Demazure Crystals for Flagged Key Polynomials

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Abstract

One definition of key polynomials is as the weight generating functions of key tableaux. Assaf and Schilling introduced a crystal structure on key tableaux and related it to Morse–Schilling crystals on reduced factorizations for permutations via weak Edelman–Greene insertion. In this paper, we consider generalizations of both crystals depending on a flag. We extend weak EG insertion to a bijection between our flagged objects and show that the recording tableau gives a crystal isomorphism. As an application, we show that flagged key tableaux have a natural Demazure crystal structure, whose characters recover Reiner and Shimozono's flagged key polynomials.

Mathematics Subject Classifications: 05E05,05E10

1 Introduction

Schur polynomials s_λ, indexed by partitions, are the weight generating functions for *semi-standard Young tableaux* of shape λ. As λ varies over all partitions with at most n parts, $\{s_\lambda\}$ forms a $\mathbb{Z}$-basis for the ring of symmetric polynomials in n variables. In representation theory, Schur polynomials are characters of irreducible polynomial representations of the general linear groups. *Key polynomials* κ_α, indexed by weak compositions, are the characters of Demazure modules, which are representations of Borel subgroups of the general linear groups [6]. Key polynomials are nonsymmetric generalizations of Schur polynomials, and $\{\kappa_\alpha\}$ forms a basis for $\mathbb{Z}[x_1, x_2, \ldots]$ as α ranges over all weak compositions. Lascoux and Schützenberger proved that key polynomials are weight generating functions of various combinatorial objects [12].

Key polynomials are also related to other important polynomials in combinatorics. For example, *Schubert polynomials* are representatives of the Schubert classes in the cohomology ring of the complete flag variety [11]. Schubert polynomials also generalize Schur polynomials in the sense that Schur polynomials are representatives of the Schubert classes indexed by *Grassmannian* permutations. Schubert polynomials are also weight generating functions for *bounded reduced factorizations*, which are certain generalizations of reduced

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words for a permutation [4, 8]; see §3 for more details. Lascoux and Schützenberger proved that Schubert polynomials expand as non-negative integer linear combinations of key polynomials; see [15, Thm. 3].

Assaf generalized semistandard Young tableaux to provide another combinatorial formula of key polynomials [1]. Namely, κ_α is the weight generating function for *(semistandard) key tableaux* of shape α. Such generalization already appeared in earlier literature such as Mason's *permuted basement semi-skyline augmented filling* in [13, §6], and one can show that Mason's generalization is equivalent to Assaf's generalization. But Assaf introduced a *weak Edelman–Greene (EG) insertion algorithm* to give a new bijective proof of the key positivity of Schubert polynomials in the later paper [2]. Assaf's algorithm is based on the *Edelman–Greene insertion*, which Edelman and Greene introduced in [7] to prove that *Stanley symmetric functions* [16] have non-negative integer coefficients in the Schur basis.

Let $\mathrm{RF}_n(w)$ be the set of all reduced factorizations for a permutation w with exactly n components. In [14], Morse and Schilling defined a $\mathfrak{gl}_n$-crystal structure on $\mathrm{RF}_n(w)$, which they related to a crystal structure on semistandard Young tableaux through EG insertion. In [3], Assaf and Schilling defined a crystal on key tableaux and proved that $\mathrm{SSKT}(\alpha)$, the set of all key tableaux with shape α, is the Demazure truncation of the highest weight crystal of the highest weight λ, where λ is the partition rearrangement of α. Then they proved that the recording tableau for weak EG insertion gives a crystal isomorphism between $\mathrm{BRF}(w)$ and $\bigsqcup_\alpha \mathrm{SSKT}(\alpha)$, where $\mathrm{BRF}(w) \subseteq \mathrm{RF}_n(w)$ is the subset of reduced factorizations that are bounded in the sense of Definition 29.

A *flag* is a weakly increasing function $\varphi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $\varphi(n) \geqslant n$ for all $n \in \mathbb{Z}_{>0}$. If $\varphi(i) = i$ for all i, then we say that φ is the *standard flag*. In both $\mathrm{BRF}(w)$ and $\mathrm{SSKT}(\alpha)$, the entries in the i-th part of the elements are bounded by i which is the i-th value of the standard flag. A natural way of generalizing these sets is to replace the standard flag by an arbitrary flag. In a *φ-flagged reduced factorization*, a given letter i may appear in the first $\varphi(i)$ parts instead of the first i parts. The i-th row of a *φ-flagged key tableau* is bounded above by $\varphi(i)$ instead of i. We denote these φ-flagged generalizations as $\mathrm{BRF}(w, \varphi)$ and $\mathrm{SSKT}(\alpha, \varphi)$, respectively.

Our main results concern crystal structures on these sets. Let φ be any flag. The set $\mathrm{BRF}(w, \varphi)$ inherits a crystal structure from [14], and we show how to extend the existing crystal structure on $\mathrm{SSKT}(\alpha)$ to the larger set $\mathrm{SSKT}(\alpha, \varphi)$. Assaf's weak EG insertion algorithm can be evaluated on flagged factorizations, and we are able to prove the following generalization of [3, Thm. 5.10]:

Proposition 1 (See Propositions 32 and 41)**.** *The weak EG insertion algorithm induces a crystal isomorphism from* $\mathrm{BRF}(w, \varphi)$ *to* $\bigsqcup_\alpha \mathrm{SSKT}(\alpha, \varphi)$.

One of our other main results gives an explicit relation between the crystals $\mathrm{BRF}(w, \varphi)$ and $\mathrm{BRF}(w)$. Let $\mathcal{B}$ be any crystal with raising operators $e_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$. The i-th *Demazure operator* $\mathfrak{D}_i^{\mathcal{B}}$ acts on subsets $X \subseteq \mathcal{B}$ by

$$\mathfrak{D}_i^{\mathcal{B}} X := \{b \in \mathcal{B} : e_i^k(b) \in X \text{ for some } k \geqslant 0\}.$$

When $\mathcal{B} = \mathrm{RF}_n(w)$, applying arbitrary sequences of these operators to highest weight elements generates a family of subcrystals called *Demazure crystals*; see Definition 36.

Theorem 2 (See Theorem 37)**.** *Let n be sufficiently large such that $\mathrm{BRF}(w, \varphi) \subseteq \mathrm{RF}_n(w)$. Then $\mathrm{BRF}(w, \varphi) = \mathfrak{D}_{i_1}^{\mathrm{RF}_n(w)} \cdots \mathfrak{D}_{i_k}^{\mathrm{RF}_n(w)} \mathrm{BRF}(w)$ for certain $i_1, \ldots, i_k \in \mathbb{Z}_{>0}$ depending only on φ.*

Corollary 3. *Each $\mathrm{BRF}(w, \varphi)$ is a disjoint union of Demazure crystals and its weight generating function is key-positive.*

It is shown in [3] that $\mathrm{SSKT}(\alpha)$ is a Demazure crystal. We can upgrade this to any flag:

Theorem 4 (See Theorem 42)**.** *Each $\mathrm{SSKT}(\alpha, \varphi)$ is a Demazure crystal.*

Taking the character of $\mathrm{SSKT}(\alpha, \varphi)$ gives a flagged generalization of key polynomials. One reason that our flagged constructions are natural to consider is that such *flagged key polynomials* have already appeared in the literature in a different form. Reiner and Shimozono considered flagged key polynomials in the context of a flagged Littlewood-Richardson rule [15, Thm. 20]. Their original definition does not involve key tableaux; however, we can prove the following:

Theorem 5. *Reiner and Shimozono's flagged key polynomial $\kappa_{(\alpha,\varphi)}$ from [15] is equal to*

$$\sum_{T \in \mathrm{SSKT}(\alpha,\varphi)} x^{\mathsf{wt}(T)}.$$

Reiner and Shimozono observed that $\kappa_{(\alpha,\varphi)} = \kappa_\beta$ for some β. Our results imply a crystal analog:

Corollary 6 (See Corollary 44)**.** *The Demazure crystal $\mathrm{SSKT}(\alpha, \varphi) \cong \mathrm{SSKT}(\beta)$ for some β.*

Other recent work [10] has also considered crystal structures on flagged objects such as flagged reversed plane partitions; however, these objects are different from what we consider in this paper.

The structure of the paper is as follows. In §2, we review the definitions of (flagged) key polynomials and key tableaux. In §3, we review Assaf's weak EG insertion algorithm and extend it to arbitrary flags. In §4, we review the Morse–Schilling crystal on $\mathrm{RF}_n(w)$ and extend the Assaf–Schilling crystal operators to $\mathrm{SSKT}(\alpha, \varphi)$. Our main results are proved in §4.

2 Preliminaries

In this section, we review some preliminaries on key polynomials and key tableaux from [3, 15, 12].

2.1 Key polynomials

Let S_∞ be the group of permutations of $\mathbb{Z}_{>0} = \{1, 2, \cdots\}$ that fix all but finitely many points. Write $s_i \in S_\infty$ for the adjacent transposition that exchanges i and $i+1$. A *word* is a finite sequence of positive integers. A minimal-length word $i_1 \cdots i_n$ with $w = s_{i_1} \cdots s_{i_n}$ is a *reduced word* for $w \in S_\infty$. Let $R(w)$ be the set of all reduced words for w. The *length* of a permutation $w \in S_\infty$ is the common length of any of its reduced words. In this article, whenever we say "reduced word" we mean an element of $R(w)$ for some $w \in S_\infty$.

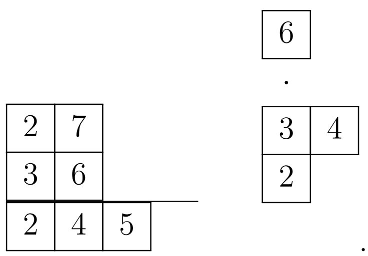
The group S_∞ acts on $\mathbb{Z}[x_1, x_2, \ldots]$ by permuting indeterminates. One has *divided difference operators* ∂_i and π_i on $\mathbb{Z}[x_1, x_2, \ldots]$ by the formulas $\partial_i(f) := \frac{f - s_i \cdot f}{x_i - x_{i+1}}$ and $\pi_i(f) := \partial_i(x_i f)$. When f is symmetric in x_i and x_{i+1} (so that $s_i \cdot f = f$), we have $\partial_i(f) = 0$ and $\pi_i(f) = f$. Both divided difference operators satisfy the braid relations for S_∞ along with $\partial_i^2 = 0$, $\pi_i^2 = \pi_i$ for all i. This means that we can define $\partial_w := \partial_{i_1} \cdots \partial_{i_n}$ and $\pi_w := \pi_{i_1} \cdots \pi_{i_n}$ for any $i_1 \cdots i_n \in R(w)$.

For $m \in \mathbb{Z}_{\geqslant 0}$, a *weak composition* α of m is an infinite tuple of non-negative integers $(\alpha_1, \alpha_2, \ldots)$ with finitely many $\alpha_i > 0$ and $\sum_{i=1}^{\infty} \alpha_i = m$. We identify finite tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with the zero-padded infinite tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$. The *length* $\ell(\alpha)$ is the minimal $n \in \mathbb{Z}_{\geqslant 0}$ such that $\alpha_i = 0$ for all $i > n$. The group S_∞ acts on weak compositions on the right by permuting components. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a weak composition such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$.

Given a weak composition α, let w be the minimal-length permutation such that $\alpha \cdot w = \lambda$, where λ is a partition. The *key polynomial* associated to α is $\kappa_\alpha := \pi_w x^\lambda$, where $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$. For example, $\kappa_{1201} = \pi_1 \pi_3 (x^{2110}) = x^{2110} + x^{1210} + x^{2101} + x^{1201}$. We often use the following recursive property of key polynomials: if $\alpha_i > \alpha_{i+1}$ then $\pi_i(\kappa_\alpha) = \kappa_{\alpha \cdot s_i}$ and otherwise $\pi_i(\kappa_\alpha) = \kappa_\alpha$.

2.2 Key tableaux

A *diagram* is a finite set of left-aligned boxes in the right half plane $\mathbb{Z} \times \mathbb{Z}_{>0}$. Most of the diagrams appearing in this paper will be in $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. If not, we draw a horizontal line between the rows indexed by 0 and 1. Throughout this paper, we use French notation when we draw diagrams, where the columns are indexed from left to right and the rows are indexed from bottom to top. The *shape* of a diagram is the weak composition recording its number of boxes in each row whose index is positive. A *tableau* is a filling of a diagram with positive integers. For example, the tableau on the left has a row with a non-positive index, and the tableau on the right is in $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$:



The *weight* of a word u is the weak composition $\mathsf{wt}(u) := (\beta_1, \beta_2, \dots)$ of $\ell(u)$, where β_i is the multiplicity of i in u. The *row reading word* of a tableau T is the word $\mathsf{row}(T) := \cdots r^{(2)}r^{(1)}r^{(0)}r^{(-1)}\cdots$, where $r^{(i)}$ is the i-th row of T read from left to right. The *weight* of a tableau T is $\mathsf{wt}(T) := \mathsf{wt}(\mathsf{row}(T))$. For example, the left tableau has row reading word 2736245 and weight $(0, 2, 1, 1, 1, 1, 1)$.

A *semistandard Young tableau* is a tableau of a partition shape whose rows are weakly increasing and whose columns are strictly increasing. A *standard Young tableau* is a semistandard Young tableau whose entries are the numbers $1, 2, \dots, n$ with no repetitions for some integer $n \geqslant 0$. The following definition of a *key tableau* is identical to [3, Def. 3.1]. Here, given a tableau T and a box (i, j) which it contains, we write $T_{i,j}$ for the entry in box (i, j).

Definition 7. A *key tableau* T is a tableau such that

(a) each row is weakly decreasing, and each column has distinct entries; and

(b) if $i < k$ and $T_{i,j} > T_{k,j}$, then T contains $(i, j+1)$ and $T_{i,j+1} > T_{k,j}$.

A *standard key tableau* is a key tableau filled by $1, 2, \dots, n$ with no repetitions for some $n \in \mathbb{Z}_{\geqslant 0}$. For example, the tableaux

$$\begin{array}{|c|c|c|c}\hline 5 & 4 & 3 & \\ \hline \multicolumn{1}{c}{\cdot} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \\ \hline 7 & 6 & 5 & \multicolumn{1}{c|}{5} \\ \hline 1 & 2 & \multicolumn{1}{c}{} & \\ \cline{1-2} \end{array} \qquad \text{and} \qquad \begin{array}{|c|c|}\hline 5 & 4 \\ \hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

are key tableaux of shapes $(2, 4, 0, 3)$ and $(2, 1, 2)$.

Fix a flag φ. We say that a key tableau T is *φ-flagged* if $T_{i,j} \leqslant \varphi(i)$ whenever (i, j) is a box in T. We denote the set of all φ-flagged key tableaux of shape α as $\mathrm{SSKT}(\alpha, \varphi)$. If ϕ is another flag such that $\varphi(i) = \phi(i)$ for all $1 \leqslant i \leqslant \ell(\alpha)$, then $\mathrm{SSKT}(\alpha, \varphi) = \mathrm{SSKT}(\alpha, \phi)$. When φ is the standard flag with $\varphi(i) = i$ for all i, we omit φ in our notation and write $\mathrm{SSKT}(\alpha, \varphi)$ as $\mathrm{SSKT}(\alpha)$. The elements of $\mathrm{SSKT}(\alpha)$ are what Assaf and Schilling called *semistandard key tableaux* in [3, Def. 3.2].

Theorem 8 ([1, Prop. 2.6])**.** *If α is a weak composition then $\kappa_\alpha = \sum_{T \in \mathrm{SSKT}(\alpha)} x^{\mathsf{wt}(T)}$.*

Example 9. If $\alpha = (1, 2, 0, 1)$ then $\kappa_\alpha = x^{2110} + x^{1210} + x^{2101} + x^{1201}$ while the set $\mathrm{SSKT}(\alpha)$ consists of four key tableaux:

$$\begin{array}{|c|c}\hline 3 & \\ \hline \multicolumn{1}{c}{\cdot} & \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline 1 & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c}\hline 3 & \\ \hline \multicolumn{1}{c}{\cdot} & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline 1 & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c}\hline 4 & \\ \hline \multicolumn{1}{c}{\cdot} & \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline 1 & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c}\hline 4 & \\ \hline \multicolumn{1}{c}{\cdot} & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline 1 & \\ \cline{1-1} \end{array}.$$

Key polynomials generalize *Schur polynomials* in the following way. Let λ be a partition such that $\ell(\lambda) \leqslant n$ and let $\mathrm{SSYT}_n(\lambda)$ be the set of semistandard Young tableaux of shape λ filled by $1, 2, \dots, n$. The *Schur polynomial* indexed by λ with n variables is $s_\lambda(x_1, \dots, x_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda)} x^{\mathsf{wt}(T)}$. It is well-known that $s_\lambda(x_1, \dots, x_n)$ is S_n-invariant [17, Thm. 7.10.2].

Suppose α is the weak composition $(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$. For this choice of α, any $T \in \mathrm{SSKT}(\alpha)$ is weakly decreasing along each row and strictly increasing along each column. We have a weight-reversing bijection from $\mathrm{SSKT}(\alpha)$ to $\mathrm{SSYT}_n(\lambda)$ sending a key tableau T to the unique semistandard Young tableau S with $S_{i,j} = n + 1 - T_{n+1-i,j}$. Hence

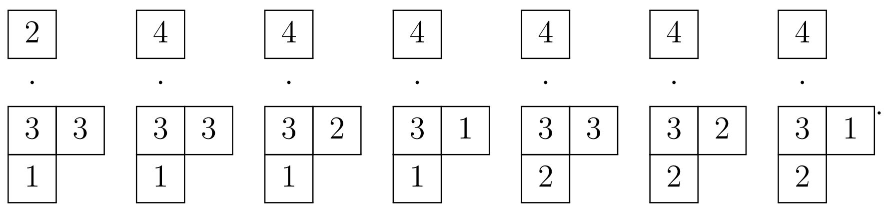
$$\kappa_\alpha = s_\lambda(x_n, x_{n-1}, \dots, x_1) = s_\lambda(x_1, \dots, x_n)$$

and so every Schur polynomial is a key polynomial.

Theorem 8 suggests a natural flagged generalization of key polynomials. Our main results imply that these flagged generalizations coincide with the flagged key polynomials defined in [15].

Definition 10. The *φ-flagged key polynomial* of a weak composition α is $\kappa_{(\alpha,\varphi)} := \sum_{T \in \mathrm{SSKT}(\alpha,\varphi)} x^{\mathsf{wt}(T)}$.

Example 11. Let $\alpha = (1, 2, 0, 1)$ as above. Let φ be the non-standard flag such that $\varphi(1) = 2$, $\varphi(2) = 3$ and $\varphi(3) = \varphi(4) = 4$. There are 7 key tableaux in $\mathrm{SSKT}(\alpha, \varphi)$ that are not in $\mathrm{SSKT}(\alpha)$:



In this case $\kappa_{(\alpha,\varphi)} = x^{2110} + x^{1210} + x^{2101} + x^{1201} + x^{1120} + x^{1021} + 2x^{1111} + x^{2011} + x^{0121} + x^{0211}$.

3 Insertion algorithms

In this section, we review Assaf's definition of *weak EG insertion* and the *lift operation* [2]. The main goal is to prove Proposition 32, which describes a flagged version of the weak EG correspondence. Most of the material in this section is either a review of [2, 3] or a mild generalization of results therein. However, Corollary 24 and Lemma 31 were not presented in [3].

3.1 Partial orders and weak EG insertion tableaux

Before we describe weak EG insertion, we need to define a partial order $\leqslant$ on $R(w)$.

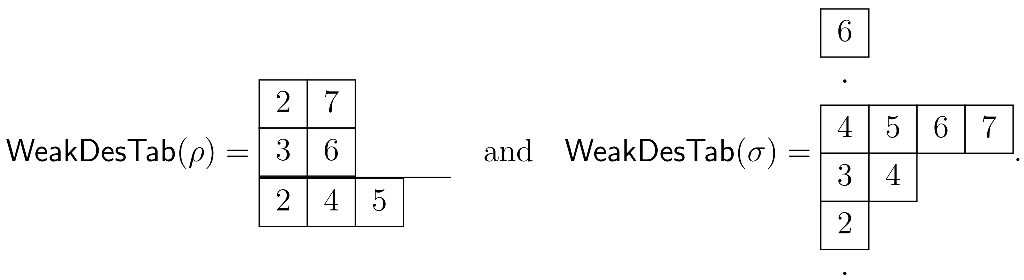
Definition 12. An *increasing factorization* $\rho^{(\bullet)}$ of a reduced word ρ partitions ρ into consecutive subwords $(\rho^{(k)}|\rho^{(k-1)}|\cdots|\rho^{(1)})$ that are each strictly increasing. Some of the subwords $\rho^{(j)}$ in $\rho^{(\bullet)}$ may be empty. We say that $\rho^{(\bullet)}$ is a *reduced factorization* for w if $\rho \in R(w)$. We denote the i-th letter of the j-th component in the factorization $\rho^{(\bullet)}$ by $\rho_i^{(j)}$. The index i counts from left to right, but the index j counts from right to left, following the convention in [2].

There are at least two canonical increasing factorizations associated to a reduced word ρ. First, the *run factorization* of ρ is the increasing factorization obtained by dividing ρ into increasing subwords of maximal length. Second, the *trivial factorization* of ρ is the increasing factorization $(\rho^{(k)}|\rho^{(k-1)}|\cdots|\rho^{(1)})$ with $k = \ell(\rho)$ in which each subword $\rho^{(j)}$ has length one. For example, the run factorization of $\rho = 2736245$ is $(27|36|245)$ and the trivial factorization is $(2|7|3|6|2|4|5)$.

Definition 13. The *weak descent tableau* of a reduced word ρ, denoted $\mathsf{WeakDesTab}(\rho)$, is the tableau constructed as follows. Suppose ρ has run factorization $(\rho^{(k)}|\rho^{(k-1)}|\cdots|\rho^{(1)})$. Place $\rho^{(k)}$ into row $\rho_1^{(k)}$. Then iterating over $i = k-1, \ldots, 2, 1$, we either place $\rho^{(i)}$ into row $\rho_1^{(i)}$ if this is below the row containing $\rho^{(i+1)}$ or place $\rho^{(i)}$ into the row directly below $\rho^{(i+1)}$ otherwise.

This may result in a tableau with boxes in rows with non-positive index.

Example 14. Suppose $\rho = 2736245$ and $\sigma = 64567342$. Then



The horizontal line in $\mathsf{WeakDesTab}(\rho)$ divides the positive and non-positive rows.

Remark 15. In [2], Assaf defined both a *descent tableau* [2, Def. 2.3] and a *weak descent tableau* [2, Def. 2.8] of a reduced word ρ. These are (accidentally) both denoted as $\mathbb{D}(\rho)$. Definition 13 refers to [2, Def. 2.8]. In general, the descent tableau and the weak descent tableau of a reduced word are different. The descent tableau of ρ can be obtained by removing all empty rows in $\mathsf{WeakDesTab}(\rho)$.

A reduced word ρ is *virtual* if $\mathsf{WeakDesTab}(\rho)$ occupies a row with a non-positive index. Otherwise, we say ρ is *non-virtual*. The *weak descent composition* of a non-virtual reduced word ρ is the shape of $\mathsf{WeakDesTab}(\rho)$, denoted by $\operatorname{des}(\rho)$. When ρ is virtual, we define $\operatorname{des}(\rho) = \varnothing$.

Example 16. Continuing from Example 14, the reduced word $\rho = 2736245$ is virtual and $\operatorname{des}(\rho) = \varnothing$ since the row consisting of 245 in the weak descent tableau has a non-positive index. But the reduced word $\sigma = 64567342$ is non-virtual and $\operatorname{des}(\sigma) = (0, 1, 2, 4, 0, 1)$.

The *Coxeter–Knuth equivalence relation* is the transitive closure of the relations on reduced words with $\mathbf{a}xyz\mathbf{b} \sim \mathbf{a}xzy\mathbf{b}$ for $y < x < z$, with $\mathbf{a}xyz\mathbf{b} \sim \mathbf{a}yxz\mathbf{b}$ for $y < z < x$, and with $\mathbf{a}i(i+1)i\mathbf{b} \sim \mathbf{a}(i+1)i(i+1)\mathbf{b}$ whenever $x, y, z, i \in \mathbb{Z}_{>0}$ and $\mathbf{a}$ and $\mathbf{b}$ are possibly empty subwords.

For weak compositions μ and σ of n, we write $\mu \leqslant \sigma$ if $\sum_{i=1}^{k} \mu_i \leqslant \sum_{i=1}^{k} \sigma_i$ for all $k \geqslant 1$. Following [2], if ρ and τ are non-virtual reduced words, then we write $\rho \leqslant \tau$ if $\rho \sim \tau$ and $\mathrm{des}(\rho) \leqslant \mathrm{des}(\tau)$ as weak compositions. This is a partial order, and each Coxeter–Knuth equivalence class of $R(w)$ contains a unique minimal non-virtual element by [2, Thm. 4.23]. A reduced word is *Yamanouchi* [2, Def. 4.13] if it is the minimal element of its Coxeter–Knuth equivalence class under $\leqslant$.

Definition 17. The *weak Edelman–Greene insertion tableau* $\hat{P}(\rho)$ of a reduced word ρ is the weak descent tableau of the unique Yamanouchi reduced word $\hat{\rho}$ with $\rho \sim \hat{\rho}$. We define $\hat{P}(\rho^{(\bullet)}) := \hat{P}(\rho)$.

Definition 17 is a generalization of the ordinary *Edelman–Greene insertion* [7, Def. 6.20], which can be defined similarly: the *Edelman–Greene insertion tableau* is the unique increasing semistandard Young tableau whose row reading word is Coxeter–Knuth equivalent to ρ. Both tableaux can be computed by explicit inductive algorithms, as we explain in the next section.

3.2 Assaf's lift operation

Here, we review Assaf's *lift operation* [2, §4.2], an algorithm to compute Yamanouchi reduced words.

The lift operation involves the following pairing procedure. Fix two increasing words $\tau = \tau_1\tau_2\cdots\tau_s$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_t$. If all letters of σ are greater than τ_s, then the pairing procedure ends and all letters of σ are unpaired. Otherwise, we pair τ_s with the largest letter σ_i such that $\tau_s \geqslant \sigma_i$. Then we repeat the pairing process with the subwords $\tau_1\cdots\tau_{s-1}$ and $\sigma_1\cdots\sigma_{i-1}$. If the unpaired letters in σ are $x_1 < \cdots < x_k$, then we can arrange τ on top of σ as

$$\begin{array}{ccccccccc} \tau^{(0)} & \tau^{(1)} & & \tau^{(2)} & & \cdots & & \tau^{(k+1)} \\ & \sigma^{(1)} & x_1 & \sigma^{(2)} & x_2 & \cdots & x_k & \sigma^{(k+1)} \end{array}$$

where $\tau^{(i)}$ and $\sigma^{(i)}$ are possibly empty consecutive subwords with the same length, whose corresponding entries are paired together.

Definition 18 ([2, Def. 4.17])**.** If $\tau^{(i)}$, x_i, and $\sigma^{(i)}$ are as above then we define

$$\mathsf{lift}(\tau,\sigma) = (\tau^{(0)}\tau^{(1)}x_1\tau^{(2)}\cdots x_k\tau^{(k+1)}|\sigma^{(1)}\breve{\sigma}^{(2)}\cdots\breve{\sigma}^{(k+1)}).$$

For each $1 \leqslant j \leqslant k$, $\breve{\sigma}^{(j+1)}$ denotes the word of length $\ell(\sigma^{(j+1)})$ with

$$\breve{\sigma}^{(j+1)}_i = \begin{cases} \sigma^{(j+1)}_i - 1 & \text{for } 1 \leqslant i \leqslant b_j \\ \sigma^{(j+1)}_i & \text{for } b_j + 1 \leqslant i \leqslant \ell(\sigma^{(j+1)}) \end{cases}$$

where $b_j \in \mathbb{Z}_{\geqslant 0}$ is maximum such that $\tau^{(j+1)}_i = \sigma^{(j+1)}_i = x_j + i$ for all $1 \leqslant i \leqslant b_j$.

An important property of lift is that if τ and σ are increasing such that $\tau\sigma$ is a reduced word, then $\mathsf{lift}(\tau, \sigma)$ is a reduced word (ignoring the division into factors), and $\mathsf{lift}(\tau, \sigma) \sim \tau\sigma$ [2, Lem. 4.18].

Definition 19 ([2, Def. 4.19])**.** Let $\rho^{(\bullet)} = (\rho^{(k)}|\cdots|\rho^{(1)})$ be an increasing factorization of a reduced word ρ. Fix $i \in [k-1]$ and suppose $(\tilde{\rho}^{(i+1)}|\tilde{\rho}^{(i)}) = \mathsf{lift}(\rho^{(i+1)}, \rho^{(i)})$. If $\rho^{(i+1)}$ and $\rho^{(i)}$ are both nonempty and $\tilde{\rho}^{(i+1)}$ begins with the same letter as $\rho^{(i+1)}$, then we define $\mathsf{lift}_i(\rho^{(\bullet)}) = (\rho^{(k)}|\cdots|\tilde{\rho}^{(i+1)}|\tilde{\rho}^{(i)}|\cdots|\rho^{(1)})$. Otherwise, let $\mathsf{lift}_i(\rho^{(\bullet)}) = \rho^{(\bullet)}$.

For $i \leqslant j$, we define the *lifting sequence* as $\mathsf{lift}_{[i,j]} = \mathsf{lift}_j \circ \mathsf{lift}_{j-1} \circ \cdots \circ \mathsf{lift}_i$. We say lift_i acts *faithfully* on an increasing factorization $\rho^{(\bullet)}$ if $\mathsf{lift}_i(\rho^{(\bullet)}) \neq \rho^{(\bullet)}$. We say $\mathsf{lift}_{[i,j]}$ acts faithfully if lift_i acts *faithfully* on $\rho^{(\bullet)}$ and lift_k acts faithfully on $\mathsf{lift}_{[i,k-1]}(\rho^{(\bullet)})$ for $i < k \leqslant j$.

Definition 20. For an increasing factorization $\rho^{(\bullet)} = (\rho^{(k)}|\cdots|\rho^{(1)})$, we construct the increasing factorization $\mathsf{lift}(\rho^{(\bullet)})$ as follows. Set $(\sigma_0)^{(\bullet)} = \rho^{(\bullet)}$, and assume $(\sigma_1)^{(\bullet)}, \ldots, (\sigma_{n-1})^{(\bullet)}$ are known.

(1) If $\mathsf{lift}_i\big((\sigma_{n-1})^{(\bullet)}\big) = (\sigma_{n-1})^{(\bullet)}$ for all i, then $\mathsf{lift}(\rho^{(\bullet)}) = (\sigma_{n-1})^{(\bullet)}$.

(2) Otherwise, set $(\sigma_n)^{(\bullet)} = \mathsf{lift}_{[i_n,j_n]}\big((\sigma_{n-1})^{(\bullet)}\big)$ where j_n is the maximum $j < k$ for which there exists $i \leqslant j$ such that $\mathsf{lift}_{[i,j]}$ acts faithfully on $(\sigma_{n-1})^{(\bullet)}$, and i_n is the minimum $i \leqslant j_n$ for which $\mathsf{lift}_{[i,j_n]}$ acts faithfully on $(\sigma_{n-1})^{(\bullet)}$.

Definition 21. For an increasing Young tableau T with reduced reading word, let $\rho^{(\bullet)}$ be the run factorization of $\mathsf{row}(T)$ and define $\mathsf{lift}(T) := \mathsf{WeakDesTab}(\mathsf{lift}(\rho^{(\bullet)}))$.

Remark 22. If $\mathsf{lift}(\rho^{(\bullet)}) = (\eta^{(k)}|\eta^{(k-1)}|\cdots|\eta^{(1)})$ then $\mathsf{lift}(T)$ is obtained by just placing the word $\eta^{(i)}$ in row $\eta_1^{(i)}$. Definition 21 is intended to match [2, Def. 4.22], which uses the same notation $\mathsf{lift}(T)$. However, Assaf's definition in [2] specifies $\mathsf{lift}(T)$ as the tableau obtained by placing $\eta^{(i)}$ in row i. This appears to be a mistake; although [2, Fig. 17] matches [2, Def. 4.22], the later figure [2, Fig. 21] and associated results use Definition 21. For example, see Figure 1.

$$T = \begin{array}{|c|c|c|c|c|}\cline{1-2} 6 & 9 \\ \cline{1-3} 3 & 7 & 8 \\ \cline{1-3} 2 & 3 & 5 \\ \hline 1 & 2 & 4 & 5 & 6 \\ \hline \end{array} \qquad \mathsf{lift}(T) = \begin{array}{|c|c|c|c|c|}\cline{1-3} 6 & 7 & 9 \\ \cline{1-3} \multicolumn{1}{c}{\cdot} \\ \multicolumn{1}{c}{\cdot} \\ \cline{1-2} 3 & 8 \\ \hline 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 4 \\ \cline{1-3} \end{array}$$

Figure 1: The run factorization of $\mathsf{row}(T)$ is $(69|378|235|12456)$, and T has lift sequences $[i_0, j_0] = [3, 3]$ and $[i_1, j_1] = [1, 1]$.

For a reduced word ρ, let $P(\rho)$ be the EG insertion tableau of ρ; see [7, Def. 6.20].

Theorem 23 ([2, Thm. 4.23])**.** *If ρ is a reduced word then $\hat{P}(\rho) = \mathsf{lift}(P(\rho))$.*

We note two useful properties of $P(\rho)$. A reduced word ρ has a *descent* at i if $\rho_i > \rho_{i+1}$ where $1 \leqslant i \leqslant \ell(\rho) - 1$. A standard Young tableau has a *descent* at i if $i + 1$ appears in one of the rows above i. Edelman and Greene proved that the EG insertion preserves descents [7, Lem. 6.28]. This property combined with Theorem 23 implies:

Corollary 24. *Suppose ρ and ρxy are reduced for $x, y \in \mathbb{Z}_{>0}$. Suppose the box in $\hat{P}(\rho x) \setminus \hat{P}(\rho)$ is in column c_x and the box in $\hat{P}(\rho xy) \setminus \hat{P}(\rho x)$ is in column c_y. Then $x < y$ if and only if $c_x < c_y$.*

$$\hat{P}(\rho) = \begin{array}{|c|c|}\hline 4 & 8 \\ \hline 3 & 6 \\ \hline \multicolumn{2}{l}{\cdot} \\ \hline 1 & 2 \\ \hline \end{array} \qquad \hat{P}(\rho x) = \begin{array}{|c|c|c|}\hline 4 & 6 & 8 \\ \hline 3 & 7 \\ \cline{1-2} \multicolumn{3}{l}{\cdot} \\ \cline{1-2} 1 & 2 \\ \cline{1-2} \end{array} \qquad \hat{P}(\rho xy) = \begin{array}{|c|c|c|c|}\hline 4 & 6 & 7 & 8 \\ \hline 3 & 7 \\ \cline{1-2} \multicolumn{4}{l}{\cdot} \\ \cline{1-2} 1 & 2 \\ \cline{1-2} \end{array}$$

Figure 2: Let $\rho = 438612$, $x = 7$, and $y = 8$. Then $c_x = 3$ and $c_y = 4$.

It also holds that if i the smallest letter in ρ then $P(\rho)_{(1,1)} = i$. Therefore:

Corollary 25. *If i is the smallest letter in ρ then the first nonempty row in $\hat{P}(\rho)$ has index i.*

3.3 Weak EG recording tableaux

This section introduces a recording tableau $\hat{Q}(\rho^{(\bullet)})$ for the weak EG insertion which slightly generalizes constructions in [2, 3]. These references defined $\hat{Q}(\rho^{(\bullet)})$ when $\rho^{(\bullet)}$ is either the trivial factorization or bounded by the standard flag. Here, we extend to arbitrary flagged factorizations.

Definition 26. The *weak Edelman–Greene recording tableau* $\hat{Q}(\rho^{(\bullet)})$ is the tableau with same shape as $\hat{P}(\rho^{(\bullet)})$ having i in the all boxes that are in $\hat{P}(\rho^{(k)}|\cdots|\rho^{(i+1)}|\rho^{(i)})$ but not in $\hat{P}(\rho^{(k)}|\cdots|\rho^{(i+1)})$. We define $\hat{Q}(\rho)$ to be the weak EG recording tableau of the trivial factorization of ρ.

The weak EG recording tableau is well-defined because if ρ and ρx are reduced for $x \in \mathbb{Z}_{>0}$ then $\hat{P}(\rho x) \setminus \hat{P}(\rho)$ has exactly one box [2, Lem. 5.8].

Example 27. Let $\rho^{(\bullet)} = (3|26|56|4)$. As we insert $\rho^{(\bullet)}$ from left to right, insertion tableaux are

$$\begin{array}{l}3\\ \cdot\\ \cdot\end{array}\quad \begin{array}{l}3\\ 2\\ \cdot\end{array}\quad \begin{array}{ll}3&6\\ 2&\\ \cdot&\end{array}\quad \begin{array}{ll}3&6\\ 2&5\\ \cdot&\cdot\end{array}\quad \begin{array}{lll}3&5&6\\ 2&5&\\ \cdot&\cdot&\end{array}\quad \begin{array}{lll}6&&\\ \cdot&&\\ \cdot&&\\ 3&5&6\\ 2&4&\\ \cdot&\cdot&\end{array}.$$

The corresponding weak EG recording tableau $\hat{Q}$ at each step is as follows:

$$\begin{array}{l}4\\ \cdot\\ \cdot\end{array}\quad \begin{array}{l}4\\ 3\\ \cdot\end{array}\quad \begin{array}{ll}4&3\\ 3&\\ \cdot&\end{array}\quad \begin{array}{ll}4&3\\ 3&2\\ \cdot&\cdot\end{array}\quad \begin{array}{lll}4&3&2\\ 3&2&\\ \cdot&\cdot&\end{array}\quad \begin{array}{lll}1&&\\ \cdot&&\\ \cdot&&\\ 4&3&2\\ 3&2&\\ \cdot&\cdot&\end{array}.$$

The final tableaux on the right are $\hat{P}(\rho^{(\bullet)})$ and $\hat{Q}(\rho^{(\bullet)})$.

Let $\beta = (\beta_1, \beta_2, \dots)$ be the weight of a key tableau T. We *standardize* T by the following procedure: first replace all the 1's by $1, 2, \dots, \beta_1$ from right to left, then replace all the 2's by $\beta_1+1, \beta_1+2, \dots, \beta_1+\beta_2$, and so on. Denote the result by $\mathsf{std}_{\mathsf{key}}(T)$, which is a standard key tableau of the same shape as T. If ρ is a reduced word, then $\mathsf{std}_{\mathsf{key}}\big(\hat{Q}(\rho^{(\bullet)})\big)$ is equal to $\hat{Q}(\rho)$. If

$$T = \begin{array}{llll}3&&&\\ \cdot&&&\\ 2&2&&\\ 5&4&3&1\\ \cdot&\cdot&\cdot&\cdot\end{array} \qquad \text{then} \qquad \mathsf{std}_{\mathsf{key}}(T) = \begin{array}{llll}5&&&\\ \cdot&&&\\ 3&2&&\\ 7&6&4&1\\ \cdot&\cdot&\cdot&\cdot\end{array}.$$

Lemma 28. *The weak EG recording tableau $\hat{Q}(\rho^{(\bullet)})$ is a key tableau.*

Proof. By [2, Thm. 5.9], $\mathsf{std}_{\mathsf{key}}\big(\hat{Q}(\rho^{(\bullet)})\big)$ is a standard key tableau. The conditions in Definition 7 follow by the definition of standardization. □

Fix a flag φ, that is, a weakly increasing map $\varphi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with $\varphi(i) \geqslant i$.

Definition 29. We say that $\rho^{(\bullet)}$ is *φ-flagged* if for each $i \geqslant 1$, the first entry of block i satisfies $\varphi(\rho_1^{(i)}) \geqslant i$ when $\rho^{(i)}$ is nonempty. We denote the set of all φ-flagged reduced factorizations for w as $\mathrm{BRF}(w, \varphi)$. If φ is the standard flag, then we omit φ in the notation and write $\mathrm{BRF}(w)$, which is the same as the set of *reduced factorizations with cutoff* defined in [3, Def. 5.4].

Remark 30. Fix $n \in \mathbb{Z}_{>0}$ and let $\mathrm{BRF}_n(w, \varphi)$ be the subset of $\mathrm{BRF}(w, \varphi)$ consisting of factorizations whose nonempty components lie in the first n components. Let $t = \min\{i \in \mathbb{Z}_{>0} : \varphi(i) \geqslant n\}$. If ϕ is another flag such that $\varphi(j) = \phi(j)$ for all $1 \leqslant j \leqslant t$, then $\mathrm{BRF}_n(w, \varphi) = \mathrm{BRF}_n(w, \phi)$.

Lemma 31. *The weak EG recording tableau $\hat{Q}(\rho^{(\bullet)})$ is φ-flagged (in the sense that any entry in the i-th row is at most $\varphi(i)$) if and only if $\rho^{(\bullet)}$ is φ-flagged.*

Proof. Suppose $\hat{Q}(\rho^{(\bullet)})$ is φ-flagged. Let $\hat{Q}(\rho^{(\bullet)})|_{[i,k]}$ be the restriction to boxes with entries in $\{i, \ldots, k\}$. For $1 \leqslant i \leqslant k$, $\hat{Q}(\rho^{(\bullet)})|_{[i,k]}$ is φ-flagged. Assume $\rho^{(i)}$ is nonempty, and suppose the first nonempty row of $\hat{Q}(\rho^{(\bullet)})|_{[i,k]}$ is row l_i with the maximum entry j. By the flagged condition, we have $i \leqslant j \leqslant \varphi(l_i)$. Because the minimal letter of $\rho^{(k)}\rho^{(k-1)} \cdots \rho^{(i)}$ is l_i by Corollary 25, the first entry $\rho_1^{(i)}$ of the i-th component is at least l_i. Hence, $\varphi(\rho_1^{(i)}) \geqslant \varphi(l_i) \geqslant i$. Thus, $\rho^{(\bullet)}$ is φ-flagged.

Conversely, suppose $\rho^{(\bullet)}$ is φ-flagged. The letters in $\rho^{(k)} \cdots \rho^{(i)}$ are bounded below by l_i, where $l_i = \min\{n : \varphi(n) \geqslant i\}$. By Corollary 25, $\hat{Q}(\rho^{(\bullet)})|_{[i,k]}$ has no boxes below row l_i. If an i-entry in $\hat{Q}(\rho^{(\bullet)})$ shows up in row j, then we have $l_i \leqslant j$ and $i \leqslant \varphi(l_i) \leqslant \varphi(j)$. Hence, $\hat{Q}(\rho^{(\bullet)})$ is φ-flagged. □

Given a weak composition α, let $\mathrm{YR}_\alpha(w)$ be the set of Yamanouchi reduced words σ for w such that $\mathrm{des}(\sigma) = \alpha$. The map $\mathsf{WeakDesTab}(-) : \mathrm{YR}_\alpha(w) \to \{\hat{P}(\rho) : \rho \in R(w), \mathrm{sh}\left(\hat{P}(\rho)\right) = \alpha\}$ is a bijection and the inverse map is given by $\mathsf{row}(-)$. Here we generalize [3, Cor. 5.8] to all flags.

Proposition 32. *The weak EG insertion map $\rho^{(\bullet)} \mapsto \left(\hat{P}(\rho^{(\bullet)}), \hat{Q}(\rho^{(\bullet)})\right)$ is a weight-preserving bijection $\mathrm{BRF}(w, \varphi) \to \bigsqcup_\alpha \left(\mathrm{YR}_\alpha(w) \times \mathrm{SSKT}(\alpha, \varphi)\right)$.*

Proof. Suppose $\mathrm{YR}_\alpha(w)$ is nonempty and $(\hat{P}, \hat{Q}) \in \mathrm{YR}_\alpha(w) \times \mathrm{SSKT}(\alpha, \varphi)$. There exists a unique reduced word ρ such that $\hat{P}(\rho) = \hat{P}$ and $\hat{Q}(\rho) = \mathsf{std}_{\mathsf{key}}(\hat{Q})$ by [2, Cor. 5.12]. Let $\beta = (\beta_1, \ldots, \beta_k)$ be the weight of $\hat{Q}$ and define $\rho^{(\bullet)} = (\rho^{(k)}| \cdots |\rho^{(1)})$ to be the unique factorization of ρ with $\ell(\rho^{(i)}) = \beta_i$. The factorization is increasing by Corollary 24 and by construction $\hat{Q}(\rho^{(\bullet)}) = \hat{Q}$. Since $\hat{Q}$ is φ-flagged, Lemma 31 implies that $\rho^{(\bullet)}$ is φ-flagged. Hence, the weak EG insertion map is surjective.

Suppose $\sigma^{(\bullet)} \in \mathrm{BRF}(w, \varphi)$ such that $(\hat{P}(\rho^{(\bullet)}), \hat{Q}(\rho^{(\bullet)})) = (\hat{P}(\sigma^{(\bullet)}), \hat{Q}(\sigma^{(\bullet)}))$. Then $\mathsf{std}_{\mathsf{key}}(\hat{Q}(\rho^{(\bullet)})) = \mathsf{std}_{\mathsf{key}}(\hat{Q}(\sigma^{(\bullet)}))$, so $\rho = \sigma$ by [2, Cor. 5.12]. Hence $\rho^{(\bullet)} = \sigma^{(\bullet)}$ because $\mathsf{wt}(\hat{Q}(\rho^{(\bullet)})) = \mathsf{wt}(\hat{Q}(\sigma^{(\bullet)}))$. □

4 Crystal structures

In this section, we prove our main results from the introduction. Throughout, we fix $n \in \mathbb{Z}_{>0}$, $w \in S_\infty$, a flag φ, and a weak composition α. For a positive integer k, let $[k] := \{1, 2, \ldots, k\}$.

4.1 Crystal structure on reduced factorizations

We begin with the basics of *Morse–Schilling crystals* on reduced factorizations [14, §3.2]. All *crystals* refer to $\mathfrak{gl}_n$*-crystals* in the sense of [5]. Such a crystal consists of a finite set $\mathcal{B}$ with *raising* and *lowering operators* $e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ indexed by $1 \leqslant i \leqslant n-1$, along with a weight function wt taking values in $\mathbb{Z}^n$. It is required that for any $b, c \in \mathcal{B}$, we have $e_i(b) = c$ if and only if $f_i(c) = b$, and in this case $\mathsf{wt}(c) = \mathsf{wt}(b) + \mathbf{e}_i - \mathbf{e}_{i+1}$ where $\mathbf{e}_i$ is the i-th standard basis vector of $\mathbb{Z}^n$. Literature on crystals sometimes involves additional axioms, but we will not impose any of those here.

The *crystal graph* of $\mathcal{B}$ is a directed graph with vertices in $\mathcal{B}$ and edges labeled by $[n-1]$. For $x, y \in \mathcal{B}$, we draw an edge $x \xrightarrow{i} y$ if $f_i(x) = y$. A *connected component* of $\mathcal{B}$ is a subset of $\mathcal{B}$ whose elements form a connected component in the crystal graph of $\mathcal{B}$. The *character* of a finite crystal $\mathcal{B}$ is the Laurent polynomial $\mathrm{ch}(\mathcal{B}) := \sum_{b \in \mathcal{B}} x^{\mathsf{wt}(b)}$. A *crystal isomorphism* between crystals $\mathcal{B}$ and $\mathcal{C}$ is a weight-preserving bijection $\mathcal{B} \to \mathcal{C}$ that commutes with all raising and lowering operators.

Let $\mathrm{RF}_n(w)$ be the set of all reduced factorizations $r^{(\bullet)} = (r^{(n)}|\cdots|r^{(1)})$ for $w \in S_\infty$ (as specified in Definition 12) with exactly n components, some of which may be empty. Suppose $r^{(\bullet)} = (r^{(n)}|\cdots|r^{(1)}) \in \mathrm{RF}_n(w)$. The operators e_i and f_i applied to $r^{(\bullet)}$ only change the factors $r^{(i+1)}$ and $r^{(i)}$. The definition of these operators depends on the following pairing procedure.

Starting with the largest element b in $r^{(i)}$, pair it with the smallest element a in $r^{(i+1)}$ with $a > b$. If there is no such a then b is unpaired. Next, we pair the second-largest element b' in $r^{(i)}$ with the smallest unpaired element a' in $r^{(i+1)}$ with $a' > b'$. If there is no such a' then b' is unpaired. We continue this procedure for the remaining elements of $r^{(i)}$ in decreasing order, ignoring at each stage any elements in $r^{(i+1)}$ that have already been paired. Once the procedure ends, we define

$$\begin{aligned} R_i(r^{(\bullet)}) &= \{b \in r^{(i)} : b \text{ is unpaired in the pairing of } r^{(i+1)}r^{(i)}\}, \\ L_i(r^{(\bullet)}) &= \{a \in r^{(i+1)} : a \text{ is unpaired in the pairing of } r^{(i+1)}r^{(i)}\}. \end{aligned} \tag{4.1}$$

Then $f_i(r^{(\bullet)})$ and $e_i(r^{(\bullet)})$ are given as follows:

Definition 33. If $R_i(r^{(\bullet)}) = \varnothing$ then $f_i(r^{(\bullet)}) = 0$. Otherwise, $f_i(r^{(\bullet)})$ is obtained by replacing $r^{(i+1)}$ and $r^{(i)}$ with $\tilde{r}^{(i+1)}$ and $\tilde{r}^{(i)}$, respectively, where

$$\tilde{r}^{(i)} = r^{(i)} \setminus \{b\} \quad \text{and} \quad \tilde{r}^{(i+1)} = r^{(i+1)} \cup \{b - t\}$$

for $b = \min R_i(r^{(\bullet)})$ and $t = \min\{j \geqslant 0 : b - j - 1 \notin r^{(i)}\}$. Similarly, if $L_i(r^{(\bullet)}) = \varnothing$ then $e_i(r^{(\bullet)}) = 0$. Otherwise, $e_i(r^{(\bullet)})$ is obtained by replacing $r^{(i+1)}$ and $r^{(i)}$ with $\tilde{r}^{(i+1)}$ and $\tilde{r}^{(i)}$, respectively, where

$$\tilde{r}^{(i)} = r^{(i)} \cup \{a + s\} \quad \text{and} \quad \tilde{r}^{(i+1)} = r^{(i+1)} \setminus \{a\}$$

for $a = \max L_i(r^{(\bullet)})$ and $s = \min\{j \geqslant 0 : a + j + 1 \notin r^{(i+1)}\}$.

The pairing procedure ensures that $b-t+1,\ldots,b-1,b \in r^{(i+1)}$ and $a, a+1, \ldots, a+s-1 \in r^{(i)}$. Since r is reduced, we have $b-t \notin r^{(i+1)}$ and $a+s \notin r^{(i)}$.

Example 34. Let $r^{(\bullet)} = (268|14|345)$. Then $L_1(r^{(\bullet)}) = \{1\}$ and $R_1(r^{(\bullet)}) = \{5,4\}$, so $f_1(r^{(\bullet)}) = (268|134|35)$ and $e_1(r^{(\bullet)}) = (268|4|1345)$.

For $r^{(\bullet)} \in \mathrm{RF}_n(w)$, let $\mathsf{wt}(r^{(\bullet)}) := (\ell_1, \ell_2, \ldots, \ell_n)$ where ℓ_i is the length of $r^{(i)}$. For example, this gives $\mathsf{wt}((268|134|35)) = (2,3,3)$. The operators f_i and e_i for $1 \leqslant i < n$ and the weight function wt define a *normal* $\mathfrak{gl}_n$-crystal structure on $\mathrm{RF}_n(w)$, i.e. a $\mathfrak{gl}_n$-crystal arises from representations of the *quantized enveloping algebra* $\mathcal{U}_q(A_{n-1})$ [14, Thm. 3.5]. Additionally, if $e_i(r^{(\bullet)}) \neq 0$ then the underlying reduced words of $r^{(\bullet)}$ and $e_i(r^{(\bullet)})$ are Coxeter–Knuth equivalent by [14, Thm. 4.11].

Lemma 35. *Assume $r^{(\bullet)} \in \mathrm{RF}_n(w)$ is φ-flagged.*

(1) $e_i(r^{(\bullet)})$ is φ-flagged or zero.

(2) If $j = \varphi(i) - 1 > \varphi(i-1)$ then $f_j(r^{(\bullet)})$ is φ-flagged or zero.

Proof. Assume $r^{(\bullet)}$ is φ-flagged. Each number i only appears in the rightmost $\varphi(i)$ components. Since $e_i(r^{(\bullet)}) \neq 0$ is formed by removing a from $r^{(i+1)}$ and adding $a+s \geqslant a$ to $r^{(i)}$, it is φ-flagged.

Suppose $j = \varphi(i) - 1 > \varphi(i-1)$. The factorization $f_j(r^{(\bullet)})$ is obtained from $r^{(\bullet)}$ by removing $b = \min R_j(r^{(\bullet)})$ from block j and adding $b-t$ to block $j+1$, where $t \geqslant 0$ is minimal such that $b-t-1 \notin r^{(j)}$. Since $r^{(\bullet)}$ is φ-flagged, $f_j(r^{(\bullet)})$ can fail to be φ-flagged only if $\varphi(b-t) < j+1 = \varphi(i)$, which can only happen if $b-t < i$. However, as $i-1$ can only show up in the first $\varphi(i-1) < j$ blocks of $r^{(\bullet)}$, we must have $i-1 \notin r^{(j)}$ so $b-t \geqslant i$. Thus $f_j(r^{(\bullet)})$ is φ-flagged. □

4.2 Demazure crystals

Recall that if $\mathcal{B}$ is a crystal and $X \subseteq \mathcal{B}$, then we define $\mathfrak{D}_i^{\mathcal{B}} X := \{b \in \mathcal{B} : e_i^k(b) \in X \text{ for some } k \geqslant 0\}$. We abbreviate the *Demazure operator* $\mathfrak{D}_i^{\mathcal{B}}$ as $\mathfrak{D}_i$ if $\mathcal{B}$ is clear from the context.

Suppose $u^{(\bullet)} \in \mathrm{RF}_n(w)$ is a *highest weight element* in the sense that $e_i(u^{(\bullet)}) = 0$ for all $1 \leqslant i < n$. If $i_1 \cdots i_k, j_1 \cdots j_k \in R(\sigma)$ for some $\sigma \in S_n$ and $\mathfrak{D}_i := \mathfrak{D}_i^{\mathrm{RF}_n(w)}$, then $\mathfrak{D}_{i_1} \cdots \mathfrak{D}_{i_k}\{u^{(\bullet)}\} = \mathfrak{D}_{j_1} \cdots \mathfrak{D}_{j_k}\{u^{(\bullet)}\}$ [5, Thm. 13.5]. We can therefore define $\mathfrak{D}_\sigma\{u^{(\bullet)}\} := \mathfrak{D}_{i_1} \cdots \mathfrak{D}_{i_k}\{u^{(\bullet)}\}$ when $i_1 \cdots i_k \in R(\sigma)$ and refer to $\mathfrak{D}_\sigma\{u^{(\bullet)}\}$ as a *Demazure subcrystal* of $\mathrm{RF}_n(w)$. We view $\mathfrak{D}_\sigma\{u^{(\bullet)}\}$ as a crystal by redefining e_i and f_i to act as zero whenever they would send an element outside the subset. The Demazure character formula [5, Thm. 13.7] implies that $\mathrm{ch}(\mathfrak{D}_\sigma\{u^{(\bullet)}\}) = \pi_\sigma x^{\mathsf{wt}(u^{(\bullet)})}$.

Definition 36. A *Demazure crystal* is a crystal isomorphic to a Demazure subcrystal of $\mathrm{RF}_n(w)$ for some $w \in S_\infty$ and some $n \in \mathbb{Z}_{>0}$.

It is known (see [9]) that two Demazure crystals are isomorphic if and only if they have the same character, and that every κ_α (with $\ell(\alpha) \leqslant n$) occurs as the character of some Demazure crystal. Moreover, any Demazure crystal that can be embedded in $\mathrm{RF}_n(w)$ must be equal to $\mathfrak{D}_\sigma\{u^{(\bullet)}\}$ for some $\sigma \in S_n$ and some highest weight element $u^{(\bullet)}$.

In [3, Thm. 5.11], Assaf and Schilling showed that $\mathrm{BRF}(w)$ is a union of Demazure crystals. If φ is a flag, we denote $\varphi - \mathbf{e}_i : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ to be the function such that $(\varphi - \mathbf{e}_i)(l) = \varphi(l) - \delta_{il}$, where δ_{il} is the Kronecker delta. The following theorem is the main ingredient to prove Theorem 2.

Theorem 37. *Suppose φ is non-standard, and $i \in \mathbb{Z}_{>0}$ is minimal with $\varphi(i) > i$. Let $j = \varphi(i) - 1$. If $j \geqslant n$ then* $\mathrm{BRF}_n(w, \varphi) = \mathrm{BRF}_n(w, \varphi - \boldsymbol{e}_i)$. *Otherwise,* $\mathrm{BRF}_n(w, \varphi) = \mathfrak{D}_j^{\mathrm{RF}_n(w)}\mathrm{BRF}_n(w, \varphi - \boldsymbol{e}_i)$.

Proof. The flagged condition says that i can only show up in the first $j + 1 = \varphi(i)$ blocks of any factorization in $\mathrm{BRF}_n(w, \varphi)$, counting from right to left. However, i can only show up in the first j blocks for any factorization in $\mathrm{BRF}_n(w, \varphi - \mathbf{e}_i)$. Therefore, we have $\mathrm{BRF}_n(w, \varphi - \mathbf{e}_i) \subseteq \mathrm{BRF}_n(w, \varphi)$. When $j \geqslant n$, the desired equality holds by Remark 30. Assume $j < n$ from now on.

Suppose $r^{(\bullet)} \in \mathrm{BRF}_n(w, \varphi) \setminus \mathrm{BRF}_n(w, \varphi - \mathbf{e}_i)$. Then $r^{(j+1)}$ must start with i, but $r^{(j)}$ does not contain any number smaller than i since

$$\varphi(i - 1) = i - 1 < i \leqslant \varphi(i) - 1 = j.$$

Therefore, we will have $i \in L_j(r^{(\bullet)})$ when we pair $r^{(j+1)}$ and $r^{(j)}$. The operator e_j removes the largest element $a \in L_j(r^{(\bullet)})$ from $r^{(j+1)}$ and adds $a + s$ to $r^{(j)}$, where s is the smallest non-negative integer such that $a + s + 1 \notin r^{(j+1)}$. If $a = i$, then block $j + 1$ of $e_j(r^{(\bullet)})$ does not contain i and $e_j(r^{(\bullet)}) \in \mathrm{BRF}_n(w, \varphi - \mathbf{e}_i)$. If $a \neq i$, then we have $a + s \geqslant a > i$ so $e_j(r^{(\bullet)}) \in \mathrm{BRF}_n(w, \varphi) \setminus \mathrm{BRF}_n(w, \varphi - \mathbf{e}_i)$ and $i \in L_j\left(e_j(r^{(\bullet)})\right)$. Since we can only apply e_j a finite number of times before reaching zero, we must have $e_j^k(r^{(\bullet)}) \in \mathrm{BRF}_n(w, \varphi - \mathbf{e}_i)$ for some $k \geqslant 0$, so $\mathrm{BRF}_n(w, \varphi) \subseteq \mathfrak{D}_j\mathrm{BRF}_n(w, \varphi - \mathbf{e}_i)$.

Conversely, suppose $r^{(\bullet)} \in \mathfrak{D}_j\mathrm{BRF}_n(w, \varphi - \mathbf{e}_i)$. We want to show that $r^{(\bullet)}$ is φ-flagged. By definition, $e_j^k(r^{(\bullet)}) \in \mathrm{BRF}_n(w, \varphi - \mathbf{e}_i)$ for some $k \geqslant 0$, which is the same as saying $f_j^k(u^{(\bullet)}) = r^{(\bullet)}$ for some $u^{(\bullet)} \in \mathrm{BRF}_n(w, \varphi - \mathbf{e}_i)$. Fix $v^{(\bullet)} \in \mathrm{BRF}_n(w, \varphi)$ such that $f_j(v^{(\bullet)}) \neq 0$. Since $\mathrm{BRF}_n(w, \varphi - \mathbf{e}_i) \subseteq \mathrm{BRF}_n(w, \varphi)$, it suffices to show that $f_j(v^{(\bullet)}) \in \mathrm{BRF}_n(w, \varphi)$, which follows from Lemma 35 because $\varphi(i - 1) = i - 1$ and $j = \varphi(i) - 1 \geqslant (i + 1) - 1 = i$. □

For $n \geqslant b > a \geqslant 1$, let $\mathfrak{D}_{b\downarrow a} := \mathfrak{D}_{b-1}^{\mathrm{RF}_n(w)}\mathfrak{D}_{b-2}^{\mathrm{RF}_n(w)} \cdots \mathfrak{D}_a^{\mathrm{RF}_n(w)}$ and $\mathfrak{D}_{a\downarrow a} := \mathrm{id}$.

Corollary 38. *Let $t_i = \min\{n, \varphi(i)\}$ for $1 \leqslant i \leqslant n$. Then we have*

$$\mathrm{BRF}_n(w, \varphi) = \mathfrak{D}_{t_1\downarrow 1}\mathfrak{D}_{t_2\downarrow 2} \cdots \mathfrak{D}_{t_n\downarrow n}\mathrm{BRF}_n(w). \tag{4.2}$$

As a consequence, $\sum_{\rho^{(\bullet)} \in \mathrm{BRF}_n(w,\varphi)} x^{\mathsf{wt}\rho^{(\bullet)}}$ is key-positive.

Proof. Let $\overline{\varphi}(i) = \min\{n, \varphi(i)\}$ for all $1 \leqslant i \leqslant n$. Since $\mathrm{BRF}_n(w, \varphi) = \mathrm{BRF}_n(w, \overline{\varphi})$, (4.2) follows by repeatedly applying Theorem 37. Because $\mathrm{BRF}_n(w)$ is a disjoint union of Demazure crystals, so is $\mathrm{BRF}_n(w, \varphi)$; hence the character of $\mathrm{BRF}_n(w, \varphi)$ is key-positive. □

4.3 Crystal structures on key tableaux

Assaf and Schilling [3, §3.2] defined a $\mathfrak{gl}_n$-crystal on key tableaux, and they proved that $\mathrm{SSKT}(\alpha)$ is a Demazure crystal. Instead of reviewing their definition, we summarize some important properties.

First, the raising operator e_i specified in [3, Def. 3.7] can be applied to all key tableaux T not just the semistandard ones. If T is a key tableau, then $e_i(T)$ is obtained by changing some entries $i+1$ in the same row of T to i and then changing all i's in the same columns as these entries to $i+1$'s. All entries $i+1$ changed by e_i are in consecutive columns, and each of these entries has an i above it except for the rightmost one; see the proof of [3, Prop. 3.8].

Proposition 39. *If* $T \in \mathrm{SSKT}(\alpha, \varphi)$ *then* $e_i(T) \in \mathrm{SSKT}(\alpha, \varphi) \sqcup \{0\}$.

Proof. Assume $e_i(T) \neq 0$ and those $i+1$ changed by e_i are in row r. Then $i+1 \leqslant \varphi(r)$ by the flagged condition. Replacing an $i+1$ by i does not violate the flagged condition. For an entry i in row r' replaced by $i+1$, we have $r' > r$ by the observation in the previous paragraph. Hence, we have $i+1 \leqslant \varphi(r) \leqslant \varphi(r')$, so $e_i(T) \in \mathrm{SSKT}(\alpha, \varphi)$. □

There is a unique way to define *lowering operators* $f_i : \mathrm{SSKT}(\alpha, \varphi) \to \mathrm{SSKT}(\alpha, \varphi) \sqcup \{0\}$ such that $e_i(T) = U$ if and only if $T = f_i(U)$ for $T, U \in \mathrm{SSKT}(\alpha, \varphi)$; see [3, Def. 3.10], which also applies in our φ-flagged case. By the previous proposition, we can view the set $\mathrm{SSKT}(\alpha, \varphi)$ as a $\mathfrak{gl}_n$-crystal with raising operators e_i for all n with $\varphi(\ell(\alpha)) \leqslant n$. The bound on n is necessary and sufficient for the weight function of $\mathrm{SSKT}(\alpha, \varphi)$ to take values in $\mathbb{Z}^n$. Let λ be the partition rearrangement of α and λ^T be the transpose of λ. Inspecting [3, Def. 3.7] gives the following proposition:

Proposition 40. *The unique highest weight element in* $\mathrm{SSKT}(\alpha, \varphi)$ *is the tableau in which column* i *is filled by* $1, 2, \ldots, \lambda_i^T$ *from bottom to top.*

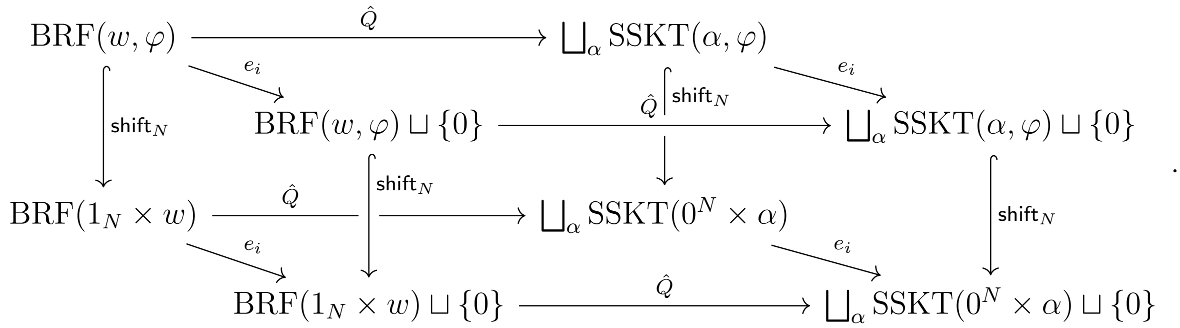
Assaf and Schilling showed that the crystal operators for $\mathrm{BRF}(w)$ and $\mathrm{SSKT}(\alpha)$ commute with the weak EG recording tableau $\hat{Q}(-)$ [3, Thm. 5.10]. Our goal is to show that this relation remains true for all flags. Before we proceed to the proof, we need to define two shifting maps: one on $\mathrm{BRF}(w, \varphi)$, and the other on $\mathrm{SSKT}(\alpha, \varphi)$. These shifting maps commute with the crystal operators e_i. Hence, the action of e_i on $\mathrm{BRF}(w, \varphi)$ and $\mathrm{SSKT}(\alpha, \varphi)$ can be computed from the action of e_i in the case of the standard flag.

Let N be a positive integer. If $w \in S_m$ then $1_N \times w \in S_{N+m}$ is the permutation fixing $[N]$ that has $(1_N \times w)(i+N) = w(i) + N$ for all $i \in \mathbb{Z}_{>0}$. We define $\mathsf{shift}_N : \mathrm{RF}_n(w) \to \mathrm{RF}_n(1_N \times w)$ by adding N to each letter of every factor. Since the flagged condition on reduced factorizations gives a lower bound for the letters in each component, the shifting map shift_N sends $\mathrm{BRF}(w, \varphi) \hookrightarrow \mathrm{BRF}(1_N \times w, \varphi)$.

For any key tableau T, we define $\mathsf{shift}_N(T)$ to be the key tableau obtained from T by shifting up N rows. By the definition, shift_N preserves the flagged condition. Then shift_N sends $\mathrm{SSKT}(\alpha, \varphi) \hookrightarrow \mathrm{SSKT}(0^N \times \alpha, \varphi)$, where $0^N \times \alpha$ is the weak composition by adding N 0's at the beginning of α. Now, we are ready to prove the φ-flagged analog of [3, Thm. 5.10].

Proposition 41. *Given* $r^{(\bullet)} \in \mathrm{BRF}(w,\varphi)$ *and any* $i > 0$*, if* $e_i(r^{(\bullet)}) \neq 0$*, then* $\hat{P}\left(e_i(r^{(\bullet)})\right) = \hat{P}(r^{(\bullet)})$ *and* $\hat{Q}(e_i(r^{(\bullet)})) = e_i\big(\hat{Q}(r^{(\bullet)})\big)$.

Proof. By [14, Thm. 4.11], e_i preserves the Coxeter–Knuth equivalence relation. Thus, we have $\hat{P}\left(e_i(r^{(\bullet)})\right) = \hat{P}(r^{(\bullet)})$. Now let N be a large positive integer such that the images of shift_N lie in the relevant subsets corresponding to the standard flag. Let $\hat{Q}(0) = 0$, $\mathsf{shift}_N(0) = 0$ and consider the following diagram:



The proposition is equivalent to saying that the top face of this diagram commutes. The bottom face commutes by [3, Thm. 5.10]. Crystal operators e_i for both reduced factorizations and key tableaux only depend on the relative order of the entries, so shift_N commutes with crystal operators e_i. Also, shift_N commutes with $\hat{Q}(-)$ because $\hat{P}(\mathsf{shift}_N(r^{(\bullet)}))$ is obtained from $\hat{P}(r^{(\bullet)})$ by shifting all boxes up by N rows and adding N to all entries. Hence, all vertical faces commute. Since the rightmost vertical arrow is injective, the top face must commute. □

The following theorem is a generalization of [3, Thm. 3.14], which asserts that $\mathrm{SSKT}(\alpha)$ is a Demazure crystal with character κ_α. For positive integers $t > s$ define $\pi_{t\downarrow s} := \pi_{t-1}\pi_{t-2}\cdots\pi_s$ and let $\pi_{t\downarrow t} := \mathrm{id}$.

Theorem 42. *Suppose* α *is a weak composition with length* k *and* φ *is any flag. Then* $\mathrm{SSKT}(\alpha,\varphi)$ *is a Demazure crystal (of type* $\mathfrak{gl}_n$ *for any* $n \geqslant \varphi(k)$*) with character* $\kappa_{(\alpha,\varphi)} = \pi_{\varphi(1)\downarrow 1}\pi_{\varphi(2)\downarrow 2}\cdots\pi_{\varphi(k)\downarrow k}(\kappa_\alpha)$.

Proof. By [3, Thm. 3.14], $\mathrm{SSKT}(\alpha)$ is a Demazure crystal. Suppose $\varphi(k) \leqslant n$ and $\mathrm{SSKT}(\alpha)$ can be embedded into $\mathrm{RF}_n(w)$ for some $w \in S_\infty$; hence, $\mathrm{YR}_\alpha(w) \neq \varnothing$. Choose any $T \in \mathrm{YR}_\alpha(w)$. Define $\mathcal{C}$ to be the set of $\rho^{(\bullet)} \in \mathrm{BRF}(w)$ with $\hat{P}(\rho^{(\bullet)}) = T$. Let $\mathcal{C}'$ be the set of $\rho^{(\bullet)} \in \mathrm{BRF}(w,\varphi)$ with $\hat{P}(\rho^{(\bullet)}) = T$. By Proposition 32 and Proposition 41, the map $\hat{Q}(-)$ is a crystal isomorphism $\mathrm{SSKT}(\alpha) \cong \mathcal{C}$ and $\mathrm{SSKT}(\alpha,\varphi) \cong \mathcal{C}'$. Therefore $\mathcal{C}$ is a Demazure crystal embedded in $\mathrm{RF}_n(w)$, so $\mathcal{C} = \mathfrak{D}_\sigma^{\mathrm{RF}_n(w)}\{u^{(\bullet)}\}$ for some $\sigma \in S_n$ and some highest weight element $u^{(\bullet)}$.

Let $\mathfrak{D}_i = \mathfrak{D}_i^{\mathrm{RF}_n(w)}$. Since the e_i operators preserve the weak EG insertion tableau, Corollary 38 implies that $\mathcal{C}' = \mathfrak{D}_{t_1\downarrow 1}\mathfrak{D}_{t_2\downarrow 2}\cdots\mathfrak{D}_{t_n\downarrow n}\mathcal{C}$ where $t_i = \min\{n, \varphi(i)\}$. Hence, $\mathcal{C}'$ is also a Demazure crystal. Because $\mathrm{RF}_n(w)$ is normal, the Demazure character formula [5, Thm. 13.7] implies that

$$\mathrm{ch}(\mathrm{SSKT}(\alpha,\varphi)) = \mathrm{ch}(\mathcal{C}') = \pi_{t_1\downarrow 1}\cdots\pi_{t_n\downarrow n}\,\mathrm{ch}(\mathcal{C}) = \pi_{t_1\downarrow 1}\cdots\pi_{t_n\downarrow n}(\kappa_\alpha). \tag{4.3}$$

Since α has length k, $\pi_{t_{k+1}\downarrow k+1} \cdots \pi_{t_n\downarrow n}(\kappa_\alpha) = \kappa_\alpha$ by properties in §2.1. Also, $t_i = \varphi(i)$ for all $1 \leqslant i \leqslant k$ since $\varphi(i) \leqslant \varphi(k) \leqslant n$. Therefore, (4.3) reduces to $\mathrm{ch}(\mathrm{SSKT}(\alpha, \varphi)) = \pi_{\varphi(1)\downarrow 1} \cdots \pi_{\varphi(k)\downarrow n}(\kappa_\alpha)$. □

As an application of the theorem, we derive a recurrence for $\kappa_{(\alpha,\varphi)}$.

Theorem 43. *Let φ be a flag. If φ is strictly increasing, then $\kappa_{(\alpha,\varphi)} = \kappa_\beta$ where $\beta_j = \alpha_i$ if $\varphi(i) = j$ for some i or 0 otherwise. If φ is not strictly increasing and i is the smallest index with $\varphi(i) = \varphi(i+1)$, then*

$$\kappa_{(\alpha,\varphi)} = \begin{cases} \kappa_{(\alpha,\varphi-\mathbf{e}_i)} & \text{if } \alpha_i \leqslant \alpha_{i+1}, \\ \kappa_{(\alpha\cdot s_i,\varphi-\mathbf{e}_i)} & \text{if } \alpha_i > \alpha_{i+1}. \end{cases} \tag{4.4}$$

Proof. By Theorem 42, we have $\kappa_{(\alpha,\varphi)} = \pi_{\varphi(1)\downarrow 1} \cdots \pi_{\varphi(n)\downarrow n}\kappa_\alpha$. If φ is strictly increasing, then $\kappa_{(\alpha,\varphi)}$ expands into κ_β directly using the recursive property in §2.1.

Assume φ is not strictly increasing and i is the smallest integer such that $\varphi(i) = \varphi(i+1) = N$. Notice that

$$\begin{aligned} \pi_{N\downarrow i}\pi_{N\downarrow(i+1)} &= (\pi_{N-1} \cdots \pi_{i+1}\pi_i)(\pi_{N-1} \cdots \pi_{i+2}\pi_{i+1}) \\ &= (\pi_{N-2} \cdots \pi_{i+1}\pi_i)(\pi_{N-1} \cdots \pi_{i+2}\pi_{i+1})\pi_i = \pi_{(N-1)\downarrow i}\pi_{N\downarrow(i+1)}\pi_i \end{aligned}$$

since both expressions give reduced words for the same permutation when every "π" is replaced by "s". Substituting this identity and noting that all subscripts in the expression $\pi_{\varphi(i+2)\downarrow(i+2)} \cdots \pi_{\varphi(n)\downarrow n}$ are at least $i+2$, we deduce that

$$\kappa_{(\alpha,\varphi)} = \pi_{\varphi(1)\downarrow 1} \cdots \pi_{(\varphi(i)-1)\downarrow i}\pi_{\varphi(i+1)\downarrow(i+1)} \cdots \pi_{\varphi(n)\downarrow n}(\pi_i\kappa_\alpha)$$

and this becomes the desired identity by the recursive property in §2.1. □

Continue to assume that $n \geqslant \varphi(\ell(\alpha))$, so that $\mathrm{SSKT}(\alpha, \varphi)$ is a $\mathfrak{gl}_n$-crystal.

Corollary 44. *Suppose φ is a non-standard flag with $i \leqslant n$ the smallest positive integer such that $\varphi(i) = \varphi(i+1)$. Then the crystal $\mathrm{SSKT}(\alpha, \varphi)$ is isomorphic to $\mathrm{SSKT}(\alpha\cdot s_i, \varphi-\mathbf{e}_i)$ if $\alpha_i > \alpha_{i+1}$ or to $\mathrm{SSKT}(\alpha, \varphi - \mathbf{e}_i)$ if $\alpha_i \leqslant \alpha_{i+1}$.*

Proof. By Theorems 42 and 43, the relevant crystals are Demazure crystals with the same character. Hence, they are isomorphic. □

Reiner and Shimozono defined *φ-flagged key polynomials* associated to α as

$$\sum_{u^\bullet \in \mathscr{W}(\alpha,\varphi)} x^{\mathsf{wt}(u^\bullet)}$$

for a certain set $\mathscr{W}(\alpha, \varphi)$; see [15]. As a final application, we can now prove Theorem 5 from the introduction.

Proof of Theorem 5. If φ is the standard flag, $\kappa_\alpha = \sum_{u^\bullet \in \mathscr{W}(\alpha,\varphi)} x^{\mathsf{wt}(u^\bullet)}$ [12]. In the proof of [15, Thm. 21], Reiner and Shimozono observed the following list of recursive properties of $\mathscr{W}(\alpha, \varphi)$. When φ is strictly increasing, we have $\mathscr{W}(\alpha, \varphi) = \mathscr{W}(\beta)$, where $\beta_j = \alpha_i$ if $j = \varphi(i)$ and $\beta_j = 0$ otherwise. When φ is not strictly increasing, assume $i \in \mathbb{Z}_{>0}$ is minimal such that $\varphi(i) = \varphi(i+1)$. If $\alpha_i \leqslant \alpha_{i+1}$ then $\mathscr{W}(\alpha, \varphi) = \mathscr{W}(\alpha, \varphi - \mathbf{e}_i)$. If $\alpha_i > \alpha_{i+1}$ then there is a bijection between $\mathscr{W}(\alpha, \varphi)$ and $\mathscr{W}(\alpha \cdot s_i, \varphi) = \mathscr{W}(\alpha \cdot s_i, \varphi - \mathbf{e}_i)$. Hence, $\mathscr{W}(\alpha, \varphi)$ satisfies the same recursive relations involving the flag φ as $\mathrm{SSKT}(\alpha, \varphi)$ in Corollary 44. □

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