Generalized Ramsey Numbers at the Linear and Quadratic Thresholds

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Abstract

The generalized Ramsey number $f(n,p,q)$ is the smallest number of colors needed to color the edges of the complete graph K_n so that every p-clique spans at least q colors. Erdős and Gyárfás showed that $f(n,p,q)$ grows linearly in n when p is fixed and $q = q_{\text{lin}}(p) := \binom{p}{2} - p + 3$, but $f(n,p,q-1)$ is sublinear. Similarly they showed that $f(n,p,q)$ is quadratic in n when p is fixed and $q = q_{\text{quad}}(p) := \binom{p}{2} - \frac{p}{2} + 2$, but $f(n,p,q-1)$ is subquadratic. In this note we improve on the known estimates for $f(n,p,q_{\text{lin}})$ and $f(n,p,q_{\text{quad}})$. Our proofs involve establishing a significant strengthening of a previously known connection between $f(n,p,q)$ and another extremal problem first studied by Brown, Erdős and Sós, as well as building on some recent progress on this extremal problem by Delcourt and Postle and by Shangguan. Also, our upper bound on $f(n,p,q_{\text{lin}})$ follows from an application of the recent forbidden submatchings method of Delcourt and Postle (a method which appears independently as the conflict-free matching method of Glock, Joos, Kim, M. Kühn, and Lichev).

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1 Introduction

Erdős and Shelah [12] first considered the following generalization of a classical Ramsey problem.

Definition 1. Fix integers p, q such that $p \geqslant 3$ and $2 \leqslant q \leqslant \binom{p}{2}$. A (p,q)-*coloring* of K_n is a coloring of the edges of K_n such that every p-clique has at least q distinct colors among its edges. The generalized Ramsey number $f(n,p,q)$ is the minimum number of colors such that K_n has a (p,q)-coloring.

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Erdős and Gyárfás [13] systematically studied $f(n,p,q)$ for fixed p,q as $n \to \infty$. In this paper all asymptotic statements are as $n \to \infty$. Among other results, Erdős and Gyárfás [13] proved that for arbitrary p and

$$q = q_{\text{lin}}(p) := \binom{p}{2} - p + 3,$$

$f(n,p,q)$ is linear, but $f(n,p,q-1)$ is sublinear. Similarly, they showed in [13] that for

$$q = q_{\text{quad}}(p) := \binom{p}{2} - \lfloor p/2 \rfloor + 2,$$

$f(n,p,q)$ is quadratic, but $f(n,p,q-1)$ is subquadratic. Thus for fixed p, we call the value q_{lin} the *linear threshold* and q_{quad} the *quadratic threshold*. The main goal of this note is to estimate $f(n,p,q)$ when q is at the linear or quadratic threshold. In terms of explicit general bounds, we prove the following.

Theorem 2. *For all $p \geqslant 3$ we have*

$$\frac{3p-7}{4p-10}n + o(n) \leqslant f(n,p,q_{lin}) \leqslant n + o(n). \tag{1}$$

For even $p \geqslant 6$ we have

$$\frac{2p-7}{5p-18}n^2 + o(n^2) \leqslant f(n,p,q_{quad}) \leqslant \frac{5}{12}n^2 + o(n^2). \tag{2}$$

We note that recently Gómez-Leos, Heath, Parker, Schwieder, and Zerbib independently show in [17] that $\frac{6}{7}(n-1) \leqslant f(n,5,8) \leqslant n + o(n)$.

Since the initial investigation by Erdős and Gyárfás [13], the asymptotic behavior of $f(n,p,q)$ has attracted a considerable amount of attention. Here we will just mention a few results that are near the linear or quadratic threshold, but the reader can refer to the recent paper of the first author, third author and English [6] for some more history of the problem. Axenovich [1] showed that $f(n,5,9) \leqslant n^{1+o(1)}$ (and it is at least linear since $f(n,5,8)$ is linear). It is an open question whether $f(n,5,9) = O(n)$, and more generally whether $f(n,p,q)$ could be linear for any $q > q_{\text{lin}}(p)$. Sárközy and Selkow [21] addressed this question by proving that there are at most $\log p$ many such values q. Indeed, they showed that $f(n,p,q) > n^{1+\Omega(1)}$ for $q > q_{\text{lin}}(p) + \log p$. At the quadratic threshold, Erdős and Gyárfás [13] showed that $f(n,p,q_{\text{quad}}(p)) \leqslant \left(\frac{1}{2} - \Omega(1)\right)n^2$. Above the quadratic threshold, they asked how large p needs to be before we have $f(n,p,q) = \left(\frac{1}{2} - o(1)\right)n^2$. Again, Sárközy and Selkow [22] addressed this question by proving that there are at most $\frac{1}{2}\log p$ many such values q.

Except for the trivial case of $f(n,3,3) = n + O(1)$ and some values of q larger than the quadratic threshold, there have only been two results where $f(n,p,q)$ is known with a $(1+o(1))$ multiplicative error. Erdős and Gyárfás [13] stated that it "can be easily determined" that

$$f(n,6,14) = \frac{5}{12}n^2 + O(n). \tag{3}$$

More recently, the present authors with Prałat [4] proved that $f(n,4,5) = \frac{5}{6}n + o(n)$ (a fact for which Joos and Mubayi gave a second proof in [19], and for which Joos, Mubayi and Smith gave a third proof in [20]). In this note we provide a proof for (3) and also obtain $f(n,6,14)$ exactly when $n \equiv 1,4 \pmod{12}$ (see Theorem 7). We also obtain two more explicit and asymptotically sharp estimates for generalized Ramsey numbers at the quadratic threshold.

Theorem 3. *We have*

$$f(n,8,26) = \frac{9}{22}n^2 + o(n^2) \quad \text{and} \quad f(n,10,42) = \frac{5}{12}n^2 + o(n^2).$$

The proofs of Theorems 2 and 3 will involve establishing certain connections between $f(n,p,q)$ and the following extremal problem first studied by Brown, Erdős and Sós [7].

Definition 4. Let $\mathcal{H}$ be an r-uniform hypergraph. A (s,k)-*configuration* in $\mathcal{H}$ is a set of s vertices inducing k or more edges. We say $\mathcal{H}$ is (s,k)-*free* if it has no (s,k)-configuration. Let $F^{(r)}(n;s,k)$ be the largest possible number of edges in an (s,k)-free r-uniform hypergraph with n vertices. In terms of classical extremal numbers,

$$F^{(r)}(n;s,k) = \mathrm{ex}_r(n,\mathcal{G}_{s,k}),$$

where $\mathcal{G}_{s,k}$ is the family of all r-uniform hypergraphs on s vertices and k edges.

In fact, three of the four explicit bounds in Theorem 2 follow by first bounding $f(n,p,q)$ implicitly in terms of some values $F^{(r)}(n;s,k)$ and then using explicit bounds on the latter. Thus, further improvements on the estimates for $F^{(r)}(n;s,k)$ would in some cases automatically give improved estimates for $f(n,p,q)$. In the case of the quadratic threshold (and even p), we actually show that the problem of asymptotically estimating $f(n,p,q)$ completely reduces to asymptotically estimating a certain value $F^{(r)}(n;s,k)$.

However, before we introduce that result, we need to review some recent advances in this area. Shangguan [23] and independently Delcourt and Postle [11] showed that

$$\lim_{n\to\infty} \frac{F^{(4)}\left(n;p,\frac{p}{2}-1\right)}{n^2}$$

exists. These proofs extend the recent result by Delcourt and Postle [11], which resolved a conjecture from Brown, Erdős and Sós [7] regarding the existence of a similar limit involving the function $F^{(3)}$ for 3-uniform hypergraphs. In particular, Delcourt and Postle [11] proved the existence (for fixed $\ell \geqslant 3$) of the limit

$$\lim_{n\to\infty} \frac{F^{(3)}\left(n;\ell,\ell-2\right)}{n^2}.$$

Interestingly, the proofs of Delcourt and Postle [11] and Shangguan [23] do not seem to shed much light (at least, not as much as one might hope) on how to actually find the limits whose existence they establish. However, these limits are known in two cases relevant to us. In particular, it is known due to Shangguan and Tamo [24] that

$$F^{(4)}(n;8,3) = \frac{1}{11}n^2 + o(n^2). \tag{4}$$

It is also known due to Glock, Joos, Kuhn, Kim, Lichev and Pikhurko [16] that

$$F^{(4)}(n;10,4) = \frac{1}{12}n^2 + o(n^2). \tag{5}$$

Theorem 5. *For all even $p \geqslant 6$ we have*

$$\lim_{n\to\infty} \frac{f\left(n,p,q_{quad}\right)}{n^2} = \frac{1}{2} - \lim_{n\to\infty} \frac{F^{(4)}\left(n;p,\frac{p}{2}-1\right)}{n^2}. \tag{6}$$

In particular, the limit on the left exists. Furthermore, there exist (p, q_{quad})-colorings using $f(n,p,q_{quad}) + o(n^2)$ colors that use no color more than twice.

Thus, Theorem 3 follows from Theorem 5 together with (4) and (5).

It is perhaps surprising that we need not use any color more than twice. Indeed a (p, q_{quad})-coloring is allowed to use a color up to $\frac{p}{2} - 1$ times, and it would seem more efficient to use the same color as many times as possible.

The lower bound on $f(n,p,q_{\text{lin}})$ in Theorem 2 is similar to the quadratic case, in the sense that it follows from an upper bound on $F^{(3)}(n;p,p-2)$. In particular we prove that

Theorem 6. *For all $p \geqslant 3$ we have*

$$\liminf_{n\to\infty} \frac{f\left(n,p,q_{lin}\right)}{n} \geqslant 1 - \lim_{n\to\infty} \frac{F^{(3)}\left(n;p,p-2\right)}{n^2}. \tag{7}$$

In light of Theorem 5, one might suspect that there is a matching upper bound for (7), but unfortunately this is not the case. Indeed, Glock [14] proved that

$$\lim_{n\to\infty} \frac{F^{(3)}\left(n;5,3\right)}{n^2} = \frac{1}{5},$$

which together with (7) yields $f(n,5,8) \geqslant \frac{4}{5}n + o(n)$. However this lower bound is not close to the truth. Indeed, recently it was proved that $f(n,5,8) = \frac{6}{7}n + o(n)$. The lower bound was proved by Gómez-Leos, Heath, Parker, Schwieder, and Zerbib [17] and the upper bound by the current authors [3].

1.1 Comparison to previous bounds

Here we compare the bounds in Theorem 2 to what was previously known. For the linear threshold, Erdős and Gyárfás [13] showed that

$$(n-1)/(p-2) \leqslant f(n,p,q_{\text{lin}}) \leqslant c_p n \tag{8}$$

for some coefficient c_p. The lower bound in (8) follows from the simple fact that in a (p, q_{lin})-coloring each vertex is adjacent to at most $p-2$ edges of each color. The upper bound in (8) follows from the Local Lemma. The constant c_p is not explicitly discussed in [13] but it is easy to see from their proof that $c_p \to \infty$ as $p \to \infty$. Thus we see that in Theorem 2, (1) is a significant improvement on previous bounds. Indeed, the gap between the coefficients in (8) grows without bound with p, whereas the coefficients in (1) are always between 3/4 and 1.

Likewise, for the quadratic threshold (and even p) the trivial bounds are

$$\frac{\binom{n}{2}}{\frac{p}{2}-1} \leqslant f(n, p, q_{\text{quad}}) \leqslant \binom{n}{2}.$$

The upper bound follows since we can give different colors to the different edges, and the lower bound follows from the fact that each color must be used at most $\frac{p}{2}-1$ times. Thus we see that (2) in Theorem 2 is a significant improvement.

1.2 Structure of the note

The structure of this note is as follows. In Section 2 we address the quadratic threshold. We start with a proof of a more precise version of (3). We go on to prove Theorem 5 and (2) from Theorem 2. In Section 3 we address the linear threshold. There we prove Theorem 6 and (1) from Theorem 2.

2 Quadratic Threshold

In this section we address the quadratic threshold. First we introduce some terminology. Suppose we are given a coloring of the edges of K_n. For a set of vertices S, let $c(S)$ be the number of colors appearing on edges within S, and let $r(S)$ be $\binom{|S|}{2} - c(S)$. We call $r(S)$ the *number of color repetitions (or just repeats) in* S. Sometimes it may help the reader to imagine counting $r(S)$ by examining each edge of S in some order and counting a repeat whenever we see a color we have already seen.

2.1 Estimating $f(n, 6, 14)$

In this subsection we state and prove our more precise result for $f(n, 6, 14)$. As we noted, Erdős and Gyárfás [13] stated that $f(n, 6, 14) = \frac{5}{12}n^2 + O(n)$ without proof. To help the reader gain familiarity with the concepts in this paper, we present a proof of a more precise version of this result.

Theorem 7. *We have*

$$\frac{5}{6}\binom{n}{2} \leqslant f(n, 6, 14) \leqslant \frac{5}{6}\binom{n}{2} + O(n).$$

Furthermore, the lower bound above is the exact value of $f(n, 6, 14)$ *whenever* n *is congruent to* 1 *or* 4 *modulo* 12.

Proof. Starting with the lower bound, suppose we have any $(6,14)$-coloring. Since $\binom{6}{2} = 15$, any set of 6 vertices is allowed to have only one repeat, which implies that we cannot have 3 edges of the same color. Indeed, taking the union of these edges would be a set of at most 6 vertices with more than one repeated color. This also means that there can be at most one monochromatic path on three vertices P_3, since the union of two of them would be a set of at most 6 vertices with at least two repeats. If our coloring contains a monochromatic P_3, then we remove it and get a coloring of K_{n-3}. So we have a $(6,14)$-coloring of $K_{n'}$ with $n' \in \{n-3, n\}$ with no monochromatic P_3.

Suppose the color c is used twice, say on the (nonincident) edges ab and xy. Then the other four edges in $\{a, b, x, y\}$ must all have different colors which are only used once in the whole graph. Let C_1 be the set of colors used once and C_2 the colors used twice. For each $c \in C_2$ let K_c be the set of 4 vertices consisting of both endpoints of both edges of color c. Note that for $c, c' \in C_2$ we have $|K_c \cap K_{c'}| \leqslant 1$, since otherwise $K_c \cup K_{c'}$ is a set of at most 6 vertices with too many repeats. Thus the sets K_c induce edge-disjoint 4-cliques. Thus, if we did not remove any P_3, we have that $|C_2|$, the number of such cliques, is at most

$$\frac{1}{6}\binom{n}{2}. \tag{9}$$

On the other hand, if we did remove a P_3, this would contribute one additional color to C_2 along with the restriction on the K_c. From our discussion above, we note that this P_3 is vertex disjoint from all the K_c and does not share a color with any other edges. Thus, in this case, $|C_2|$ is at most

$$1 + \frac{1}{6}\binom{n-3}{2}. \tag{10}$$

But since (9) is at least (10) for $n \geqslant 4$, we conclude that the number of colors used is at least

$$|C_1| + |C_2| = \left(\binom{n}{2} - 2|C_2|\right) + |C_2| = \binom{n}{2} - |C_2| \geqslant \frac{5}{6}\binom{n}{2}.$$

Thus we are done with the lower bound for Theorem 7. We move on to the upper bound.

If $n \equiv 1$ or $4 \mod 12$, then we are guaranteed a perfect packing of $\frac{1}{6}\binom{n}{2}$ edge-disjoint 4-cliques by Hanani's result [18]. Then for each clique in the packing, color two nonadjacent edges the same color and give a unique color to the remaining edges. Since we use exactly 5 colors for each clique, we use exactly $\frac{5}{6}\binom{n}{2}$ colors to color all the edges.

Otherwise, let $i = (n \mod 12)$ and partition the vertices into $K_{n-i+1} \cup K_{i-1}$, and find a perfect packing of edge-disjoint 4-cliques for K_{n-i+1}. Follow the same coloring as above for the perfect packing, and then color the remaining $(n-i+1)(i-1) + \binom{i-1}{2} = O(n)$ edges with a different color for each edge. Thus, we use $\frac{5}{6}\binom{n-i+1}{2} + O(n) = \frac{5}{6}\binom{n}{2} + O(n)$ colors.

Notice that in either case, the resulting coloring satisfies the $(6,14)$-coloring condition. If not, then there exists a set S of 6 vertices with more than 2 repeated colors. In our coloring, this means that S must contain two cliques from the packing. But since the cliques must be edge-disjoint, this implies that $|S| \geqslant 7$, a contradiction. □

2.2 Proof of Theorem 5

In this subsection we will prove Theorem 5 after some discussion. We consider the case of (p, q)-coloring, where

$$p = 2\ell \quad \text{and} \quad q = q_{\text{quad}}(p) = \binom{2\ell}{2} - \ell + 2.$$

This choice of parameters allows using a color $\ell - 1$ times but not ℓ times. Erdős and Gyárfás [13] showed that for this choice of parameters $f(n, p, q)$ is quadratic in n. Of course the upper bound $f(n, p, q) \leqslant \binom{n}{2}$ is trivial, but [13] also gives a nontrivial upper bound of $(1/2-\varepsilon)n^2$ for some $\varepsilon > 0$. Specifically, Erdős and Gyárfás [13] used a 4-uniform $(2\ell, \ell - 1)$-free hypergraph $\mathcal{H}$ in which for each edge of $\mathcal{H}$, a new color is used to color two pairs of vertices from that edge (and then coloring uncolored edges with distinct colors). Crucially, every color repetition in the coloring corresponds to a hyperedge of $\mathcal{H}$. We also notice that each color is used at most twice, and for any color used on two edges, the union of those two edges is a hyperedge of $\mathcal{H}$. The existence of a suitable hypergraph had already been established by Brown, Erdős and Sós [7]. The same basic connection between (p, q)-coloring near the quadratic threshold and 4-uniform (s, k)-free hypergraphs (for the appropriate s, k) was exploited by Sárközy and Selkow [22] and again by Conlon, Gishboliner, Levanzov and Shapira [9]. However, this connection as it was used in [8, 13, 22] is not precise enough to prove Theorem 5. Indeed, all these previous results give away a constant factor in the main term of their estimate of $f(n, p, q)$, while we want an asymptotically tight estimate. Thus, we will have to significantly refine these previously established connections between the Erdős-Gyárfás coloring problem and the Brown-Erdős-Sós packing problem.

Now we will define some functions related to $F^{(4)}(n; 2\ell, \ell - 1)$. The first one relaxes the problem to multi-hypergraphs.

Definition 8. Let $\mathcal{H}$ be an r-uniform *multi-hypergraph*, meaning that $\mathcal{H}$ can have edges with multiplicity (but each edge has r distinct vertices). A (s, k)-*configuration* in $\mathcal{H}$ is a set of s vertices inducing k or more edges (counted with multiplicity). Let $G^{(r)}(n; s, k)$ be the largest possible number of edges in an (s, k)-free r-uniform multi-hypergraph with n vertices.

Next we define a function that restricts the extremal problem for $F^{(4)}(n; 2\ell, \ell - 1)$ to a smaller family of hypergraphs.

Definition 9. Let $H^{(4)}(n; 2\ell, \ell-1)$ be the largest possible number of edges in a 4-uniform hypergraph $\mathcal{H}$ on n vertices which satisfies the following conditions:

1. $\mathcal{H}$ is $(2\ell, \ell - 1)$-free,
2. $\mathcal{H}$ is $(2i + 1, i)$-free for $i = 2, \dots, \ell - 2$, and
3. for every vertex v of $\mathcal{H}$, either v has degree 0 or degree at least $\ell - 1$.

Using Shangguan's notation [23], our function $H^{(4)}(n; 2\ell, \ell-1)$ defined above is the same as what Shangguan refers to as $f_r^{(t)}(n; er-(e-1)k, e)$, where 4 is substituted for r, 2 for k, 2 for t, and $\ell-1$ for e. Since $H^{(4)}$ is a restriction and $G^{(4)}$ is a relaxation, we have

$$H^{(4)}(n; 2\ell, \ell-1) \leqslant F^{(4)}(n; 2\ell, \ell-1) \leqslant G^{(4)}(n; 2\ell, \ell-1).$$

Shangguan [23] proved (see Lemma 5.5 and the discussion above it) that

Lemma 10 (Lemma 5.5 in [23])**.**

$$\lim_{n\to\infty} \frac{H^{(4)}(n; 2\ell, \ell-1)}{n^2} = \lim_{n\to\infty} \frac{F^{(4)}(n; 2\ell, \ell-1)}{n^2}. \tag{11}$$

Now we will easily see that G is likewise asymptotically the same as the others.

Claim 11.

$$\lim_{n\to\infty} \frac{G^{(4)}(n; 2\ell, \ell-1)}{n^2} = \lim_{n\to\infty} \frac{F^{(4)}(n; 2\ell, \ell-1)}{n^2}. \tag{12}$$

Proof. Let $\mathcal{H}$ be an extremal multi-hypergraph for the $G^{(4)}(n; 2\ell, \ell-1)$ problem, i.e., $\mathcal{H}$ has $G^{(4)}(n; 2\ell, \ell-1)$ edges and is $(2\ell, \ell-1)$-free. We form a new hypergraph $\mathcal{H}'$ by simply deleting all multiple edges in $\mathcal{H}$. Clearly $\mathcal{H}'$ is $(2\ell, \ell-1)$-free, so it has at most $F^{(4)}(n; 2\ell, \ell-1)$ edges.

We show that $\mathcal{H}'$ has almost the same number of edges as $\mathcal{H}$. Indeed, suppose we enumerate all the multiple edges of $\mathcal{H}$, say $\{e_1, \dots, e_a\}$ where e_i has multiplicity $m_i \geqslant 2$. Notice that, for even ℓ, taking $\ell/2$ of the e_i would give a $(2\ell, \ell)$-configuration and, for odd ℓ, taking $(\ell-1)/2$ of e_i would give a $(2\ell-2, \ell-1)$-configuration. Therefore, we have $a \leqslant \ell$. In addition is easy to see that each $m_i \leqslant \ell$ (otherwise there is a $(2\ell, \ell-1)$-configuration). Thus, we remove at most ℓ^2 edges from $\mathcal{H}$ to get $\mathcal{H}'$. Consequently, we have

$$F^{(4)}(n; 2\ell, \ell-1) \leqslant G^{(4)}(n; 2\ell, \ell-1) \leqslant F^{(4)}(n; 2\ell, \ell-1) + \ell^2$$

and (12) follows (recall we already knew that the limit on the right exists due to Lemma 10). ☐

Now we start to attack the lower bound for the coloring problem.

Claim 12.

$$f\left(n, p, q_{quad}\right) \geqslant \binom{n}{2} - G^{(4)}(n; 2\ell, \ell-1).$$

We will prove this claim directly, by using a (p, q_{quad})-coloring to construct a $(2\ell, \ell-1)$-free multi-hypergraph. Towards that end we define the following.

Definition 13. Consider any coloring C of the edges of K_n. We say a 4-uniform hypergraph $\mathcal{H}$ is a *repeat multi-hypergraph* for the coloring C if it is formed as follows. $\mathcal{H}$ has the same vertex set as K_n. For each color c used in the coloring, let $E(c) \neq \emptyset$ be the set of edges of color c and let e_c be some particular (arbitrary) edge of color c. Then $\mathcal{H}$ will have all the edges $\{e \cup e_c : e \in E(c) \setminus \{e_c\}\}$. Of course, $e \cup e_c$ might only have 3 vertices (when we claimed $\mathcal{H}$ would be 4-uniform) but we fix this by arbitrarily adding vertices to edges of size 3.

Note that $\mathcal{H}$ can have multiple edges since a single set of 4 vertices can contain, say, two red edges and also two blue edges. Also, since the construction of $\mathcal{H}$ potentially involves some arbitrary choices (in particular, the choice of the edges e_c as well as the choice of vertices used to enlarge 3-edges), in general a coloring C may give rise to several possible repeat multi-hypergraphs $\mathcal{H}$.

We now make the key observation about repeat multi-hypergraphs. Essentially it says that edges in $\mathcal{H}$ count color repetitions of C "faithfully," i.e., without under- or over-counting.

Observation 14. *Let $\mathcal{H}$ be a repeat multi-hypergraph for the coloring C. Then if S is the set of all vertices,*

$$r(S) = |E(\mathcal{H})|.$$

Proof of Observation 14. Recall that each hyperedge of $\mathcal{H}$ contains $e \cup e_c$ for some color c and some edge e that has color c. Now if S spans b hyperedges all corresponding to the same color c, then S contains $e_i \cup e_c$ for $1 \leqslant i \leqslant b$ and some edges $e_1, \ldots, e_b$ which all have color c. In particular S contains $b+1$ edges, namely $e_c, e_1, \ldots e_b$, which all have color c, i.e., S spans b repeats in the color c. Now if S spans b hyperedges (which now need not all correspond to the same color), we likewise conclude that S spans b repeats by simply summing over the colors. □

We are now ready to prove Claim 12.

Proof of Claim 12. Consider a (p, q_{quad})-coloring C of K_n that is optimal, i.e., uses $f(n, p, q_{\text{quad}})$ colors. In such a coloring, any set of $p = 2\ell$ vertices spans at most $\ell - 2$ repeats. Let $\mathcal{H}$ be a repeat multi-hypergraph for C. By Observation 14, a $(2\ell, \ell-1)$-configuration in $\mathcal{H}$ would be a set of 2ℓ vertices spanning at least $\ell - 1$ repeats. Since C is a $(2\ell, \binom{2\ell}{2} - \ell + 2)$-coloring, $\mathcal{H}$ is $(2\ell, \ell-1)$-free. In particular, $|E(\mathcal{H})| \leqslant G^{(4)}(n; 2\ell, \ell-1)$. But now applying Observation 14, we have $|E(\mathcal{H})| = \binom{n}{2} - f(n, p, q_{\text{quad}})$ since C uses $f(n, p, q_{\text{quad}})$ colors. The claim now follows from

$$\binom{n}{2} - f(n, p, q_{\text{quad}}) = |E(\mathcal{H})| \leqslant G^{(4)}(n; 2\ell, \ell - 1). \qquad \square$$

Next we attack the upper bound for the coloring problem. To get a bound that comes close to matching Claim 12, we will have to "reverse" the procedure we used to turn a coloring C into a repeat multi-hypergraph $\mathcal{H}$. We must be careful for a few reasons. First, as we discussed earlier, a single coloring C can give rise to many different $\mathcal{H}$. Second, although we saw that if C is a (p, q_{quad})-coloring then $\mathcal{H}$ must be $(2\ell, \ell-1)$-free, in general the converse does not hold. In particular, if some set of vertices S does not contain the edge e_c then S could have many repeats in the color c but not span even one edge of $\mathcal{H}$. We will get around these issues by ensuring that our coloring uses each color at most twice, and we never use the same color on adjacent edges. For such a coloring C, the repeat multi-hypergraph $\mathcal{H}$ is unique. Furthermore, such a coloring C is a (p, q_{quad})-coloring if and only if $\mathcal{H}$ is $(2\ell, \ell - 1)$-free.

Claim 15.

$$f\left(n, p, q_{quad}\right) \leqslant \binom{n}{2} - H^{(4)}(n; 2\ell, \ell-1).$$

Proof. We will construct a (p, q_{quad})-coloring that uses $\binom{n}{2} - H^{(4)}(n; 2\ell, \ell-1)$ colors. We start with an extremal graph $\mathcal{H}$ for the $H^{(4)}(n; 2\ell, \ell-1)$ problem. In other words, $\mathcal{H}$ has n vertices, $H^{(4)}(n; 2\ell, \ell-1)$ edges, and properties (1)–(3).

We construct a coloring as follows. Start with an edge $h_1 \in \mathcal{H}$ and choose two arbitrary, disjoint pairs $e_1, f_1 \subseteq h_1$ and assign them the color c_1. Assign all other pairs in h_1 a different color. Let the set of "active" pairs after step 1 be $A_1 = \{e_1, f_1\}$. Then define

$$H_1 = \{h \in E(\mathcal{H}) \setminus \{h_1\} : e \subseteq h \text{ for some } e \in A_1\}.$$

In general, assume that we have defined colors in $h_1, h_2, \ldots, h_{k-1}$ such that

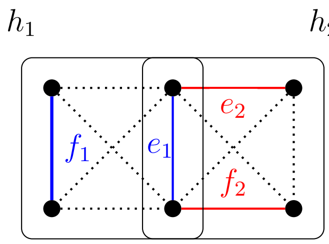

Figure 1: The coloring after step 2. Dotted edges are given all unique colors.

- $A_{k-1} = \{e_1, \ldots, e_{k-1}\} \cup \{f_1, \ldots, f_{k-1}\}$
- $H_{k-1} = \{h \in E(\mathcal{H}) \setminus \{h_1, \ldots, h_{k-1}\} : e \subseteq h \text{ for some } e \in A_{k-1}\}$, and
- $|\bigcup_{i=1}^{k-1} h_i| = 2k$.

Notice that these are true for step 1 and that $k < \ell - 1$ since otherwise there would be a set of 2ℓ vertices on $\ell - 1$ edges, violating property (1). We choose an arbitrary edge h_k from H_{k-1}. We will show that $\left(\bigcup_{i=1}^{k-1} h_i\right) \cap h_k = e$ for some $e \in A_{k-1}$. Indeed, clearly $e \subseteq \left(\bigcup_{i=1}^{k-1} h_i\right) \cap h_k$ and by property (2) if the cardinality of the intersection were 3, then $\bigcup_{i=1}^{k} h_i$ has $2k + 4 - 3 = 2k + 1$ vertices that induces at least k edges, violating the $(2k+1, k)$-free condition in $\mathcal{H}$. Thus, $\left|\bigcup_{i=1}^{k} h_i\right| = 2(k+1)$. Then pick two disjoint, uncolored pairs $e_k, f_k \subseteq h_k$ (there are two such choices) and color them c_k and color the different uncolored pairs in h_k different colors. Finally, define

$$A_k = A_{k-1} \cup \{e_k, f_k\}$$

and

$$H_k = \{h \in E(\mathcal{H}) \setminus \{h_1, \ldots, h_k\} : e \subseteq h \text{ for some } e \in A_k\}.$$

Continue in this way until $H_k = \emptyset$ for some k. Notice that $k < \ell - 1$ since otherwise there would be a set of 2ℓ vertices on $\ell - 1$ edges, violating property (1). Then repeat this process with any edge other than $h_1, \ldots, h_k$. Continue until all edges have been processed, and give any uncolored pairs a unique color. Notice that each newly chosen edge will intersect the union of any of the former edges by at most 2 by property (2). Note that for each edge in $\mathcal{H}$, the coloring has exactly one repeat, and there are $H^{(4)}(n, 2\ell, \ell - 1)$ edges. Thus when considering the coloring of the pairs, we obtain a coloring C of K_n with

$$|C| = \binom{n}{2} - H^{(4)}(n, 2\ell, \ell - 1).$$

To verify that C is a (p, q_{quad})-coloring, choose any set S of $p = 2\ell$ vertices. By property (1), in $\mathcal{H}$, the set S induces at most $\ell - 2$ hyperedges. And our coloring defines exactly one repeat per hyperedge, and none elsewhere. So the total number of distinct colors among the edges of $K_n[S]$ is at least $q_{\text{quad}} = \binom{2\ell}{2} - \ell + 2$. □

Finally observe that Theorem 5 follows from Lemma 10 and Claims 11, 12 and 15.

2.3 Proof of (2) from Theorem 2

In this subsection we establish the explicit bounds (2). They will follow from Theorem 5 together with explicit bounds for the function $F^{(4)}$.

As we mentioned before, Delcourt and Postle [10] proved some very general and powerful results to the effect that certain hypergraphs have almost-perfect matchings which avoid certain forbidden submatchings. Similar results were independently proved by Glock, Joos, Kim, Kühn and Lichev [15]. Each team of researchers was motivated in part by finding approximate designs of high "girth". In particular, it follows just as well from either [10] (Theorem 1.3) or [15] (Theorem 1.1) that for any $\ell \geqslant 3$ there exists a linear 4-uniform hypergraph $\mathcal{H}$ on n vertices with $\frac{1}{12}n^2 + o(n^2)$ edges which is also $(2\ell, \ell - 1)$-free. In other words, $F^{(4)}(n; 2\ell, \ell - 1) \geqslant \frac{1}{12}n^2 + o(n^2)$. Now the upper bound in (2) follows from Theorem 5.

We move on to the lower bound in (2), which will follow from an upper bound on $H^{(4)}(n; 2\ell, \ell - 1)$ (which of course gives an asymptotic upper bound on $F^{(4)}(n; 2\ell, \ell - 1)$ by (11)). In particular, we will be done when we prove the following:

Claim 16. *For $\ell \geqslant 2$ we have*

$$H^{(4)}(n; 2\ell, \ell - 1) \leqslant \frac{\ell - 2}{10\ell - 18} n^2 + o(n^2).$$

The proof is a straightforward adaptation of Delcourt and Postle's proof of their Lemma 1.9 in [11].

Proof. Let $\mathcal{H}$ be a 4-uniform hypergraph on n vertices which is $(2\ell, \ell - 1)$-free and $(2i+1, i)$-free for $i = 2, \ldots, \ell - 2$ (recall Definition 9).

Define a graph G with $V(G) = E(\mathcal{H})$, where $e_1 e_2 \in E(G)$ whenever $|e_1 \cap e_2| \geqslant 2$. Each component of G must have order at most $\ell - 2$ since $\mathcal{H}$ is $(2\ell, \ell - 1)$-free. Let $\{e_1, \ldots e_b\}$ be a component in G for some $1 \leqslant b \leqslant \ell - 2$. Assume that the ordering $\{e_1, \dots e_b\}$ is chosen so that for each $2 \leqslant i \leqslant b$ there is $1 \leqslant j \leqslant i-1$ such that $|e_i \cap e_j| \geqslant 2$. We claim that for each $i \geqslant 2$, e_i has two vertices (in $V(\mathcal{H})$) which are not in $e_1 \cup \dots \cup e_{i-1}$; otherwise, we would have a $(2i+1, i)$-configuration in $\mathcal{H}$. On the other hand, due to our choice of the ordering, there is an edge $e_j \in \{e_1 \cup \dots \cup e_{i-1}\}$ such that $|e_i \cap e_j| \geqslant 2$. Consequently, e_i shares only one pair of vertices with $e_1 \cup \dots \cup e_{i-1}$ and so e_i contains five pairs of vertices which are not subsets of any $e_j, j < i$. Of course e_1 contains six pairs and each edge after that has five more, so the total number of pairs contained in some $e_j, j \leqslant b$ is at least $5b+1$. Note that for two edges of $\mathcal{H}$, if they are in different components of G then they do not share any pair of vertices in $\mathcal{H}$.

For $1 \leqslant b \leqslant \ell - 2$ let C_b be the number of components of G of order b. Then we have

$$|E(\mathcal{H})| = \sum_{1 \leqslant b \leqslant \ell-2} bC_b, \tag{13}$$

which implies

$$\sum_{1 \leqslant b \leqslant \ell-2} C_b \geqslant \frac{1}{\ell-2}|E(\mathcal{H})|. \tag{14}$$

But now by summing the vertex-pairs in $\mathcal{H}$ we have

$$\binom{n}{2} \geqslant \sum_{1 \leqslant b \leqslant \ell-2} (5b+1)C_b \geqslant \left(5 + \frac{1}{\ell-2}\right)|E(\mathcal{H})|,$$

where the last inequality uses (13) and (14). It follows that

$$|E(\mathcal{H})| \leqslant \frac{\ell-2}{10\ell-18}n^2 + o(n^2),$$

which completes the proof. □

3 Linear threshold

In this section we address the linear threshold. First we prove Theorem 6 and the lower bound in (1).

3.1 Proof of Theorem 6 and the lower bound in (1)

We start by comparing $F^{(3)}$ with $G^{(3)}$. This is analogous to Claim 11.

Claim 17. *For all* $p \geqslant 3$,

$$G^{(3)}(n; p, p-2) = F^{(3)}(n; p, p-2) + O(n).$$

Proof. Let $\mathcal{H}$ be an extremal multi-hypergraph for the $G^{(3)}(n; p, p-2)$ problem. We will show that $\mathcal{H}$ has $G^{(3)}(n; p, p-2)$ edges and is $(p, p-2)$-free. Then $\mathcal{H}$ has at most Cn edges of multiplicity at least 2, where $3C = 2x$ and x is whichever of $(p-2)/2$ or $(p-1)/2$ is an integer. Suppose to the contrary. Let $\mathcal{H}_2$ be the multi-hypergraph with $V(\mathcal{H}_2) = V(\mathcal{H})$ and all edges from $\mathcal{H}$ with multiplicity at least 2. Then the average degree in $\mathcal{H}_2$ is at least $3C = 2x$. So there is a set of $2x$ edges on at most $1 + 2x$ vertices. But $2x \geqslant p - 2$ and $2x + 1 \leqslant p$, so this contradicts the fact that $\mathcal{H}$ is $(p, p-2)$-free.

Now we form $\mathcal{H}'$ by deleting the edges with multiplicity at least 2 that appear in $\mathcal{H}$. Since $\mathcal{H}'$ is also $(p, p-2)$-free, then it has at most $F^{(4)}(n; p, p-2)$ edges. In addition, we must delete at most Cn edges of $\mathcal{H}$ to obtain $\mathcal{H}'$, so

$$F^{(3)}(n; p, p-2) \leqslant G^{(3)}(n; p, p-2) \leqslant F^{(3)}(n; p, p-2) + Cn,$$

proving the claim. □

The next claim is analogous to Claim 12.

Claim 18. *For all $p \geqslant 3$*

$$f\left(n, p, q_{lin}\right) \geqslant n - 1 - \frac{1}{n} G^{(3)}(n; p, p-2).$$

Proof. Consider any (p, q_{lin})-coloring using color set C. So any set of p vertices spans at most $p - 3$ repeats. We form the 3-uniform hypergraph $\mathcal{H}$ as follows. For each vertex v and color c used on at least one edge adjacent to v, say $\{e_1, \ldots e_\ell\}$ is the set of edges adjacent to v and colored c. Then $\mathcal{H}$ has the edges $e_1 \cup e_i$ for $i = 2, \ldots, \ell$.

$\mathcal{H}$ is $(p, p-2)$-free, but it might have multiple edges which come from monochromatic triangles. Therefore

$$|E(\mathcal{H})| \leqslant G^{(3)}(n; p, p-2).$$

Each hyperedge of $\mathcal{H}$ corresponds to two edges of the same color sharing a vertex v, and so some particular vertex v plays that role at most

$$\frac{1}{n}|E(\mathcal{H})| \leqslant \frac{1}{n} G^{(3)}(n; p, p-2)$$

times. But these hyperedges of $\mathcal{H}$ count all of the color repeats on edges incident with v. Thus the number of different colors used on edges adjacent with v is at least

$$n - 1 - \frac{1}{n} G^{(3)}(n; p, p-2).$$

□

Theorem 6 now follows from Claims 17 and 18. In turn, the lower bound in (1) follows from Theorem 6 and Lemma 1.9 from Delcourt and Postle [11], which states that

$$F^{(3)}(n, p, p-2) \leqslant \frac{p-3}{4p-10} n^2 + o(n^2).$$

3.2 Proof of the upper bound in (1)

We now turn to the upper bound at the linear threshold found in Theorem 2. We use the forbidden submatchings method of Delcourt and Postle [10]. This method, and the very similar method of Glock, Joos, Kim, Kühn and Lichev [15], consists of some theorems which guarantee "large" matchings avoiding certain submatchings (i.e. not containing certain sets of edges) in "nice" hypergraphs. Joos and Mubayi [19] first applied the method from [15] to generalized Ramsey problems. Since then, both of the two similar methods from [15] and [11] have been applied to several more generalized Ramsey problems (see [2, 3, 5, 17, 20]).

To introduce this method, we require the following definitions from [10].

Definition 19.

(i) For $r \geqslant 1$, we say a hypergraph is r-bounded if every edge has size at most r. The i-degree of a vertex v of H, denoted $d_{H,i}(v)$, is the number of edges of H of size i containing v. The maximum i-degree of H, denoted $\Delta_i(H)$, is the maximum of $d_{H,i}(v)$ over all vertices v of H.

(ii) Let G be a hypergraph. We say a hypergraph H is a configuration hypergraph for G if $V(H) = E(G)$ and $E(H)$ consists of a set of matchings of G of size at least 2. We say a matching of G is H-avoiding if it spans no edge of H.

(iii) We say a hypergraph $G = (A, B)$ is *bipartite with parts* A *and* B if $V(G) = A \cup B$ and every edge of G contains exactly one vertex from A. We say a matching of G is A*-perfect* if every vertex of A is in an edge of the matching. We say a matching in G is H*-avoiding* if it contains no edge of H.

(iv) Let H be a hypergraph. The *maximum* (k, ℓ)*-codegree* of H is

$$\Delta_{k,\ell}(H) = \max_{S \in \binom{V(H)}{\ell}} |\{e \in E(H) : S \subset e, |e| = k\}|$$

We will use a slightly simplified version of Theorem 1.16 of Delcourt and Postle [10]. In particular the full version allows H to have edges of size 2, at the cost of having to check some extra conditions. For our application we will avoid edges of size 2 in H.

Theorem 20 (Delcourt and Postle [10])**.** *For all integers $r \geqslant 2, g \geqslant 3$ and real $\beta \in (0, 1)$, there exist an integer $D_\beta > 0$ and real $\alpha > 0$ such that following holds for all $D \geqslant D_\beta$:*

Let $G = (A, B)$ be a bipartite r-bounded hypergraph such that

(G1) every pair of vertices is in at most $D^{1-\beta}$ edges, and

(G2) every vertex in A has degree at least $(1 + D^{-\alpha})D$ and every vertex in B has degree at most D.

Let H be a g-bounded configuration hypergraph of G whose edges all have size at least 3 such that

(H1) $\Delta_i(H) \leqslant \alpha \cdot D^{i-1} \log D$ *for all* $3 \leqslant i \leqslant g$*;*

(H2) $\Delta_{k,\ell}(H) \leqslant D^{k-\ell-\beta}$ *for all* $3 \leqslant \ell < k \leqslant g$*; and*

Then there exists an H*-avoiding* A*-perfect matching of* G*.*

Proof of upper bound in (1). Fix some $\beta \in (0,1)$, set $r = 3$ and $g = \binom{p}{2}$ and let $\alpha > 0$ be the value guaranteed by Theorem 20. Fix some ε with $0 < \varepsilon < \alpha$. Let C be a set of $n + n^{1-\varepsilon}$ colors. For each $c \in C$ and $v \in V(K_n)$, we define the vertex v_c. Let $G = (A, B)$ be a bipartite hypergraph with parts $A = E(K_n)$ and $B = \{v_c : v \in V(K_n), c \in C\}$, and with edge set

$$E(G) = \{\{uv, u_c, v_c\} : u, v \in V(K_n), c \in C\}.$$

Note that G is 3-uniform (and thus 3-bounded). We intend to find an A-perfect matching M in G, which will give us a $(p, p-2)$-coloring as follows. For each edge $\{uv, u_c, v_c\} \in M$ we just color the edge uv with the color c. Since M is A-perfect, every edge of K_n gets exactly one color. Note that since M is a matching, no two incident edges uv and vw in K_n can get the same color c.

We now define H, our configuration hypergraph of G. Of course we let $V(H) = E(G)$. Suppose $S \subseteq E(G) = V(H)$ is a matching, so S corresponds to a coloring c_S of some of the edges of K_n. We will let S be an edge of H if we have that

> the number of vertices of K_n spanned by edges that are colored by c_S is $v(S)$ where $4 \leqslant v(S) \leqslant p$ and the number of color repetitions in c_S is $R(S) \geqslant v(S) - 2$.

If any edge in $E(H)$ is not minimal (i.e., it properly contains another edge) we remove it. When $k = p$, an edge of H corresponds to a violation of the (p, q_{lin})-condition in K_n. Including the edges of H corresponding to $4 \leqslant k \leqslant p-1$ is important to verify the conditions Theorem 20. It is easy to see that H is $\binom{p}{2}$-bounded. Also, note that for all $e \in E(H)$, $|e| \geqslant 4$.

We now verify that condition (G1) holds. Define $D = n$. Let $x, y \in V(G)$. Clearly, if $x, y \in A$, then the codegree is zero since there is exactly one member of A in each edge of G. If $x \in A$ and $y \in B$ then $x = uv$ for some $u, v \in K_n$. If $y = u_c$ or v_c, then the codegree is 1. Otherwise, the codegree is 0. Finally, if $x, y \in B$, then the codegree is either 0 or 1, depending on whether they share the same color subscript c. Thus, all codegrees in G are at most 1, verifying condition (G1).

Next, we verify that condition (G2) holds. For any vertex $uv \in A$, the degree of uv in G is exactly $|C| = n + n^{1-\varepsilon} = D(1 + D^{-\varepsilon}) \geqslant D(1 + D^{-\alpha})$. In addition, for any vertex $u_c \in B$, the degree of u_c in G is exactly $n - 1 \leqslant D$. So condition (G2) is verified.

Next, we verify that condition (H1) holds. Let $e = \{uv, u_c, v_c\} \in V(H) = E(G)$. We count edges I of H of size i with $e \in I$. For some $4 \leqslant k \leqslant p$, the number $v(I)$ of vertices of K_n spanned by edges of I in G is k and the number of color repetitions $R(I)$ is at least $k - 2$. So besides u and v, $v(I)$ must count exactly $k - 2$ other vertices of K_n. Let x be the number of colors induced by I other than c. We count $R(I)$ by taking the difference between the number of edges colored by c_S and the number of distinct colors used. Thus we have $i-(1+x) \geqslant k-2$, so $x \leqslant i-k+1$. There are at most $\binom{p}{2} = O(1)$ choices for remaining, unaccounted for, colored edges of K_n edges in I. I is determined by choosing $k-2$ vertices of K_n and at most $i-k+1$ colors, so

$$\Delta_i(H) = O\left(\sum_{k=4}^{p} n^{k-2} \cdot n^{i-k+1}\right) = O(n^{i-1}) \leqslant \alpha \cdot D^{i-1} \log D$$

for all $2 \leqslant i \leqslant g$, verifying condition (H1).

To verify condition (H2), fix k and ℓ, and let $L \subseteq V(H)$ have size ℓ. We count the number of $K \in E(H)$ such that $L \subset K$ and $|K| = k$. If $v(L) > p$ there is no possible K, so we assume $v(L) \leqslant p$. If $v(L)$ is 2 or 3 then $R(L) = 0$. Otherwise we have $v(L) \geqslant 4$. If $R(L) \geqslant v(L) - 2$ then there is no possible $K \subseteq L$ since we removed nonminimal edges from H, so assume $R(L) \leqslant v(L) - 3$. Let us count possible edges K such that $v(K) = t$. Since K is an edge of H, we have $R(K) = t - 2$. Then $v(L) - 3 \geqslant R(L) = \ell - |C_L|$, so $|C_L| \geqslant \ell - v(L) + 3$. Suppose there are $\ell - R(L)$ many colors used by the coloring c_L, and say there are x colors used by c_K that are not used by c_L. Then the number of colors used by c_K is

$$x + \ell - R(L) \leqslant k - R(K) = k - t + 2$$

and so

$$x \leqslant R(L) - \ell + k - t + 2 \leqslant v(L) - \ell + k - t - 1.$$

To determine K we choose $t - v(L)$ vertices of K_n which are not touched by the coloring c_L, and then we choose x many colors. Given that choice there are only a constant number of ways to choose which edges are colored and which colors they get. Therefore,

$$\Delta_{k,\ell}(H) \leqslant O\left(\sum_{t \leqslant p} n^{t-v(L)} \cdot n^{v(L)-\ell+k-t-1}\right) = O(n^{k-\ell-1}) \leqslant D^{k-\ell-\beta}$$

for all $2 \leqslant \ell < k \leqslant g$, verifying condition (H2).

Therefore, there exists an H-avoiding A-perfect matching of G, which corresponds to a $\left(p, \binom{p}{2} - p + 3\right)$-coloring of K_n using our set C of $n + n^{1-\varepsilon}$ colors. □

4 Concluding remarks

We previously conjectured that $f(n, 5, 8) \geqslant \frac{7}{8}n + o(n)$. However there was an error in our proof, and recently in [3] we proved that instead we have $f(n, 5, 8) = \frac{6}{7}n + o(n)$, matching the lower bound of Gómez-Leos, Heath, Parker, Schwieder, and Zerbib [17].

It is plausible to conjecture the following. Currently it is known only for $p = 3, 4$.

Conjecture 21. The limit

$$\lim_{n \to \infty} \frac{f(n, p, q_{\text{lin}})}{n}$$

exists for all $p \geqslant 3$.

Finally, at the quadratic threshold, recall that we only proved that the analogous limit exists when p is even. The same should likely also hold when p is odd, as well as when q is above the quadratic threshold.

Conjecture 22. The limit

$$\lim_{n\to\infty} \frac{f(n,p,q)}{n^2}$$

exists for all $p \geqslant 4$ and $q \geqslant q_{\text{quad}}$.

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