The Linear q-Hypergraph Process

Sayok Chakravarty[a] Nicholas Spanier[b]

Submitted: Aug 15, 2023; Accepted: Dec 10, 2024; Published: Feb 14, 2025

Abstract

We analyze a random greedy process to construct q-uniform linear hypergraphs using the differential equation method. We show for $q = o(\sqrt{\log n})$, that this process yields a hypergraph with $\frac{n(n-1)}{q(q-1)}(1 - o(1))$ edges. We also give some bounds for maximal linear hypergraphs.

1 Introduction

1.1 F-free processes and the Differential Equation Method

The differential equation method for graph processes was popularized by Wormald in 1999 [13] to analyze random graph processes. The survey [1] provides an accessible introduction to the differential equation method. A common application of the differential method is the analysis of the $\mathcal{F}$-free process where $\mathcal{F}$ is a family of graphs. This is random process which creates a graph G_i on n vertices by adding edges uniformly at random one at a time so G_i contains no subgraph in the family $\mathcal{F}$. The case in which $\mathcal{F}$ is a single graph has been studied for graphs including K_3 and K_4 to give lower bounds on the Ramsey numbers $r(3, t)$ and $r(4, t)$ [3, 5, 7, 11].

This paper uses the differential equation method to construct approximate partial Steiner systems. An (n, q, t) *Steiner system* is a family $\mathcal{H} \subset \binom{[n]}{q}$ so that any t subset is contained in exactly one element of $\mathcal{H}$. When each t subset is contained in at most one element in $\mathcal{H}$, the family is called a *partial Steiner system*. It is easy to see that $|\mathcal{H}| \leqslant \frac{\binom{n}{t}}{\binom{q}{t}}$ for a partial Steiner system $\mathcal{H}$ with equality when the family is a Steiner system. When $|\mathcal{H}| = \frac{\binom{n}{t}}{\binom{q}{t}}(1 - o(1))$ as $n \to \infty$ we say $\mathcal{H}$ is an *approximate Steiner system*. The existence of approximate Steiner systems for q constant is proven by Rödl in [12]. A design with parameters (n, q, r, λ) is a collection S of q sets from $[n]$ where each r subset of $[n]$ is

[a]Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, IL (`schakr31@uic.edu`).
[b]Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, IL (`spanier2@uic.edu`).

contained in exactly λ sets in S. In the groundbreaking paper [9] Keevash proves that designs exist given the necessary divisibility conditions. Furthermore, in [10] this result was generalized to the setting of subset sums in lattices with coordinates indexed by labelled faces of simplicial complexes.

Theorem 7.1 of [13] uses the differential equation method to show that a greedy matching of a k-uniform hypergraph will use almost all of the vertices given certain degree conditions are satisfied. The problem of finding an (n, q, t) partial Steiner system on $[n]$ can be viewed as finding a matching in a particular k-uniform hypergraph. In [13], Wormald analyzes the greedy packing process to construct a hypergraph matching. He comments that while the proof only works for fixed k, one should be able to let k be a function of the number of vertices and get an analogous result. Wormald's result suggests the greedy packing process could construct a $(n, q, 2)$ approximate partial Steiner system for $q = o(\sqrt[4]{\log n})$. We explain the connection between Wormald's result and our result in Section 1.2. Bohman, Frieze, and Lubetzky studied a random triangle removal process which constructs a partial Steiner triple system [4]. We analyze a process that is equivalent to randomly removing K_q subgraphs. Bennett, Dudek, and Zerbib analyze a similar process they call the online triangle packing process to prove Tuza's conjecture for $G(n, m)$ under certain conditions [2]. Our main contribution is that we show that an $\mathcal{F}$-free process constructs an $(n, q, 2)$ approximate partial Steiner system for $q = o(\sqrt{\log n})$.

Bohman and Warnke showed there exists approximate partial Steiner triple systems with high girth by analyzing an F-free process [6]. Our work uses their approach to analyze the process of q-uniform graphs where q may depend on n.

1.2 Notation

For a sequence of events $\{E_n\}$ we say that this sequence occurs with high probability if

$$\lim_{n\to\infty} \mathbb{P}(E_n) = 1.$$

We abbreviate with high probability as whp. We will use the notation $f(n) \ll g(n)$ interchangeably with $f(n) = o(g(n))$ to mean $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. For a sequence of random variables $\{X_i\}_{i=1}^n$, let $\Delta X_i = X_{i+1} - X_i$.

1.3 *q*-Linear Process

A hypergraph $\mathcal{H}$ is called *linear* if for any $A, B \in E(\mathcal{H})$ we have $|A \cap B| \leqslant 1$. In other words, any pair of vertices appears in at most one edge. Suppose that $\mathcal{H}$ is q-uniform. Then $|E(\mathcal{H})| \leqslant \frac{\binom{n}{2}}{\binom{q}{2}} = \frac{n(n-1)}{q(q-1)}$ as the $\binom{q}{2}$ pairs in each edge are distinct.

Consider the following simple randomized greedy algorithm for constructing a maximal q-uniform linear hypergraph where we add one edge at each step. Let $\mathcal{H}_i$ be the hypergraph at step i and let e_i be the edge added at step i.

1. Let $\mathcal{H}_0$ be the empty q-uniform hypergraph on $[n]$.

2. For $i \geqslant 1$, in step i pick e_i uniformly at random from the set

$$\left\{ e \in \binom{[n]}{q} : |e \cap e_j| \leqslant 1 \text{ for } 1 \leqslant j \leqslant i-1 \right\}$$

and form $\mathcal{H}_i$ by adding e_i to $\mathcal{H}_{i-1}$.

We call this the *q-linear process*. We analyze the stopping time of this algorithm for various values of q that may depend on n. Our first result gives a lower bound on sizes of maximal linear hypergraphs. Note that such lower bounds give a lower bound on the stopping time of the q-linear process.

Proposition 1. *Let $\mathcal{H}$ be a q-uniform linear hypergraph on $[n]$ that is maximal (i.e. no edge can be added to $\mathcal{H}$ while maintaining linearity). Then* $e(\mathcal{H}) \geqslant \frac{n(n-q+1)}{q(q-1)^2} = \frac{n^2}{q^3}(1-o(1))$.

Observe that Proposition 1 shows that any maximal partial $(n, q, 2)$ Steiner system asymptotically has at least $\frac{n^2}{q^3}$ edges. In particular the q-linear process must continue for at least $\frac{n^2}{q^3}(1-o(1))$ steps.
In addition to Proposition 1, notice that there is a trivial lower bound for the size of a maximal $(n, q, 2)$ partial Steiner system on the order of $\frac{n}{q}$ by a counting argument. Notice that if $q > \sqrt{n}$, we have that $\frac{n}{q} > \frac{n^2}{q^3}$ and so in this case, the trivial lower is better than the bound from Proposition 1. Furthermore, when the trivial lower bound is better than the bound from Proposition 1, Proposition 2 says exactly how long the process continues asymptotically.

Proposition 2. *Let $\mathcal{H}$ be a q-uniform linear hypergraph on $[n]$ with $q \geqslant \sqrt{2n}$. Then $e(\mathcal{H}) < q$. Further, if $\mathcal{H}$ is maximal then $e(\mathcal{H}) = \Theta(\frac{n}{q})$.*

The problem of finding an (n, q, t) partial Steiner system on $[n]$ can be viewed as finding a matching in a $\binom{q}{t}$-uniform hypergraph H where $V(H) = \binom{[n]}{t}$ and for each $S \in \binom{[n]}{q}$ H has an edge which corresponds to all of the t-sets in S. In [13] Wormald defines the greedy packing process on a hypergraph H as the process which picks an edge from H one at a time uniformly at random and then deletes all the vertices in the chosen edge and continues until there are no edges remaining. We state Wormald's result below:

Theorem 3. *Let H be a k-uniform hypergraph with ν vertices where k is a fixed constant. Assume $\nu < r^C$ for some constant $C, \delta = o(r^{1/3})$ and $r = o(\nu)$. Also if $d(v)$ is the degree of vertex v in H then assume $|d(v) - r| \leqslant \delta$. Then for any $\epsilon_0 < \frac{1}{9k(k-1)+3}$ a.a.s. at most $\frac{\nu}{r^{\epsilon_0}}$ vertices remain at the end of the greedy packing process applied to H.*

Based on the connection between partial Steiner systems on $[n]$ and matchings in the hypergraph H, the number of unused vertices in the greedy packing process is the number of unused pairs at the end of the q-linear process. Note that the correspondences between the greedy packing process and the q-linear process is given by $\nu = \binom{n}{2}$ and $k = \binom{q}{2}$. Hence, for the partial Steiner system to have $(1-o(1))\frac{n(n-1)}{q(q-1)}$

edges, Wormald's result suggests that if k were allowed to grow as a function of ν then we would need that $\frac{\nu}{r^{\epsilon_0}} = o(n^2)$.Then using the fact that $r = o(\nu) = o(n^2)$ and $\epsilon_0 = O(\frac{1}{k^2}) = O(\frac{1}{q^4})$, we would need that $q = o(\sqrt[4]{\log n})$.
Our main result allows us to still get almost all of the edges until q is $o(\sqrt{\log n})$, giving an improvement over the expected result from Wormald 1999 [13].

Theorem 4. *Let $q = o(\sqrt{\log n})$ and let $\mathcal{H}_n$ be a q-uniform hypergraph on $[n]$ obtained from the q-linear process. Then whp $|E(\mathcal{H}_n)| \geqslant \frac{n(n-1)}{q(q-1)}(1 - o(1))$.*

Note that for q between $\sqrt{\log n}$ and $\sqrt{2n}$ all we know about the q-linear process is the lower bound from Proposition 1.

In Section 2 of this paper, we will prove Proposition 1 and Proposition 2 and in Section 3 we will prove Theorem 4.

2 Auxiliary Results

We will begin by proving Proposition 1

Proof. Let $\mathcal{H}$ be a q-uniform linear hypergraph on $[n]$ that is maximal. Consider the graph G on $[n]$ whose edge set is pairs that are not present in any edge of $\mathcal{H}$. Then G is K_q-free as a K_q in G corresponds to an edge that can be added to $\mathcal{H}$. Then $\binom{n}{2} = e(\mathcal{H})\binom{q}{2} + e(G)$. Hence, by Turán's Theorem

$$\begin{aligned} e(\mathcal{H}) &= \binom{q}{2}^{-1}\left(\binom{n}{2} - e(G)\right) \\ &\geqslant \binom{q}{2}^{-1}\left(\binom{n}{2} - \left(1 - \frac{1}{q-1}\right)\frac{n^2}{2}\right) \\ &= \frac{n(n-q+1)}{q(q-1)^2}. \end{aligned}$$

□

Next, we prove Proposition 2.

Proof. Let $\mathcal{H}$ be a q-uniform linear hypergraph on $[n]$. Let $e(\mathcal{H}) = m$ and let $E(\mathcal{H}) = \{e_i : i \in [m]\}$. Now define $\mathcal{H}_i$ as the q-uniform hypergraph on $[n]$ with $E(\mathcal{H}_i) = \{e_j : j \in [i]\}$. Define $V_i = [n] \setminus \cup_{j=1}^{i} e_j$ be the set of vertices not used by any edge in $\mathcal{H}_i$. Now notice that since $\mathcal{H}$ is linear, $|e_{i+1} \cap \cup_{j=1}^{i} e_j| \leqslant i$, so $|V_i \setminus V_{i+1}| \geqslant q - i$.

Now let $V_0 = [n]$ and notice that for all $i \in [m]$

$$\begin{aligned}|V_i| &= n - \sum_{j=1}^{i} |V_{j-1} \setminus V_j| \\ &\leqslant n - \sum_{j=1}^{i} q - (j-1) \\ &= n - iq + \frac{1}{2} i(i-1).\end{aligned}$$

Now let $f(x) = n - qx + \frac{1}{2}x(x-1)$ and notice that $f(i) \geqslant |V_i|$ for all $i \in [m]$. Also, notice that $|V_i|$ is a non-negative integer for all $i \in [m]$ since it is the number of vertices not used in any edge of $\mathcal{H}_i$. Then notice that $f(q) = n - \frac{1}{2}q^2 - \frac{1}{2}q < 0$ since $q \geqslant \sqrt{2n}$, so if $m \geqslant q$ this would lead to a contradiction since $|V_q| \leqslant f(q) < 0$ but $|V_q| \geqslant 0$. Thus $m < q$.

Now notice that $\sum_{i=1}^{m} |V_{i-1} \backslash V_i| \geqslant \sum_{i=1}^{m} (q - (i-1)) = mq - \frac{1}{2}m(m-1)$ and further, $\sum_{i=1}^{m} |V_{i-1} \backslash V_i| \leqslant n$. Thus we get that $m(q - \frac{1}{2}(m-1)) \leqslant n$ but since $m - 1 < q$ then $m(q - \frac{1}{2}q) \leqslant n$. Thus $m \leqslant \frac{2n}{q}$ so $m = O(\frac{n}{q})$.

Next, assume $\mathcal{H}_i$ is maximal and notice that every new edge uses at most q vertices not used by other edges, and there cannot be q unused vertices because $\mathcal{H}$ is maximal. Thus $m > \frac{n-q}{q} = \Omega(\frac{n}{q})$. Thus $e(\mathcal{H}) = \Theta(\frac{n}{q})$. $\square$

3 Analysis of the q-Linear Process

We prove Theorem 4 using the differential equation method.

3.1 Trajectories and Definitions

To understand q-linear process we need to track the codegree of sets $A \subset [n]$. The *codegree* of A at step i is the number of $B \subset [n]$ with $A \cap B = \emptyset$ so that $A \cup B$ can be added to $\mathcal{H}_{i-1} = \{e_1, \ldots, e_{i-1}\}$. Towards this end, for each $J \subset [n]$ with $|J| = j \in \{0\} \cup [q-1]$ consider the sets

$$\begin{aligned}H(i) &:= \left\{ e \in \binom{[n]}{q} : |e \cap e_k| \leqslant 1 \text{ for } 1 \leqslant k \leqslant i-1 \right\} \\ P_j(i) &:= \left\{ J \in \binom{[n]}{j} : J \subset e \text{ for some } e \in H(i) \right\} \\ Y_J(i) &:= \begin{cases} \left\{ K \in \binom{[n]\backslash J}{q-j} : J \cup K \in H(i) \right\} & J \in P_j(i) \\ Y_J(i-1) & J \notin P_j(i). \end{cases}\end{aligned}$$

Here $P_j(i)$ represents j sets which can still be a subset of a new edge at step i, and $|Y_J(i)|$ represents the codegree of a set in $P_j(i)$ with the convention that if $J \notin P_j(i)$

then we freeze Y_J at its current value. We are particularly interested in $|Y_\emptyset(i)|$ as this gives the number of available edges at step i, and we give this set another name $H(i)$ for clarity.

Next we will define trajectory functions which we expect the random variables $|Y_J(i)|$ to follow. Observe that after i steps the proportion of pairs that are not in any edge is $\frac{\binom{n}{2}-i\binom{q}{2}}{\binom{n}{2}} = 1 - \frac{iq(q-1)}{n(n-1)}$. Our heuristic is that the probability that a pair is not in any edge at step i is $1 - \frac{iq(q-1)}{n(n-1)}$ and the events that distinct pairs are not in any edges are mutually independent. Now we will define a continuous time variable t which relates to discrete steps by

$$t(i) = t_i = \frac{i}{n(n-1)}$$

and define the following functions

$$\begin{aligned}
p(t) &:= 1 - q(q-1)t \\
y_j(t) &:= \binom{n-j}{q-j} p^{\binom{q}{2}-\binom{j}{2}} \text{ for all } j \in [q-1] \cup \{0\} \\
h(t) &:= \binom{n}{q} p^{\binom{q}{2}}.
\end{aligned}$$

Notice that p is a real-valued function which matches our heuristic for the probability that a pair is not in an edge at step i when $t = \frac{i}{n(n-1)}$. Further, notice that if our heuristic is close to true, then $y_j(t_i)$ gives the approximate size of $Y_J(i)$ when $t = \frac{i}{n(n-1)}$ given that $J \in P_j$. Also note that $h(t) = y_0(t)$.

Now, we define our targeted stopping time m_0 and the errors ϵ_j we allow on these trajectories as

$$\begin{aligned}
f &:= (\log\log n)^2 \\
\beta &:= \frac{1}{6q^2} \\
m_0 &:= \left\lfloor \frac{n(n-1)}{q(q-1)}(1 - n^{-\beta}) \right\rfloor \\
\epsilon_j(t) &:= \binom{n-j}{q-j} n^{-1+3\beta\binom{q}{2}} q^f p^{-\binom{j}{2}-2\binom{q}{2}} \\
\epsilon_H(t) &:= \epsilon_0(t).
\end{aligned}$$

Now to prove Theorem 4, we prove the following lemma:

Lemma 5. *For all* $0 \leqslant i \leqslant m_0$*, and for all* $j \in [q-1] \cup \{0\}$ *we have that*

$$\begin{aligned} ||H(i)| - h(t_i)| &\leqslant \epsilon_H(t_i) \\ ||Y_J(i)| - y_j(t_i)| &\leqslant \epsilon_j(t_i) \text{ for all } J \in P_j(i) \end{aligned}$$

whp.

We now prove Theorem 4 assuming Lemma 5.

Proof. Now notice that if $\epsilon_H(t) = o(h(t))$ and $\epsilon_j(t) = o(y_j(t))$, Lemma 5 will show that whp $H(i) \sim h(t_i)$ and $Y_J(i) \sim y_j(t_i)$, which since $h(t_{m_0}) \gg 1$ will prove Theorem 4.

To see that $\epsilon_j(t) = o(y_j(t))$ for all $j \in [q-1] \cup \{0\}$ notice that for all $t \in [0, \frac{m_0}{n(n-1)}]$

$$\begin{aligned} \frac{\epsilon_j(t)}{y_j(t)} &= \frac{\binom{n-j}{q-j} n^{-1+3\beta\binom{q}{2}} q^f p^{-\binom{j}{2}} - 2\binom{q}{2}}{\binom{n-j}{q-j} p^{\binom{q}{2}-\binom{j}{2}}} \\ &= q^f n^{-1+3\beta\binom{q}{2}} p^{-3\binom{q}{2}} \\ &= q^f n^{-1+3\beta\binom{q}{2}} (1 - q(q-1)t)^{-3\binom{q}{2}} \\ &\leqslant q^f n^{-1+3\beta\binom{q}{2}} \left(1 - q(q-1)\frac{m_0}{n(n-1)}\right)^{-3\binom{q}{2}} \\ &\leqslant q^f n^{-1+6\beta\binom{q}{2}} = o(1), \end{aligned}$$

where the last statement in the above follows from our choices of f and β. ☐

Notice that since we want $n^{-\beta} = o(1)$ then we need $\beta = \omega\left(\frac{1}{\log n}\right)$, and we also need $\beta = \Theta\left(q^{-2}\right)$. Thus we need that $\binom{q}{2} = o(\log n)$ which holds since we assumed $q = o(\sqrt{\log n})$.

3.2 Expected One-Step Change

Let $\Delta Y_J(i) = |Y_J(i+1)| - |Y_J(i)|$ and let $\mathcal{F}_i$ be the natural filtration of the process at step i. We refer to $\Delta Y_J(i)$ as the one step change of $Y_J(i)$.

Lemma 6. *The one step change* $\mathbb{E}[\Delta Y_J(i)|\mathcal{F}_i]$ *is given by*

$$-\frac{1}{|H(i)|}\left(\sum_{K \in Y_J(i)} \left(\sum_{S \subset J, T \subset K, |S|+|T| \geqslant 2, |T| \geqslant 1} (-1)^{|S|+|T|}(|S|+|T|-1)|Y_{S\cup T}(i)|\right)\right). \quad (1)$$

Proof. Observe that $\Delta Y_J(i)$ is the number of elements in the codegree of $Y_J(i)$ that are made unavailable by the addition of e_i to $\mathcal{H}_{i-1}$. We show that for fixed $K \in Y_J(i)$ the number of edges e_i that causes $K \notin Y_J(i+1)$ is

$$\sum_{S \subset J, T \subset K, |S|+|T| \geqslant 2, |T| \geqslant 1} (-1)^{|S|+|T|}(|S|+|T|-1)|Y_{S\cup T}(i)|. \quad (2)$$

Suppose $e \in H(i)$ with $|e \cap J| = k$ and $|e \cap K| = \ell$ such that $k + \ell \geqslant 2$. We will use the following Lemma to show that if $k = 0, 1$ then e is counted once in (2) and that if $k \geqslant 2$ then e is counted 0 times in (2).

Lemma 7. *The following identities hold*

$$\sum_{m=2}^{\ell} \binom{\ell}{m} (-1)^m (m-1) = 1$$

$$\sum_{m=2}^{\ell} \binom{\ell}{m} (-1)^m (m-1) + \sum_{m=1}^{\ell} \binom{\ell}{m} (-1)^{m+1} m = 1$$

$$\sum_{0 \leqslant m_1 \leqslant k, 1 \leqslant m_2 \leqslant \ell, m_1 + m_2 \geqslant 2} \binom{k}{m_1} \binom{\ell}{m_2} (-1)^{m_1 + m_2} (m_1 + m_2 - 1) = 0.$$

We leave the proof of Lemma 7 to the appendix.
Let $k = 0$. Then e is counted

$$\sum_{m=2}^{\ell} \binom{\ell}{m} (-1)^m (m-1) = 1$$

times in (2).

Let $k = 1$. Then e is counted

$$\sum_{m=2}^{\ell} \binom{\ell}{m} (-1)^m (m-1) + \sum_{m=1}^{\ell} \binom{\ell}{m} (-1)^{m+1} m = 1$$

times in (2).

Let $k \geqslant 2$. Then e is counted

$$\sum_{0 \leqslant m_1 \leqslant k, 1 \leqslant m_2 \leqslant \ell, m_1 + m_2 \geqslant 2} \binom{k}{m_1} \binom{\ell}{m_2} (-1)^{m_1 + m_2} (m_1 + m_2 - 1) = 0$$

times in (2).

□

3.3 Supermartingale and Submartingale Properties

Let $\mathcal{G}_i$ be the event that all the bounds in Lemma 5 hold for all $j \leqslant i$. To prove Lemma 5, we will define the following random variable where $J \in \binom{[n]}{j}$ for $j \in [q-1] \cup \{0\}$ as

$$Y_J^{\pm}(i) = \begin{cases} |Y_J(i)| - (y_j(t_i) \pm \epsilon_j(t_i)), & \text{if } \mathcal{G}_{i-1} \text{ holds and } J \in P_j(i) \\ Y_J^{\pm}(i-1), & \text{otherwise.} \end{cases}$$

Lemma 8. *Let $n \geqslant n_0$ for some sufficiently large constant n_0. For all $J \subseteq [n]$ with $|J| \leqslant q-1$, $\{Y_J^+(i)\}_{i=0}^{m_0}$ is a supermartingale and $\{Y_J^-(i)\}_{i=0}^{m_0}$ is a submartingale.*

Proof. We first note that

$$\Delta Y_J^+(i) = (Y_J(i+1) - Y_J(i)) - (y_j(t_{i+1}) - y_j(t_i)) - (\epsilon_j(t_{i+1}) - \epsilon_j(t_i)).$$

Since by Taylor's theorem

$$y_j(t_{i+1}) - y_j(t_i) = \frac{y_j'(t_i)}{n(n-1)} + \frac{1}{2}\frac{y_j''(c)}{n^2(n-1)^2}$$

for some $c \in [t_i, t_{i+1}]$, and similarly

$$\epsilon_j(t_{i+1}) - \epsilon_j(t_i) = \frac{\epsilon_j'(t_i)}{n(n-1)} + \frac{1}{2}\frac{\epsilon_j''(c)}{n^2(n-1)^2}$$

for some $c \in [t_i, t_{i+1}]$, we have that $\mathbb{E}[\Delta Y_J^+(i)|\mathcal{F}_i]$ is at most

$$\mathbb{E}[\Delta Y_J(i)|\mathcal{F}_i] - \frac{y_j'(t_i)}{n(n-1)} - \frac{\epsilon_j'(t_i)}{n(n-1)} + \frac{\sup_{s\in[0,\frac{m_0}{n(n-1)}]}|y_j''(s)|}{2n^2(n-1)^2} + \frac{\sup_{s\in[0,\frac{m_0}{n(n-1)}]}|\epsilon_j''(s)|}{2n^2(n-1)^2}. \quad (3)$$

Note that if $\mathcal{G}_{i-1}$ does not hold then $\Delta Y_J^+(i) = 0$. In the event $\mathcal{G}_{i-1}$, by Lemma 6 we get

$$\begin{aligned}\mathbb{E}[\Delta Y_J(i)|\mathcal{F}_i] &\leqslant \frac{y_j(t_i)+\epsilon_j(t_i)}{h(t_i)-\epsilon_h(t_i)}\sum_{m=2}^{q}\left(\binom{q}{m}-\binom{j}{m}\right)(m-1)((-1)^{m+1}y_m(t_i)+\epsilon_m(t_i))\\ &\sim \sum_{m=2}^{q}\left(\binom{q}{m}-\binom{j}{m}\right)(m-1)\left((-1)^{m+1}\frac{y_j(t_i)y_m(t_i)}{h(t_i)} + \frac{y_j(t_i)\epsilon_m(t_i)}{h(t_i)}\right).\end{aligned} \quad (4)$$

To show that $\left\{Y_J^+(i)\right\}$ is a supermartingale, we need to verify that $\mathbb{E}[\Delta Y_J^+(i)|\mathcal{F}_i] \leqslant 0$. We do this by showing that the negative terms in (3) are larger than the positive terms. We do this by showing the following lemma.

Lemma 9. *The following hold for all large enough n and $t \in [0, m_0/(n(n-1))]$*

$$\left(\binom{q}{2}-\binom{j}{2}\right)\frac{y_j(t)y_2(t)}{h(t)} = -\frac{y_j'(t)}{n(n-1)}$$

$$\binom{q}{2}\frac{y_j(t)\epsilon_2(t)}{h(t)} \leqslant \frac{1}{2}\frac{\epsilon_j'(t)}{n(n-1)}$$

$$\binom{q}{m}\frac{y_j(t)y_m(t)}{h(t)} \ll \frac{1}{q}\frac{\epsilon_j'(t)}{n(n-1)} \text{ for } m \geqslant 3$$

$$\binom{q}{m}\frac{y_j(t)\epsilon_m(t)}{h(t)} \ll \frac{1}{q}\frac{\epsilon_j'(t)}{n(n-1)} \text{ for } m \geqslant 3$$

$$\frac{\sup_{s\in[0,\frac{m_0}{n(n-1)}]}|y_j''(s)|}{2n^2(n-1)^2} + \frac{\sup_{s\in[0,\frac{m_0}{n(n-1)}]}|\epsilon_j''(s)|}{2n^2(n-1)^2} \ll \frac{\epsilon_j'(t)}{n(n-1)}.$$

We leave the proofs of Lemma 9 for the appendix. Note that the first equation in Lemma 9 shows that the $-\left(\binom{q}{2}-\binom{j}{2}\right)\frac{y_j(t_i)y_2(t_i)}{h}$ term from (4) cancels completely in (3) with $-\frac{y_j'(t_i)}{n(n-1)}$. The second, third, and fourth equations in Lemma 9 show that the absolute value of the rest of the terms in (4) is smaller than the absolute value of the $-\frac{\epsilon_j'(t_i)}{n(n-1)}$ term in (3). Finally the last term in Lemma 9 shows the absolute value of the second derivative terms is smaller than the absolute value of the $-\frac{\epsilon_j'(t_i)}{n(n-1)}$ term in (3). For showing $\{Y_J^-(i)\}$ is a submartingale, notice that the main term from $\mathbb{E}[\Delta Y_J(i)|\mathcal{F}_i]$ and the $y_j'(t_i)$ term still cancel, so the $\epsilon_j'(t_i)$ term is still sufficiently larger than all the other terms, but the $\epsilon'(t_i)$ term is positive in $\mathbb{E}[\Delta Y_J^-(i)|\mathcal{F}_i]$, so $\mathbb{E}[\Delta Y_J^-(i)|\mathcal{F}_i] \geqslant 0$.

□

3.4 Absolute Bound on One-Step Change

Lemma 10. *For all $J \subseteq [n]$ with $|J| \leqslant q-1$ and all $i \in [m_0]$,*

$$|\Delta Y_J^+(i)| \leqslant (q-1)\binom{n-j}{q-j-1}(1+o(1))$$

$$|\Delta Y_J^-(i)| \leqslant (q-1)\binom{n-j}{q-j-1}(1+o(1)).$$

Proof. First, notice that

$$|\Delta Y_J^+(i)| \leqslant |\Delta Y_J(i)| + \sup_{t\in[t_i,t_{i+1}]} \frac{|y_j'(t)|}{n(n-1)} + \sup_{t\in[t_i,t_{i+1}]} \frac{|\epsilon_j'(t)|}{n(n-1)}. \tag{5}$$

We will bound each of the terms in (5). We will start by bounding $\Delta|Y_J(i)|$. Notice that since Y_J only changes when an available edge containing J becomes unavailable, then existing edges can only cause the absolute change in $Y_J(i)$ to be smaller since sets that would have been removed from the codegree of J were already not in the codegree of J. Thus without loss of generality we may assume $i=0$, and now we will consider three types of edges, e, which can be added, edges where $|e\cap J| \geqslant 2$, edges where $|e\cap J|=1$, and edges where $|e\cap J|=0$. First, since we freeze Y_J once J has an intersection with an existing edge of size at least 2, in the case where $|e\cap J|\geqslant 2$ we get that $\Delta Y_J(0)=0$. Next, when $|e\cap J|=1$ then the number of $f\in Y_J(0)\setminus Y_J(1)$ is the number of edges which contain J and at least one vertex in $e\setminus J$. To upperbound this quantity, we pick one of the $q-1$ vertices in $e\setminus J$ and then just pick the rest of the vertices of f as any $q-j-1$ vertices in $[n]\setminus J$, which gives an upper bound on $|\Delta Y_J(0)|$ of

$$|\Delta Y_J(0)| \leqslant (q-1)\binom{n-j}{q-j-1}.$$

We will now show that $(q-1)\binom{n-j}{q-j-1}$ is the largest term in (5) using the following standard lower bound

$$\binom{n-j}{q-j-1} \geqslant \left(\frac{n-j}{q-j-1}\right)^{q-j-1} = \Omega(n^{q-j-1}q^{-q+j+1}). \tag{6}$$

When $|e \cap J| = 0$, the only element of $Y_J(0)$ that are not in $Y_J(1)$ are those with at least 2 vertices in e. It follows that when $|e \cap J| = 0$,

$$\begin{aligned}
|\Delta Y_J(0)| &\leqslant \binom{q}{2}\binom{n-j}{q-j-2} \\
&\leqslant \frac{1}{2}q^2\left(\frac{(n-j)e}{q-j-2}\right)^{q-j-2} \\
&= O(n^{q-j-2}e^{q-j-2}q^2) \\
&= o\left((q-1)\binom{n-j}{q-j-1}\right)
\end{aligned}$$

where the last line follows from (6). Thus for all J and for all i we have that $|\Delta Y_J(i)| \leqslant (q-1)\binom{n-j}{q-j-1}$.

Next, we will bound $\sup_{t\in[t_i,t_{i+1}]} \frac{|y_j'(t)|}{n(n-1)}$. Notice that for all $t \in [0, m_0]$

$$\begin{aligned}
\left|\frac{y_j'(t)}{n(n-1)}\right| &= \left|\frac{\binom{n-j}{q-j}p^{\binom{q}{2}-\binom{j}{2}-1}(\binom{q}{2}-\binom{j}{2})(-q(q-1))}{n(n-1)}\right| \\
&= O\left(\left(\frac{ne}{q-j}\right)^{q-j}q^4n^{-2}\right) \\
&= O(n^{q-j-2}q^4e^{q-j}) \\
&= o\left((q-1)\binom{n-j}{q-j-1}\right)
\end{aligned}$$

where the last line follows from (6). Finally we will bound $\sup_{t\in[t_i,t_{i+1}]} \frac{|\epsilon_j'(t)|}{n(n-1)}$. Now notice that for all $t \in [0, m_0/n(n-1)]$ we get that

$$
\begin{aligned}
\left|\frac{\epsilon_j'(t)}{n(n-1)}\right| &= \left|\frac{\binom{n-j}{q-j}n^{-1+3\beta\binom{q}{2}}q^f p^{-\binom{j}{2}-2\binom{q}{2}-1}(-\binom{j}{2}-2\binom{q}{2})(-q(q-1))}{n(n-1)}\right| \\
&= O\left(\left(\frac{ne}{q-j}\right)^{q-j} n^{-1+3\beta\binom{q}{2}}q^f\left(1-q(q-1)\frac{m_0}{n(n-1)}\right)^{-\binom{j}{2}-2\binom{q}{2}-1} n^{-2}q^4\right) \\
&= O(n^{q-j-3+3\beta\binom{q}{2}+\beta(\binom{j}{2}+2\binom{q}{2}+1)}q^{4+f}e^{q-j}) \\
&= o\left((q-1)\binom{n-j}{q-j-1}\right)
\end{aligned}
$$

where the last line follows from (6). Thus $|\Delta Y_J^+(i)| \leqslant (q-1)\binom{n-j}{q-j-1}(1+o(1))$ for all J and i. A similar proof bounds $|\Delta Y_J^-(i)|$. ☐

3.5 Freedman's Inequality

To finish the proof of Lemma 5 we use Freedman's Inequality which we state below [8].

Theorem 11. *Let $\{S(i)\}_{i\geqslant 0}$ be a supermartingale with respect to the filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\geqslant 0}$. If $\max_{i\geqslant 0}|\Delta S(i)| \leqslant C$ and $\sum_{i\geqslant 0}\mathbb{E}(|\Delta S(i)| \mid \mathcal{F}_i) \leqslant V$, then for any $z>0$*

$$
\mathbb{P}\left(S(i) \geqslant S(0)+z \text{ for some } i \geqslant 0\right) \leqslant \exp\left\{-\frac{z^2}{2C(V+z)}\right\}.
$$

We now prove Lemma 5.

Proof. We first apply Theorem 11 to $\{Y_J^+\}$ to prove the upper bound in Lemma 5. Observe that if we set $S = Y_J^+$ and $z_j = -Y_J^+(0) = \epsilon_j(0)$ and show that $\frac{z_j^2}{2C_j(V_j+z_j)} \to \infty$ as $n \to \infty$ then we will have shown that $\mathbb{P}(Y_J^+(i) < 0$ for all $i)$ goes to 1 as n goes to infinity. This along with the analogous statement of Y_J^- and a union bound argument will show that the inequalities in Lemma 5 hold.

We now compute C_j and V_j. From the absolute bound on the one step change in Y_J^+ we know that

$$
|\Delta Y_J^+(i)| \leqslant \binom{q-1}{1}\binom{n-j}{q-j-1}(1+o(1)).
$$

So we can take $C_j = \binom{q-1}{1}\binom{n-j}{q-j-1}(1+o(1))$ Furthermore, Lemma 9 together with $m_0 \leqslant$

$\frac{n(n-1)}{q(q-1)}$ implies that

$$\begin{aligned}
&\sum_{i\geqslant 0}\mathbb{E}(|\Delta Y_J^+(i)|\,|\,\mathcal{F}_i)\\
&\leqslant \frac{n(n-1)}{q(q-1)}O\left(\frac{\sup_{t\in[0,t_{m_0}]}|\epsilon_j'(t)|}{n(n-1)}\right)\\
&\leqslant \frac{n(n-1)}{q(q-1)}O\left(\left|\frac{\binom{n-j}{q-j}n^{-1+3\beta\binom{q}{2}}q^f n^{\beta\left(\binom{j}{2}+2\binom{q}{2}+1\right)}(-\binom{j}{2}-2\binom{q}{2})(-q(q-1))}{n(n-1)}\right|\right)\\
&= O\left(\binom{n-j}{q-j}q^{f+2}n^{-1+6\beta\binom{q}{2}}\right).
\end{aligned}$$

Hence, we can set $V_j = O\left(\binom{n-j}{q-j}q^{f+2}n^{-1+6\beta\binom{q}{2}}\right)$.

Set $z_j=\epsilon_j(0)=\binom{n-j}{q-j}n^{-1+3\beta\binom{q}{2}}q^f$. Notice that $z_j\ll V_j$, so to verify that $\frac{z_j^2}{2C_j(V_j+z_j)}\gg 1$, it suffices to check that $\frac{z_j^2}{C_jV_j}\gg 1$. Observe that

$$\begin{aligned}
\frac{z_j^2}{C_jV_j} &= \frac{\binom{n-j}{q-j}^2 n^{-2+6\beta\binom{q}{2}}q^{2f}}{O\left(\binom{n-j}{q-j}q^{f+2}n^{-1+6\beta\binom{q}{2}}\right)\cdot\binom{q-1}{1}\binom{n-j}{q-j-1}}\\
&= \Omega\left(\frac{n-q+1}{q-j}n^{-1}q^{f-3}\right)\\
&= \Omega\left(\left(1-\frac{q}{n}+\frac{1}{n}\right)q^{f-4}\right)\\
&= \Omega(q^{f-4}).
\end{aligned}$$

By Theorem 11, for all J

$$\mathbb{P}(Y_J^+(i)\geqslant 0 \text{ for some } i)\leqslant \exp\left(-\Omega(q^{f-4})\right).$$

Since $\{Y_J^-\}$ is a submartingale, $\{-Y_J^-\}$ is a supermartingale, so by applying Theorem 11 to $\{-Y_J^-\}$ with $S=-Y_J^-$ and $z=Y_J^-(0)=\epsilon_j(0)$, a similar argument shows for all J

$$\mathbb{P}(-Y_J^-(i)\geqslant 0 \text{ for some } i)\leqslant \exp\left(-\Omega(q^{f-4})\right).$$

To get the conclusion of Lemma 5, we show $\mathbb{P}(\mathcal{G}_{m_0}^c)\to 0$ as $n\to\infty$. Observe that

$$\begin{aligned}
\mathbb{P}(\mathcal{G}_{m_0}^c) &\leqslant \mathbb{P}\left(\bigcup_{J\subset[n],|J|<q}\left\{Y_J^+(i)\geqslant 0 \text{ for some } i\geqslant 0\right\}\cup\left\{Y_J^-(i)\leqslant 0 \text{ for some } i\geqslant 0\right\}\right)\\
&\leqslant 2q\binom{n}{q}e^{-\Omega(q^{f-4})}\\
&\leqslant 2qn^q e^{-\Omega(q^{f-4})}.
\end{aligned}$$

Observe that

$$\log\left(2qn^q e^{-\Omega(q^{f-4})}\right) = \log 2 + \log q + q\log n - \Omega(q^{f-4}).$$

Since the largest term is $q\log n$ we need to verify that that $q\log n \ll q^{f-4}$. To see this note that $f \gg \frac{\log\log n}{\log q}$. Hence, $2qn^q e^{-\Omega(q^{f-4})} = o(1)$ and we have $\mathbb{P}(\mathcal{G}_{m_0}) \to 1$ as $n \to \infty$ which proves Lemma 5. $\square$

3.6 Appendix

3.6.1 Proof of Lemma 9

We now prove Lemma 9.

Proof. Throughout this section, we will use the derivatives $y_j'(t)$ and $\epsilon_j'(t)$ which are given by

$$\begin{aligned} y_j'(t) &= -\left(\binom{q}{2} - \binom{j}{2}\right)\binom{n-j}{q-j} p^{\binom{q}{2}-\binom{j}{2}-1} q(q-1) \\ \epsilon_j'(t) &= \left(\binom{j}{2} + 2\binom{q}{2}\right)\binom{n-j}{q-j} n^{-1+3\beta\binom{q}{2}} q^f p^{-\binom{j}{2}-2\binom{q}{2}-1} q(q-1). \end{aligned}$$

We first show that $\left(\binom{q}{2} - \binom{j}{2}\right)\frac{y_j(t)y_2(t)}{h(t)} = -\frac{y_j'(t)}{n(n-1)}$. Observe that

$$\begin{aligned} \left(\binom{q}{2} - \binom{j}{2}\right)\frac{y_j(t)y_2(t)}{h(t)} &= \left(\binom{q}{2} - \binom{j}{2}\right)\frac{\binom{n-j}{q-j}p^{\binom{q}{2}-\binom{j}{2}}\binom{n-2}{q-2}p^{\binom{q}{2}-1}}{\binom{n}{q}p^{\binom{q}{2}}} \\ &= \left(\binom{q}{2} - \binom{j}{2}\right)\binom{n-j}{q-j}\frac{q(q-1)}{n(n-1)}p^{\binom{q}{2}-\binom{j}{2}-1} \\ &= -\frac{y_j'(t)}{n(n-1)}. \end{aligned}$$

Next, we show that $\binom{q}{2}\frac{y_j(t)\epsilon_2(t)}{h(t)} \leqslant \frac{1}{2}\frac{\epsilon_j'(t)}{n(n-1)}$. Indeed

$$\begin{aligned} \binom{q}{2}\frac{y_j(t)\epsilon_2(t)}{h(t)}\left(\frac{\epsilon_j'(t)}{n(n-1)}\right)^{-1} &= \frac{\frac{\binom{q}{2}\binom{n-j}{q-j}p^{\binom{q}{2}-\binom{j}{2}}\binom{n-2}{q-2}n^{-1+3\beta\binom{q}{2}}q^f p^{-\binom{2}{2}-2\binom{q}{2}}}{\binom{n}{q}p^{\binom{q}{2}}}}{\frac{\binom{n-j}{q-j}n^{-1+3\beta\binom{q}{2}}q^f p^{-\binom{j}{2}-2\binom{q}{2}-1}(\binom{j}{2}+2\binom{q}{2})(q(q-1))}{n(n-1)}} \\ &= \frac{\binom{q}{2}}{\binom{j}{2}+2\binom{q}{2}} \leqslant \frac{1}{2}. \end{aligned}$$

To prove the rest of Lemma 9, we will first give a lower bound on $\frac{\epsilon_j'(t)}{n(n-1)}$ and then prove this is asymptotically larger than all the remaining terms. First notice that

$$\begin{aligned}\frac{\epsilon_j'(t)}{n(n-1)} &= \frac{\binom{n-j}{q-j} n^{-1+3\beta\binom{q}{2}} q^f p^{-\binom{j}{2}-2\binom{q}{2}-1}\left(\binom{j}{2}+2\binom{q}{2}\right)(q(q-1))}{n(n-1)} \\ &= \Omega\left(\binom{n-j}{q-j} n^{-3+3\beta\binom{q}{2}} q^{f+4} p^{-\binom{j}{2}-2\binom{q}{2}-1}\right).\end{aligned}$$

Now, we will compute $q\binom{q}{m}\frac{y_j(t)y_m(t)}{h(t)}$ where $3 \leqslant m \leqslant q-1$ and show that each of these terms is $o(\binom{n-j}{q-j} n^{-3+3\beta\binom{q}{2}} q^{f+4} p^{-\binom{j}{2}-2\binom{q}{2}-1})$. Observe that

$$\begin{aligned}q\binom{q}{m}\frac{y_j(t)y_m(t)}{h(t)} &= q\binom{q}{m}\frac{\binom{n-j}{q-j}p^{\binom{q}{2}-\binom{j}{2}}\binom{n-m}{q-m}p^{\binom{q}{2}-\binom{m}{2}}}{\binom{n}{q}p^{\binom{q}{2}}} \\ &\leqslant \binom{n-j}{q-j} q \left(\frac{qe}{m}\right)^m \left(\frac{ne}{q-m}\right)^{q-m} \left(\frac{q}{n}\right)^q p^{-\binom{q}{2}} \\ &\leqslant \binom{n-j}{q-j} n^{-m+\beta\binom{q}{2}} q^{2m+1} e^q m^{-m} \\ &= o\left(\binom{n-j}{q-j} n^{-3+3\beta\binom{q}{2}} q^{f+4} p^{-\binom{j}{2}-2\binom{q}{2}-1}\right).\end{aligned}$$

Since $\epsilon_m(t) = o(y_m(t))$

$$q\binom{q}{m}\frac{y_j(t)\epsilon_m(t)}{h(t)} = o\left(q\binom{q}{m}\frac{y_j(t)y_m(t)}{h(t)}\right) = o\left(\binom{n-j}{q-j} n^{-3+3\beta\binom{q}{2}} q^{f+4} p^{-\binom{j}{2}-2\binom{q}{2}-1}\right).$$

Lastly, we need to verify that $\frac{\sup_{s\in[0,m_0/(n(n-1))]}|y_j''(s)|}{2n^2(n-1)^2}$ and $\frac{\sup_{s\in[0,m_0/(n(n-1))]}|\epsilon_j''(s)|}{2n^2(n-1)^2}$ are also both $o(\binom{n-j}{q-j} n^{-3+3\beta\binom{q}{2}} q^{f+4} p^{-\binom{j}{2}-2\binom{q}{2}-1})$.

$$\begin{aligned}\frac{\sup_{s\in[0,\frac{m_0}{n(n-1)}]}|y_j''(s)|}{2n^2(n-1)^2} &= \frac{\binom{n-j}{q-j}(p(0))^{\binom{q}{2}-\binom{j}{2}-2}\left(\binom{q}{2}-\binom{j}{2}\right)\left(\binom{q}{2}-\binom{j}{2}-1\right)(q^2(q-1)^2)}{2n^2(n-1)^2} \\ &= O\left(\binom{n-j}{q-j} n^{-4} q^8\right) \\ &= o\left(\binom{n-j}{q-j} n^{-3+3\beta\binom{q}{2}} q^{f+4} p^{-\binom{j}{2}-2\binom{q}{2}-1}\right).\end{aligned}$$

Similarly, we compute

$$\frac{\sup_{s\in[0,\frac{m_0}{n(n-1)}]}|\epsilon_j''(s)|}{2n^2(n-1)^2} \leqslant O\left(\frac{\binom{n-j}{q-j}n^{-1+3\beta\binom{q}{2}}q^{f+8}\left(p\left(\frac{m_0}{n(n-1)}\right)\right)^{-\binom{j}{2}-2\binom{q}{2}-2}}{n^2(n-1)^2}\right)$$
$$= O\left(\binom{n-j}{q-j}n^{-5+\beta(6\binom{q}{2}+2)}q^{f+8}\right)$$
$$= o\left(\binom{n-j}{q-j}n^{-3+3\beta\binom{q}{2}}q^{f+4}p^{-\binom{j}{2}-2\binom{q}{2}-1}\right).$$

This completes the proof of Lemma 9. □

3.6.2 Proof of Combinatorial Identities

We now prove Lemma 7

Proof. We first show that $\sum_{m=2}^{\ell}\binom{\ell}{m}(-1)^m(m-1)=1$. Observe that

$$\frac{(1+x)^\ell-1}{x}=\sum_{m=1}^{\ell}\binom{\ell}{m}x^{m-1}.$$

This means

$$\frac{d}{dx}\left[\frac{(1+x)^\ell-1}{x}\right]=\sum_{m=2}^{\ell}\binom{\ell}{m}(m-1)x^{m-2}.$$

Letting $x=-1$ yields $\sum_{m=2}^{\ell}\binom{\ell}{m}(-1)^m(m-1)=1$.

We now show that $\sum_{m=2}^{\ell}\binom{\ell}{m}(-1)^m(m-1)+\sum_{m=1}^{\ell}\binom{\ell}{m}(-1)^{m+1}m=1$. Observe that

$$(1+x)^\ell=\sum_{m=0}^{\ell}\binom{\ell}{m}x^m.$$

Differentiating and letting $x=-1$ yields $\sum_{m=1}^{\ell}\binom{\ell}{m}(-1)^{m+1}m=0$, so we have $\sum_{m=2}^{\ell}\binom{\ell}{m}(-1)^m(m-1)+\sum_{m=1}^{\ell}\binom{\ell}{m}(-1)^{m+1}m=1$.

We now show that

$$\sum_{0\leqslant m_1\leqslant k,1\leqslant m_2\leqslant \ell,m_1+m_2\geqslant 2}\binom{k}{m_1}\binom{\ell}{m_2}(-1)^{m_1+m_2}(m_1+m_2-1)=0.$$

We show this by verifying that

$$\sum_{0\leqslant m_1\leqslant k,0\leqslant m_2\leqslant \ell,m_1+m_2\geqslant 2}\binom{k}{m_1}\binom{\ell}{m_2}(-1)^{m_1+m_2}(m_1+m_2-1)=1$$

and the $m_2 = 0$ part

$$\sum_{0\leqslant m_1\leqslant k, m_1\geqslant 2} \binom{k}{m_1}\binom{\ell}{0}(-1)^{0+m_1}(0+m_1-1) = 1.$$

This second equality is the same as the first identity we proved. For the first one, observe that

$$\frac{(1+x)^{k+\ell}-1}{x} = \frac{(1+x)^k(1+x)^\ell-1}{x} = \sum_{0\leqslant m_1\leqslant k, 0\leqslant m_2\leqslant \ell, m_1+m_2\geqslant 1} \binom{k}{m_1}\binom{\ell}{m_2} x^{m_1+m_2-1}.$$

This implies

$$\frac{d}{dx}\left[\frac{(1+x)^{k+\ell}-1}{x}\right] = \sum_{0\leqslant m_1\leqslant k, 0\leqslant m_2\leqslant \ell, m_1+m_2\geqslant 2} \binom{k}{m_1}\binom{\ell}{m_2}(m_1+m_2-1)x^{m_1+m_2-2}.$$

Substituting $x = -1$ yields

$$\sum_{0\leqslant m_1\leqslant k, 0\leqslant m_2\leqslant \ell, m_1+m_2\geqslant 2} \binom{k}{m_1}\binom{\ell}{m_2}(-1)^{m_1+m_2}(m_1+m_2-1) = 1. \qquad \square$$

Acknowledgements

We would like to thank Dhruv Mubayi for the ideas used in Proposition 2 and his guidance throughout. This work was funded by NSF DMS-1763317. We would also like to thank the referees for their many helpful comments.

References

[1] Patrick Bennett and Andrzej Dudek. A gentle introduction to the differential equation method and dynamic concentration. *Discrete Math.*, 345(12):113071, 2022.

[2] Patrick Bennett, Andrzej Dudek, and Shira Zerbib. Large triangle packings and Tuza's conjecture in sparse random graphs. *Combin. Probab. Comput.*, 29(5):757–779, 2020.

[3] Tom Bohman. The triangle-free process. *Adv. Math.*, 221(5):1653–1677, 2009.

[4] Tom Bohman, Alan Frieze, and Eyal Lubetzky. Random triangle removal. *Adv. Math.*, 280:379–438, 2015.

[5] Tom Bohman and Peter Keevash. Dynamic concentration of the triangle-free process. *Random Structures Algorithms*, 58(2):221–293, 2021.

[6] Tom Bohman and Lutz Warnke. Large girth approximate Steiner triple systems. *J. Lond. Math. Soc. (2)*, 100(3):895–913, 2019.

[7] Paul Erdős, Stephen Suen, and Peter Winkler. On the size of a random maximal graph. *Random Structures Algorithms*, 6(2-3):309–318, 1995.

[8] David A. Freedman. On tail probabilities for martingales. *Ann. Probability*, 3:100–118, 1975.

[9] Peter Keevash. The existence of designs. arXiv:1401.3665, 2014.

[10] Peter Keevash. The existence of designs ii. arXiv:1802.05900, 2018.

[11] Jeong Han Kim. The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$. *Random Structures Algorithms*, 7(3):173–207, 1995.

[12] Vojtěch Rödl. On a packing and covering problem. *European J. Combin.*, 6(1):69–78, 1985.

[13] Nicholas C Wormald. The differential equation method for random graph processes and greedy algorithms. *Lectures on approximation and randomized algorithms*, 73:73–155, 1999.