled poset $P = (D, \gamma)$ is a map $\kappa : V(D) \to \mathbb{P}$ with

(a) $\kappa(u) \leqslant \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) < \gamma(v)$, and

(b) $\kappa(u) < \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) > \gamma(v)$.

Define $\Gamma(D, \gamma) = \sum_{\kappa} x^{\kappa}$ where the sum is over all P-partitions for $P = (D, \gamma)$.

Given directed acyclic graph D, let γ^{decr} denote an arbitrary *decreasing labeling* of D, meaning a injective map $\gamma^{\mathrm{decr}} : V(D) \to \mathbb{Z}$ with

$$\gamma^{\mathrm{decr}}(x) > \gamma^{\mathrm{decr}}(y) \text{ whenever } x \to y \in E(D). \tag{2.1}$$

Then define $\Gamma(D) := \Gamma(D, \gamma^{\mathrm{decr}}) \in \mathbb{N}[\![x_1, x_2, \ldots]\!]$.

Finally, for any graph G, let $\mathsf{AO}(G)$ be its set of *acyclic orientations*, that is, the directed acyclic graphs that recover G when we ignore edge orientations. The following is well-known [1] and can be seen as a special case of Theorem 37.

Proposition 15. *There is a commutative diagram of morphisms*

$$(\mathsf{Graphs}, \zeta_{\mathsf{Graphs}}) \lhook\joinrel\longrightarrow (\mathsf{DAGs}, \zeta_{\mathsf{DAGs}}) \lhook\joinrel\longrightarrow (\mathsf{LPosets}, \zeta_{\mathsf{LPosets}})$$
$$(\mathsf{QSym}, \zeta_{\mathsf{Q}})$$

in which the horizontal maps send $G \mapsto \sum_{D \in \mathsf{AO}(G)} D$ *and* $D \mapsto (D, \gamma^{\mathrm{decr}})$, *and the* QSym-*valued maps send* $G \mapsto X_G$ *and* $D \mapsto \Gamma(D)$ *and* $(D, \gamma) \mapsto \Gamma(D, \gamma)$.

This proposition recovers the well-known identity $X_G = \sum_{D \in \mathsf{AO}(G)} \Gamma(D)$.

3 K-theoretic generalizations

We now explain how the results in the previous can be extended "K-theoretically" to construct interesting quasisymmetric functions of unbounded degree, including $\overline{X}_G$. This requires a brief discussion of *linearly compact modules*.

3.1 Linearly compact modules

Let A and B be $\mathbb{K}$-modules with a $\mathbb{K}$-bilinear form $\langle \cdot, \cdot \rangle : A \times B \to \mathbb{K}$. Assume A is free and $\langle \cdot, \cdot \rangle$ is *nondegenerate* in the sense that $b \mapsto \langle \cdot, b \rangle$ is a bijection $B \to \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$.

Fix a basis $\{a_i\}_{i \in I}$ for A. For each $i \in I$, there exists a unique $b_i \in B$ with $\langle a_i, b_j \rangle = \delta_{ij}$ for all $i, j \in I$, and we identify $b \in B$ with the formal linear combination $\sum_{i \in I} \langle a_i, b \rangle b_i$. We call $\{b_i\}_{i \in I}$ a *pseudobasis* for B.

We give $\mathbb{K}$ the discrete topology. Then the *linearly compact topology* [6, §I.2] on B is the coarsest topology in which the maps $\langle a_i, \cdot \rangle : B \to \mathbb{K}$ are all continuous. This topology depends on $\langle \cdot, \cdot \rangle$ but not on the choice of basis for A. For a basis of open sets in the linearly compact topology, see [17, Eq. (3.1)].

Definition 16. A *linearly compact* (or *LC* for short) $\mathbb{K}$-module is a $\mathbb{K}$-module B with a nondegenerate bilinear form $A \times B \to \mathbb{K}$ for some free $\mathbb{K}$-module A, given the linearly compact topology; in this case we say that B is the *dual* of A. Morphisms between such modules are continuous $\mathbb{K}$-linear maps.

Let B and B' be linearly compact $\mathbb{K}$-modules dual to the free $\mathbb{K}$-modules A and A'. Let $\langle \cdot, \cdot \rangle$ denote both of the associated forms. Every linear map $\phi : A' \to A$ has a unique adjoint $\psi : B \to B'$ such that $\langle \phi(a), b \rangle = \langle a, \psi(b) \rangle$. A linear map $B \to B'$ is continuous when it is the adjoint of a linear map $A' \to A$.

Definition 17. Define $B \,\overline{\otimes}\, B' := \mathrm{Hom}_{\mathbb{K}}(A \otimes A', \mathbb{K})$ and give this the LC-topology from the pairing $(A \otimes A') \times \mathrm{Hom}_{\mathbb{K}}(A \otimes A', \mathbb{K}) \to \mathbb{K}$.

If $\{b_i\}_{i \in I}$ and $\{b'_j\}_{j \in J}$ are pseudobases for B and B', then we can realize the *completed tensor product* $B \,\overline{\otimes}\, B'$ concretely as the linearly compact $\mathbb{K}$-module with the set of tensors $\{b_i \otimes b'_j\}_{(i,j) \in I \times J}$ as a pseudobasis.

Example 18. Let $A = \mathbb{K}[x]$ and $B = \mathbb{K}[\![x]\!]$. Define $\langle \cdot, \cdot \rangle : A \times B \to \mathbb{K}$ to be the nondegenerate $\mathbb{K}$-bilinear form $\left\langle \sum_{n \geqslant 0} a_n x^n, \sum_{n \geqslant 0} b_n x^n \right\rangle := \sum_{n \geqslant 0} a_n b_n$. Then the set $\{x^n\}_{n \geqslant 0}$ is a basis for A and a pseudobasis for B, and we have

$$\mathbb{K}[\![x]\!] \otimes \mathbb{K}[\![y]\!] \neq \mathbb{K}[\![x]\!] \,\overline{\otimes}\, \mathbb{K}[\![y]\!] \cong \mathbb{K}[\![x, y]\!].$$

Suppose $\nabla : B \,\overline{\otimes}\, B \to B$ and $\iota : B \to \mathbb{K}$ are continuous linear maps which are the adjoints of linear maps $\epsilon : \mathbb{K} \to A$ and $\Delta : A \to A \otimes A$. We say that (B, ∇, ι) is an *LC-algebra* if (A, Δ, ϵ) is a $\mathbb{K}$-coalgebra. Similarly, we say that $\Delta : B \to B \,\overline{\otimes}\, B$ and $\epsilon : B \to \mathbb{K}$ make B into an *LC-coalgebra* if Δ and ϵ are the adjoints of the product and unit maps of a $\mathbb{K}$-algebra on A. We define *LC-bialgebras* and *LC-Hopf algebras* analogously; see [17]. If B is an LC-Hopf algebra then its *antipode* is the adjoint of the antipode of A.

3.2 Combinatorial LC-Hopf algebras

Following [17], define a *combinatorial LC-Hopf algebra* to be a pair (H, ζ) where

(a) H is an LC-Hopf algebra;

(b) $\zeta : H \to \mathbb{K}[\![t]\!]$ is a homomorphism of linearly compact algebras;

(c) the counit of H coincides with $\zeta(\cdot)|_{t \mapsto 0}$.

A morphism of combinatorial LC-Hopf algebras $\Psi : (H, \zeta) \to (H', \zeta')$ is a LC-Hopf algebra morphism $\Psi : H \to H'$ with $\zeta = \zeta' \circ \Psi$.

Example 19. Let $\mathfrak{m}\mathsf{QSym}$ be the set of all quasisymmetric power series in $\mathbb{K}[\![x_1, x_2, \dots,]\!]$ of unbounded degree. The (co)product, (co)unit, and antipode QSym all extend to continuous $\mathbb{K}$-linear maps that make $\mathfrak{m}\mathsf{QSym}$ into an LC-Hopf algebra, with $\{M_\alpha\}$ as a pseudobasis. Then $(\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_\mathsf{Q})$ is a combinatorial LC-Hopf algebra when $\overline{\zeta}_\mathsf{Q}$ is the map $\overline{\zeta}_\mathsf{Q} : f \mapsto f(t, 0, 0, \dots)$, which sends

$$\overline{\zeta}_\mathsf{Q} : M_\alpha \mapsto \begin{cases} t^{|\alpha|} & \text{if } \alpha \in \{\emptyset, (1), (2), (3), \dots\} \\ 0 & \text{otherwise.} \end{cases}$$

The preceding example is an instance of a general construction. If A is a free $\mathbb{K}$-module with basis S, then its *completion* $\overline{A}$ is the set of arbitrary $\mathbb{K}$-linear combinations of S. We view $\overline{A}$ as a linearly compact $\mathbb{K}$-module with S as a pseudobasis, relative to the nondegenerate bilinear form $A \times \overline{A} \to \mathbb{K}$ making S orthonormal.

Suppose $H = \bigoplus_{n \in \mathbb{N}} H_n$ is a connected Hopf algebra that is graded as an algebra with finite graded dimension, and filtered as a coalgebra in the sense that if $h \in H_n$ then $\Delta(h) \in \bigoplus_{i+j \geqslant n} H_i \otimes H_j$. If $\zeta : H \to \mathbb{K}$ is any linear map, then we write $\overline{\zeta} : \overline{H} \to \mathbb{K}[\![t]\!]$ for the unique continuous linear map satisfying

$$\overline{\zeta}(h) := \zeta(h) t^n \quad \text{for } n \in \mathbb{N} \text{ and } h \in H_n. \tag{3.1}$$

There is a unique way of extending the (co)unit and (co)product of H to continuous linear maps on its completion $\overline{H}$, for which the following holds.

Lemma 20. *The structures just given make $\overline{H}$ into an LC-Hopf algebra. If (H, ζ) is a combinatorial Hopf algebra (so the coproduct is graded rather than just filtered), then $(\overline{H}, \overline{\zeta})$ is a combinatorial LC-Hopf algebra, and the unique morphism $(H, \zeta) \to (\mathsf{QSym}, \zeta_\mathsf{Q})$ extends to a morphism $(\overline{H}, \overline{\zeta}) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_\mathsf{Q})$.*

Proof. In this setup, the graded dual $H^* = \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_\mathbb{K}(H_n, \mathbb{K})$ is a well-defined Hopf algebra, which is graded as a coalgebra with finite graded dimension, and filtered as an algebra in the sense that the product sends

$$\mathrm{Hom}_\mathbb{K}(H_i, \mathbb{K}) \otimes \mathrm{Hom}_\mathbb{K}(H_j, \mathbb{K}) \to \bigoplus_{n \leqslant i+j} \mathrm{Hom}_\mathbb{K}(H_n, \mathbb{K}).$$

Our definition of $\overline{H}$ is an LC-Hopf algebra because it is the (ungraded) dual of H^* in the sense of Definition 16, via the natural bilinear form $H^* \times \overline{H} \to \mathbb{K}$.

Now suppose (H, ζ) is a combinatorial Hopf algebra. Since ζ is an algebra homomorphism, the map $\overline{\zeta}$ is also an algebra homomorphism. As H is connected, we may assume that $H_0 = \mathbb{K}$ and that the counit map $H \to H_0 = \mathbb{K}$ is the natural projection. Then the algebra homomorphism $\zeta : H \to \mathbb{K}$ must restrict to the identity map on $H_0 = \mathbb{K}$, so the counit ε of $\overline{H}$ evaluated at a generic element $h = \sum_{n \in \mathbb{N}} h_n \in \overline{H}$ with $h_n \in H_n$ has the formula $\varepsilon(h) = h_0 = \zeta(h_0) = \overline{\zeta}(h)|_{t=0}$. We conclude that $(\overline{H}, \overline{\zeta})$ is a combinatorial LC-Hopf algebra.

The morphism $(H, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{Q}})$ is a graded linear map, so it extends to a morphism of LC-Hopf algebras $\Psi : \overline{H} \to \mathfrak{m}\mathsf{QSym}$. This map satisfies

$$\overline{\zeta}_{\mathsf{Q}}(\Psi(h)) = \sum_{n \in \mathbb{N}} \overline{\zeta}_{\mathsf{Q}}(\Psi(h_n)) = \sum_{n \in \mathbb{N}} \zeta_{\mathsf{Q}}(\Psi(h_n))t^n = \sum_{n \in \mathbb{N}} \zeta(h_n)t^n = \overline{\zeta}(h)$$

for any $h = \sum_{n \in \mathbb{N}} h_n \in \overline{H}$ with $h_n \in H_n$, so we have $\overline{\zeta}_{\mathsf{Q}} \circ \Psi = \overline{\zeta}$. $\qquad\qquad\square$

The pair $(\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ is a final object in the category of combinatorial LC-Hopf algebras, meaning there is a unique morphism $(H, \zeta) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ for any combinatorial LC-Hopf algebra. Specifically, if H has coproduct Δ, then define $\Delta^{(0)} = \mathrm{id}_H$ and

$$\Delta^{(k)} = (\Delta^{(k-1)} \overline{\otimes} \mathrm{id}) \circ \Delta = (\mathrm{id} \overline{\otimes} \Delta^{(k-1)}) \circ \Delta : H \to H^{\overline{\otimes}(k+1)} \quad \text{for } k \in \mathbb{P}.$$

For each nonempty composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, let $\zeta_\alpha : H \to \mathbb{K}$ be the map sending $h \in H$ to the coefficient of $t^{\alpha_1} \otimes t^{\alpha_2} \otimes \cdots \otimes t^{\alpha_k}$ in $\zeta^{\otimes k} \circ \Delta^{(k-1)}(h) \in \mathbb{K}[\![t]\!]$. When $\alpha = \emptyset$ is empty let $\zeta_\emptyset := \zeta(\cdot)|_{t \mapsto 0}$ be the counit of H.

Theorem 21 ([16, Thm. 2.8]). *If (H, ζ) is a combinatorial LC-Hopf algebra then the map $\Psi_{H,\zeta} : h \mapsto \sum_\alpha \zeta_\alpha(h)M_\alpha$ is the unique morphism $(H, \zeta) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$.*

Remark 22. Let $\mathfrak{m}\mathsf{Sym}$ be the LC-Hopf subalgebra of symmetric functions in $\mathfrak{m}\mathsf{QSym}$. When H cocommutative, it is apparent from the formula given above that the morphism $\Psi_{H,\zeta}$ has its image in $\mathfrak{m}\mathsf{Sym}$.

3.3 Graphs and weighted graphs

For each graph G define

$$\blacktriangle(G) := \sum_{S \cup T = V(G)} G|_S \otimes G|_T. \qquad (3.2)$$

This differs from our other coproduct in allowing vertex decompositions that are not disjoint. Use the continuous linear extension of (3.2) to replace the natural coproduct in $\overline{\mathsf{Graphs}}$ and denote the resulting modified structure as $\mathfrak{m}\mathsf{Graphs}$. Also let $\overline{\zeta}_{\mathsf{Graphs}} : \mathfrak{m}\mathsf{Graphs} \to \mathbb{K}[\![t]\!]$ be the continuous linear map with

$$\overline{\zeta}_{\mathsf{Graphs}}(G) := 0^{|E(G)|} t^{|G|}$$

so that $\overline{\zeta}_{\mathsf{Graphs}}$ extends ζ_{Graphs} via the formula (3.1).

The kromatic symmetric function $\overline{X}_G$ emerges from these constructions via the theory in the previous section, in the following sense:

Proposition 23. *The pair* $(\mathfrak{mGraphs}, \overline{\zeta}_{\mathsf{Graphs}})$ *is a combinatorial LC-Hopf algebra, and the unique morphism* $(\mathfrak{mGraphs}, \overline{\zeta}_{\mathsf{Graphs}}) \to (\mathfrak{mQSym}, \overline{\zeta}_{\mathsf{Q}})$ *assigns each graph* G *to its kromatic symmetric function* $\overline{X}_G$.

We skip the proof of this result and instead derive a slight generalization. A *weighted graph* is a pair (G, ω) consisting of a (finite, undirected) graph G and a map $\omega : V(G) \to \mathbb{P}$ assigned positive integer weights to all vertices. We do not distinguish between two weighted graphs (G_1, ω_1) and (G_2, ω_2) if there is a graph isomorphism $\phi : G_1 \to G_2$ with $\omega_1 = \omega_2 \circ \phi$.

Definition 24 ([4]). The *kromatic symmetric function* of a weighted graph (G, ω) is $\overline{X}_{(G,\omega)} = \sum_\kappa \prod_{v \in V(G)} \left(\prod_{i \in \kappa(v)} x_i \right)^{\omega(v)} \in \mathfrak{mSym}$ where the sum is over all proper set-valued colorings $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$.

Recall from Example 6 that $\left\{ {r \atop n} \right\}$ is the Stirling number of the second kind. For a partition ν, write $m_\nu = \sum_{\mathrm{sort}(\alpha)=\nu} x^\alpha \in \mathsf{Sym}$ for the usual monomial symmetric function.

Example 25. Crew, Pechenik, and Spirkl [4, §3.2] define the *K-theoretic augmented monomial symmetric function* of a partition λ to be $\widetilde{m}_\lambda := \overline{X}_{(G,\omega)}$ for the complete graph $G = K_{\ell(\lambda)}$ with weight map $\omega : i \mapsto \lambda_i$. Let $r_i(\lambda)$ be the number of parts of λ equal to i. One has $\widetilde{m}_\emptyset = 1$, and if λ is nonempty then

$$\widetilde{m}_\lambda = \sum_\nu \left(\prod_{i \in \mathbb{P}} \left\{ {r_i(\nu) \atop r_i(\lambda)} \right\} \cdot r_i(\lambda)! \right) m_\nu$$

where the sum is over all integer partitions ν that have the same set of parts $\{\nu_1, \nu_2, \dots\} = \{\lambda_1, \lambda_2, \dots\}$ as λ and that satisfy $r_i(\nu) \geqslant r_i(\lambda)$ for all $i \in \mathbb{P}$.

It is natural to consider this generalization of the kromatic symmetric function since one can expand $\overline{X}_G$ into a sum of $\widetilde{m}_\lambda$'s via a deletion-contraction relation [4, §3.3]. This is perhaps the most efficient method available to actually compute $\overline{X}_G$.

We can again interpret $\overline{X}_{(G,\omega)}$ via combinatorial LC-Hopf algebras. Let $\mathfrak{mWGraphs}$ be the linearly compact $\mathbb{K}$-module with a pseudobasis provided by all weighted graphs. The LC-Hopf algebra structure on $\mathfrak{mGraphs}$ extends to $\mathfrak{mWGraphs}$ in the following way. The product is again given by disjoint union, the unit element is the unique weighted graph with no vertices, and the counit sends $(G, \omega) \mapsto 0^{|G|}$. Extend the coproduct $\blacktriangle$ of $\mathfrak{mGraphs}$ by setting

$$\blacktriangle((G, \omega)) := \sum_{S \cup T = V(G)} (G|_S, \omega|_S) \otimes (G|_T, \omega|_T) \tag{3.3}$$

for each weighted graph (G, ω), and let

$$\overline{\zeta}_{\mathsf{WGraphs}}((G, \omega)) := 0^{|E(G)|} \prod_{v \in V(G)} t^{\omega(v)}. \tag{3.4}$$

This extends to a continuous algebra homomorphism $\mathfrak{mWGraphs} \to \mathbb{K}[\![t]\!]$. It is important to require our vertex weightings $\omega : V(G) \to \mathbb{P}$ to take positive values since if we allowed $\omega : V(G) \to \mathbb{N}$ then $\overline{\zeta}_{\mathsf{WGraphs}}$ would not be well-defined when applied to infinite linear combinations of weighted graphs.

Proposition 26. *The pair* $(\mathfrak{mWGraphs}, \overline{\zeta}_{\mathsf{WGraphs}})$ *is a combinatorial LC-Hopf algebra, and the unique morphism* $(\mathfrak{mWGraphs}, \overline{\zeta}_{\mathsf{WGraphs}}) \to (\mathfrak{mQSym}, \overline{\zeta}_{\mathsf{Q}})$ *assigns each weighted graph* (G, ω) *to its kromatic symmetric function* $\overline{X}_{(G,\omega)}$.

We recover Proposition 23 by identifying $\mathfrak{mGraphs}$ with the LC-Hopf subalgebra with a pseudobasis given by the weighted graphs (G, ω) with $\omega : G \to \{1\}$.

Proof. Restricting the (co)unit and (co)product of $\mathfrak{mWGraphs}$ to the $\mathbb{K}$-span of all weighted graphs gives a Hopf algebra satisfying the conditions in Lemma 20: the relevant axioms are straightforward to check.

For example, the product is clearly associative and gives a graded algebra structure of finite graded dimension when we set the degree of (G, ω) to be $\sum_{v \in V(G)} \omega(v)$. For this grading, the coproduct similarly defines a filtered coalgebra.

The compatibility axiom for the product and coproduct holds since if $\tau : M \otimes N \to N \otimes M$ is the usual twist morphism for $\mathbb{K}$-modules, then both

$$\blacktriangle \circ \nabla \quad \text{and} \quad (\nabla \otimes \nabla) \circ (1 \otimes \tau \otimes 1) \circ (\blacktriangle \otimes \blacktriangle)$$

applied to the tensor $(G_1, \omega_1) \otimes (G_2, \omega_2)$ of two weighted graphs expand as

$$\sum_{\substack{S_1 \cup T_1 = V(G_1) \\ S_2 \cup T_2 = V(G_2)}} ((G_1|_{S_1}, \omega_1|_{S_1}) \sqcup (G_2|_{S_2}, \omega_2|_{S_2})) \otimes ((G_1|_{T_1}, \omega_1|_{T_1}) \sqcup (G_2|_{T_2}, \omega_2|_{T_2})).$$

We conclude from Lemma 20 that $\mathfrak{mGraphs}$ is an LC-Hopf algebra. The map $\overline{\zeta}_{\mathsf{WGraphs}}$ is a continuous algebra homomorphism with $\overline{\zeta}_{\mathsf{WGraphs}}(\cdot)|_{t \mapsto 0}$ equal to the counit of $\mathfrak{mWGraphs}$, so $(\mathfrak{mWGraphs}, \overline{\zeta}_{\mathsf{WGraphs}})$ is a combinatorial LC-Hopf algebra.

Let $\zeta = \overline{\zeta}_{\mathsf{WGraphs}}$ and $\Psi := \Psi_{\mathfrak{mWGraphs}, \overline{\zeta}_{\mathsf{WGraphs}}}$ as in Theorem 21. For a weighted graph (G, ω) and a subset $S \subseteq V(G)$, let $|S|_\omega := \sum_{i \in S} \omega(i)$. Then

$$\zeta^{\otimes k} \circ \blacktriangle^{(k-1)}((G, \omega)) = \sum t^{|S_1|_\omega} \otimes t^{|S_2|_\omega} \otimes \cdots \otimes t^{|S_k|_\omega}$$

for each $k \in \mathbb{K}$, where the sum is over all (not necessarily disjoint) decompositions $S_1 \cup S_2 \cup \cdots \cup S_k = V(G)$ such that $G|_{S_i}$ is a discrete graph for all $i \in [k]$, so that $E(G|_{S_i}) = \varnothing$. Therefore

$$\Psi((G, \omega)) = \sum_{k \in \mathbb{N}} \sum_{\substack{S_1 \cup S_2 \cup \cdots \cup S_k = V(G) \\ E(G|_{S_i}) = \varnothing \ \forall i \in [k]}} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k} x_{i_1}^{|S_1|_\omega} x_{i_2}^{|S_2|_\omega} \cdots x_{i_k}^{|S_k|_\omega}.$$

The monomial in the innermost sum is exactly $\prod_{v \in V(G)} \left(\prod_{i \in \kappa(v)} x_i \right)^{\omega(v)}$ for the proper set-valued coloring of G with $\kappa(v) = \{i_j : j \in [k] \text{ with } v \in S_j\}$ for $v \in V(G)$, and as the parameters indexing the sums vary, the corresponding κ ranges over all proper set-valued colorings of G. Thus $\Psi((G, \omega)) = \overline{X}_{(G,\omega)}$. $\qquad \square$

Recall that $r_i(\lambda)$ is the number of parts equal to i in a partition λ. If λ and ν are partitions with $r_i(\lambda) \leqslant r_i(\nu)$ for all i then let $\nu \ominus \lambda$ be the partition with $r_i(\nu \ominus \lambda) = r_i(\nu) - r_i(\lambda)$ for all i. Let $\nu \oplus \lambda$ be the partition with $r_i(\nu \oplus \lambda) = r_i(\nu) + r_i(\lambda)$ for all i.

Corollary 27. *If ν is a partition then*

$$\Delta(\overline{\widetilde{m}}_\nu) = \sum_{\lambda,\mu} \left(\prod_{i \in \mathbb{P}} \binom{r_i(\nu)}{r_i(\nu \ominus \lambda),\ r_i(\nu \ominus \mu),\ r_i((\lambda \oplus \mu) \ominus \nu)} \right) \overline{\widetilde{m}}_\lambda \otimes \overline{\widetilde{m}}_\mu$$

where the sum is over all partitions λ and μ with

$$0 \leqslant r_i(\lambda) \leqslant r_i(\nu) \quad and \quad 0 \leqslant r_i(\mu) \leqslant r_i(\nu) \leqslant r_i(\lambda) + r_i(\mu) \quad for\ all\ i \in \mathbb{P}.$$

Proof. Proposition 26 implies that

$$\Delta(\overline{X}_{(G,\omega)}) = \sum_{S \cup T = V(G)} \overline{X}_{(G|_S, \omega|_S)} \otimes \overline{X}_{(G|_S, \omega|_S)} \tag{3.5}$$

for all weighted graphs (G, ω). Applying this to Example 25 gives

$$\Delta(\overline{\widetilde{m}}_\nu) = \sum_{S \cup T = [\ell(\nu)]} \overline{\widetilde{m}}_{\nu_S} \otimes \overline{\widetilde{m}}_{\nu_T} \tag{3.6}$$

where if $S = \{i_1 < i_2 < \dots\}$ then ν_S means the partition $(\nu_{i_1} \geqslant \nu_{i_2} \geqslant \dots)$. It is a straightforward exercise to check that the number of not necessarily disjoint unions $S \cup T = [\ell(\nu)]$ that give $\nu_S = \lambda$ and $\nu_T = \mu$ is the product of multinomial coefficients $\prod_{i \in \mathbb{P}} \binom{r_i(\nu)}{r_i(\nu \ominus \lambda),\ r_i(\nu \ominus \mu),\ r_i((\lambda \oplus \mu) \ominus \nu)}$. $\qquad\square$

3.4 Set-valued P-partitions

If $S, T \in \mathsf{Set}(\mathbb{P})$ then write $S \prec T$ if $\max(S) < \min(T)$ and $S \preceq T$ if $\max(S) \leqslant \min(T)$. For a labeled poset (D, γ) define

$$\overline{\Gamma}(D, \gamma) = \sum_\kappa x^\kappa \in \mathbb{K}[\![x_1, x_2, \dots]\!]$$

where the sum is over all maps $\kappa : V(D) \to \mathsf{Set}(\mathbb{P})$ that are *set-valued P-partitions* for $P = (D, \gamma)$ in the sense of [15, 16], meaning

   (a) $\kappa(u) \preceq \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) < \gamma(v)$, and

   (b) $\kappa(u) \prec \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) > \gamma(v)$.

Example 28. If $D = (1 \to 2 \to 3 \to \cdots \to n)$ is an n-element chain and S is the set of $i \in [n-1]$ with $\gamma(i) > \gamma(i+1)$ then we define $\overline{L}_{n,S} := \overline{\Gamma}(D, \gamma)$. Following [15], we construct the *multifundamental quasisymmetric function* of a composition α as the power series $\overline{L}_\alpha := \overline{L}_{n,S}$ where $n = |\alpha|$ and $S = I(\alpha)$ for

$$I(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots\} \setminus \{n\}.$$

These elements form another pseudobasis for $\mathfrak{mQSym}$ [15]. We say that a power series in $\mathfrak{mQSym}$ is *multifundamental positive* if it can be expressed as a possibly infinite sum of (not necessarily distinct) $\overline{L}_\alpha$'s.

For a directed acyclic graph D, let $\overline{\Gamma}(D) = \overline{\Gamma}(D, \gamma^{\mathrm{decr}}) = \sum_\kappa x^\kappa$ where the sum is over all maps $\kappa : V(D) \to \mathsf{Set}(\mathbb{P})$ with $\kappa(u) \prec \kappa(v)$ if $u \to v \in E(D)$.

Example 29. If $D = (1 \to 2 \to 3 \to \cdots \to n)$ is an n-element path then define $\overline{e}_n := \overline{\Gamma}(D) = \sum_{k=n}^{\infty} \binom{k-1}{n-1} e_k$. For each partition λ let $\overline{e}_\lambda := \overline{e}_{\lambda_1} \overline{e}_{\lambda_2} \cdots$. These functions are a pseudobasis for $\mathfrak{m}\mathsf{Sym}$.

Recall from Example 6 that $\overline{X}_{K_n} = n! \sum_{r=n}^{\infty} \left\{ {r \atop n} \right\} e_r$

Proposition 30. *If $n \in \mathbb{P}$ then $\overline{X}_{K_n} = n! \sum_{r=n}^{\infty} \left\{ {r-1 \atop n-1} \right\} \overline{e}_r$.*

Proof. Substituting the e-expansion of $\overline{e}_n$ from Example 29 gives

$$\sum_{s=n}^{\infty} \left\{ {s-1 \atop n-1} \right\} \overline{e}_s = \sum_{s=n}^{\infty} \sum_{r=s}^{\infty} \left\{ {s-1 \atop n-1} \right\} \binom{r-1}{s-1} e_r = \sum_{r=n}^{\infty} \left(\sum_{s=n-1}^{r-1} \left\{ {s \atop n-1} \right\} \binom{r-1}{s} \right) e_r.$$

One has $\sum_{s=n-1}^{r-1} \left\{ {s \atop n-1} \right\} \binom{r-1}{s} = \sum_{S \subseteq [r-1]} \left\{ {|S| \atop n-1} \right\} = \left\{ {r \atop n} \right\}$ since to form an n-block set partition of $[r]$, one can first choose any subset of $[r-1]$ to form the complement of the block containing r, and then divide this set into $n-1$ blocks. Thus, we have $n! \sum_{s=n}^{\infty} \left\{ {s-1 \atop n-1} \right\} \overline{e}_s = n! \sum_{r=n}^{\infty} \left\{ {r \atop n} \right\} e_r = \overline{X}_{K_n}.$ $\qquad\square$

A *multilinear extension* of a directed acyclic graph D with n vertices is a sequence $w = (w_1, w_2, \ldots, w_N)$ with $V(D) = \{w_1, w_2, \ldots, w_N\}$ such that $i < j$ whenever $w_i \to w_j \in E(D)$, and $w_i \neq w_{i+1}$ for all $i \in [N-1]$. If $\mathcal{M}(D)$ is the set of all multilinear extensions of D and $\gamma : V(D) \to \mathbb{Z}$ is injective, then

$$\overline{\Gamma}(D, \gamma) = \sum_{w \in \mathcal{M}(D)} \overline{L}_{\ell(w), \mathrm{Des}(w, \gamma)} \tag{3.7}$$

where $\mathrm{Des}(w, \gamma) := \{ i \in [\ell(w) - 1] : \gamma(w_i) > \gamma(w_{i+1}) \}$ for $w \in \mathcal{M}(D)$ [15].

3.5 Acyclic multi-orientations

Let G be a graph. An *acyclic multi-orientation* of G is an acyclic orientation of the α-clan graph $\mathsf{Clan}_\alpha(G)$ from Remark 7 for some $\alpha : V(G) \to \mathbb{P}$, such that for each $v \in V(G)$ and each $i \in [\alpha(v) - 1]$ there are at least two directed paths from the vertex $(v, i+1) \in V(\mathsf{Clan}_\alpha(G))$ to the vertex $(v, i) \in V(\mathsf{Clan}_\alpha(G))$.

Departing from our usual convention, we consider two acyclic multi-orientations to be the same if and only if they have the same vertices and the same directed edges, rather than if they are merely isomorphic. Let $\mathfrak{m}\mathsf{AO}(G)$ be the set of all acyclic multi-orientations of G. We say that $D \in \mathfrak{m}\mathsf{AO}(G)$ has type α if D is an acyclic orientation of $\mathsf{Clan}_\alpha(G)$.

Proposition 31. *Let D be an acyclic orientation of $\mathsf{Clan}_\alpha(G)$. Then $D \in \mathfrak{m}\mathsf{AO}(G)$ if and only if for each $v \in V(G)$ both of the following properties hold:*

(a) If $i, j \in [\alpha(v)]$ have $i > j$ then $(v, i) \to (v, j)$ is a directed edge in D.

(b) If $i \in [\alpha(v) - 1]$ then there exists a directed path in D involving no edges of the form $(v, j) \to (v, k)$ from $(v, i + 1)$ to (v, i).

Proof. If D has property (a) then $(v, i + 1) \to (v, i)$ is a directed path in D, and if D also has property (b) then there is a second directed path from $(v, i + 1)$ to (v, i), so $D \in \mathfrak{mAO}(G)$.

Conversely, assume $D \in \mathfrak{mAO}(G)$. Suppose $i, j \in [\alpha(v)]$ and $i > j$. Then D contains a directed path from (v, i) to (v, j), so $(v, j) \to (v, i)$ cannot be an edge in D as this would create a cycle. As $\{(v, i), (v, j)\}$ is an edge in $\mathsf{Clan}_\alpha(G)$, the edges of D must contain $(v, i) \to (v, j)$ instead.

Since D is acyclic, no directed path in D passing through the vertex (v, j) can ever reach (v, k) for $k \geqslant j$. Thus, a directed path in D from $(v, i + 1)$ to (v, i) can only involve an edge of the form $(v, j) \to (v, k)$ if the path has length one and $j = i + 1$ and $k = i$. As D contains a second directed path from $(v, i + 1)$ to (v, i) in addition to this one, property (b) must hold. $\square$

Example 32. If $|E(G)| = 0$ then $\mathfrak{mAO}(G) = \{G\}$ and otherwise $\mathfrak{mAO}(G)$ is infinite. If $G = K_2$ then $\mathfrak{mAO}(G)$ has two (isomorphic but considered to be distinct) elements with k vertices for each $k \geqslant 2$; see Figure 1.
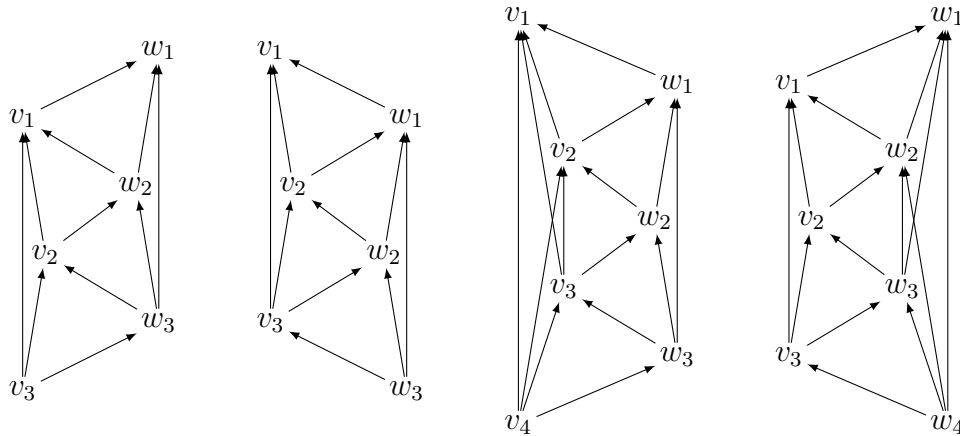

Figure 1: The distinct acyclic multi-orientations with 6 and 7 vertices for the complete graph with vertex set $\{v, w\}$. Here, we have written the vertices of these directed graphs as v_i and w_i instead of (v, i) and (w, i) to save space.

A *source* in a directed graph is a vertex with no incoming edges. One can relate the coefficients in the $\overline{e}$-expansion of $\overline{X}_G$ to the source counts of its acyclic multi-orientations, generalizing a result of Stanley [20, Thm. 3.3].

Remark 33. For this type of statement to make sense, the relevant coefficients must be uniquely defined integers, so we should assume that the integral domain $\mathbb{K}$ contains $\mathbb{Z}$. Since we need to divide by 2 in the next lemma, for the rest of this subsection we make the slightly strongly assumption that $\mathbb{K} \supseteq \mathbb{Q}$.

Define $\varphi : \mathsf{mQSym} \to \mathbb{K}[\![t]\!]$ to be the continuous linear map with

$$\varphi(\overline{L}_{n,S}) = \begin{cases} t(\frac{t-1}{2})^i & \text{if } S = [n-1] \setminus [i] \text{ for some } 0 \leqslant i < n \\ 0 & \text{otherwise} \end{cases}$$

for $n \in \mathbb{P}$ and $S \subseteq [n-1]$. We also set $\varphi(\overline{L}_{0,\varnothing}) = 1$. The next result generalizes the key *Claim* in the proof of [20, Thm. 3.3], and follows by a similar argument.

Lemma 34. *If D is a directed acyclic graph then $\varphi(\overline{\Gamma}(D)) = t^m$ where m is the number of source vertices in D.*

Proof. Choose a decreasing labeling $\gamma^{\mathrm{decr}} : V(D) \to \mathbb{Z}$ for D. Then by (3.7) we have $\overline{\Gamma}(D) = \sum_{w \in \mathcal{M}(D)} \overline{L}_{\ell(w),\mathrm{Des}(w,\gamma^{\mathrm{decr}})}$.

The only way to obtain a multilinear extension $w = (w_1, w_2, \ldots, w_n)$ of D with $\mathrm{Des}(w, \gamma^{\mathrm{decr}}) = [n-1] \setminus [i]$ is as follows. Let $w_{i+1} \in V(D)$ be the vertex maximizing $\gamma^{\mathrm{decr}}(w_{i+1})$; this element must be a source vertex. Then define $w_1, w_2, \ldots, w_i$ to be any i distinct sources vertices other than w_{i+1} listed in increasing order of their labels under γ^{decr}. Finally, choose a subset of $\{w_1, w_2, \ldots, w_i\}$ and define $w_{i+2}, w_{i+3}, \ldots, w_n$ to consist of these vertices plus all elements of $V(D) \setminus \{w_1, w_2, \ldots, w_{i+1}\}$ arranged in decreasing order of their γ^{decr} labels.

If D has m source vertices then there are $\binom{m-1}{i}$ choices for $(w_1, w_2, \ldots, w_i)$ and 2^i choices for the subset of these vertices repeated in $(w_{i+2}, w_{i+3}, \ldots, w_n)$, so using the binomial formula we get $\varphi(\overline{\Gamma}(D)) = t \sum_{i=0}^{m-1} \binom{m-1}{i} 2^i (\frac{t-1}{2})^i = t^m$. $\square$

Proposition 35. *Let G be a graph and suppose $\overline{X}_G = \sum_\lambda c_\lambda \overline{e}_\lambda$ for some coefficients $c_\lambda \in \mathbb{Z}$. Then the number of acyclic multi-orientations of G with exactly j sources and k vertices is $\sum_{\ell(\lambda)=j,|\lambda|=k} c_\lambda \in \mathbb{N}$.*

Proof. Let $\mathsf{source}(G, j, k)$ be the number of acyclic multi-orientations of G with exactly j sources and k vertices. Applying φ to $\overline{X}_G = \sum_{D \in \mathsf{mAO}(G)} \overline{\Gamma}(D)$ using Lemma 34 gives $\varphi(\overline{X}_G) = \sum_{D \in \mathsf{mAO}(G)} \mathsf{source}(G, j, k) t^j$. On the other hand, $\overline{e}_\lambda = \overline{\Gamma}(D)$ where D is the disjoint union of directed chains of sizes $\lambda_1, \lambda_2, \ldots$, so $\varphi(\overline{e}_\lambda) = t^{\ell(\lambda)}$. Therefore we also have $\varphi(\overline{X}_G) = \sum_\lambda c_\lambda t^{\ell(\lambda)}$. The result follows by taking the coefficients of t^j in both formulas. $\square$

As noted in [4], the coefficients c_λ appearing in $\overline{X}_G = \sum_\lambda c_\lambda \overline{e}_\lambda$ can be negative, even when G in the incomparability graph of a $(3+1)$-free poset.

Example 36. Fix $n > 0$. Then every $D \in \mathsf{mAO}(K_n)$ is an acyclic orientation of a nonempty complete graph, so has a single source vertex. Consulting Proposition 30, we see that the number of k-vertex acyclic multi-orientations of the complete graph K_n is zero if $k < n$ and $n! \left\{ {k-1 \atop n-1} \right\}$ if $k \geqslant n$. For $n = 2$ this number is always 2, which matches the description of $\mathsf{mAO}(K_2)$ in Example 32.

3.6 Diagrams and isomorphisms

For each directed acyclic graph D and labeled poset $P = (D, \gamma)$, define

$$\blacktriangle(D) = \sum D|_S \otimes D|_T \quad \text{and} \quad \blacktriangle(P) = \sum (D|_S, \gamma|_S) \otimes (D|_T, \gamma|_T), \qquad (3.8)$$

where both sums are over all (not necessarily disjoint) vertex decompositions $S \cup T = V(D)$ in which S is a lower set, T is an upper set, and $S \cap T$ is an antichain.

Use the continuous linear extensions of the operations (3.8) to replace the coproducts in the completions of DAGs and $\mathsf{LPosets}$, and denote the resulting structures as $\mathfrak{m}\mathsf{DAGs}$ and $\mathfrak{m}\mathsf{LPosets}$. Also let $\overline{\zeta}_{\mathsf{DAGs}}$ and $\overline{\zeta}_{\mathsf{LPosets}}$ be the continuous linear maps to $\mathbb{K}[\![t]\!]$ extending ζ_{DAGs} and ζ_{LPosets} as in (3.1).

Theorem 37. *Each of the pairs*

$$(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}}), \quad (\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}}), \quad \text{and} \quad (\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}})$$

is a combinatorial LC-Hopf algebra, and there is a commutative diagram of morphisms of combinatorial LC-Hopf algebras

$$(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}}) \longrightarrow (\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}}) \longrightarrow (\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}})$$

$$(\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$$

in which the horizontal maps send graphs and DAGs to

$$G \mapsto \sum_{D \in \mathfrak{m}\mathsf{AO}(G)} D \quad \text{and} \quad D \mapsto (D, \gamma^{\mathrm{decr}}),$$

and the $\mathfrak{m}\mathsf{QSym}$-valued maps send $G \mapsto \overline{X}_G$, $D \mapsto \overline{\Gamma}(D)$, and $(D, \gamma) \mapsto \overline{\Gamma}(D, \gamma)$.

Proof. We observed in Proposition 23 that $(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}})$ is a combinatorial LC-Hopf algebra and that the bottom left map sends $G \mapsto \overline{X}_G$.

For a proof that $(\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}})$ is a combinatorial LC-Hopf algebra, see [16, §3.1]. In [16], the scalar ring for $\mathfrak{m}\mathsf{LPosets}$ is $\mathbb{Z}[\beta]$ rather than a generic integral domain $\mathbb{K}$, and the definition of a labeled poset (P, γ) assumes that P is (the Hasse diagram of) a finite poset rather than any finite DAG, but all arguments carry over unchanged to our slightly more general setting. With the same caveats, the claim that the given map $\mathfrak{m}\mathsf{LPosets} \to \mathfrak{m}\mathsf{QSym}$ is the unique morphism $(\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}}) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ is [16, Thm. 3.8].

The pair $(\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}})$ is a combinatorial LC-Hopf algebra as its structure maps are the unique ones compatible with the inclusion $\mathfrak{m}\mathsf{DAGs} \hookrightarrow \mathfrak{m}\mathsf{LPosets}$ sending $D \mapsto (D, \gamma^{\mathrm{decr}})$. The two right-most arrows in the diagram are morphisms of combinatorial LC-Hopf algebras, and their composition gives the vertical arrow by definition.

It remains to show that linear map acting on graphs as $G \mapsto \sum_{D \in \mathfrak{m}\mathsf{AO}(G)} D$ is a morphism $(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}}) \to (\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}})$. Denote this map by ψ. If a graph G

has at least one edge then so does every $D \in \mathfrak{mAO}(G)$ and if G has no edges then $\mathfrak{mAO}(G) = \{G\}$. In either case we have $\overline{\zeta}_{\mathsf{DAGs}} \circ \psi = \overline{\zeta}_{\mathsf{Graphs}}$. Likewise, ψ commutes with the (co)unit maps of $\mathfrak{mGraphs}$ and $\mathfrak{mDAGs}$ since $G = \emptyset$ is empty if and only if $\mathfrak{mAO}(G) = \{G\} = \{\emptyset\}$.

Since $\mathfrak{mAO}(G \sqcup H) = \{D_1 \sqcup D_2 : (D_1, D_2) \in \mathfrak{mAO}(G) \times \mathfrak{mAO}(H)\}$ for two graphs G and H, we also have $\psi \circ \nabla_{\mathsf{mGraphs}} = \nabla_{\mathsf{mDAGs}} \circ (\psi \overline{\otimes} \psi)$. To check that $(\psi \overline{\otimes} \psi) \circ \Delta_{\mathsf{mGraphs}} = \Delta_{\mathsf{mDAGs}} \circ \psi$, we must show for any graph G that

$$(\psi \overline{\otimes} \psi) \circ \Delta_{\mathsf{mGraphs}}(G) = \sum_{S \cup T = V(G)} \sum_{D_S \in \mathfrak{mAO}(G|_S)} \sum_{D_T \in \mathfrak{mAO}(G|_T)} D_S \otimes D_T \qquad (3.9)$$

is equal to

$$\Delta_{\mathsf{mDAGs}} \circ \psi(G) = \sum_{\tilde{D} \in \mathfrak{mAO}(G)} \sum_{\tilde{S} \cup \tilde{T} = V(\tilde{D})} \tilde{D}|_{\tilde{S}} \otimes \tilde{D}|_{\tilde{T}} \qquad (3.10)$$

where in the last sum $\tilde{S}$ must be a lower set of $\tilde{D}$, $\tilde{T}$ must be an upper set, and $\tilde{S} \cap \tilde{T}$ must be an antichain. For this, it suffices to construct a bijection from the set of tuples (S, T, D_S, D_T) indexing the summands in the first expression to the set of triples $(\tilde{D}, \tilde{S}, \tilde{T})$ indexing the summands in the second expression such that $D_S \cong \tilde{D}|_{\tilde{S}}$ and $D_T \cong \tilde{D}|_{\tilde{T}}$. Such a bijection is constructed as follows.

Fix a graph G. Suppose S and T are (not necessarily disjoint) sets such that $S \cup T = V(G)$. Choose $D_S \in \mathfrak{mAO}(G|_S)$ and $D_T \in \mathfrak{mAO}(G|_T)$.

Suppose D_T has type $\alpha : T \to \mathbb{P}$. Define $\alpha(v) = 0$ for $v \in S \setminus T$. For each $v \in S$, let $d_v \in \mathbb{N}$ be either $\alpha(v) - 1$ if $v \in T$ and $(v, 1)$ is a sink vertex in D_S and $(v, \alpha(v))$ is a source vertex in D_T, or else $\alpha(v)$. Then set

$$\tilde{S} = \{(v, i + d_v) : (v, i) \in V(D_S)\} \quad \text{and} \quad \tilde{T} = V(D_T).$$

Notice that $\tilde{S} \cap \tilde{T}$ is the set of pairs $(v, \alpha(v))$ for $v \in S \cap T$ with $d_v = \alpha(v) - 1$, so in $\tilde{S} \cup \tilde{T}$ the sink vertices in D_S of the form $(v, 1)$ are merged with the source vertices in D_T of the form $(v, \alpha(v))$.

Finally define $\tilde{D}$ to be the directed graph with vertex set $\tilde{S} \cup \tilde{T}$ and edges

$$
\begin{aligned}
(u, i + d_u) \to (v, j + d_v) \qquad & \text{if } (u, i) \to (v, j) \in E(D_S), \\
(u, i) \to (v, j) \qquad & \text{if } (u, i) \to (v, j) \in E(D_T), \\
(u, i) \to (v, j) \qquad & \text{if } (u, i) \in \tilde{S} \setminus \tilde{T}, (v, j) \in \tilde{T}, \\
& \text{and } u = v \text{ or } \{u, v\} \in E(G).
\end{aligned}
\qquad (3.11)
$$

Example 38. Let G be the complete graph on two elements v and w. Suppose $S = T = V(G) = \{v, w\}$. Let $D_S \in \mathfrak{mAO}(G|_S)$ be the leftmost directed graph in Figure 1 and let $D_T \in \mathfrak{mAO}(G|_T)$ be the rightmost directed graph in the same figure. Writing v_i and w_i instead of (v, i) and (w, i) as in Figure 1, we have

$$\tilde{S} = \{v_5, v_4, v_3, w_6, w_5, w_4\} \quad \text{and} \quad \tilde{T} = \{v_3, v_2, v_1, w_4, w_3, w_3, w_1\}$$

so $\tilde{S} \cap \tilde{T} = \{v_3, w_4\}$. The directed graph $\tilde{D}$ on $\tilde{S} \cup \tilde{T}$ is shown in Figure 2.

Figure 2: The directed graph $\tilde{D}$ corresponding to Example 38, with some edges omitted and with vertices (v, i) and (w, j) written as v_i and w_j. The boxed vertices are the elements of $\tilde{S} \setminus \tilde{T}$ while the unboxed vertices are the elements of $\tilde{T}$, and $\tilde{S} \cap \tilde{T} = \{v_3, w_4\}$. This picture does not show all of $\tilde{D}$, which has additional directed edges from each boxed vertex to each unboxed vertex.

One can check as follows that $\tilde{S}$ is a lower set in $\tilde{D}$ while $\tilde{T}$ is an upper set. First, notice that an edge of the first type in (3.11) can only go from $\tilde{T}$ to $\tilde{S}$ in the event that $(u, i) \to (v, j)$ is an edge in D_S and $(u, j + d_u) \in \tilde{S} \cap \tilde{T}$. However, this never occurs since we can only have $(u, j + d_u) \in \tilde{S} \cap \tilde{T}$ when $j = 1$ and $(u, 1)$ is a sink vertex in D_S.

Similarly, an edge of the second type in (3.11) can only go from $\tilde{T}$ to $\tilde{S}$ in the event that $(u, i) \to (v, j)$ is an edge in D_T and $(v, j) \in \tilde{S} \cap \tilde{T}$. However, this never occurs since we can only have $(v, j) \in \tilde{S} \cap \tilde{T}$ when $j = \alpha(v)$ and $(v, \alpha(v))$ is a source vertex in D_T.

As edges of the third type in (3.11) all start from $\tilde{S} \setminus \tilde{T}$, we conclude that $\tilde{S}$ is a lower set in $\tilde{D}$ and $\tilde{T}$ is an upper set.

To finish our proof of Theorem 37, it suffices to derive the following lemma.

Lemma 39. *The map* $(S, T, D_S, D_T) \mapsto (\tilde{D}, \tilde{S}, \tilde{T})$ *is a bijection from the set of tuples*

$$(S, T, D_S, D_T),$$

in which S and T are sets with $S \cup T = V(G)$ and $D_S \in \mathfrak{mAO}(G|_S)$ and $D_T \in \mathfrak{mAO}(G|_T)$, to the set of triples

$$(\tilde{D}, \tilde{S}, \tilde{T}),$$

where $\tilde{D} \in \mathfrak{mAO}(G)$ and where $\tilde{S}$ and $\tilde{T}$ are respectively a lower set and an upper set in $\tilde{D}$ such that $\tilde{S} \cup \tilde{T} = V(\tilde{D})$ and $\tilde{S} \cap \tilde{T}$ is an antichain. This bijection satisfies $D_S \cong \tilde{D}|_{\tilde{S}}$ and $D_T \cong \tilde{D}|_{\tilde{T}}$.

Proof. We first check that $\tilde{D} \in \mathfrak{mAO}(G)$. The directed graph $\tilde{D}$ is acyclic since D_S and D_T are acyclic. Fix $v \in V(G)$ and $i \in \mathbb{P}$ such that $(v, i+1)$ and (v, i) are both vertices

in $\tilde{D}$. If these vertices are both in $\tilde{S}$ (respectively, $\tilde{T}$), then any two directed paths in D_S (respectively, D_T) from $(v, i+1)$ and (v, i) lift to paths in $\tilde{D}$.

Suppose $(v, i+1) \in \tilde{S}$ and $(v, i) \in \tilde{T}$. Then $i = \alpha(v)$ and either $(v, i+1)$ is not a sink in $\tilde{D}|_{\tilde{S}} \cong D_S$ or (v, i) is not a source in $\tilde{D}|_{\tilde{T}} = D_T$. In the first case $\tilde{D}$ must have an edge of the form $(v, i+1) \to (w, j) \in \tilde{T}$ where $v \neq w$, and in the second $\tilde{D}$ must have an edge of the form $(v, i) \leftarrow (w, j) \in \tilde{S}$ where $v \neq w$. Either way, we have $\{v, w\} \in E(G)$, so $(v, i+1) \to (w, j) \to (v, i)$ is a second directed path in $\tilde{D}$ besides $(v, i+1) \to (v, i)$. We conclude that $\tilde{D} \in \mathfrak{mAO}(G)$.

We have already seen that $\tilde{S}$ is a lower set in $\tilde{D}$ and $\tilde{T}$ is an upper set with $\tilde{S} \cup \tilde{T} = V(\tilde{D})$. No single edge of $\tilde{D}$ may connect two vertices in $\tilde{S} \cap \tilde{T}$, since then the vertices could not both be sources in D_T. This implies that no directed path in $\tilde{D}$ may connect two vertices in $\tilde{S} \cap \tilde{T}$, since every vertex in this path would have to be in $\tilde{S} \cap \tilde{T}$ since $\tilde{S}$ is a lower set in $\tilde{D}$ and $\tilde{T}$ is an upper set. Thus $\tilde{S} \cap \tilde{T}$ is an antichain in $\tilde{D}$.

We conclude that every image of the given operation is in the described codomain. This operation is a bijection as it is straightforward to construct its inverse. Namely, given $(\tilde{D}, \tilde{S}, \tilde{T})$ we first recover D_T and α as

$$D_T = \tilde{D}|_{\tilde{T}} \quad \text{and} \quad \alpha : v \mapsto \max\{i : (v, i) \in \tilde{T}\}.$$

Then we recover D_S from the (isomorphic) induced subgraph $\tilde{D}|_{\tilde{S}}$ by replacing each vertex (v, i) by $(v, i+1-\alpha(v))$, and finally we obtain S and T as

$$S = \{v : (v, i) \in V(D_S)\} \quad \text{and} \quad T = \{v : (v, i) \in V(D_T)\}.$$

We have $S \cup T = \{v : (v, i) \in V(\tilde{D})\} = V(G)$. Since D_S and D_T are formed by restricting $\tilde{D} \in \mathfrak{mAO}(G)$ to a lower set or an upper set, it holds that $D_S \in \mathfrak{mAO}(G|_S)$ and $D_T \in \mathfrak{mAO}(G|_T)$ as needed. $\qquad\square$

The proof of Lemma 39 completes the proof of Theorem 37. $\qquad\square$

Outside the special case when $\omega : V(G) \to \{1\}$, the kromatic symmetric function $\overline{X}_{(G,\omega)}$ of a weighted graph is not always multifundamental positive, nor is its expansion into $\overline{L}_\alpha$'s finite. However, since the diagram in Theorem 37 commutes, (3.7) implies that:

Corollary 40. *If G is a graph then $\overline{X}_G = \sum_{D \in \mathfrak{mAO}(G)} \overline{\Gamma}(D)$ and consequently this symmetric power series is multifundamental positive.*

Theorem 37 also leads to a coproduct formula for $\overline{e}_n$ from Example 29.

Corollary 41. *If n is a positive integer then*

$$\Delta(\overline{e}_n) = \sum_{\substack{i,j \in \mathbb{N} \\ i+j=n}} \overline{e}_i \otimes \overline{e}_j + \sum_{\substack{i,j \in \mathbb{P} \\ i+j=n+1}} \overline{e}_i \otimes \overline{e}_j.$$

Proof. Let $D = (1 \to 2 \to 3 \to \cdots \to n)$ be an n-element path. By Theorem 37 we have $\Delta(\bar{e}_n) = \Delta(\overline{\Gamma}(D)) = \sum_{S \cup T = V(D)} \overline{\Gamma}(D|_S) \otimes \overline{\Gamma}(D|_T)$ where the sum is over all lower sets S and upper sets T with $S \cap T$ an antichain in D. To be an antichain, the intersection $S \cap T$ must either be empty or the singleton set $\{i\}$ for some $i \in [n]$.

Either way, the choices for S and T are limited. If $S \cap T$ is empty then we must have $S = [i]$ and $T = [n] \setminus [i]$ for some $0 \leqslant i \leqslant n$, and then $\overline{\Gamma}(D|_S) \otimes \overline{\Gamma}(D|_T) = \bar{e}_i \otimes \bar{e}_{n-i}$. If $S \cap T = \{i\}$ then we must have $S = [i]$ and $T = [n] \setminus [i-1]$, and then $\overline{\Gamma}(D|_S) \otimes \overline{\Gamma}(D|_T) = \bar{e}_i \otimes \bar{e}_{n+1-i}$. Adding up these terms gives the result. $\qquad\square$

Fix a directed acyclic graph D. When $\alpha : V(D) \to \mathbb{P}$ is any map, define $\mathsf{Clan}_\alpha^{\mathsf{dag}}(D)$ to be the directed acyclic graph with vertices $\mathsf{Clan}_\alpha(V(D))$ and edges

$$(v,i) \to (w,j) \quad \text{whenever} \quad v \to w \in E(D) \text{ or both } v = w \text{ and } i > j.$$

When $\gamma : V(D) \to \mathbb{Z}$ is injective, let $\tilde{\gamma}$ be any labeling of $\mathsf{Clan}_\alpha^{\mathsf{dag}}(D)$ with

$$\tilde{\gamma}(v,i) < \tilde{\gamma}(w,j) \quad \text{if and only if} \quad \gamma(v) < \gamma(w) \text{ or both } v = w \text{ and } i < j,$$

and then define $\mathsf{Clan}_\alpha^{\mathsf{dag}}(D, \gamma) = (\mathsf{Clan}_\alpha^{\mathsf{dag}}(D), \tilde{\gamma})$.

Recall that for each $\mathcal{H} \in \{\mathsf{Graphs}, \mathsf{DAGs}, \mathsf{LPosets}\}$ we use the symbols $\mathfrak{m}\mathcal{H}$ and $\overline{\mathcal{H}}$ to denote the same sets with different LC-Hopf algebra structures. Let

$$\Psi_{\mathsf{Graphs}} : \mathfrak{m}\mathsf{Graphs} \to \overline{\mathsf{Graphs}},$$
$$\Psi_{\mathsf{DAGs}} : \mathfrak{m}\mathsf{DAGs} \to \overline{\mathsf{DAGs}}, \text{ and}$$
$$\Psi_{\mathsf{LPosets}} : \mathfrak{m}\mathsf{LPosets} \to \overline{\mathsf{LPosets}}$$

be the continuous linear maps sending

$$G \mapsto \sum_\alpha \tfrac{1}{\alpha!} \mathsf{Clan}_\alpha(G), \quad D \mapsto \sum_\alpha \mathsf{Clan}_\alpha^{\mathsf{dag}}(D), \quad (D, \gamma) \mapsto \sum_\alpha \mathsf{Clan}_\alpha^{\mathsf{dag}}(D, \gamma) \qquad (3.12)$$

for each graph G, directed acyclic graph D, and labeled poset (D, γ), where the sums are over all maps $\alpha : V \to \mathbb{P}$ for $V = V(G)$ or $V(D)$ as appropriate. The map Ψ_{Graphs} is only defined if $\mathbb{Q} \subseteq \mathbb{K}$, which we assume in the following theorem.

Theorem 42. *When defined, the maps Ψ_{Graphs}, Ψ_{DAGs}, and Ψ_{LPosets} are isomorphisms of LC-Hopf algebras, and there is a commutative diagram*

$$
\begin{array}{ccccc}
(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}}) & \longrightarrow & (\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}}) & \longrightarrow & (\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}}) \\
\downarrow {\scriptstyle \Psi_{\mathsf{Graphs}}} & & \downarrow {\scriptstyle \Psi_{\mathsf{DAGs}}} & & \downarrow {\scriptstyle \Psi_{\mathsf{LPosets}}} \\
(\overline{\mathsf{Graphs}}, \overline{\zeta}_{\mathsf{Graphs}}) & \longrightarrow & (\overline{\mathsf{DAGs}}, \overline{\zeta}_{\mathsf{DAGs}}) & \longrightarrow & (\overline{\mathsf{LPosets}}, \overline{\zeta}_{\mathsf{LPosets}})
\end{array}
$$

of combinatorial LC-Hopf algebra morphisms, in which the horizontal arrows extend the maps in the diagrams in Proposition 15 and Theorem 37.

Proof. The vertical maps are invertible since each has a unitriangular matrix in the standard pseudobases of the source and target ordered by number of vertices. It is straightforward to see that $\overline{\zeta}_{\mathsf{Graphs}} = \Psi_{\mathsf{Graphs}} \circ \overline{\zeta}_{\mathsf{Graphs}}$ and $\overline{\zeta}_{\mathsf{DAGs}} = \Psi_{\mathsf{DAGs}} \circ \overline{\zeta}_{\mathsf{DAGs}}$ and $\overline{\zeta}_{\mathsf{LPosets}} = \Psi_{\mathsf{LPosets}} \circ \overline{\zeta}_{\mathsf{LPosets}}$, and that each of the vertical maps is compatible with the relevant product, unit, and counit. It is also easy to see that the right square in the given diagram commutes.

The nontrivial things to check are that Ψ_{Graphs} and Ψ_{LPosets} are coalgebra morphisms, and that the left square in the given diagram commutes. This amounts to verifying three identities:

(a) If G is a graph then the sum

$$\sum_{S \cup T = V(G)} \sum_{\substack{\alpha':S \to \mathbb{P} \\ \alpha'':T \to \mathbb{P}}} \tfrac{1}{\alpha'!\alpha''!} \mathsf{Clan}_{\alpha'}(G|_S) \otimes \mathsf{Clan}_{\alpha'}(G|_T)$$

is equal to

$$\sum_{\alpha:V(G) \to \mathbb{P}} \sum_{S \sqcup T = V(\mathsf{Clan}_\alpha(G))} \tfrac{1}{\alpha!} \mathsf{Clan}_\alpha(G)|_S \otimes \mathsf{Clan}_\alpha(G)|_T.$$

(b) If (D, γ) is a labeled poset then the sum

$$\sum_{\substack{S \cup T = V(D) \\ S \text{ lower set} \\ T \text{ upper set} \\ S \cap T \text{ antichain}}} \sum_{\substack{\alpha':S \to \mathbb{P} \\ \alpha'':T \to \mathbb{P}}} \mathsf{Clan}^{\mathsf{dag}}_{\alpha'}(D|_S, \gamma|_S) \otimes \mathsf{Clan}^{\mathsf{dag}}_{\alpha''}(D|_T, \gamma|_T)$$

is equal to

$$\sum_{\alpha:V(D) \to \mathbb{P}} \sum_{\substack{S \sqcup T = V(\mathsf{Clan}^{\mathsf{dag}}_\alpha(D)) \\ S \text{ lower set} \\ T \text{ upper set}}} (\mathsf{Clan}^{\mathsf{dag}}_\alpha(D)|_S, \tilde{\gamma}|_S) \otimes (\mathsf{Clan}^{\mathsf{dag}}_\alpha(D)|_T, \tilde{\gamma}|_T).$$

(c) If G is a graph then

$$\sum_{D \in \mathfrak{m}\mathsf{AO}(G)} \sum_{\alpha:V(D) \to \mathbb{P}} \mathsf{Clan}^{\mathsf{dag}}_\alpha(D) = \sum_{\alpha:V(G) \to \mathbb{P}} \sum_{D \in \mathfrak{m}\mathsf{AO}(\mathsf{Clan}_\alpha(G))} \tfrac{1}{\alpha!} \mathsf{Clan}_\alpha(D).$$

Each of these identities follows as a straightforward, if slightly cumbersome, exercises in algebra. We omit the details. $\square$

The preceding result reduces the natural question of finding antipode formulas for $\mathfrak{m}\mathsf{Graphs}$, $\mathfrak{m}\mathsf{DAGs}$, and $\mathfrak{m}\mathsf{LPosets}$ (see, e.g., [16, Problem 3.5]) to the more familiar graded Hopf algebras Graphs, DAGs, and $\mathsf{LPosets}$, where some formulas are already known [13].

4 Kromatic quasisymmetric functions

In this section we study two quasisymmetric refinements of $\overline{X}_G$. Our goal in these constructions to find interesting common generalizations of [4] and [18].

4.1 Hopf algebras for chromatic quasisymmetric functions

Let G be an *ordered graph*, that is, a graph with a total order $<$ on its vertex set $V(G)$. One can think of the ordering on $V(G)$ as defining an acyclic orientation on the edges of G, and we do not distinguish between G and another ordered graph H if the corresponding directed acyclic graphs are isomorphic.

Fix an invertible element $q \in \mathbb{K}$. If $\mathsf{OGraphs}_n$ is the free $\mathbb{K}$-module spanned by all (isomorphism classes of) ordered graphs with n vertices, then $\mathsf{OGraphs} := \bigoplus_{n \in \mathbb{N}} \mathsf{OGraphs}_n$ has a graded connected Hopf algebra structure in which the product is disjoint union and the coproduct is the $\mathbb{K}$-linear map Δ_q satisfying

$$\Delta_q(G) = \sum_{S \sqcup T = V(G)} q^{\mathrm{asc}_G(S,T)} G|_S \otimes G|_T \in \mathsf{OGraphs} \,\overline{\otimes}\, \mathsf{OGraphs} \tag{4.1}$$

for each ordered graph G, where

$$\mathrm{asc}_G(S,T) := |\{(s,t) \in S \times T : \{s,t\} \in E(G) \text{ and } s < t\}|.$$

This structure becomes a combinatorial Hopf algebra when paired with the algebra morphism $\zeta_{\mathsf{OGraphs}} : \mathsf{OGraphs} \to \mathbb{K}$ sending $G \mapsto 0^{|E(G)|}$. It follows from [1] that the unique morphism $(\mathsf{OGraphs}, \zeta_{\mathsf{OGraphs}}) \to (\mathsf{QSym}, \zeta_{\mathsf{Q}})$ sends an ordered graph G to the chromatic quasisymmetric function $X_G(q)$ of Shareshian and Wachs from Definition 3.

We have not been able to find a natural K-theoretic generalization of the coproduct Δ_q that leads to a q-analogue of the LC-Hopf algebra $\mathfrak{m}\mathsf{Graphs}$. The simplest idea would be to replace the disjoint union $\sqcup$ in (4.1) by arbitrary union $\cup$, but this does not lead to a co-associative map on the completion of $\mathsf{OGraphs}$. This problem remains if we change the q-power exponent $\mathrm{asc}_G(S,T)$ to other forms that are equivalent in the homogeneous case, like $\mathrm{asc}_G(S-T,T)$, $\mathrm{asc}_G(S,T-S)$, or $\mathrm{asc}_G(S-T,T-S)$.

4.2 A quasisymmetric kromatic function

Nevertheless, there do appear to be interesting ways of defining a simultaneous K-theoretic generalization of $X_G(q)$ and q-analogue of $\overline{X}_G$. We examine two such constructions, the first of which is given as follows.

Remark 43. For the rest of this section we assume $\mathbb{K} \supseteq \mathbb{Z}$ and let q be a formal parameter. We will consider the polynomial and power series rings

$$\mathsf{Sym}[q] \subset \mathfrak{m}\mathsf{QSym}[q] \subset \mathfrak{m}\mathsf{QSym}[\![q]\!].$$

Recall for $\kappa : V(G) \to \mathbb{P}$ that $\mathrm{asc}_G(\kappa)$ and $\mathrm{des}_G(\kappa)$ respectively count the edges $\{u,v\} \in E(G)$ with $u < v$ and $\kappa(u) < \kappa(v)$ or $\kappa(u) > \kappa(v)$.

Definition 44. For an ordered graph G define

$$\overline{L}_G(q) = \sum_\kappa q^{\mathrm{asc}_G(\max \circ \kappa)} x^\kappa \in \mathfrak{m}\mathsf{QSym}[q]$$

where the sum is over all proper set-valued colorings $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$.

One recovers $X_G(q)$ from $\overline{L}_G(q)$ as its homogeneous component of lowest degree in the x_i variables, setting $\deg x_i = 1$ and $\deg q = 0$. Despite the discussion in Section 4.1, it is clear that if $G = G_1 \sqcup G_2$ is the disjoint union of two ordered graphs then $\overline{L}_G(q) = \overline{L}_{G_1}(q)\overline{L}_{G_2}(q)$. In addition to the following example, see Table 1 for some computations of $\overline{L}_G(q)$.

Example 45. Suppose $G = K_n$ is the complete graph on the vertex set $[n]$. There are $\left\{{r \atop n}\right\}$ ways of partitioning r colors into n nonempty subsets. For each way of assigning these subsets to the vertices of G to form a proper set-valued coloring, there is a unique permutation $\sigma \in S_n$ whose letters appear in the same relative order as $\max(\kappa(n)), \max(\kappa(n-1))$, ..., $\max(\kappa(1))$. The inversion set of σ coincides with $\mathrm{asc}(\max \circ \kappa)$, and it is well-known (see [18, (A.2)]) that $\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} = [n]_q!$ as defined Example 4. We conclude that

$$\overline{L}_{K_n}(q) = [n]_q! \sum_{r=n}^\infty \left\{{r \atop n}\right\} e_r = \frac{[n]_q!}{n!} \overline{X}_{K_n} = [n]_q! \sum_{r=n}^\infty \left\{{r-1 \atop n-1}\right\} \overline{e}_r$$

using Proposition 30 for the last equality.

Let us clarify the seeming asymmetry in Definition 44. Form $\overline{L}_G^{\mathrm{des,min}}(q)$ by replacing "asc" by "des" and "max" by "min" in Definition 44. Construct

$$\overline{L}_G^{\mathrm{asc,min}}(q), \quad \overline{L}_G^{\mathrm{asc,max}}(q), \quad \text{and} \quad \overline{L}_G^{\mathrm{des,max}}(q)$$

likewise, so that we have $\overline{L}_G(q) = \overline{L}_G^{\mathrm{asc,max}}(q)$. Now let ρ be the continuous linear map $\mathfrak{m}\mathsf{QSym} \to \mathfrak{m}\mathsf{QSym}$ sending

$$\rho : M_{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \mapsto M_{(\alpha_k,\ldots,\alpha_2,\alpha_1)}. \tag{4.2}$$

This map extends to a ring involution of $\mathfrak{m}\mathsf{QSym}[q]$ fixing q. Finally let τ be the $\mathfrak{m}\mathsf{QSym}$-linear map $\mathfrak{m}\mathsf{QSym}[q] \to \mathfrak{m}\mathsf{QSym}[q]$ acting on nonzero elements as

$$\tau : f(q) \mapsto q^{\deg_q(f)} f(q^{-1}). \tag{4.3}$$

In other words, τ reverses the order of the coefficients of $f \neq 0$, viewed as a polynomial in q with coefficients in $\mathfrak{m}\mathsf{QSym}$.

Proposition 46. *If G is an ordered graph then*

$$\overline{L}_G(q) = \rho\left(\overline{L}_G^{\mathrm{des,min}}(q)\right) = \tau\left(\overline{L}_G^{\mathrm{des,max}}(q)\right) = \rho \circ \tau\left(\overline{L}_G^{\mathrm{asc,min}}(q)\right).$$

Proof. Fix an ordered graph G. Given any map $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$, let

$$S := \bigcup_{v \in V(G)} \kappa(v) = \{i_0 < i_1 < \cdots < i_m\}$$

and let $\sigma : S \to S$ be the permutation sending $i_j \mapsto i_{m-j}$ and then define $\kappa^* : V(G) \to \mathsf{Set}(\mathbb{P})$ by $\kappa^*(v) = \{\sigma(i) : i \in \kappa(v)\}$. The operation $\kappa \mapsto \kappa^*$ is an involution of the set of proper set-valued colorings of G with

$$\mathrm{asc}_G(\max \circ \kappa) = \mathrm{des}_G(\min \circ \kappa^*),$$

and if $x^\kappa = x_{i_0}^{a_0} x_{i_1}^{a_1} \cdots x_{i_m}^{a_m}$ then $x^{\kappa^*} = \rho(x^\kappa) := x_{i_0}^{a_m} x_{i_1}^{a_{m-1}} \cdots x_{i_m}^{a_0}$. Thus

$$\overline{L}_G(q) = \sum_\kappa q^{\mathrm{asc}_G(\max \circ \kappa)} x^\kappa = \sum_\kappa q^{\mathrm{des}_G(\min \circ \kappa^*)} \rho(x^{\kappa^*}) = \rho\left(\overline{L}_G^{\mathrm{des,min}}(q)\right).$$

Since $\max(\kappa(u)) \neq \max(\kappa(v))$ if κ is proper and $\{u, v\} \in E(G)$, it is clear that $\overline{L}_G(q) = q^{|E(G)|} \cdot \overline{L}_G^{\mathrm{des,max}}(q^{-1})$. The latter expression is the same as what we get by applying τ since $|E(G)|$ is the degree of $\overline{L}_G(q)$ as a polynomial in q: this number is an upper bounded for the degree by definition, and if we write $V(G) = \{v_1 < v_2 < \cdots < v_n\}$ then $\mathrm{asc}_G(\kappa) = |E(G)|$ for the proper set-valued coloring with $\kappa(v_i) = \{i\}$.

It follows by a similar argument that $\overline{L}_G^{\mathrm{des,max}}(q) = \tau\left(\overline{L}_G^{\mathrm{asc,min}}(q)\right)$, which gives the third desired equality. $\qquad\square$

Fix $D \in \mathfrak{mAO}(G)$. Each vertex in D has the form (v, i) for some $v \in V(G)$ and $i \in \mathbb{P}$. Define $\mathsf{align}(D) := |\{(u, i) \to (v, j) \in E(D) : u < v \text{ and } i = j = 1\}|$.

Proposition 47. *If G is an ordered graph then*

$$\overline{L}_G(q) = \sum_{D \in \mathfrak{mAO}(G)} q^{\mathsf{align}(D)} \overline{\Gamma}(D).$$

This power series is multifundamental positive in the sense of being a possibly infinite $\mathbb{N}[q]$-linear combination of multifundamental quasisymmetric functions.

Proof. Fix $D \in \mathfrak{mAO}(G)$ and consider the maps $\kappa : V(D) \to \mathsf{Set}(\mathbb{P})$ that have $\kappa(u) \prec \kappa(v)$ whenever $u \to v$ is an edge in D, so that $\overline{\Gamma}(D) = \sum_\kappa x^\kappa$. If $v \in V(G)$ and $i > j$ are positive integers with $(v, i), (v, j) \in V(D)$ then $\kappa((v, i)) \prec \kappa((v, j))$ by Proposition 31. Define a map $\eta : V(G) \to \mathsf{Set}(\mathbb{P})$ by setting $\eta(v) = \bigsqcup_i \kappa((v, i))$ where the union is over all $i \in \mathbb{P}$ with $(v, i) \in D$.

This is a proper set-valued coloring of G since if $\{u, v\} \in E(G)$ then every vertex of the form $(u, i) \in V(D)$ is connected by an edge to every vertex of the form $(v, j) \in V(D)$. Observe that $q^{\mathrm{asc}_G(\max \circ \eta)} = q^{\mathsf{align}(D)}$ and $x^\eta = x^\kappa$.

We claim that every proper set-valued coloring of G arises as η for some $D \in \mathfrak{mAO}(G)$ and $\kappa : V(D) \to \mathsf{Set}(\mathbb{P})$ as above. For such a coloring η, let

$$\eta(v) = \eta(v)_1 \sqcup \eta(v)_2 \sqcup \cdots \sqcup \eta(v)_k$$

be the smallest disjoint decomposition into nonempty subsets such that for each $i \in [k-1]$ an edge $\{v, w\} \in E(G)$ exists with a color $c \in \eta(w)$ that has $\eta(v)_i \succ \{c\} \succ \eta(v)_{i+1}$; then let $\alpha(v) = k$. Form an acyclic multi-orientation $D \in \mathsf{mAO}(G)$ of type α that has the edge $(v, i) \to (w, j)$ if and only if $v = w$ and $i > j$, or $\{v, w\} \in E(G)$ and $\eta(v)_i \prec \eta(w)_j$. Then the coloring η arises from the map $\kappa : V(D) \to \mathsf{Set}(\mathbb{P})$ given by $\kappa((v, i)) = \eta(v)_i$.

As we already know that $\overline{X}_G = \sum_{D \in \mathsf{mAO}(G)} \overline{\Gamma}(D)$ from Corollary 40, we conclude that the operator $(D, \kappa) \mapsto \eta$ is a bijection to proper set-valued colorings of G. Thus $\overline{L}_G(q) = \sum_{\eta} q^{\mathsf{asc}_G(\eta)} x^{\kappa} = \sum_{D \in \mathsf{mAO}(G)} q^{\mathsf{align}(D)} \overline{\Gamma}(D)$. $\qquad \square$

We can make this expansion more explicit, generalizing a result in [18]. Suppose D is a directed acyclic graph. Write $<_D$ for the transitive closure of the relation on $V(D)$ with $x <_D y$ if $x \to y \in E(D)$.

Lemma 48. *Suppose $G = \mathsf{inc}(P)$ for a finite poset P and $D \in \mathsf{AO}(G)$. Then there exists a decreasing labeling γ of D such that $\gamma(x) < \gamma(y)$ for all vertices x and y that are incomparable under $<_D$ but have $x <_P y$.*

Proof. If we change the condition "$x <_P y$" to "$x >_P y$" then the lemma becomes the *Claim* established in the proof of [18, Thm. 3.1]. This alternate version of the lemma is equivalent to the one stated since we can replace P by its dual P^* without changing G, and $x >_{P^*} y$ holds if and only if $x <_P y$.

More directly, following the argument in the proof of [18, Thm. 3.1], the way to construct γ is as follows. All vertices that are maximal under $<_D$ are comparable in P, and we let x_1 be the P-smallest such vertex. Likewise, assuming $x_1, x_2, \ldots, x_k$ have been defined, all vertices in $V(D) \setminus \{x_1, x_2, \ldots, x_k\}$ that are maximal under $<_D$ are comparable in P, and we let x_{k+1} be the P-smallest such vertex. After letting k range from 1 to $|D| - 1$, we set $\gamma(x_i) = i$.

It is clear that γ is a decreasing labeling of D. Checking that $\gamma(x) < \gamma(y)$ when x and y are incomparable under $<_D$ with $x <_P y$ requires justification. For this, one can adapt the argument in the proof of [18, Thm. 3.1]. $\qquad \square$

Following [15], a *multipermutation* of $n \in \mathbb{N}$ is a word $w = w_1 w_2 \cdots w_m$ with

$$\{w_1, w_2, \ldots, w_m\} = \{1, 2, \ldots, n\} \quad \text{and} \quad w_i \neq w_{i+1} \text{ for all } i \in [m-1].$$

Let $S_n^{\mathfrak{m}}$ be the set of all multipermutations of n.

For $w = w_1 w_2 \cdots w_m \in S_n^{\mathfrak{m}}$ let $\mathrm{Inv}(w)$ be the set of pairs (w_i, w_j) with $i < j$ and $w_i > w_j$ and $\{w_1, w_2, \ldots, w_{i-1}\} \cap \{w_i\} = \{w_1, w_2, \ldots, w_{j-1}\} \cap \{w_j\} = \varnothing$. If P is a poset on $[n]$ and $G = \mathsf{inc}(P)$ is its incomparability graph, then set

$$\begin{aligned} \mathrm{inv}_G(w) &:= |\{(a, b) \in \mathrm{Inv}(w) : \{a, b\} \in E(G)\}|, \\ S(w, P) &:= \{m - i : i \in [m-1] \text{ and } w_i \not\succ_P w_{i+1}\}. \end{aligned} \tag{4.4}$$

The following generalizes [18, Thm. 3.1]:

Theorem 49. *If $G = \mathsf{inc}(P)$ for a poset P on $[n]$ then*

$$\overline{L}_G(q) = \sum_{w \in S_n^{\mathfrak{m}}} q^{\mathsf{inv}_G(w)} \overline{L}_{\ell(w),S(w,P)}.$$

Proof. Let $G = \mathsf{inc}(P)$ for a poset P on $[n]$. By (3.7) and Proposition 47,

$$\overline{L}_G(q) = \sum_{D \in \mathfrak{mAO}(G)} \sum_{v \in \mathcal{M}(D)} q^{\mathsf{align}(D)} \overline{L}_{\ell(v),\mathrm{Des}(v,\gamma_D)} \tag{4.5}$$

where for each $D \in \mathfrak{mAO}(G)$ the map γ_D is any decreasing labeling of D.

Fix $D \in \mathfrak{mAO}(G)$ and $v = (v_1, v_2, \cdots, v_m) \in \mathcal{M}(D)$. Each entry of v has the form $v_j = (w_j, i_j)$ for some $w_j \in V(G) = [n]$ and $i_j \in \mathbb{P}$. If $a := w_j = w_k$ for some $1 \leqslant j < k \leqslant n$ then we must have $i_j \geqslant i_k$, since if $i_j < i_k$ then a directed path must connect (a, i_k) to (a, i_j) in D which would imply that $k < j$. We can never have $w_j = w_{j+1}$ since $(a, i+1) <_D (a, i)$ is never a covering relation when D is an acyclic multi-orientation. Hence, the sequence $w = w_1 w_2 \cdots w_m$ belongs to $S_n^{\mathfrak{m}}$.

Let $w^{\mathbf{r}} = w_m \cdots w_2 w_1 \in S_n^{\mathfrak{m}}$ and observe that

$$\mathsf{align}(D) = \mathsf{inv}_G(w^{\mathbf{r}}). \tag{4.6}$$

Note that D is an acyclic orientation of the incomparability graph of the partial order on $V(D) \subset [n] \times \mathbb{P}$ that has $(x, i) < (y, j)$ if and only if $x <_P y$. By Lemma 48, we can therefore choose γ_D to be a decreasing labeling of D with $\gamma_D(x, i) < \gamma_D(y, j)$ for all $(x, i), (y, i) \in V(D)$ that are incomparable under $<_D$ but have $x <_P y$. For this labeling, we have

$$\mathrm{Des}(v, \gamma_D) = \{j \in [m-1] : w_j \not<_P w_{j+1}\} = S(w^{\mathbf{r}}, P) \tag{4.7}$$

since for each $j \in [m-1]$ one of the following occurs:

(1) If $\{w_j, w_{j+1}\} \in E(G)$ then v_j and v_{j+1} are comparable in $<_D$ by the definition of an acyclic multi-orientation. In this case $v_j <_D v_{j+1}$ as we cannot have $v_{j+1} <_D v_j$ for $v \in \mathcal{M}(D)$, so $\gamma_D(v_j) > \gamma_D(v_{j+1})$ and $j \in \mathrm{Des}(v, \gamma_D)$, while at the same time $w_j \not<_P w_{j+1}$ holds as $G = \mathsf{inc}(P)$.

(2) If $\{w_j, w_{j+1}\} \notin E(G)$ then $v_j \to v_{j+1}$ and $v_{j+1} \to v_j$ are not edges in D, so v_j and v_{j+1} are incomparable in $<_D$, as any vertices in a directed path from v_j to v_{j+1} in D would have to appear between two consecutive elements v, which is impossible. In this case, since w_j and w_{j+1} are comparable in P, we have $j \in \mathrm{Des}(v, \gamma_D)$ if and only if $w_j \not<_P w_{j+1}$.

Now observe that $(D, v) \mapsto w$ is a bijection from the set of pairs (D, v) with $D \in \mathfrak{mAO}(G)$ and $v \in \mathcal{M}(D)$ to $S_n^{\mathfrak{m}}$. To recover $v = (v_1, v_2, \ldots, v_m)$ from $w = w_1 w_2 \cdots w_m$, we set $v_j = (w_j, i_j)$ where either $i_j = 1$ if $w_j \notin \{w_{j+1}, \ldots, w_m\}$, or if there is a smallest index k with $j < k \leqslant m$ such that $w_j = w_k$, then

$$i_j = \begin{cases} i_k & \text{if } w_{j+1}, w_{j+2}, \ldots, w_{k-1} \text{ are all comparable to } w_j = w_k \text{ in } P; \\ i_k + 1 & \text{if one of } w_{j+1}, w_{j+2}, \ldots, w_{k-1} \text{ is incomparable to } w_j = w_k \text{ in } P. \end{cases}$$

In turn, we recover D as the directed graph with vertex set $\{v_1, v_2, \ldots, v_m\}$ that has a directed edge $v_j \to v_k$ whenever either $w_j = w_k$ and $i_j > i_k$, or $\{w_j, w_k\} \in E(G)$ and $\{l \in [m] : v_l = v_j\} \prec \{l \in [m] : v_l = v_k\}$.

Combining (4.6) and (4.7) with this observation turns (4.5) into

$$\overline{L}_G(q) = \sum_{w \in S_n^{\mathfrak{m}}} q^{\operatorname{inv}_G(w^{\mathbf{r}})} \overline{L}_{\ell(w^{\mathbf{r}}), S(w^{\mathbf{r}}, P)}$$

which is equivalent to the desired formula. $\qquad\square$

Unlike $X_G(q)$, the power series $\overline{L}_G(q)$ is rarely symmetric. However, we can exactly characterize the ordered graphs G for which $\overline{L}_G(q) \in \mathfrak{m}\mathsf{Sym}$.

Theorem 50. *One has $\overline{L}_G(q) \in \mathfrak{m}\mathsf{Sym}[q]$ if and only if G is a* cluster graph, *that is, a disjoint union of complete graphs.*

The proof of the theorem will use three lemmas. For an ordered graph G, let $\min G$ be the set of vertices $v \in V(G)$ such that if $\{v, w\} \in E(G)$ then $v < w$, and let $\max G$ be the set of vertices $v \in V(G)$ such that if $\{u, v\} \in E(G)$ then $u < v$. Both $\min G$ and $\max G$ are independent sets. Also let $W_G := X_G(0) = \sum_\kappa x^\kappa$ where the sum is over proper colorings $\kappa : V(G) \to \mathbb{P}$ with $\operatorname{asc}_G(\kappa) = \varnothing$.

Lemma 51. *If $W_G \in \mathsf{Sym}[q]$ then $|\min G| = |\max G|$.*

Proof. If $f = \sum_\alpha c_\alpha M_\alpha$ is a quasisymmetric function then let $[M_\alpha]f := c_\alpha$. Notice that this is also the coefficient of $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ in f.

Now let $p = |\min G|$, $m = |\max G|$, and $n = |G|$. If $p > m$ then

$$[M_{(p, 1^{n-p})}]W_G = 0 < [M_{(1^{n-p}, p)}]W_G$$

since in this case there is no ascent-less proper coloring of G that assigns the color 1 to p distinct vertices (as to have no ascents every 1-colored vertex must be in $\max G$), but there is an ascent-less proper coloring that assigns the same color $n - p + 1$ to all p minimal vertices while assigning the distinct colors $1, 2, \ldots, n - p$ to the non-minimal vertices (if these vertices are totally ordered as $v_1 > v_2 > \cdots > v_{n-p}$ then let vertex v_i have color i).

Alternatively, if $p < m$ then

$$[M_{(1^{n-m}, m)}]W_G = 0 < [M_{(m, 1^{n-m})}]W_G$$

since in this case there is no ascent-less proper coloring of G that assigns the distinct colors $1, 2, \ldots, n - m$ some set of vertices while assigning the same color $n - m + 1$ to the m vertices that remain (as to have no ascents every $(n - m + 1)$-colored vertex must be in $\min G$), but there is an ascent-less proper coloring that assigns the same color 1 to all m maximal vertices while assigning the distinct colors $2, 3, \ldots, n - m + 1$ to the non-maximal vertices (if these vertices are totally ordered as $v_1 > v_2 > \cdots > v_{n-m}$ then let vertex v_i have color $i + 1$).

In either case, we have $[M_\alpha]W_G \neq [M_\beta]W_G$ for two strict compositions α and β that sort to the same partition, so W_G is not symmetric. We conclude that if $W_G \in \mathsf{Sym}$ is symmetric then $p = m$. $\qquad\square$

Lemma 52. *If $\overline{L}_G(q) \in \mathfrak{mSym}[q]$ then $W_G \in \mathsf{Sym}$ and every independent subset of G has size at most $|\max G|$.*

Proof. As W_G is the homogeneous component of degree $|G|$ in $\overline{L}_G(0)$, if we have $\overline{L}_G(q) \in \mathfrak{mSym}$ then $W_G \in \mathsf{Sym}$. Let $n := |G|$. If S is an independent subset of G of size $s := |S| > |\min G|$, then

$$[M_{(1^n, s)}]\overline{L}_G(0) = 0 < [M_{(s, 1^n)}]\overline{L}_G(0)$$

since in this case there is no ascent-less proper set-valued coloring of G that assigns its largest color to s distinct vertices (as to have no ascents, each vertex whose colors include the largest appearing color must be in $\min G$), but if the vertices of G are totally ordered as $v_1 > v_2 > \cdots > v_n$ then the set-valued coloring $\kappa : G \to \mathsf{Set}(\mathbb{P})$ with

$$\kappa(v_i) = \begin{cases} \{1, i+1\} & \text{if } v_i \in S \\ \{i+1\} & \text{if } v_i \notin S \end{cases}$$

is ascent-less and proper, and this contributes at least one monomial term $x_1^s x_2 x_3 \cdots x_{n+1}$ to the monomial-positive expansion of $\overline{L}_G(0)$.

Thus if G has an independent subset with more than $|\max G|$ elements then $\overline{L}_G(0)$ is not symmetric. Therefore if $\overline{L}_G(q) \in \mathfrak{mSym}$, then W_G is symmetric so $|\min G| = |\max G|$ by Lemma 51, and $\overline{L}_G(0)$ is symmetric so every independent subset of G has size at most $|\min G| = |\max G|$. $\qquad\square$

Lemma 53. *Suppose $W_G \in \mathsf{Sym}$ and every independent subset of G has size at most $|\max G|$. Then G is a cluster graph.*

Proof. Suppose G has an isolated vertex v. Let K be the subgraph formed by removing this vertex. Then $W_G = e_1 W_K$ so W_K is symmetric, and if S is an independent subset of K then $|S| \leqslant |\max K|$ since $S \sqcup \{v\}$ is an independent subset of G and $\max G = \max K \sqcup \{v\}$. In this case, we may assume by induction that K is a cluster graph, and then G is also a cluster graph.

Assume G has no isolated vertices so that $\min G$ and $\max G$ are disjoint. Let $m = |\min G|$ and define H to be the subgraph G induced on the vertex set $V(G) \setminus \min G$. If $m = 0$ then G is empty so there is nothing to prove. Therefore, assume $m > 0$ so that H is a proper subgraph.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is any strict composition of $|H|$ then

$$[M_\alpha]W_H = [M_{(\alpha_1, \alpha_2, \ldots, \alpha_l, m)}]W_G$$

since in any ascent-less proper coloring of G assigning the color i to α_i vertices and the color $l+1$ to m vertices, the set of vertices colored by $l+1$ must be exactly $\min G$, so restriction defines a bijection from such colorings to ascent-less proper colorings of H that assign the color i to α_i vertices. Since $W_G \in \mathsf{Sym}$, it follows that $W_H \in \mathsf{Sym}$.

Since G has no isolated vertices, one has $\max H = \max G$. Therefore $|\min H| = |\max H| = |\min G| = |\max G| = m$. If S is an independent subset of H then S is also

an independent subset of G, so $|S| \leqslant |\max G| = |\max H|$. Thus, we may assume by induction that H is a disjoint union of m complete graphs.

We now make some observations. Each element of $\min H$ must be connected by an edge in G to at least one element of $\min G$ since $\min H$ and $\min G$ are disjoint. Any two elements of $\min H$ must be connected by edges in G to at least two different elements of $\min G$, since if $b, c \in \min H$ were connected to the same element $a \in \min G$ and to no other elements of $\min G$, then $(\min G \setminus \{a\}) \sqcup \{b, c\}$ would be an independent subset of G of the impossible size $|\min G| + 1 = |\max G| + 1$. Therefore, if $p = |H| = |G| - m$, and $k_1, k_2, \ldots, k_m$ are the sizes of the m connected components of H, then

$$[M_{(1^p, m)}]W_G = \binom{p}{k_1, k_2, \ldots, k_m} \geqslant [M_{(1^{p-1}, m, 1)}]W_G$$

with equality if and only if each element of $\min H$ is connected by an edge in G to exactly one element of $\min G$. This last property must hold as W_G is symmetric.

Let $\min G = \{u_1, u_2, \ldots, u_m\}$ and $\min H = \{v_1, v_2, \ldots, v_m\}$. By the properties in the paragraph above, we may assume that $\{u_i, v_i\} \in E(G)$ for each $i \in [m]$ and that $\{u_i, v_j\} \notin E(G)$ for $i \neq j$. For each $i \in [m]$, every vertex $w \in V(H)$ with $\{v_i, w\} \in E(G)$ must have $\{u_i, w\} \in E(G)$, since otherwise $(\min H \setminus \{v_i\}) \sqcup \{u_i, w\}$ would be an independent subset of G of the impossible size $m + 1 = |\max G| + 1$.

To conclude that G is a cluster graph, suppose $w \in V(H)$ has $\{v_j, w\} \in E(G)$. We must show that $\{u_i, w\} \notin E(G)$ for all $i \neq j$. This is trivial if $m = 1$, and if $m > 1$ then no such edge can exist as then the equalities

$$[M_{(1^p, m)}]W_G = [M_{(1^a, m, 1^b)}]W_G = [M_{(m, 1^p)}]W_G$$

cannot hold for all $a, b \in \mathbb{P}$ with $a + b = p$, contradicting $W_G \in \mathsf{Sym}$. $\qquad \square$

Proof of Theorem 50. If G is a disjoint union of m cluster graphs of sizes $n_1, n_2, \ldots, n_m$, then $\overline{L}_G(q) = \prod_{i=1}^m \overline{L}_{K_{n_i}}(q) \in \mathfrak{mSym}[q]$ by Example 45. If $\overline{L}_G(q) \in \mathfrak{mSym}[q]$, then G is a cluster graph by Lemmas 52 and 53. $\qquad \square$

4.3 Another quasisymmetric analogue

The preceding results indicate that $\overline{L}_G(q)$ is an interesting quasisymmetric q-analogue of $\overline{X}_G$ and K-theoretic analogue of $X_G(q)$. However, there is another candidate for such a generalization. Continue to let G be an ordered graph. Following [14], an *ascent* (respectively, *descent*) of a set-valued map $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$ is a tuple (u, v, i, j) with $\{u, v\} \in E(G)$, $i \in \kappa(u)$, $j \in \kappa(v)$, $u < v$, and $i < j$ (respectively, $i > j$). Let $\mathrm{asc}_G(\kappa)$ denote the number of ascents, and let $\mathrm{des}_G(\kappa)$ be the number of descents.

Definition 54. Given an ordered graph G, define

$$\overline{X}_G(q) = \sum_{\kappa} q^{\mathrm{asc}_G(\kappa)} x^{\kappa} \in \mathfrak{mQSym}[\![q]\!]$$

where the sum is over all proper set-valued colorings $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$.

As with $\overline{L}_G(q)$, it is clear that if $G = G_1 \sqcup G_2$ is the disjoint union of two ordered graphs, then we have a factorization $\overline{X}_G(q) = \overline{X}_{G_1}(q)\overline{X}_{G_2}(q)$.

The power series $\overline{X}_G(q)$ is closely related to the quasisymmetric functions $X_G(\mathbf{x}, q, \mu)$ studied in [14]. For each map $\mu : V(G) \to \mathbb{N}$, Hwang [14] defines

$$X_G(\mathbf{x}, q, \mu) := \sum_{\kappa} q^{\mathrm{asc}_G(\kappa)} x^{\kappa} \tag{4.8}$$

where the sum is over all proper set-valued colorings κ of G with $|\kappa(v)| = \mu(v)$. Evidently $\overline{X}_G(q) = \sum_{\mu : V(G) \to \mathbb{P}} X_G(\mathbf{x}, q, \mu)$, and, as noted in [14, Rem. 2.2],

$$X_G(\mathbf{x}, q, \mu) = \tfrac{1}{[\mu]_q!} X_{\mathsf{Clan}_\mu(G)}(q) \quad \text{where } [\mu]_q! := \prod_{v \in V(G)} [\mu(v)]_q!. \tag{4.9}$$

Here, we view $\mathsf{Clan}_\mu(G)$ as an ordered graph with $(v, i) < (w, j)$ if either $v < w$ or $v = w$ and $i < j$.

Using (4.9), various positive or alternating expansions of $X_G(q)$ (e.g., into fundamental quasisymmetric functions [18, Thm. 3.1], Schur functions [18, Thm. 6.3], power sum symmetric functions [2, Thm. 3.1], or elementary symmetric functions [18, Conj. 5.1]) can be extended in a straightforward way to $X_G(\mathbf{x}, q, \mu)$ and $\overline{X}_G(q)$. See Hwang's results [14, Thms. 3.3, 4.10, and 4.19] and his conjecture [14, Conj. 3.10].

Some of these statements require G to be isomorphic to the incomparability graph of a *natural unit interval order*, meaning a poset P on a finite subset of $\mathbb{P}$ such that if $x <_P z$ then $x < z$ and every y incomparable in P to both x and z has $x < y < z$ [18, Prop. 4.1].

Remark 55. A finite poset is isomorphic to a (unique) natural unit interval order if and only if it is both $(3 + 1)$- and $(2 + 2)$-free [18, Prop. 4.2]. This does not mean that every $(3 + 1)$- and $(2 + 2)$-free poset on a finite subset of $\mathbb{P}$ is a natural unit interval order. For example, the poset $P = \{1 < 2\} \sqcup \{3\}$ is $(3 + 1)$- and $(2 + 2)$-free but not a natural unit interval order; however, it is isomorphic to the natural unit interval order $Q = \{1 < 3\} \sqcup \{2\}$.

If G is the incomparability graph of a natural unit interval order, then so are all of its α-clans, and the following holds:

Lemma 56. *If G is the incomparability graph of a natural unit order interval, then $\overline{X}_G(q)$ is symmetric and $\overline{X}_G(q) = \sum_{\kappa} q^{\mathrm{des}_G(\kappa)} x^{\kappa}$ where the sum is over all proper set-valued colorings $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$.*

Proof. In this case $X_G(\mathbf{x}, q, \mu)$ is symmetric by [14, Thm. 3.8], so the same is true of $\overline{X}_G(q)$, and the alternate formula for $\overline{X}_G(q)$ holds by [14, Prop. 2.1]. $\square$

Some sample computations of $\overline{X}_G(q)$ are shown in Tables 2 and 3. One of the few cases where we have explicit formulas is discussed in the following example.

Example 57. If K_n is the complete graph on $[n]$ then

$$\overline{X}_{K_n}(q) = \sum_{r=n}^{\infty} F_r^{(n)} e_r \quad \text{for} \quad F_r^{(n)} := \sum_{\substack{k_1, k_2, \ldots, k_n \in \mathbb{P} \\ k_1 + k_2 + \cdots + k_n = r}} \binom{r}{k_1, k_2, \ldots, k_n}_q$$

where $(q)_n := \prod_{i \in [n]}(1 - q^i)$ and $\binom{r}{k_1, k_2, \dots, k_n}_q = \frac{(q)_r}{(q)_{k_1}(q)_{k_2}\cdots(q)_{k_n}}$.

When q is a prime power, $F_r^{(n)}$ counts the *strictly increasing flags* of $\mathbb{F}_q$-subspaces $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{F}_q^r$. The article [22] contains a recurrence for the *generalized Galois number* $G_r^{(n)}$, which counts all increasing flags $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{F}_q^r$. An inclusion-exclusion argument shows that

$$G_r^{(n)} = \sum_{i=0}^n \binom{n}{i} F_r^{(i)} \quad \text{and} \quad F_r^{(n)} = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} G_r^{(i)} \tag{4.10}$$

where $F_r^{(0)} = G_r^{(0)} = 0$ if $r > 0$. Vinroot shows in [22] that

$$G_{r+1}^{(n)} = \sum_{i=0}^{n-1} \binom{n}{i+1}(-1)^i \frac{(q)_r}{(q)_{r-i}} G_{r-i}^{(n)} \tag{4.11}$$

by using a recurrence for the q-multinomial coefficients, namely:

$$\binom{r}{k_1, k_2, \dots, k_n}_q = \sum_{\varnothing \neq J \subseteq [n]} (-1)^{|J|-1} \frac{(q)_{r-1}}{(q)_{r-|J|}} \binom{r-|J|}{\underline{k}-\underline{e}_J}_q \tag{4.12}$$

where $\underline{k} = (k_1, k_2, \dots, k_n)$ and where $\underline{e}_J = (e_1, e_2, \dots, e_n)$ is the n-tuple with $e_i = |\{i\} \cap J|$. A more complicated recurrence holds for $F_r^{(n)}$.

Proposition 58. *For each $r \in \mathbb{N}$ and $n \in \mathbb{P}$ it holds that*

$$F_{r+1}^{(n)} = \sum_{i=0}^{n-1} \sum_{j=n-1-i}^{n} \binom{n}{j}\binom{j}{n-1-i}(-1)^i \frac{(q)_r}{(q)_{r-i}} F_{r-i}^{(j)}$$

under the convention that $F_0^{(0)} = 1$ and $F_r^{(n)} = 0$ if $r < n$ or $r > 0 = n$.

Proof. Starting from the definition, we have

$$\begin{aligned}
F_{r+1}^{(n)} &= \sum_{\underline{k} \in \mathbb{P}^n} \binom{r+1}{\underline{k}}_q = \sum_{\underline{k} \in \mathbb{P}^n} \sum_{\varnothing \neq J \subseteq [n]} (-1)^{|J|-1} \frac{(q)_r}{(q)_{r-|J|+1}} \binom{r+1-|J|}{\underline{k}-\underline{e}_J}_q \\
&= \sum_{\varnothing \neq J \subseteq [n]} \sum_{\underline{k} \in \mathbb{P}^n} (-1)^{|J|-1} \frac{(q)_r}{(q)_{r-|J|+1}} \binom{r+1-|J|}{\underline{k}-\underline{e}_J}_q \\
&= \sum_{i=0}^{n-1} \sum_{\substack{J \subseteq [n] \\ |J|=i+1}} \sum_{\underline{k} \in \mathbb{P}^n} (-1)^i \frac{(q)_r}{(q)_{r-i}} \binom{r-i}{\underline{k}-\underline{e}_J}_q.
\end{aligned}$$

As $\underline{k}$ varies over all sequences in $\mathbb{P}^n$ and J varies over all $i+1$ element subsets of $[n]$, the modified sequence $\underline{k} - \underline{e}_J$ varies over all sequences in $\mathbb{P}^n$ as well as some elements of $\mathbb{N}^n$. Let $\mathsf{reduce}(\underline{k} - \underline{e}_J)$ be the sequence in $\mathbb{P}^m$ for $m \leqslant n$ formed by removing all zero entries. Each element of $\mathbb{P}^n$ arises as $\underline{k} - \underline{e}_J = \mathsf{reduce}(\underline{k} - \underline{e}_J)$ for exactly $\binom{n}{i+1}$ different pairs $(\underline{k}, J)$. More generally, if $j \in [i+1]$ then each of element of $\mathbb{P}^{n-j}$ arises as $\mathsf{reduce}(\underline{k} - \underline{e}_J)$ for exactly $\binom{n}{j}\binom{n-j}{i+1-j}$ pairs $(\underline{k}, J)$. So, continuing from the displayed equation, we get

$$F_{r+1}^{(n)} = \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} \sum_{\underline{k}' \in \mathbb{P}^{n-j}} \binom{n}{j}\binom{n-j}{i+1-j}(-1)^i \frac{(q)_r}{(q)_{r-i}} \binom{r-i}{\underline{k}'}_q.$$

Now substitute the identities $\binom{n}{j} = \binom{n}{n-j}$ and $F_{r-i}^{(n-j)} = \sum_{\underline{k}' \in \mathbb{P}^{n-j}} \binom{r-i}{\underline{k}'}_q$. $\qquad\square$

The result of applying the coproduct of $\mathfrak{mQSym}$ to $\overline{X}_G(q)$ cannot be expressed as a finite linear combination of terms $\overline{X}_{G_1}(q) \otimes \overline{X}_{G_2}(q)$, and $\overline{X}_G(q)$ does not seem to naturally arise as the image in $\mathfrak{mQSym}$ of a combinatorial LC-Hopf algebra. In general $\overline{X}_G(q)$ is neither $\overline{e}$-positive nor multifundamental-positive (see the examples in Tables 2 and 3). However, $\overline{X}_G(q)$ does have a nontrivial positivity property that is not shared by $X_G(\mathbf{x}, q, \mu)$.

A *set-valued tableau* T of shape λ is an assignment of sets $T_{ij} \in \mathsf{Set}(\mathbb{P})$ to the cells (i,j) in the *Young diagram* $\mathsf{D}_\lambda = \{(i,j) \in \mathbb{P} \times \mathbb{P} : 1 \leqslant j \leqslant \lambda_i\}$ of a partition λ. We write $(i,j) \in T$ to indicate that (i,j) belongs to the shape of T. A set-valued tableau T is *semistandard* if $T_{ij} \preceq T_{i,j+1}$ and $T_{ij} \prec T_{i+1,j}$ for all relevant positions. Let

$$x^T := \prod_{(i,j)\in T} \prod_{k \in T_{ij}} x_k \quad \text{and} \quad |T| := \sum_{(i,j)\in T} |T_{ij}|.$$

Definition 59. The *symmetric Grothendieck function* of a partition λ is the power series $\overline{s}_\lambda := \sum_{T \in \mathsf{SetSSYT}(\lambda)} (-1)^{|T|-|\lambda|} x^T \in \mathbb{Z}[\![x_1, x_2, \ldots]\!]$ where $\mathsf{SetSSYT}(\lambda)$ is the set of all semistandard set-valued tableaux of shape λ.

Each $\overline{s}_\lambda$ is in $\mathfrak{mSym}$ and the set of all symmetric Grothendieck functions is another pseudobasis for $\mathfrak{mSym}$ [3]. When restricted to n variables, these symmetric functions are the K-theory classes for the Schubert cells in the type A Grassmannian; see the discussion in [4, §2.2].

We write $\mu \subseteq \lambda$ for two partitions if $\mathsf{D}_\mu \subseteq \mathsf{D}_\lambda$ and set $\mathsf{D}_{\lambda/\mu} := \mathsf{D}_\lambda \setminus \mathsf{D}_\mu$. In the definition below, a *semistandard tableau* of shape λ/μ means a filling of $\mathsf{D}_{\lambda/\mu}$ by positive integers such that each row is weakly increasing (as column indices increase) and each column is strict increasing (as row indices increase).

Definition 60 ([4, Def. 3.8]). Suppose P is a finite poset and λ is a partition. A *Grothendieck P-tableau* of shape λ is a pair $T = (U, V)$ with both of these properties:

(a) U is a filling of D_μ by elements of P for some partition $\mu \subseteq \lambda$, such that each element of P is in at least one cell, and for each $(i,j) \in \mathsf{D}_\mu$ one has $U_{ij} <_P U_{i,j+1}$ if $(i, j+1) \in \mathsf{D}_\mu$ and $U_{ij} \not>_P U_{i+1,j}$ if $(i+1, j) \in \mathsf{D}_\mu$;

(b) V is a semistandard tableau of shape λ/μ, whose entries in each row i are all less than i (so $\mathsf{D}_{\lambda/\mu}$ must have no cells in the first row).

Let $\mathscr{G}_P$ be the set of Grothendieck P-tableaux. Let $\lambda(T)$ be the shape of $T \in \mathscr{G}_P$.

Example 61. When the elements of P do not include any positive integers, we can draw a Grothendieck P-tableau $T = (U, V)$ of shape λ as simply a filling of D_λ in English notation, as there is no ambiguity about which boxes belong to U and which belong to V. Suppose P is the poset on $\{a, b, c, d, e\}$ with $a < c < d < e$ and $b < c < d < e$, while a and b are incomparable. Then

$$
\begin{array}{|c|c|c|c|}
\hline
a & c & d & e \\
\hline
b & d & e & 1 \\
\hline
c & 1 & 2 \\
\cline{1-3}
3 \\
\cline{1-1}
\end{array}
\quad \text{and} \quad
\begin{array}{|c|c|c|c|}
\hline
a & c & d & e \\
\hline
b & c & d & e \\
\hline
a & e & 2 \\
\cline{1-3}
1 \\
\cline{1-1}
\end{array}
$$

are both Grothendieck P-tableaux of shape $\lambda = (4, 4, 3, 1)$.

Suppose P is a finite poset on a subset of $\mathbb{Z}$, and let $G = \mathsf{inc}(P)$. Choose some $T = (U, V) \in \mathscr{G}_P$ and let μ be the shape of U. A *G-inversion* of T is a pair of cells $(i,j), (k,l) \in \mathsf{D}_\mu$ containing incomparable elements of P such that $i < k$ and $U_{ij} > U_{kl}$. Let $\mathrm{inv}_G(T)$ be the number of all G-inversions of T.

Example 62. Define P as in Example 61 and set $a = -1$, $b = -2$, $c = -3$, $d = -4$, and $e = -5$. Then the two Grothendieck P-tableaux in Example 61 both have only a single G-inversion, given by the pair of cells $(1,1)$ and $(2,1)$.

One of the main results of [4] establishes that if G is the incomparability graph $\mathsf{inc}(P)$ of a $(3+1)$-free poset P then $\overline{X}_G = \sum_{T \in \mathscr{G}_P} \overline{s}_{\lambda(T)}$. This is a K-theoretic generalization of a theorem of Gasharov [7] showing that X_G is Schur positive for the same family of *claw-free incomparability graphs*.

Shareshian and Wachs [18] discovered a refinement of Gasharov's result, which has the following K-theoretic extension. Both our theorem and its predecessor in [18] only apply to incomparability graphs of natural unit interval orders, which make up a proper subset of all claw-free incomparability graphs. This loss of generality is to be expected, since we need P to be a natural unit interval order to deduce from Lemma 56 that $\overline{X}_{\mathsf{inc}(P)}(q)$ is symmetric.

Theorem 63. *If $G = \mathsf{inc}(P)$ for a natural unit interval order P then*

$$\overline{X}_G(q) = \sum_{T \in \mathscr{G}_P} q^{\mathrm{inv}_G(T)} \overline{s}_{\lambda(T)}.$$

Table 4 shows some examples of these positive $\overline{s}$-expansions.

Proof. Our argument is a hybrid of the proofs of [4, Thm. 3.9] and [18, Thm. 6.3]. To remove any ambiguity in the following definitions, we assume that the elements of P are all negative integers, so that as a set P is disjoint from $\mathbb{P} = \{1, 2, 3, \dots\}$. Fix a partition λ and let $n = |\lambda|$ and $N \geqslant 2n$.

Following [4], define a *Grothendieck P-array of type λ* to be a pair (π, A) where $\pi \in S_N$ and A is a map $\mathbb{P} \times \mathbb{P} \to P \sqcup \{\emptyset\} \sqcup \mathbb{P}$, written $(i,j) \mapsto A_{ij}$, with all of the following properties:

- $A_{ij} = \emptyset$ unless $i \leqslant \ell(\lambda)$ and $j \leqslant \lambda_{\pi(i)} - \pi(i) + i$;

- for each $p \in P$ there is some $(i,j) \in \mathbb{P} \times \mathbb{P}$ with $A_{ij} = p$;

- if $A_{ij} \in P$ when $j > 1$, then $A_{i(j-1)} \in P$ and $A_{i(j-1)} <_P A_{ij}$;

- if $A_{ij} \in \mathbb{P}$, then $A_{ij} \leqslant \pi(i) - 1$; and

- if $A_{ij} \in \mathbb{P}$ when $j > 1$, then $A_{i(j-1)} \in P$, or $A_{i(j-1)} \in \mathbb{P}$ and $A_{i(j-1)} \leqslant A_{ij}$.

One can think of A as a partial filling of an infinite matrix by elements of the poset P (viewing $A_{ij} = \emptyset$ as an unfilled position), with left-justified rows that each consist of an increasing chain in P followed by a weakly increasing sequence of positive integers.

If (π, A) is a Grothendieck P-array of type λ, then define $\mathrm{inv}_G(A)$ to be the set of pairs of positions $(i,j), (k,l) \in \mathbb{P} \times \mathbb{P}$ with $i < k$, such that A_{ij} and A_{kl} are incomparable elements of P with $A_{ij} > A_{kl}$.

Write $\overline{X}_G(q) = \sum_\lambda c_\lambda(q) \overline{s}_\lambda$ where $c_\lambda(q) \in \mathbb{Z}[\![q]\!]$. Then, repeating the part of the proof of [4, Thm. 3.9] through [4, Eq. (3.4)], using the formula $\overline{X}_G(q) = \sum_\kappa q^{\mathrm{des}_G(\kappa)} x^\kappa$ from Lemma 56 in place of $\overline{X}_G$, shows that

$$c_\lambda(q) = \sum_{(\pi, A)} \mathrm{sgn}(\pi) q^{\mathrm{inv}_G(A)} \tag{4.13}$$

where the sum ranges over all Grothendieck P-arrays of type λ.

As in [4], we refer to a position (i,j) with $i \geqslant 2$ as a *flaw* of a Grothendieck P-array (π, A) if one of the following occurs:

- $A_{ij} \in P$ and either $A_{(i-1)j} \notin P$ or $A_{ij} <_P A_{(i-1)j}$; or

- $A_{ij} \in \mathbb{P}$ and either $A_{(i-1)j} = \emptyset$ or $A_{(i-1)j} \in \mathbb{P}$ with $A_{ij} \leqslant A_{(i-1)j}$.

One can check that a Grothendieck P-array (π, A) has no flaws if and only if $\pi = 1$ is the identify permutation and $A \in \mathscr{G}_P$ corresponds to a Grothendieck P-tableau.

The goal now is to produce a sign-reversing, inversion-preserving involution Ψ of the set of all Grothendieck P-arrays (π, A) that have flaws. The existence of such a map will imply that (4.13) reduces to $c_\lambda(q) = \sum_{T \in \mathscr{G}_P, \lambda(T)=\lambda} q^{\mathrm{inv}_G(A)}$. The involution used in the proof of [4, Thm. 3.9] is sign-reversing but not inversion-preserving. We can correct this by incorporating ideas in the proof of [18, Thm. 6.3].

Given a flawed Grothendieck P-array (π, A), let c be the minimal column in which a flaw occurs, and let r be the maximal row such that (r, c) is a flaw. Let $C_{r-1}(A)$ be the set of elements of P that appear in row $r-1$ weakly to the right of column c. Let $C_r(A)$ be the set of elements of P that appear in row r strictly to the right of column c. Then let $H(A)$ be the subgraph of G induced on $C_{r-1}(A) \cup C_r(A)$. For $i \in \{r, r+1\}$ let $O_i(A)$ be the set of $x \in C_i(A)$ that belong to a connected component of odd size in $H(A)$. All elements of P that belong to both $C_{r-1}(A)$ and $C_r(A)$ are comparable in P to all vertices in $H(A)$, so must belong to connected components of size one, and are therefore in both $O_{r-1}(A)$ and $O_r(A)$.

Set $E_i(A) := C_i(A) \setminus O_i(A)$. Then let $I_{r-1}(A)$ be the set of elements of P that appear in row $r-1$ strictly to the left of column c, and let $I_r(A)$ be the set of elements of P in row r weakly to the left of column c. By repeating the argument in the proof of [18, Thm. 6.3], which uses the hypothesis that P is $(3+1)$-free as a natural unit interval order, one checks that $I_{r-1}(A) \cup E_{r-1}(A) \cup O_r(A)$ is a chain in P in which $I_{r-1}(A)$ is an initial chain, and that $I_r(A) \cup E_r(A) \cup O_{r-1}(A)$ is a chain in P in which $I_r(A)$ is an initial chain. Therefore we can define $\Psi(\pi, A) = (\pi', A')$ where π' is the product of π and the transposition $(r-1\, r)$, where A' is formed from A by changing row $r-1$ and r as follows:

- replace row $r-1$ by the chain $I_{r-1}(A) \cup E_{r-1}(A) \cup O_r(A)$ followed by any integers strictly to the right of column c in row r of A;

- replace row r by the chain $I_r(A) \cup E_r(A) \cup O_{r-1}(A)$ followed by any integers weakly to the right of column c in row $r - 1$ of A.

Then one can check that (π', A') is a Grothendieck P-array also with a flaw at (r, c), and that $\Psi(\pi', A') = (\pi, A)$. The details are similar to what is presented in the proof of [4, Thm. 3.9].

Finally, we explain why $\mathrm{inv}_G(A) = \mathrm{inv}_G(A')$. This relies on the fact that the connected components in $H(A)$ are paths with numerically increasing vertices [18, Lem. 4.4]. In each such component of even size exactly half the elements must appear in row $r - 1$ of A and exactly half must appear in row r of A. In the components of odd size, every other element must appear in row $r - 1$ of A with the elements in between appearing in row r of A. Thus in any given odd sized component with $2k + 1 > 1$ vertices, either vertices 1, 3, 5, ..., $2k - 1$ contribute inversions to $\mathrm{inv}_G(A)$ while vertices 2, 4, 6, ..., $2k$ contribute inversions to $\mathrm{inv}_G(A')$, or vice versa. Either way, we have $\mathrm{inv}_G(A) = \mathrm{inv}_G(A')$. $\square$

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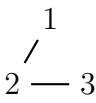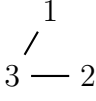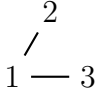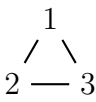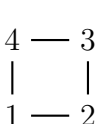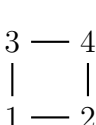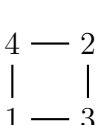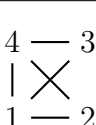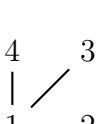
Graph G	$\overline{L}_G(q)$ in the monomial basis $\{M_\alpha\}$	$\overline{L}_G(q)$ in the basis $\{\overline{L}_\alpha\}$
$\begin{array}{c}1\\ /\\ 2 — 3\end{array}$	$(1 + 4q + q^2)M_{111} + qM_{12} + qM_{21} + (6 + 24q + 6q^2)M_{1111} + 5qM_{112} + 5qM_{121} + (1 + 3q + q^2)M_{211} + \cdots$	$(1+2q+q^2)\overline{L}_{111}+q\overline{L}_{12}+q\overline{L}_{21}+(2+5q+2q^2)\overline{L}_{1111}+3q\overline{L}_{112} + 3q\overline{L}_{121} + (1 + q + q^2)\overline{L}_{211} + \cdots$
$\begin{array}{c}1\\ /\\ 3 — 2\end{array}$	$(2 + 2q + 2q^2)M_{111} + M_{12} + q^2 M_{21} + (12 + 12q + 12q^2)M_{1111} + 5M_{112} + (2 + 3q^2)M_{121} + (2q + 3q^2)M_{211} + \cdots$	$(1+2q+q^2)\overline{L}_{111}+\overline{L}_{12}+q^2\overline{L}_{21}+(2+4q+3q^2)\overline{L}_{1111}+3\overline{L}_{112} + (1+2q^2)\overline{L}_{121} + (2q + q^2)\overline{L}_{211} + \cdots$
$\begin{array}{c}2\\ /\\ 1 — 3\end{array}$	$(2 + 2q + 2q^2)M_{111} + q^2 M_{12} + M_{21} + (12 + 12q + 12q^2)M_{1111} + 5q^2 M_{112} + (3 + 2q^2)M_{121} + (3 + 2q)M_{211} + \cdots$	$(1+2q+q^2)\overline{L}_{111}+q^2\overline{L}_{12}+\overline{L}_{21}+(3+4q+2q^2)\overline{L}_{1111}+3q^2\overline{L}_{112} + (2+q^2)\overline{L}_{121} + (1 + 2q)\overline{L}_{211} + \cdots$
$\begin{array}{c}1\\ /\ \backslash\\ 2 — 3\end{array}$	$(1+2q+2q^2+q^3)M_{111}+(6+12q+12q^2+6q^3)M_{1111}+\cdots$	$(1+2q+2q^2+q^3)\overline{L}_{111}+(3+6q+6q^2+3q^3)\overline{L}_{1111}+\cdots$
$\begin{array}{c}4 — 3\\ |\quad |\\ 1 — 2\end{array}$	$(1+8q+6q^2+8q^3+q^4)M_{1111}+(2q+2q^3)M_{112}+(2q+2q^3)M_{121}+(2q+2q^3)M_{211}+(q+q^3)M_{22}+(10+80q+60q^2+80q^3+10q^4)M_{11111}+(18q+18q^3)M_{1112}+(18q+18q^3)M_{1121}+(1+14q+6q^2+14q^3+q^4)M_{1211}+(1+14q+6q^2+14q^3+q^4)M_{2111}+(4q+4q^3)M_{122}+(4q+4q^3)M_{212}+(4q+4q^3)M_{221}+\cdots$	$(1+3q+6q^2+3q^3+q^4)\overline{L}_{1111}+(q+q^3)\overline{L}_{112}+(2q+2q^3)\overline{L}_{121}+(q+q^3)\overline{L}_{211}+(q+q^3)\overline{L}_{22}+(4+16q+24q^2+16q^3+4q^4)\overline{L}_{11111}+(7q+7q^3)\overline{L}_{1112}+(9q+9q^3)\overline{L}_{1121}+(1+5q+6q^2+5q^3+q^4)\overline{L}_{1211}+(1+3q+6q^2+3q^3+q^4)\overline{L}_{2111}+(3q+3q^3)\overline{L}_{122}+(2q+2q^3)\overline{L}_{212}+(3q+3q^3)\overline{L}_{221}+\cdots$
$\begin{array}{c}3 — 4\\ |\quad |\\ 1 — 2\end{array}$	$(2+4q+12q^2+4q^3+2q^4)M_{1111}+4q^2 M_{112}+(1+2q^2+q^4)M_{121}+4q^2 M_{211}+2q^2 M_{22}+(20+40q+120q^2+40q^3+20q^4)M_{11111}+36q^2 M_{1112}+(7+22q^2+7q^4)M_{1121}+(3+4q+22q^2+4q^3+3q^4)M_{1211}+(2+4q+24q^2+4q^3+2q^4)M_{2111}+8q^2 M_{122}+8q^2 M_{212}+(1+6q^2+q^4)M_{221}+\cdots$	$(1+4q+4q^2+4q^3+q^4)\overline{L}_{1111}+2q^2\overline{L}_{112}+(1+2q^2+q^4)\overline{L}_{121}+2q^2\overline{L}_{211}+2q^2\overline{L}_{22}+(5+16q+22q^2+16q^3+5q^4)\overline{L}_{11111}+14q^2\overline{L}_{1112}+(4+10q^2+4q^4)\overline{L}_{1121}+(1+4q+8q^2+4q^3+q^4)\overline{L}_{1211}+(1+4q+4q^2+4q^3+q^4)\overline{L}_{2111}+6q^2\overline{L}_{122}+4q^2\overline{L}_{212}+(1+4q^2+q^4)\overline{L}_{221}+\cdots$
$\begin{array}{c}4 — 2\\ |\quad |\\ 1 — 3\end{array}$	$(4+4q+8q^2+4q^3+4q^4)M_{1111}+(2+2q^4)M_{112}+4q^2 M_{121}+(2+2q^4)M_{211}+(1+q^4)M_{22}+(40+40q+80q^2+40q^3+40q^4)M_{11111}+(18+18q^4)M_{1112}+(4+28q^2+4q^4)M_{1121}+(8+4q+12q^2+4q^3+8q^4)M_{1211}+(10+4q+8q^2+4q^3+10q^4)M_{2111}+(4+4q^4)M_{122}+(4+4q^4)M_{212}+(2+4q^2+2q^4)M_{221}+\cdots$	$(1 + 4q + 4q^2 + 4q^3 + q^4)\overline{L}_{1111} + (1 + q^4)\overline{L}_{112} + 4q^2\overline{L}_{121} + (1 + q^4)\overline{L}_{211} + (1 + q^4)\overline{L}_{22} + (6 + 16q + 20q^2 + 16q^3 + 6q^4)\overline{L}_{11111} + (7 + 7q^4)\overline{L}_{1112} + (1 + 16q^2 + q^4)\overline{L}_{1121} + (3 + 4q + 4q^2 + 4q^3 + 3q^4)\overline{L}_{1211} + (1 + 4q + 4q^2 + 4q^3 + q^4)\overline{L}_{2111} + (3 + 3q^4)\overline{L}_{122} + (2 + 2q^4)\overline{L}_{212} + (1 + 4q^2 + q^4)\overline{L}_{221} + \cdots$
$\begin{array}{c}4 — 3\\ |\times|\\ 1 — 2\end{array}$	$(2+4q+6q^2+6q^3+4q^4+2q^5)M_{1111}+(q^2+q^3)M_{112}+(1+q^5)M_{121}+(q^2+q^3)M_{211}+(20+40q+60q^2+60q^3+40q^4+20q^5)M_{11111}+(9q^2+9q^3)M_{1112}+(7+2q^2+2q^3+7q^5)M_{1121}+(3+2q+4q^2+4q^3+2q^4+3q^5)M_{1211}+(2q+7q^2+7q^3+2q^4)M_{2111}+\cdots$	$(1+4q+4q^2+4q^3+4q^4+q^5)\overline{L}_{1111}+(q^2+q^3)\overline{L}_{112}+(1+q^5)\overline{L}_{121}+(q^2+q^3)\overline{L}_{211}+(6+20q+22q^2+22q^3+20q^4+6q^5)\overline{L}_{11111}+(6q^2+6q^3)\overline{L}_{1112}+(5+q^2+q^3+5q^5)\overline{L}_{1121}+(1+2q+3q^2+3q^3+2q^4+q^5)\overline{L}_{1211}+(2q+4q^2+4q^3+2q^4)\overline{L}_{2111}+\cdots$
$\begin{array}{c}4\quad 3\\ |\ /\\ 1 — 2\end{array}$	$(6+6q+6q^2+6q^3)M_{1111}+(3q^2+3q^3)M_{112}+(3+3q^3)M_{121}+(3+3q)M_{211}+q^3 M_{13}+M_{31}+(60+60q+60q^2+60q^3)M_{11111}+(27q^2+27q^3)M_{1112}+(21+6q^2+27q^3)M_{1121}+(21+18q+15q^3)M_{1211}+(21+21q+12q^2)M_{2111}+7q^3 M_{113}+(4+3q^3)M_{131}+(4+3q)M_{311}+6q^3 M_{122}+6q^2 M_{212}+6M_{221}+\cdots$	$(1 + 3q + 3q^2 + q^3)\overline{L}_{1111} + (3q^2 + 2q^3)\overline{L}_{112} + (2 + 2q^3)\overline{L}_{121} + (2 + 3q)\overline{L}_{211} + q^3\overline{L}_{13} + \overline{L}_{31} + (7 + 12q + 9q^2 + 3q^3)\overline{L}_{11111} + (12q^2 + 8q^3)\overline{L}_{1112} + (7 + 3q^2 + 11q^3)\overline{L}_{1121} + (7 + 12q + 2q^3)\overline{L}_{1211} + (5 + 9q + 6q^2)\overline{L}_{2111} + 5q^3\overline{L}_{113} + (3 + 2q^3)\overline{L}_{131} + (2 + 3q)\overline{L}_{311} + 5q^3\overline{L}_{122} + 6q^2\overline{L}_{212} + 5\overline{L}_{221} + \cdots$

Table 1: Partial computations of $\overline{L}_G(q)$ for some small graphs G. The ellipses "..." mask higher order terms indexed by compositions α with $|\alpha| > |V(G)| + 1$.

Graph G	$\overline{X}_G(q)$ in the monomial basis $\{M_\alpha\}$	$\overline{X}_G(q)$ in the basis $\{\overline{L}_\alpha\}$
$\begin{array}{c} 1 \\ / \\ 2 \!-\! 3 \end{array}$	$(1+4q+q^2)M_{111}+qM_{12}+qM_{21}+(3+12q+16q^2+4q^3+q^4)M_{1111}+(2q+3q^2)M_{112}+(2q+3q^2)M_{121}+(2q+3q^2)M_{211}+\dots$	$(1+2q+q^2)\overline{L}_{111}+q\overline{L}_{12}+q\overline{L}_{21}+(4q^2+4q^3+q^4)\overline{L}_{1111}+3q^2\overline{L}_{112}+3q^2\overline{L}_{121}+3q^2\overline{L}_{211}+\dots$
$\begin{array}{c} 1 \\ / \\ 3 \!-\! 2 \end{array}$	$(2+2q+2q^2)M_{111}+M_{12}+q^2M_{21}+(8+8q+10q^2+8q^3+2q^4)M_{1111}+(3+2q)M_{112}+(2+q^2+2q^3)M_{121}+(2q^2+2q^3+q^4)M_{211}+\dots$	$(1+2q+q^2)\overline{L}_{111}+\overline{L}_{12}+q^2\overline{L}_{21}+(4q^2+4q^3+q^4)\overline{L}_{1111}+(1+2q)\overline{L}_{112}+(1+2q^3)\overline{L}_{121}+(2q^3+q^4)\overline{L}_{211}+\dots$
$\begin{array}{c} 2 \\ / \\ 1 \!-\! 3 \end{array}$	$(2+2q+2q^2)M_{111}+q^2M_{12}+M_{21}+(8+8q+10q^2+8q^3+2q^4)M_{1111}+(2q^2+2q^3+q^4)M_{112}+(2+q^2+2q^3)M_{121}+(3+2q)M_{211}+\dots$	$(1+2q+q^2)\overline{L}_{111}+q^2\overline{L}_{12}+\overline{L}_{21}+(4q^2+4q^3+q^4)\overline{L}_{1111}+(2q^3+q^4)\overline{L}_{112}+(1+2q^3)\overline{L}_{121}+(1+2q)\overline{L}_{211}+\dots$
$\begin{array}{c} 1 \\ / \ \backslash \\ 2 \!-\! 3 \end{array}$	$(1+2q+2q^2+q^3)M_{111}+(3+6q+9q^2+9q^3+6q^4+3q^5)M_{1111}+\dots$	$(1+2q+2q^2+q^3)\overline{L}_{111}+(3q^2+6q^3+6q^4+3q^5)\overline{L}_{1111}+\dots$
$\begin{array}{c} 4 \!-\! 3 \\ | \quad\ | \\ 1 \!-\! 2 \end{array}$	$(1+8q+6q^2+8q^3+q^4)M_{1111}+(2q+2q^3)M_{112}+(2q+2q^3)M_{121}+(2q+2q^3)M_{211}+(q+q^3)M_{22}+(4+32q+56q^2+56q^3+56q^4+32q^5+4q^6)M_{11111}+(6q+9q^2+6q^3+9q^4+6q^5)M_{1112}+(6q+9q^2+6q^3+9q^4+6q^5)M_{1121}+(6q+9q^2+6q^3+9q^4+6q^5)M_{1211}+(6q+9q^2+6q^3+9q^4+6q^5)M_{2111}+(q+2q^2+2q^3+2q^4+q^5)M_{122}+(2q+2q^2+2q^4+2q^5)M_{212}+(q+2q^2+2q^3+2q^4+q^5)M_{221}+\dots$	$(1+3q+6q^2+3q^3+q^4)\overline{L}_{1111}+(q+q^3)\overline{L}_{112}+(2q+2q^3)\overline{L}_{121}+(q+q^3)\overline{L}_{211}+(q+q^3)\overline{L}_{22}+(2q^2+24q^3+22q^4+12q^5+4q^6)\overline{L}_{11111}+(5q^2+q^3+5q^4+3q^5)\overline{L}_{1112}+(7q^2-q^3+7q^4+5q^5)\overline{L}_{1121}+(7q^2-q^3+7q^4+5q^5)\overline{L}_{1211}+(5q^2+q^3+5q^4+3q^5)\overline{L}_{2111}+(2q^2+q^3+2q^4+q^5)\overline{L}_{122}+(2q^2-2q^3+2q^4+2q^5)\overline{L}_{212}+(2q^2+q^3+2q^4+q^5)\overline{L}_{221}+\dots$
$\begin{array}{c} 3 \!-\! 4 \\ | \quad\ | \\ 1 \!-\! 2 \end{array}$	$(2+4q+12q^2+4q^3+2q^4)M_{1111}+4q^2M_{112}+(1+2q^2+q^4)M_{121}+4q^2M_{211}+2q^2M_{22}+(10+20q+50q^2+80q^3+50q^4+20q^5+10q^6)M_{11111}+(9q^2+18q^3+9q^4)M_{1112}+(3+2q+6q^2+14q^3+6q^4+2q^5+3q^6)M_{1121}+(3+2q+6q^2+14q^3+6q^4+2q^5+3q^6)M_{1211}+(9q^2+18q^3+9q^4)M_{2111}+(2q^2+4q^3+2q^4)M_{122}+(2q^2+4q^3+2q^4)M_{212}+(2q^2+4q^3+2q^4)M_{221}+\dots$	$(1+4q+4q^2+4q^3+q^4)\overline{L}_{1111}+2q^2\overline{L}_{112}+(1+2q^2+q^4)\overline{L}_{121}+2q^2\overline{L}_{211}+2q^2\overline{L}_{22}+(10q^2+12q^3+22q^4+16q^5+4q^6)\overline{L}_{11111}+(-q^2+10q^3+5q^4)\overline{L}_{1112}+(1+2q-2q^2+10q^3+2q^4+2q^5+3q^6)\overline{L}_{1121}+(1+2q-2q^2+10q^3+2q^4+2q^5+3q^6)\overline{L}_{1211}+(-q^2+10q^3+5q^4)\overline{L}_{2111}+(4q^3+2q^4)\overline{L}_{122}+(-2q^2+4q^3+2q^4)\overline{L}_{212}+(4q^3+2q^4)\overline{L}_{221}+\dots$
$\begin{array}{c} 4 \!-\! 2 \\ | \quad\ | \\ 1 \!-\! 3 \end{array}$	$(4+4q+8q^2+4q^3+4q^4)M_{1111}+(2+2q^4)M_{112}+4q^2M_{121}+(2+2q^4)M_{211}+(1+q^4)M_{22}+(24+24q+48q^2+48q^3+48q^4+24q^5+24q^6)M_{11111}+(10+4q+4q^2+4q^4+4q^5+10q^6)M_{1112}+(4+10q^2+8q^3+10q^4+4q^6)M_{1121}+(4+10q^2+8q^3+10q^4+4q^6)M_{1211}+(10+4q+4q^2+4q^4+4q^5+10q^6)M_{2111}+(2+2q^2+2q^4+2q^6)M_{122}+(4+4q^6)M_{212}+(2+2q^2+2q^4+2q^6)M_{221}+\dots$	$(1+4q+4q^2+4q^3+q^4)\overline{L}_{1111}+(1+q^4)\overline{L}_{112}+4q^2\overline{L}_{121}+(1+q^4)\overline{L}_{211}+(1+q^4)\overline{L}_{22}+(8q^2+16q^3+20q^4+16q^5+4q^6)\overline{L}_{11111}+(1+4q+2q^2-q^4+4q^5+4q^6)\overline{L}_{1112}+(1+8q^3+7q^4+2q^6)\overline{L}_{1121}+(1+8q^3+7q^4+2q^6)\overline{L}_{1211}+(1+4q+2q^2-q^4+4q^5+4q^6)\overline{L}_{2111}+(1+2q^2+q^4+2q^6)\overline{L}_{122}+(2-2q^4+4q^6)\overline{L}_{212}+(1+2q^2+q^4+2q^6)\overline{L}_{221}+\dots$

Table 2: Partial computations of $\overline{X}_G(q)$ for some small graphs G. The ellipses "..." mask higher order terms indexed by compositions α with $|\alpha| > |V(G)| + 1$.

Graph G	$\overline{X}_G(q)$ in the monomial basis $\{M_\alpha\}$	$\overline{X}_G(q)$ in the basis $\{\overline{L}_\alpha\}$
$\begin{matrix}4 - 3\\ \vert\times\\ 1 - 2\end{matrix}$	$(2 + 4q + 6q^2 + 6q^3 + 4q^4 + 2q^5)M_{1111} + (q^2 + q^3)M_{112} + (1 + q^5)M_{121} + (q^2 + q^3)M_{211} + (10 + 20q + 34q^2 + 44q^3 + 48q^4 + 38q^5 + 28q^6 + 14q^7 + 4q^8)M_{11111} + (3q^2 + 5q^3 + 6q^4 + 3q^5 + q^6)M_{1112} + (3 + 2q + q^2 + 3q^3 + 2q^4 + q^5 + 3q^6 + 2q^7 + q^8)M_{1121} + (3 + 2q + q^2 + 3q^3 + 2q^4 + q^5 + 3q^6 + 2q^7 + q^8)M_{1211} + (3q^2 + 5q^3 + 6q^4 + 3q^5 + q^6)M_{2111} + \ldots$	$(1 + 4q + 4q^2 + 4q^3 + 4q^4 + q^5)\overline{L}_{1111} + (q^2 + q^3)\overline{L}_{112} + (1+q^5)\overline{L}_{121} + (q^2+q^3)\overline{L}_{211} + (10q^2 + 12q^3 + 16q^4 + 26q^5 + 20q^6 + 10q^7 + 2q^8)\overline{L}_{11111} + (2q^3 + 6q^4 + 3q^5 + q^6)\overline{L}_{1112} + (1 + 2q + 2q^3 + 2q^4 - q^5 + 3q^6 + 2q^7 + q^8)\overline{L}_{1121} + (1 + 2q + 2q^3 + 2q^4 - q^5 + 3q^6 + 2q^7 + q^8)\overline{L}_{1211} + (2q^3 + 6q^4 + 3q^5 + q^6)\overline{L}_{2111} + \ldots$
$\begin{matrix}4 \quad\; 3\\ \vert \diagup\\ 1 - 2\end{matrix}$	$(6 + 6q + 6q^2 + 6q^3)M_{1111} + (3q^2 + 3q^3)M_{112} + (3 + 3q^3)M_{121} + (3 + 3q)M_{211} + q^3M_{13} + M_{31} + (42 + 42q + 48q^2 + 48q^3 + 48q^4 + 6q^5 + 6q^6)M_{11111} + (15q^2 + 15q^3 + 18q^4 + 3q^5 + 3q^6)M_{1112} + (15 + 3q^2 + 18q^3 + 15q^4 + 3q^6)M_{1121} + (18 + 15q + 3q^3 + 18q^4)M_{1211} + (18 + 18q + 18q^2)M_{2111} + (3q^3 + 3q^4 + q^6)M_{113} + (3 + q^3 + 3q^4)M_{131} + (4 + 3q)M_{311} + 6q^4M_{122} + 6q^2M_{212} + 6M_{221} + \ldots$	$(1 + 3q + 3q^2 + q^3)\overline{L}_{1111} + (3q^2 + 2q^3)\overline{L}_{112} + (2 + 2q^3)\overline{L}_{121} + (2 + 3q)\overline{L}_{211} + q^3\overline{L}_{13} + \overline{L}_{31} + (6q^2 + 12q^3 + 9q^4 + 3q^5 + q^6)\overline{L}_{11111} + (6q^3 + 9q^4 + 3q^5 + 2q^6)\overline{L}_{1112} + (2 + 8q^3 + 9q^4 + 2q^6)\overline{L}_{1121} + (5 + 9q - 2q^3 + 9q^4)\overline{L}_{1211} + (2 + 6q + 12q^2)\overline{L}_{2111} + (q^3 + 3q^4 + q^6)\overline{L}_{113} + (2 + 3q^4)\overline{L}_{131} + (2 + 3q)\overline{L}_{311} + (-q^3 + 6q^4)\overline{L}_{122} + 6q^2\overline{L}_{212} + 5\overline{L}_{221} + \ldots$

Table 3: More partial computations of $\overline{X}_G(q)$ for some small graphs G. The ellipses "$\ldots$" mask higher order terms indexed by compositions α with $|\alpha| > |V(G)| + 1$.

Order P	$G = \mathsf{inc}(P)$	$\overline{X}_G(q)$ in the Grothendieck basis $\{\overline{s}_\lambda\}$
$\begin{array}{ll} 3 & \\ \uparrow & 2 \\ 1 & \end{array}$	$\begin{array}{c} 3 \\ \backslash \\ 1 \!-\! 2 \end{array}$	$(1+2q+q^2)\overline{s}_{111} + q\overline{s}_{21} + (6+12q+10q^2+4q^3+q^4)\overline{s}_{1111} + (4q+3q^2)\overline{s}_{211} + q\overline{s}_{22} + (12q+15q^2+6q^3+2q^4)\overline{s}_{2111} + (4q+4q^2)\overline{s}_{221} + (12q+20q^2+8q^3+3q^4)\overline{s}_{2211} + (4q+5q^2)\overline{s}_{222} + (12q+25q^2+11q^3+4q^4)\overline{s}_{2221} + (12q+29q^2+15q^3+5q^4)\overline{s}_{2222} + \ldots$
$1\ \ 2\ \ 3$	$\begin{array}{c} 3 \\ / \ \backslash \\ 1 \!-\! 2 \end{array}$	$(1+2q+2q^2+q^3)\overline{s}_{111} + (6+12q+15q^2+12q^3+6q^4+3q^5)\overline{s}_{1111} + \ldots$
$\begin{array}{ll} 3 & 4 \\ \uparrow\!\nearrow\!\uparrow \\ 1 & 2 \end{array}$	$\begin{array}{c} 4 \!-\! 3 \\ \mid \\ 1 \!-\! 2 \end{array}$	$(1+3q+3q^2+q^3)\overline{s}_{1111} + (2q+2q^2)\overline{s}_{211} + (q+q^2)\overline{s}_{22} + (8+24q+30q^2+20q^3+8q^4+2q^5)\overline{s}_{11111} + (12q+18q^2+10q^3+2q^4)\overline{s}_{2111} + (6q+10q^2+4q^3)\overline{s}_{221} + (48q+90q^2+82q^3+46q^4+16q^5+6q^6)\overline{s}_{21111} + (24q+52q^2+42q^3+17q^4+3q^5)\overline{s}_{2211} + (10q+21q^2+12q^3+3q^4)\overline{s}_{222} + (80q+204q^2+228q^3+166q^4+76q^5+32q^6+8q^7)\overline{s}_{22111} + (40q+105q^2+102q^3+59q^4+18q^5+4q^6)\overline{s}_{2221} + (128q+385q^2+492q^3+426q^4+236q^5+123q^6+40q^7+12q^8)\overline{s}_{22211} + (56q+163q^2+183q^3+131q^4+56q^5+21q^6+3q^7+q^8)\overline{s}_{2222} + (192q+635q^2+908q^3+898q^4+580q^5+363q^6+144q^7+64q^8+12q^9+4q^{10})\overline{s}_{22221} + (256q+891q^2+1381q^3+1514q^4+1099q^5+797q^6+379q^7+205q^8+65q^9+28q^{10}+4q^{11}+3q^{12})\overline{s}_{22222} + \ldots$
$\begin{array}{ll} 3 & 4 \\ \uparrow\!\nearrow \\ 1 & 2 \end{array}$	$\begin{array}{c} 4 \!-\! 3 \\ \backslash\ \mid \\ 1 \!-\! 2 \end{array}$	$(1+3q+4q^2+3q^3+q^4)\overline{s}_{1111} + (q+2q^2+q^3)\overline{s}_{211} + (8+24q+37q^2+37q^3+25q^4+13q^5+5q^6+q^7)\overline{s}_{11111} + (6q+14q^2+14q^3+7q^4+3q^5)\overline{s}_{2111} + (q+3q^2+2q^3)\overline{s}_{221} + (24q+62q^2+82q^3+69q^4+45q^5+23q^6+10q^7+3q^8)\overline{s}_{21111} + (6q+20q^2+23q^3+13q^4+6q^5)\overline{s}_{2211} + (q+4q^2+3q^3)\overline{s}_{222} + (24q+86q^2+126q^3+116q^4+80q^5+42q^6+20q^7+6q^8)\overline{s}_{22111} + (6q+26q^2+34q^3+21q^4+11q^5)\overline{s}_{2221} + (24q+110q^2+183q^3+181q^4+134q^5+72q^6+36q^7+12q^8)\overline{s}_{22211} + (6q+31q^2+46q^3+31q^4+18q^5)\overline{s}_{2222} + (24q+133q^2+252q^3+267q^4+210q^5+117q^6+61q^7+22q^8)\overline{s}_{22221} + (24q+151q^2+318q^3+363q^4+300q^5+181q^6+98q^7+37q^8)\overline{s}_{22222} + \ldots$
$\begin{array}{ll} 4 & \\ \uparrow & 2\ \ 3 \\ 1 & \end{array}$	$\begin{array}{c} 4 \!-\! 3 \\ \times\ \mid \\ 1 \!-\! 2 \end{array}$	$(1+3q+5q^2+5q^3+3q^4+q^5)\overline{s}_{1111} + (q^2+q^3)\overline{s}_{211} + (8+24q+44q^2+54q^3+48q^4+34q^5+18q^6+8q^7+2q^8)\overline{s}_{11111} + (5q^2+9q^3+6q^4+4q^5)\overline{s}_{2111} + (q^2+q^3)\overline{s}_{221} + (17q^2+41q^3+46q^4+42q^5+23q^6+14q^7+5q^8)\overline{s}_{21111} + (5q^2+9q^3+7q^4+5q^5)\overline{s}_{2211} + (q^2+q^3)\overline{s}_{222} + (17q^2+41q^3+52q^4+50q^5+29q^6+18q^7+7q^8)\overline{s}_{22111} + (5q^2+9q^3+8q^4+6q^5)\overline{s}_{2221} + (17q^2+41q^3+58q^4+58q^5+36q^6+23q^7+9q^8)\overline{s}_{22211} + (5q^2+9q^3+9q^4+7q^5)\overline{s}_{2222} + (17q^2+41q^3+64q^4+66q^5+44q^6+29q^7+11q^8)\overline{s}_{22221} + (17q^2+41q^3+69q^4+73q^5+53q^6+36q^7+13q^8)\overline{s}_{22222} + \ldots$
$1\ 2\ 3\ 4$	$\begin{array}{c} 4 \!-\! 3 \\ \mid\!\times\!\mid \\ 1 \!-\! 2 \end{array}$	$(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)\overline{s}_{1111} + (8+24q+44q^2+60q^3+64q^4+56q^5+40q^6+24q^7+12q^8+4q^9)\overline{s}_{11111} + \ldots$

Table 4: Partial Grothendieck expansions of $\overline{X}_G(q)$ for incomparability graphs $G = \mathsf{inc}(P)$ of some natural unit interval orders. The ellipses "$\ldots$" mask higher order terms indexed by partitions λ with $\ell(\lambda) > |V(G)| + 1$.