Permanental Inequalities for Totally Positive Matrices

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Abstract

We characterize ratios of permanents of (generalized) submatrices which are bounded on the set of all totally positive matrices. This provides a permanental analog of results of Fallat, Gekhtman, and Johnson [*Adv. Appl. Math.* **30** no. 3, (2003) pp. 442–470] concerning ratios of matrix minors. We also extend work of Drake, Gerrish, and the first author [*Electron. J. Combin.,* **11** no. 1, (2004) Note 6] by characterizing the differences of monomials in $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$ which evaluate positively on the set of all totally positive $n \times n$ matrices.

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1 Introduction

Given an $n \times n$ matrix $A = (a_{i,j})$ and subsets $I, J \subseteq [n] := \{1, \ldots, n\}$, let $A_{I,J} = (a_{i,j})_{i \in I, j \in J}$ denote the (I, J)-*submatrix* of A. For $|I| = |J|$, call $\det(A_{I,J})$ the (I, J)-*minor* of A. A real $n \times n$ matrix A is called *totally positive* (*totally nonnegative*) if every minor of A is positive (nonnegative). Let $\mathcal{M}_n^{\mathrm{TP}} \subset \mathcal{M}_n^{\mathrm{TNN}}$ denote these sets of matrices.

These and the set $\mathcal{M}_n^{\mathrm{HPS}}$ of $n \times n$ Hermitian positive semidefinite matrices arise in many areas of mathematics, and for more than a century mathematicians have been studying inequalities satisfied by their matrix entries. (See, e.g., [8].) Many such inequalities involve minors and permanents. For instance inequalities of Fischer [9], Fan [4], and Lieb [15] state that for all matrices $A \in \mathcal{M}_n^{\mathrm{TNN}} \cup \mathcal{M}_n^{\mathrm{HPS}}$, and for all $I \subseteq [n]$ and $I^c := [n] \smallsetminus I$, we have

$$\begin{aligned} \det(A) &\leqslant \det(A_{I,I}) \det(A_{I^c,I^c}), \\ \mathrm{per}(A) &\geqslant \mathrm{per}(A_{I,I}) \,\mathrm{per}(A_{I^c,I^c}). \end{aligned} \tag{1}$$

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Koteljanskii's inequality [13], [14] states that for $A \in \mathcal{M}_n^{\mathrm{TNN}} \cup \mathcal{M}_n^{\mathrm{HPS}}$ and for all $I, J \subseteq [n]$ we have

$$\det(A_{I\cup J,I\cup J})\det(A_{I\cap J,I\cap J}) \leqslant \det(A_{I,I})\det(A_{J,J}). \tag{2}$$

Many open questions about inequalities exist and seem difficult. For instance, it is known which 8-tuples $(I, J, K, L, I', J', K', L')$ of subsets satisfy

$$\det(A_{I,I'})\det(A_{J,J'}) \leqslant \det(A_{K,K'})\det(A_{L,L'}) \tag{3}$$

for all $A \in \mathcal{M}_n^{\mathrm{TNN}}$ [7], [16], but few permanental analogs of such inequalities are known. While some of these 8-tuples also satisfiy

$$\mathrm{per}(A_{I,I'})\,\mathrm{per}(A_{J,J'}) \geqslant \mathrm{per}(A_{K,K'})\,\mathrm{per}(A_{L,L'}), \tag{4}$$

this second inequality is not true in general. For example, the natural permanental analog

$$\mathrm{per}(A_{I\cup J,I\cup J})\,\mathrm{per}(A_{I\cap J,I\cap J}) \geqslant \mathrm{per}(A_{I,I})\,\mathrm{per}(A_{J,J}), \tag{5}$$

of (2) holds neither for all $A \in \mathcal{M}_n^{\mathrm{HPS}}$ nor for all $A \in \mathcal{M}_n^{\mathrm{TNN}}$. (See [17, §6] for a counterexample with $n = 3$.)

Let us put aside $\mathcal{M}_n^{\mathrm{HPS}}$ and consider conjectured inequalities of the form

$$\mathrm{product}_1 \leqslant \mathrm{product}_2 \tag{6}$$

involving minors and permanents of matrices in $\mathcal{M}_n^{\mathrm{TNN}}$ and $\mathcal{M}_n^{\mathrm{TP}}$. One strategy for studying (6) is to view the difference $\mathrm{product}_2 - \mathrm{product}_1$ as a polynomial

$$f(x) := f(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) \in \mathbb{Z}[x] := \mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}] \tag{7}$$

in matrix entries. Then the validity of the inequality (6) is equivalent to the statement that for all $A = (a_{i,j}) \in \mathcal{M}_n^{\mathrm{TNN}}$, we have

$$f(A) := f(a_{1,1}, a_{1,2}, \ldots, a_{n,n}) \geqslant 0. \tag{8}$$

We call a polynomial (7) with this property a *totally nonnegative polynomial*. Since $\mathcal{M}_n^{\mathrm{TP}}$ is dense in $\mathcal{M}_n^{\mathrm{TNN}}$, the inequality (8) holds for all $A \in \mathcal{M}_n^{\mathrm{TP}}$ if and only if it holds for all $A \in \mathcal{M}_n^{\mathrm{TNN}}$.

A second strategy for studying (variations of) a potential inequality (6) is to ask for which positive constants k_1, k_2 the modified inequalities

$$k_1 \cdot \mathrm{product}_1 \leqslant \mathrm{product}_2 \leqslant k_2 \cdot \mathrm{product}_1 \tag{9}$$

hold for all $A \in \mathcal{M}_n^{\mathrm{TNN}}$. Bounds of $k_1 = 1$ or $k_2 = 1$ imply the inequality (6) or its reverse to hold; other bounds give information not apparent in the proof or disproof of (6). Equivalently, we may view the ratio of $\mathrm{product}_2$ to $\mathrm{product}_1$ as a rational function

$$R(x) := R(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) \in \mathbb{Q}(x) := \mathbb{Q}(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) \tag{10}$$

in matrix entries, and we may ask for upper and lower bounds as x varies over $\mathcal{M}_n^{\mathrm{TP}}$. While a ratio (10) is not defined everywhere on $\mathcal{M}_n^{\mathrm{TNN}}$, the density of $\mathcal{M}_n^{\mathrm{TP}}$ in $\mathcal{M}_n^{\mathrm{TNN}}$ allows us to restrict our attention to $\mathcal{M}_n^{\mathrm{TP}}$: we have

$$k_1 \leqslant R(x) \leqslant k_2 \tag{11}$$

for all $x \in \mathcal{M}_n^{\mathrm{TP}}$ if and only if the same inequalities hold for all $x \in \mathcal{M}_n^{\mathrm{TNN}}$ such that $R(x)$ is defined. Clearly the lower bound k_1 is interesting only when positive, since products of minors and permanents of totally nonnegative matrices are trivially bounded below by 0.

A characterization of all ratios of the form

$$\frac{\det(x_{I,I'})\det(x_{J,J'})}{\det(x_{K,K'})\det(x_{L,L'})}, \qquad I, I', \ldots, L, L' \subset [n], \tag{12}$$

which are bounded above and/or nontrivially bounded below on $\mathcal{M}_n^{\mathrm{TP}}$ follows from work in [7] and [16]. Each ratio (12) is bounded above and/or below by 1, and for each n, factors as a product of elements of a finite set of indecomposable ratios. This result was extended in [11] to include ratios of products of arbitrarily many minors

$$\frac{\det(x_{I_1,I'_1})\cdots\det(x_{I_p,I'_p})}{\det(x_{J_1,J'_1})\cdots\det(x_{J_p,J'_p})}. \tag{13}$$

Again, each of these factors as a product of elements belonging to a finite set of indecomposable ratios. For $n = 3$, each ratio (13) is bounded above and/or below by 1; for $n \geqslant 4$, such bounds are conjectured [3].

While the permanental version (5) of Koteljanskii's inequality is false, we will show in Section 3 that the corresponding ratio is bounded above and nontrivially below. Specifically,

$$\frac{1}{|I \cup J|!\,|I \cap J|!} \leqslant \frac{\mathrm{per}(x_{I,I})\,\mathrm{per}(x_{J,J})}{\mathrm{per}(x_{I\cup J,I\cup J})\,\mathrm{per}(x_{I\cap J,I\cap J})} \leqslant |I|!\,|J|! \tag{14}$$

for all $I, J \subseteq [n]$ and $x \in \mathcal{M}_n^{\mathrm{TP}}$. The failure of (5), combined with (14), exposes a difference between ratios of minors and of permanents: unlike the bounded ratios in (12), *not* all bounded ratios of permanents are bounded by 1. Thus it is natural to ask which ratios

$$R(x) = \frac{\mathrm{per}(x_{I_1,I'_1})\mathrm{per}(x_{I_2,I'_2})\cdots\mathrm{per}(x_{I_r,I'_r})}{\mathrm{per}(x_{J_1,J'_1})\mathrm{per}(x_{J_2,J'_2})\cdots\mathrm{per}(x_{J_q,J'_q})} \tag{15}$$

are bounded above and/or nontrivially below as real-valued functions on $\mathcal{M}_n^{\mathrm{TP}}$, and to state bounds.

In Section 2 we describe a multigrading of the coordinate ring $\mathbb{Z}[x]$ of $n \times n$ matrices. Extending work in [6], we define a partial order on the monomials in $\mathbb{Z}[x]$ which characterizes the differences $\prod x_{i,j}^{c_{i,j}} - \prod x_{i,j}^{d_{i,j}}$ which are totally nonnegative polynomials. This leads to our main results in Section 3 which characterize ratios (15) which are bounded above and nontrivially below as real-valued functions on $\mathcal{M}_n^{\mathrm{TP}}$. We provide some such bounds, which are not necessarily tight. We finish in Section 4 with some open questions.

2 A multigrading of the coordinate ring and the total nonnegativity order

We will find it convenient to view degree-r monomials in $\mathbb{Z}[x]$ in terms of permutations in the symmetric group $\mathfrak{S}_r$ and multisets of $[n]$. In particular, given permutations $v, w \in \mathfrak{S}_r$ define the monomial

$$x^{v,w} := x_{v_1,w_1} \cdots x_{v_r,w_r}.$$

Define an *r-element multiset* of $[n]$ to be a nondecreasing r-tuple of elements of $[n]$. In exponential notation, we write i^k to represent k consecutive occurrences of i in such an r-tuple, e.g.,

$$(1,1,2,3) = 1^2 2^1 3^1, \qquad (1,2,2,2) = 1^1 2^3. \tag{16}$$

Two r-element multisets

$$M = (m_1, \ldots, m_r) = 1^{\alpha_1} \cdots n^{\alpha_n}, \qquad O = (o_1, \ldots, o_r) = 1^{\beta_1} \cdots n^{\beta_n}, \tag{17}$$

determine a *generalized submatrix* $x_{M,O}$ of x by $(x_{M,O})_{i,j} := x_{m_i,o_j}$. For example, when $n = 3$, we have the 4×4 generalized submatrix and monomial

$$x_{1123,1222} = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{2,1} & x_{2,2} & x_{2,2} & x_{2,2} \\ x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} \end{bmatrix}, \qquad (x_{1123,1222})^{1234,4312} = x_{1,2}x_{1,2}x_{2,1}x_{3,2}. \tag{18}$$

The ring $\mathbb{Z}[x]$ has a natural multigrading

$$\mathbb{Z}[x] = \bigoplus_{r \geqslant 0} \bigoplus_{M,O} \mathcal{A}_{M,O}, \tag{19}$$

where the second direct sum is over pairs (M, O) of r-element multisets of $[n]$,

$$\mathcal{A}_{M,O} := \mathrm{span}_{\mathbb{Z}}\{(x_{M,O})^{e,w} \mid w \in \mathfrak{S}_r\}, \tag{20}$$

and e is the identity element of $\mathfrak{S}_r$. More precisely, for M, O as in (17), a basis for $\mathcal{A}_{M,O}$ is given by all monomials

$$\prod_{i,j \in [n]} x_{i,j}^{c_{i,j}} \tag{21}$$

with $C = (c_{i,j}) \in \mathrm{Mat}_{n \times n}(\mathbb{N})$ satisfying

$$c_{i,1} + \cdots + c_{i,n} = \alpha_i, \quad c_{1,j} + \cdots + c_{n,j} = \beta_j \quad \text{for } i, j = 1, \ldots, n. \tag{22}$$

One may express a monomial (21) in the form $(x_{M,O})^{e,w}$ by the following algorithm.

Algorithm 2.1. Given a monomial (21) in $\mathcal{A}_{M,O}$ with M, O as in (17),

(i) Define the rearrangement $u = u_1 \cdots u_r$ of O by writing (21) with variables in lexicographic order as $x_{m_1,u_1} \cdots x_{m_r,u_r}$.

(ii) Let $j_1 < \cdots < j_{\beta_1}$ be the positions of the β_1 ones in u, let $j_{\beta_1+1} < \cdots < j_{\beta_1+\beta_2}$ be the positions of the β_2 twos in u, etc.

(iii) For $i = 1, \ldots, r$, define $w_{j_i} = i$.

(iv) Call the resulting word $w = w(C)$.

For example, it is easy to check that for $n = 3$ and multisets

$$(1123, 1222) = (1^2 2^1 3^1, 1^1 2^3 3^0)$$

of $\{1, 2, 3\}$, the graded component $\mathcal{A}_{1123,1222}$ of $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{3,3}]$ is spanned by monomials (21), where $C = (c_{i,j})$ is one of the matrices

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{23}$$

having row sums $(2, 1, 1)$ and column sums $(1, 3, 0)$. These are

$$x_{1,1}x_{1,2}x_{2,2}x_{3,2}, \quad x_{1,2}^2 x_{2,1}x_{3,2}, \quad x_{1,2}^2 x_{2,2}x_{3,1}, \tag{24}$$

with column index sequences equal to the rearrangements 1222, 2212, 2221 of 1222. Algorithm 2.1 then produces permutations 1234, 2314, 2341 in $\mathfrak{S}_4$, and we may express the monomials (24) as

$$(x_{1123,1222})^{1234,1234}, \quad (x_{1123,1222})^{1234,2314}, \quad (x_{1123,1222})^{1234,2341}. \tag{25}$$

For r-element multisets M, O of $[n]$, the monomials in $\mathcal{A}_{M,O}$ are closely related to parabolic subgroups of $\mathfrak{S}_r$ with standard generators $s_1, \ldots, s_{r-1}$, and double cosets of the form $W_{\iota(M)} w W_{\iota(O)}$ where w belongs to $\mathfrak{S}_r$, W_J is the subgroup of $\mathfrak{S}_r$ generated by J, and

$$\begin{aligned} \iota(M) &:= \{s_1, \ldots, s_{r-1}\} \smallsetminus \{s_{\alpha_1}, s_{\alpha_1+\alpha_2}, \ldots, s_{r-\alpha_n}\} = \{s_j \,|\, m_j = m_{j+1}\}, \\ \iota(O) &:= \{s_1, \ldots, s_{r-1}\} \smallsetminus \{s_{\beta_1}, s_{\beta_1+\beta_2}, \ldots, s_{r-\beta_n}\} = \{s_j \,|\, o_j = o_{j+1}\}. \end{aligned} \tag{26}$$

It is easy to see that the map $M \mapsto \iota(M)$ is bijective: one recovers $M = 1^{\alpha_1} \cdots n^{\alpha_n}$ from the generators not in $\iota(M)$ as in (26). It is known that each double coset has unique minimal and maximal elements with respect to the *Bruhat order on* $\mathfrak{S}_r$, defined by declaring $v \leqslant w$ if each reduced expression $s_{i_1} \cdots s_{i_\ell}$ for w contains a subword which is a reduced expression for v. (See, e.g., [2], [5].) Let $W_{\iota(M)} \backslash W / W_{\iota(O)}$ denote the set of all double cosets of $W = \mathfrak{S}_r$ determined by r-element multisets M, O.

Proposition 1. *Fix r-element multisets M, O of $[n]$ as in (17). The double cosets $W_{\iota(M)} \backslash W / W_{\iota(O)}$ satisfy the following.*

(i) Each double coset has a unique Bruhat-minimal element u satisfying $su > u$ for all $s \in \iota(M)$ and $us > u$ for all $s \in \iota(O)$; it has a unique Bruhat-maximal element u' satisfying $su' < u'$ for all $s \in \iota(M)$ and $u's < u'$ for all $s \in \iota(O)$.

(ii) We have $W_{\iota(M)}vW_{\iota(O)} = W_{\iota(M)}wW_{\iota(O)}$ *if and only if* $(x_{M,O})^{e,v} = (x_{M,O})^{e,w}$.

(iii) The cardinality $|W_{\iota(M)}\backslash W/W_{\iota(O)}|$ *is the dimension of* $\mathcal{A}_{M,O}$*, equivalently, the number of matrices in* $\mathrm{Mat}_{n\times n}(\mathbb{N})$ *with row sums* $(\alpha_1,\ldots,\alpha_n)$ *and column sums* $(\beta_1,\ldots,\beta_n)$.

(iv) Each permutation w *produced by Algorithm 2.1 is the unique Bruhat-minimal element of its coset* $W_{\iota(M)}wW_{\iota(O)}$.

Proof. (i) See [5].
(ii) The dimension of $\mathcal{A}_{M,O}$ is the cardinality of the set $\{(x_{M,O})^{e,w} \mid w \in \mathfrak{S}_r\}$. But we have $(x_{M,O})^{e,v} = (x_{M,O})^{e,w}$ if and only if when we partition the $r \times r$ permutation matrices $P(v)$, $P(w)$ of v, w into blocks by drawing bars after rows $\alpha_1, \alpha_1+\alpha_2, \ldots, r-\alpha_n$ and after columns $\beta_1, \beta_1+\beta_2, \ldots, r-\beta_n$, the corresponding blocks of $P(v)$ and $P(w)$ contain equal numbers of ones. It follows that for fixed $w \in \mathfrak{S}_r$, the set $\{v \in \mathfrak{S}_r \mid (x_{M,O})^{e,v} = (x_{M,O})^{e,w}\}$ is $W_{\iota(M)}wW_{\iota(O)}$.
(iii) This follows from (ii), where $c_{i,j}$ is the number of ones in block (i,j) of the permutation matrix of any permutation belonging to the double coset.
(iv) By Step (i) of the algorithm, subwords $w_1 \cdots w_{\alpha_1}$, $w_{\alpha_1+1} \cdots w_{\alpha_1+\alpha_2}$, etc., of $w(C)$ are increasing. It follows that for any generator $s \in \iota(M)$ we have $sw > w$. By Step (ii) of the algorithm, letters $1,\ldots,\beta_1$ appear in increasing order in $w(C)$, as do $\beta_1+1,\ldots,\beta_1+\beta_2$, etc. It follows that for any generator $w \in \iota(O)$ we have $ws > w$. □

For any subsets I, J of generators of $\mathfrak{S}_r$, the Bruhat order on $\mathfrak{S}_r$ induces a poset structure on $W_I\backslash W/W_J$ as follows. We declare $W_IvW_J \leqslant W_IwW_J$ if elements of the cosets satisfy any of the three (equivalent) inequalities in the Bruhat order on $\mathfrak{S}_r$. (See [5, Lemma 2.2].)

(i) The minimal element of W_IvW_J is less than or equal to the minimal element of W_IwW_J.

(ii) The maximal element of W_IvW_J is less than or equal to the maximal element of W_IwW_J.

(iii) At least one element of W_IvW_J is less than or equal to at least one element of W_IwW_J.

Call this the *Bruhat order on* $W_I\backslash W/W_J$. A fourth equivalent inequality can be stated in terms of matrices associated to monomials in $\mathcal{A}_{M,O}$. (See, e.g., [12, Proposition 3].) Given a matrix $C = (c_{i,j}) \in \mathrm{Mat}_{n\times n}(\mathbb{N})$, define the matrix $C^* = (c^*_{i,j}) \in \mathrm{Mat}_{n\times n}(\mathbb{N})$ by

$$c^*_{i,j} = \text{sum of entries of } C_{[i],[j]}. \tag{27}$$

Proposition 2. *Fix monomials*

$$(x_{M,O})^{e,v} = \prod_{i,j} x_{i,j}^{c_{i,j}}, \quad (x_{M,O})^{e,w} = \prod_{i,j} x_{i,j}^{d_{i,j}},$$

in $\mathcal{A}_{M,O}$ and define matrices C^*, D^* as in (27). Then we have $W_{\iota(M)} v W_{\iota(O)} \leqslant W_{\iota(M)} w W_{\iota(O)}$ in the Bruhat order if and only if $C^* \geqslant D^*$ in the componentwise order.

The Bruhat order on $W_{\iota(M)} \backslash W / W_{\iota(O)}$ is closely related to certain totally nonnegative polynomials in $\mathcal{A}_{M,O}$. Indeed, when $M = O = 1^n$, totally nonnegative polynomials of the form $x^{e,v} - x^{e,w}$ are characterized by the Bruhat order on $\mathfrak{S}_n$ [6, Theorem 2].

Theorem 3. *For $v, w \in \mathfrak{S}_n$, the polynomial $x^{e,v} - x^{e,w}$ is totally nonnegative if and only if $v \leqslant w$ in the Bruhat order.*

We will now extend this result to all monomials in $\mathbb{Z}[x]$. Let us define a partial order $\leqslant_T$ on all monomials in $\mathbb{Z}[x]$ by declaring $(x_{M,O})^{e,v} \leqslant_T (x_{P,Q})^{e,w}$ if $(x_{P,Q})^{e,w} - (x_{M,O})^{e,v}$ is a totally nonnegative polynomial. We call this the *total nonnegativity order* on monomials in $\mathbb{Z}[x]$. It is not hard to show that the total nonnegativity order is a disjoint union of its restrictions to the multigraded components (19) of $\mathbb{Z}[x]$.

Lemma 4. *Monomials*

$$\prod_{i,j} x_{i,j}^{c_{i,j}}, \quad \prod_{i,j} x_{i,j}^{d_{i,j}} \tag{28}$$

are comparable in the total nonnegativity order only if they belong to the same multigraded component of $\mathbb{Z}[x]$.

Proof. For $k, \ell \in [n]$ and $t \in \mathbb{R}_{\geqslant 0}$, define the $n \times n$ matrix $E^{k,\ell}(t) = (e_{i,j}^{k,\ell})_{i,j \in [n]}$ by

$$e_{i,j}^{k,\ell} = \begin{cases} t & \text{if } i \leqslant k \text{ and } j \leqslant \ell, \\ 1 & \text{otherwise.} \end{cases} \tag{29}$$

This matrix is totally nonnegative if $t \geqslant 1$, or $k = n$, or $\ell = n$.

Suppose that the monomials belong to components $\mathcal{A}_{M,O}$ and $\mathcal{A}_{M',O'}$ of $\mathbb{Z}[x]$, with M, O, as in (17) and

$$M' = 1^{\alpha'_1} \cdots n^{\alpha'_n}, \qquad O' = 1^{\beta'_1} \cdots n^{\beta'_n}.$$

If $M \neq M'$, then let $k \in [n]$ be the least index appearing with different multiplicities in the two multisets. Evaluating the monomials (28) at $E^{k,n}(t)$ yields $t^{\alpha_1 + \cdots + \alpha_k}$ and $t^{\alpha'_1 + \cdots + \alpha'_k}$. The difference of these can be made positive or negative by choosing $t < 1$ or $t > 1$. On the other hand, if $O \neq O'$, then the evalulation of the monomials (28) at matrices of the form $E^{n,\ell}(t)$ leads to a similar conclusion. □

Theorem 5. *Fix r-element multisets $M = 1^{\alpha_1} \cdots n^{\alpha_n}$, $O = 1^{\beta_1} \cdots n^{\beta_n}$ as in (17), and matrices $C, D \in \mathrm{Mat}_{n \times n}(\mathbb{N})$ with row and column sums $(\alpha_1, \ldots, \alpha_n)$, $(\beta_1, \ldots, \beta_n)$, and define the polynomial*

$$f(x) = \prod_{i,j} x_{i,j}^{c_{i,j}} - \prod_{i,j} x_{i,j}^{d_{i,j}}$$

in $\mathcal{A}_{M,O}$. Then the following are equivalent.

(i) $f(x)$ *is totally nonnegative.*

(ii) $C^* \geqslant D^*$ *in the componentwise order.*

(iii) $w(C) \leqslant w(D)$ *in the Bruhat order on* $\mathfrak{S}_r$*.*

(iv) $f(x)$ *is equal to a sum of products of the form* $\det(x_{I,J})x_{u_1,v_1}\cdots x_{u_{r-2},v_{r-2}}$ *in* $\mathcal{A}_{M,O}$ *with* $|I| = |J| = 2$*.*

Proof. (i $\Rightarrow$ ii) Suppose $C^* \not\geqslant D^*$ in the componentwise order and let (k,ℓ) the the lexicographically least pair satisfying $c^*_{k,\ell} < d^*_{k,\ell}$. Now choose $t > 1$ and evaluate $f(x)$ at the totally nonnegative matrix $E^{k,\ell}(t)$ to obtain $t^{c^*_{k,\ell}} - t^{d^*_{k,\ell}} < 0$. It follows that $f(x)$ is not a totally nonnegative polynomial.
(ii $\Rightarrow$ iii) This follows from Proposition 2 and the definition of the Bruhat order on double cosets.
(iii $\Rightarrow$ iv) Suppose that $w(C) \leqslant w(D)$ and let $p = \ell(w(D)) - \ell(w(C))$. Then there exist a sequence

$$w(C) = y^{(0)} < y^{(1)} < \cdots < y^{(p-1)} < y^{(p)} = w(D)$$

of permutations and a sequence $((i_0,j_0),\ldots,(i_{p-1},j_{p-1}))$ of transpositions in $\mathfrak{S}_r$ such that we have

$$y^{(k)} = (i_{k-1},j_{k-1})y^{(k-1)}, \qquad \ell(y^{(k)}) = \ell(y^{(k-1)}) + 1$$

for $k = 1,\ldots,p$. We may thus write $f(x) = (x_{M,O})^{e,w(C)} - (x_{M,O})^{e,w(D)}$ as the telescoping sum

$$\Big((x_{M,O})^{e,y^{(0)}} - (x_{M,O})^{e,y^{(1)}}\Big) + \Big((x_{M,O})^{e,y^{(1)}} - (x_{M,O})^{e,y^{(2)}}\Big) + \cdots + \Big((x_{M,O})^{e,y^{(p-1)}} - (x_{M,O})^{e,y^{(p)}}\Big),$$

where each parenthesized difference either has the desired form or is 0.
(iv $\Rightarrow$ i) A sum of products of minors is a totally nonnegative polynomial. ☐

For example, let us revisit the monomials (24) – (25) in the graded component $\mathcal{A}_{1123,1222}$ of $\mathbb{Z}[x_{1,1},x_{1,2},\ldots,x_{3,3}]$. It is easy to see that $1234 < 2314 < 2341$ in the Bruhat order on $\mathfrak{S}_4$ and that the application of (27) to the corresponding matrices in (23) yields the componentwise comparisons

$$\begin{bmatrix}1&2&2\\1&3&3\\1&4&4\end{bmatrix} \geqslant \begin{bmatrix}0&2&2\\1&3&3\\1&4&4\end{bmatrix} \geqslant \begin{bmatrix}0&2&2\\0&3&3\\1&4&4\end{bmatrix}. \tag{30}$$

Thus we have $(x_{1123,1222})^{1234,1234} \geqslant_T (x_{1123,1222})^{1234,2314} \geqslant_T (x_{1123,1222})^{1234,2341}$, i.e.,

$$x_{1,1}x_{1,2}x_{2,2}x_{3,2} \ \geqslant_T \ x_{1,2}^2x_{2,1}x_{3,2} \ \geqslant_T \ x_{1,2}^2x_{2,2}x_{3,1}.$$

Furthermore, the chain $1234 < 2134 < 2314 < 2341$ in the Bruhat order on $\mathfrak{S}_4$ with

$$2134 = (1,2)1234, \quad 2314 = (2,3)2134, \quad 2341 = (3,4)2314 \tag{31}$$

allows us to write $x_{1,1}x_{1,2}x_{2,2}x_{3,2} - x_{1,2}^2 x_{2,1}x_{3,2}$ as

$$\Big((x_{1123,1222})^{1234,1234} - (x_{1123,1222})^{1234,2134}\Big) + \Big((x_{1123,1222})^{1234,2134} - (x_{1123,1222})^{1234,2314}\Big)$$
$$= \det\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{1,1} & x_{2,2} \end{bmatrix} x_{2,2}x_{3,2} + x_{1,2}\det\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} x_{3,2},$$

and to write $x_{1,2}^2 x_{2,1}x_{3,2} - x_{1,2}^2 x_{2,2}x_{3,1}$ as

$$\Big((x_{1123,1222})^{1234,2314} - (x_{1123,1222})^{1234,2341}\Big) = x_{1,2}^2 \det\begin{bmatrix} x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{bmatrix}.$$

3 Main results

Let $\mathcal{M}_n^{\mathrm{TP}}$ be the set of totally positive $n \times n$ matrices. To characterize ratios of products of permanents which are bounded above and/or nontrivially bounded below on the set $\mathcal{M}_n^{\mathrm{TP}}$, we first consider necessary conditions on the multisets of rows and columns appearing in such ratios. Let

$$R(x) = \frac{\mathrm{per}(x_{I_1,I'_1})\mathrm{per}(x_{I_2,I'_2}) \cdots \mathrm{per}(x_{I_r,I'_r})}{\mathrm{per}(x_{J_1,J'_1})\mathrm{per}(x_{J_2,J'_2}) \cdots \mathrm{per}(x_{J_q,J'_q})}, \tag{32}$$

be such a ratio, where

$$(I_1, \dots, I_r), \quad (I'_1, \dots, I'_r), \quad (J_1, \dots, J_q), \quad (J'_1, \dots, J'_q) \tag{33}$$

are multisets of $[n]$ satisfying $|I_k| = |I'_k|$, $|J_k| = |J'_k|$ for all k. In order for $R(x)$ to be bounded above or nontrivially bounded below on $\mathcal{M}_n^{\mathrm{TP}}$ the multisets (33) must be related in terms of an operation which we call *multiset union*. Given multisets $M = 1^{\alpha_1} \cdots n^{\alpha_n}$, $O = 1^{\beta_1} \cdots n^{\beta_n}$ of $[n]$, define their multiset union to be

$$M \Cup O := 1^{\alpha_1+\beta_1} \cdots n^{\alpha_n+\beta_n}. \tag{34}$$

For example, $1124 \Cup 1233 = 11122334$.

Proposition 6. *Given multiset sequences as in (33), a ratio (32) can be bounded above or nontrivially bounded below on* $\mathcal{M}_n^{\mathrm{TP}}$ *only if we have the multiset equalities*

$$I_1 \Cup \cdots \Cup I_r = J_1 \Cup \cdots \Cup J_q, \qquad I'_1 \Cup \cdots \Cup I'_r = J'_1 \Cup \cdots \Cup J'_q. \tag{35}$$

Proof. Given a multiset K, let $\mu_i(K)$ denote the multiplicity of i in K, and define

$$\alpha_i = \sum_{k=1}^r \mu_i(I_k), \quad \beta_i = \sum_{k=1}^r \mu_i(I'_k), \quad \alpha'_i = \sum_{k=1}^q \mu_i(J_k), \quad \beta'_i = \sum_{k=1}^q \mu_i(J'_k). \tag{36}$$

Assume that the multiset equalities (35) do not hold, e.g., for some i we have $\alpha_i \neq \alpha'_i$. Let A be a totally positive matrix and construct a family of matrices $(A(t))_{t>0}$ by scaling row i of A by t. Clearly, each matrix $A(t)$ is totally positive, since each minor $\det(A(t)_{I,J})$

equals either $\det(A_{I,J})$ or t times this. Furthermore, we have $R(A(t)) = t^{\alpha_i - \alpha'_i} R(A)$, since each permanent with row multiset K containing i is scaled by $t^{\mu_i(K)}$. Thus, by letting t approach 0 or $+\infty$, we can make $R(A(t))$ arbitrarily large or arbitrarily close to 0. The same is true if we have $\beta_i \neq \beta'_i$. □

To state sufficient conditions for the boundedness of ratios (32) we observe that it is possible to bound the permanent above and below as follows.

Proposition 7. *For any $n \times n$ totally nonnegative matrix $A = (a_{i,j})$ we have*

$$a_{1,1} \cdots a_{n,n} \leqslant \mathrm{per}(A) \leqslant n! \cdot a_{1,1} \cdots a_{n,n}. \tag{37}$$

Proof. The first inequality follows from the fact that $a_{1,w_1} \cdots a_{n,w_n} > 0$ for all $w \in \mathfrak{S}_n$. The second inequality follows from the fact (Theorem 3) that $a_{1,w_1} \cdots a_{n,w_n} \leqslant a_{1,1} \cdots a_{n,n}$ for all $w \in \mathfrak{S}_n$. □

Now we state our main result, which characterizes ratios $R(x)$ as in (32) which are bounded above for $x \in \mathcal{M}_n^{\mathrm{TP}}$.

Theorem 8. *Let rational function*

$$R(x) = \frac{\mathrm{per}(x_{I_1,I'_1})\mathrm{per}(x_{I_2,I'_2}) \cdots \mathrm{per}(x_{I_r,I'_r})}{\mathrm{per}(x_{J_1,J'_1})\mathrm{per}(x_{J_2,J'_2}) \cdots \mathrm{per}(x_{J_q,J'_q})} \tag{38}$$

have index sets which satisfy (35), and define matrices $C = (c_{i,j})$, $C^* = (c^*_{i,j})$, $D = (d_{i,j})$, $D^* = (d^*_{i,j})$ by

$$(x_{I_1,I'_1})^{e,e} \cdots (x_{I_r,I'_r})^{e,e} = \prod x_{i,j}^{d_{i,j}}, \qquad (x_{J_1,J'_1})^{e,e} \cdots (x_{J_q,J'_q})^{e,e} = \prod x_{i,j}^{c_{i,j}}, \tag{39}$$

and (27). Then $R(x)$ is bounded above on the set of totally positive matrices if and only if $D^* \leqslant C^*$ in the componentwise order. In this case, it is bounded above by $|I_1|! \cdots |I_r|!$.

Proof. Suppose that $D^* \nleqslant C^*$. Then for some indices (k,ℓ) we have $d^*_{k,\ell} > c^*_{k,\ell}$.

Then, we have $R(E^{k,\ell}(t)) = \frac{p(t)}{q(t)}$ where $\deg(p(t)) = d^*_{i,j} > c^*_{i,j} = \deg(q(t))$. Thus we have

$$\lim_{t \to \infty} R(E^{k,\ell}(t)) = t^{d^*_{i,j} - c^*_{i,j}} = \infty.$$

Assume therefore that we have $D^* \leqslant C^*$ and let A be any $n \times n$ totally positive matrix. Applying the inequalities of Proposition 7 to the numerator and denominator of $R(A)$ respectively, we see that $R(A)$ is at most

$$\frac{|I_1|!(A_{I_1,I'_1})^{e,e} \cdots |I_r|!(A_{I_r,I'_r})^{e,e}}{(A_{J_1,J'_1})^{e,e} \cdots (A_{J_q,J'_q})^{e,e}} = \frac{|I_1|! \cdots |I_r|! \prod a_{i,j}^{d_{i,j}}}{\prod a_{i,j}^{c_{i,j}}}. \tag{40}$$

By Theorem 5 we have that $\Pi a_{i,j}^{d_{i,j}} < \Pi a_{i,j}^{c_{i,j}}$. Thus the right-hand side of (40) is at most $|I_1|! \cdots |I_r|!$. □

Observe that Theorem 8 guarantees no nontrivial lower bound for $R(x)$ and gives an upper bound which is sometimes tight. Indeed the ratio

$$\frac{x_{1,2}x_{2,1}}{x_{1,1}x_{2,2}}$$

attains all values in the open interval $(0, 1)$ as x varies over matrices in $\mathcal{M}_2^{\mathrm{TP}}$. On the other hand, special cases of the ratios in Theorem 8 can be shown to have both upper and nontrivial lower bounds.

Corollary 9. *For ratio $R(x)$ and matrices C, D defined as in Theorem 8, if $C = D$, then $R(x)$ is bounded above and below by*

$$\frac{1}{|J_1|! \cdots |J_q|!} \leqslant R(x) \leqslant |I_1|! \cdots |I_r|!\ , \tag{41}$$

for $x \in \mathcal{M}_n^{\mathrm{TP}}$.

For example, consider the ratio

$$R(x) = \frac{\mathrm{per}(x_{12,34})\mathrm{per}(x_{34,12})}{x_{1,3}x_{2,4}x_{3,1}x_{4,2}} \tag{42}$$

with $|I_1| = |I_2| = 2$, $|J_1| = |J_2| = |J_3| = |J_4| = 1$, and

$$C = D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{43}$$

By Corollary 9, we have

$$\frac{1}{1!^4} \leqslant R(x) \leqslant 2!^2.$$

It is easy to see that $R(x)$ attains values arbitrarily close to 4 as x approaches the matrix of all ones. It is also possible to show that $R(x)$ attains values arbitrarily close to 1. Indeed, consider the matrix $A = A(\epsilon) = (a_{i,j})$ defined by

$$A(\epsilon) = \begin{bmatrix} 1 & 1 & \epsilon & \epsilon^3 \\ 1 & 2 & 1 & \epsilon \\ \epsilon & 1 & 2 & 1 \\ \epsilon^3 & \epsilon & 1 & 1 \end{bmatrix}, \tag{44}$$

where ϵ is positive and close to 0. To see that $A(\epsilon)$ is totally positive, it suffices to verify the positivity of the sixteen minors $\det(A_{[a_1,b_1],[a_2,b_2]})$ indexed by pairs of intervals, at least one of which contains 1 [10, Theorem 9]. Observe that we have $a_{1,j} > 0$ and $a_{i,1} > 0$ for all i, j. Also,

$$\begin{aligned}
&\det(A_{12,12}) = 1,\\
&\det(A_{12,23}) = \det(A_{23,12}) = 1 - 2\epsilon,\\
&\det(A_{12,34}) = \det(A_{34,12}) = \epsilon^2 - \epsilon^3,\\
&\det(A_{123,123}) = 1 + 2\epsilon - 2\epsilon^2,\\
&\det(A_{123,234}) = \det(A_{234,123}) = 1 - 4\epsilon + \epsilon^2 + 3\epsilon^3,\\
&\det(A) = 4\epsilon - 6\epsilon^2 - 2\epsilon^3 + 9\epsilon^4 - 2\epsilon^5 - 3\epsilon^6.
\end{aligned}$$

It follows that we have

$$\lim_{\epsilon\to 0^+} R(A) = \lim_{\epsilon\to 0^+} \frac{(\epsilon^2 - \epsilon^3)^2}{\epsilon^4} = \lim_{\epsilon\to 0^+} 1 - 2\epsilon + \epsilon^2 = 1.$$

In the case that all submatrices in (38) are principal, the necessary condition (35) for boundedness is in fact sufficient to guarantee the existence of upper and nontrivial lower bounds.

Corollary 10. *For ratio R as in Theorem 8, if all submatrices in $R(x)$ are principal, ($I_k = I'_k$, $J_k = J'_k$ for all k), then $R(x)$ is bounded above and below as in (41).*

Proof. For principal submatrices $x_{I_1,I_1}, \ldots$, the condition (35) implies the equality of the matrices C and D in (39): this matrix is diagonal with (i,i) entry equal to the multiplicity of i in $I_1 \Cup \cdots \Cup I_r$. □

For example, consider the ratio

$$\frac{\mathrm{per}(x_{I,I})\,\mathrm{per}(x_{J,J})}{\mathrm{per}(x_{I\cup J,I\cup J})\,\mathrm{per}(x_{I\cap J,I\cap J})} \tag{45}$$

coming from the (false) permanental version (5) of Koteljanskii's inequality (2). By Corollary 10, the four principal submatrices of x imply that the exponent matrices C and D are equal and diagonal with (i,i) entry equal to the multiplicity of i in $I \Cup J$. Thus Corollary 9 gives the lower and upper bounds

$$\frac{1}{|I\cup J|!\,|I\cap J|!}, \qquad |I|!\,|J|! \tag{46}$$

as claimed in (14). These bounds are not in general tight. Consider the special case

$$\frac{1}{3!1!} \leqslant \frac{\mathrm{per}(x_{12,12})\mathrm{per}(x_{23,23})}{\mathrm{per}(x_{123,123})\mathrm{per}(x_{2,2})} \leqslant (2!)^2 \tag{47}$$

with

$$C = D = \begin{bmatrix} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{48}$$

We improve (47) as follows.

Proposition 11. *For $x \in \mathcal{M}_3^{\mathrm{TP}}$ we have*

$$\frac{1}{2} \leqslant \frac{\mathrm{per}(x_{12,12})\mathrm{per}(x_{23,23})}{\mathrm{per}(x_{123,123})\mathrm{per}(x_{2,2})} \leqslant 2. \tag{49}$$

Proof. The first inequality follows from expanding

$$2 \cdot \mathrm{per}(x_{12,12})\mathrm{per}(x_{23,23}) - \mathrm{per}(x_{123,123})\mathrm{per}(x_{2,2})$$

and grouping terms as

$$\begin{aligned}&(x_{11}x_{22}^2x_{33} - x_{13}x_{22}^2x_{31}) + (x_{12}x_{21}x_{22}x_{33} - x_{12}x_{22}x_{23}x_{31}) + (x_{12}x_{21}x_{23}x_{32} - x_{13}x_{21}x_{22}x_{32})\\ &\quad + x_{11}x_{22}x_{23}x_{32} + x_{12}x_{21}x_{23}x_{32}\\ &= \det(x_{13,13})x_{22}^2 + \det(x_{23,13})x_{12}x_{22} + \det(x_{12,23})x_{21}x_{32} + x_{11}x_{22}x_{23}x_{32} + x_{12}x_{21}x_{23}x_{32}.\end{aligned}$$

Similarly, the second inequality follows from expanding

$$2 \cdot \mathrm{per}(x_{123,123})\mathrm{per}(x_{2,2}) - \mathrm{per}(x_{12,12})\mathrm{per}(x_{23,23})$$

and grouping terms as

$$\begin{aligned}&x_{11}x_{22}^2x_{33} + \det(x_{23,23})x_{12}x_{21} + 2x_{12}x_{22}x_{23}x_{31}\\ &\qquad + x_{11}x_{22}x_{23}x_{32} + 2x_{13}x_{21}x_{22}x_{32} + 2x_{13}x_{22}^2x_{31}.\end{aligned}$$ □

The authors believe that even these bounds are not tight. The smallest and greatest values for (49) that we have found are 2/3 and 121/114, respectively.

4 Future directions

It would be interesting to characterize the ratios (15) which are bounded by 1, i.e., to solve the following problem.

Problem 12. Characterize the differences

$$\mathrm{per}(x_{J_1,J_1'}) \cdots \mathrm{per}(x_{J_q,J_q'}) - \mathrm{per}(x_{I_1,I_1'}) \cdots \mathrm{per}(x_{I_r,I_r'}) \tag{50}$$

which are totally nonnegative polynomials.

To consider a special case, it is possible to show that for small n, the sets $I = [2n] \smallsetminus 2\mathbb{Z}$, $J = [2n] \cap 2\mathbb{Z}$ define a totally nonnegative polynomial

$$\mathrm{per}(x_{[n],[n]})\,\mathrm{per}(x_{[n+1,2n],[n+1,2n]}) - \mathrm{per}(x_{I,I})\,\mathrm{per}(x_{J,J}). \tag{51}$$

If this polynomial is totally nonnegative in general, then it provides a permanental analog of the known totally nonnegative polynomial

$$\det(x_{I,I})\det(x_{J,J}) - \det(x_{[n,n]})\det(x_{[n+1,2n],[n+1,2n]}).$$

Conjecture 13. The polynomial (51) is totally nonnegative for all n.

Other families of possible inequalities are suggested by the known inequalities appearing in Proposition 7. In particular, the first inequality there compares the permanent to a product of permanents of 1×1 matrices. Comparing further to products of permanents of the form

$$\mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r}), \tag{52}$$

we obtain polynomials of the forms

$$\mathrm{per}(x) - \mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r}), \qquad \mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r}) - x_{1,1} \cdots x_{n,n}, \tag{53}$$

which are totally nonnegative because they belong to $\mathrm{span}_{\mathbb{N}}\{x^{e,w} \mid w \in \mathfrak{S}_n\}$. It is natural to ask if the cardinalities of the index sets determine whether a difference of the form (50) is totally nonnegative, but this is not the case. It is natural then to ask how *averages* of such products compare to one another. This problem is open. (See [1], [17, Problem 5.3].)

Problem 14. Characterize the pairs of partitions $\lambda = (\lambda_1, \dots, \lambda_r)$, $\mu = (\mu_1, \dots, \mu_q)$ such that

$$\sum_{\substack{(I_1,\dots,I_r)\\ |I_k|=\lambda_k}} \frac{\mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r})}{\binom{n}{\lambda_1,\dots,\lambda_r}} - \sum_{\substack{(J_1,\dots,J_q)\\ |J_k|=\mu_k}} \frac{\mathrm{per}(x_{J_1,J_1}) \cdots \mathrm{per}(x_{J_q,J_q})}{\binom{n}{\mu_1,\dots,\mu_r}} \tag{54}$$

is totally nonnegative.

Now consider generalizing the second inequality in Proposition 7 to products of permanents of the form (52). Differences of the form

$$\frac{\mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r})}{|I_1|! \cdots |I_r|!} - \frac{\mathrm{per}(x)}{n!} \tag{55}$$

are *not* totally nonnegative, while differences of the form

$$x_{1,1} \cdots x_{n,n} - \frac{\mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r})}{|I_1|! \cdots |I_r|!} \tag{56}$$

are (by Proposition 7). It is natural then to ask about the averages of differences (55), over all set partitions $(I_1, \dots, I_r)$ of a partition λ.

Problem 15. Decide if for fixed $\lambda = (\lambda_1, \dots, \lambda_r)$, the polynomial

$$\sum_{\substack{(I_1,\dots,I_r)\\ |I_k|=\lambda_k}} \mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r}) - \mathrm{per}(x)$$

is totally nonnegative.

To illustrate (55) and Problem 15, let us consider the case that $n = 3$. It is straightforward to show that

$$\frac{\mathrm{per}(x_{12,12})x_{3,3}}{2!1!} - \frac{\mathrm{per}(x)}{3!}, \tag{57}$$

equivalently, $3\mathrm{per}(x_{12,12})x_{3,3} - \mathrm{per}(x)$, is totally nonnegative because the latter expression equals a sum of matrix minors. Similarly,

$$\frac{\mathrm{per}(x_{23,23})x_{1,1}}{2!1!} - \frac{\mathrm{per}(x)}{3!}, \tag{58}$$

is totally nonnegative. On the other hand,

$$\frac{\mathrm{per}(x_{13,13})x_{2,2}}{2!1!} - \frac{\mathrm{per}(x)}{3!}, \tag{59}$$

is not, because its evaluation at

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is negative. However, two times the sum of the three differences (57) – (59) is

$$2x_{1,1}x_{2,2}x_{3,3} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2},$$

which is totally nonnegative by Theorem 3.

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