Independent removable edges in cubic bricks

Fuliang Lu[a] Jianguo Qian[b,c]

Submitted: Nov 8, 2023; Accepted: Jan 23, 2025; Published: Feb 14, 2025

Abstract

An edge e in a matching covered graph G is *removable* if $G-e$ is matching covered, which was introduced by Lovász and Plummer in connection with ear decompositions of matching covered graphs. A *brick* is a non-bipartite matching covered graph without non-trivial tight cuts. The importance of bricks stems from the fact that they are building blocks of matching covered graphs. Improving Lovász's result, Carvalho et al. [Ear decompositions of matching covered graphs, *Combinatorica*, 19(2):151-174, 1999] showed that each brick other than K_4 and $\overline{C_6}$ has $\Delta-2$ removable edges, where Δ is the maximum degree of G. In this paper, we show that every cubic brick G other than K_4 and $\overline{C_6}$ has a matching of size at least $|V(G)|/8$, each edge of which is removable in G.

Mathematics Subject Classifications: 05C70,05C75

1 Introduction

We consider only undirected simple graphs. A connected graph G is *matching covered*, also referred to as *1-extendable*, if each edge lies in some perfect matching of G. For the terminologies related to matching covered graphs, we follow Lovász and Plummer [10].

For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. The degree of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v. For two disjoint non-empty vertex subsets $X, Y \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and by $E_G(X,Y)$ the set of the edges joining one vertex in X and one in Y. In particular, we call $E_G(X,\overline{X})$ an *edge cut* of G and denote by $\partial_G(X)$, or simply by $\partial(X)$, where $\overline{X} = V(G)\setminus X$. We refer to X and $\overline{X}$ as the *shores* of $\partial(X)$. An edge cut $\partial(X)$ is *trivial* if either $|X|=1$ or $|\overline{X}|=1$. For an edge cut $\partial(X)$, we denote the graph obtained from G by contracting X to a single vertex x by $G/(X\to x)$, or simply G/X. Further, we call G/X the $\partial(X)$-*contraction* of G and x

[a]School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, PR China (fliangl u@163.com).
[b]School of Mathematics and Statistics, Qinghai Minzu University, Xining, PR China (jgqian@xmu.edu.cn).
[c]Corresponding author.

the *contracted vertex*. An edge cut $\partial(X)$ of G is *tight* if $|\partial(X) \cap M| = 1$ for every perfect matching M of G and is *separating* if, for any $e \in E(G)$, G has a perfect matching M_e such that $e \in M_e$ and $|\partial(X) \cap M_e| = 1$. Obviously, if G is matching covered then every tight cut $\partial(X)$ is separating and, hence both G/X and $G/\overline{X}$ are matching covered. We call a matching covered graph G that contains no non-trivial tight cuts a *brick* if G is non-bipartite, and a *brace* otherwise. Edmonds et al. [6] showed that a graph G is a *brick* if and only if G is 3-connected and $G - \{x, y\}$ has a perfect matching for any two distinct vertices $x, y \in V(G)$ (bicritical). Further, Lovász [9] proved that any matching covered graph can be decomposed into a unique list of bricks and braces by a procedure called the tight cut decomposition.

An edge e of a matching covered graph G is *removable* if $G - e$ is matching covered. For $\{e, f\} \subseteq E(G)$, we say that $\{e, f\}$ is a *removable doubleton* of G if neither e nor f is removable and $G - \{e, f\}$ is matching covered. Removable edges and removable doubletons are generally called *removable classes*. The notion of removable class was introduced by Lovász and Plummer, which arises in connection with ear decompositions of matching covered graphs.

Lovász [8] proved that every brick distinct from K_4 and $\overline{C_6}$ (the triangular prism) has a removable edge. Further, Carvalho et al. [2] showed that each brick G other than K_4 and $\overline{C_6}$ has at least $\Delta - 2$ removable edges and the lower bound is attained by the cubic brick R_8 as shown Figure 2(a), where Δ is the maximum degree of G. In [14] Zhai et al. proved that the number of removable ears in every matching covered graph G is not less than the minimum number of the perfect matchings needed to cover all edges of G. Carvalho and Little [1] showed that every matching covered graph, except the cycle, has at least three removable classes.

In this paper we consider the number of pair-wise non-adjacent removable edges in cubic bricks in terms of the number of vertices.

Theorem 1.1. Let G be a cubic brick other than K_4 and $\overline{C_6}$. Then G has a matching of size at least $|V(G)|/8$, each edge of which is removable in G.

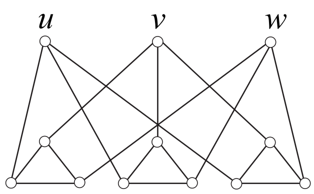


Figure 1: A 3-connected cubic matching covered graph with no removable edges.

We note that the size $|V(G)|/8$ in Theorem 1.1 is attained by the graph R_8. Further, though Theorem 1.1 gives the least numbers of removable edges for cubic bricks, this is not the case in general cubic matching covered graphs. For example, the graph shown in Figure 1 is a 3-connected cubic matching covered graph, which is not bicritical (the removal of any two vertices in $\{u, v, w\}$, the resulting graph does not have a perfect matching), and contains no removable edges at all.

In the following section, we present some basic properties concerning removable edges. In Section 3, we give a proof of Theorem 1.1.

2 Preliminaries

In this section, we recall some known results and present some basic properties concerning removable edges that will be used in our proof of the main result.

2.1 Removable doubletons in edge cut contractions

Let G be a connected graph with a perfect matching. A nonempty subset X of $V(G)$ is a *barrier* if $o(G-X) = |X|$, where $o(G-X)$ denotes the number of odd components of $G-X$. It follows from the well-known Tutte's Perfect Matching Theorem that if $uv \in E(G)$ and G has a barrier that contains both u and v, then no perfect matchings of G contains the edge uv. The following result is proved by Tutte.

Lemma 2.1. *[13]* Every 2-edge-connected cubic graph is matching covered.

An edge cut with k edges is called a *k-cut*. The following proposition follows directly from Lemma 2.1.

Proposition 2.2. *[4]* Every 3-cut of a 2-edge-connected cubic graph is a separating cut.

For a matching covered bipartite graph, we have the following theorem.

Theorem 2.3. *(Theorem 4.1.1 in [10])* Let G be a matching covered bipartite graph with color classes A and B. Then $G - \{u, v\}$ has a perfect matching for any $u \in A, v \in B$.

We call an edge cut of a graph G *good* if it is separating but not tight, and call a vertex *covered by a matching M* if it is incident with some edge in M. We note that if G has a good edge cut, then G has a perfect matching that contains at least three edges in this cut. Then every nontrivial separating cut in a brick is good. Carvalho et. al proved the following lemma.

Lemma 2.4. *(Lemma 3.1 in [3])* Let $C := \partial(X)$ be a good cut of a matching covered graph G and let $H := G/\overline{X}$. Suppose that H is a brick, and let R be a removable doubleton of H. If $R \cap C = \emptyset$ or if the edge of $R \cap C$ is removable in G/X then $R - C$ contains an edge which is removable in G.

The following corollary follows directly from Lemma 2.4.

Corollary 2.5. Let G be a cubic brick different from K_4, $u_1u_2u_3$ be a triangle of G and $u_1v_1 \in E(G)$. If uv_1 is removable in $G/(\{u_1, u_2, u_3\} \to u)$ and $\partial(\{u_1, u_2, u_3\})$ is good, then u_2u_3 is removable in G.

2.2 Essentially 4-edge-connected cubic graphs

A cubic graph is *essentially 4-edge-connected* if it is 3-edge-connected and free of non-trivial 3-cuts. Kothari et al. showed the following theorems.

Theorem 2.6. *[7]* Every essentially 4-edge-connected cubic graph is either a brick or a brace.

Theorem 2.7. *[7]* In an essentially 4-edge-connected cubic brick, each edge is either removable or lies in a removable doubleton.

For the removability of edges in a brace, we have the following.

Theorem 2.8. *[5]* Each edge in a brace with at least six vertices is removable.

For a bipartite graph $G(A, B)$, as usual we also call A and B the color classes of G. Further, if $|A| = |B|$ then we call $G(A, B)$ *balanced*.

Proposition 2.9. Let G be an essentially 4-edge-connected cubic brick other than K_4, let $\mathscr{E}_0$ be the collection of all the removable doubletons of G, and let E_0 be the set of the edges in the removable doubletons of $\mathscr{E}_0$. Then the following statements hold.
(i). *[11]* If $|\mathscr{E}_0| \geqslant 2$, then G can be decomposed into balanced bipartite vertex-induced subgraphs G_i ($i = 1, 2, \ldots, |\mathscr{E}_0|$) satisfying $E_G(V(G_j), V(G_k))$ is a removable doubleton of G if $|j-k| \equiv 1 \pmod{|\mathscr{E}_0|}$ and $E_G(V(G_j), V(G_k)) = \emptyset$ otherwise.
(ii). G has a perfect matching M such that $M \cap E_0 = \emptyset$.

Proof of (ii). Let $s = |\mathscr{E}_0|$. If $s = 0$, that is $E_0 = \emptyset$, then every perfect matching of G is that we need. We consider the case when $s = 1$. Assume that $\{uv, xy\}$ is the only removable doubleton of G. Since $G - \{uv, xy\}$ is matching covered, the result follows directly by choosing any perfect matching M in $G - \{uv, xy\}$.

We now consider the case when $s > 1$. For $i = 1, 2, \ldots, s$, let $u_{i-1}y_i$ and $v_{i-1}x_i$ denote the two edges of $E(V(G_{i-1}), V(G_i))$, where $x_i, y_i \in V(G_i)$, and the subscript is taken modulo s. By (i), the pair $R := \{u_iy_{i+1}, v_ix_{i+1}\}$ is a removable doubleton of G. As G is a brick, the graph $G - R$ is bipartite by Lemma 3.4 in [9]. Moreover, the graph $G - R$ is matching covered by the definition of removable doubletons. By Theorem 2.3, the graph $G - R - \{u_{i-1}, v_{i-1}\}$ has a perfect matching, say, N. Then, $M_i := N \cap E(G_i)$ is a perfect matching of G_i. Thus, G_i has a perfect matching. This conclusion holds for $i = 1, 2, \ldots, s$. The assertion holds, by taking $M := \cup_i M_i$. □

The following is a direct consequence of Theorem 2.8 and Proposition 2.9 (ii).

Corollary 2.10. Let G be an essentially 4-edge-connected cubic graph other than K_4. Then G has a perfect matching consisting of removable edges of G.

2.3 The splicing of two graphs

Let G and H be two vertex-disjoint graphs. Let $u \in V(G)$ and $v \in V(H)$ be two vertices with the same degree. Let E_1 be edges of G incident with u and let E_2 be edges of H incident with v, and σ be a bijection between E_1 and E_2. The *splicing of G and H at u and v* (with respect to the bijection σ), denote by $G(u) \odot_\sigma H(v)$ (or simply $G(u) \odot H$ or $G \odot H$ if no confusion occurs), is the graph obtained from $G - u$ and $H - v$ by joining, for edge e in E_1, the end of e in $G - u$ to the end of $\sigma(e)$ in $H - v$. Obviously, in the case of any edge-transitive graph, there is no need to state the bijection (for the purposes of the paper, the bijection is irrelevant). The two vertices u and v are called the *splicing* *vertices* of G and H, respectively; every edge in $\partial(V(G) \setminus \{u\})$ is called the *splicing edge* of $G \odot H$. Obviously, the $\partial(V(G) \setminus \{u\})$-contractions of $G \odot H$ are G and H.

If $H = K_4$ and u is a vertex of degree 3, then the splicing operation $G(u) \odot K_4$ can be intuitively viewed as the operation that 'inserts a triangle' at u. In this sense, we also call such an operation the *triangle-insertion* at u of G and denote $G(u) \odot K_4$ simply by $G\langle u\rangle$. For a 3-cut $\partial(X)$ of a graph G, we denote $G^{\Delta}(X) = (G/(\overline{X} \to v))\langle v\rangle$. We call $G^{\Delta}(X)$ the Δ-*replacement* of X in G and call the triangle inserted at v the *replacement-triangle*. In particular, if $|\overline{X}| = 1$ then the splicing operation $G^{\Delta}(X) = G(u) \odot K_4$; if $G[\overline{X}]$ is a triangle then $G^{\Delta}(X) = G$.

We note that every edge (or vertex) in $G \odot H$ corresponds to uniquely an edge (or vertex other than the splicing vertex) in G or H. With a mild abuse of language, we will use the same label of the edge (or vertex) in $G \odot H$ as it is in G or H, and vice versa. The following propositions are about the splicing of two graphs.

Proposition 2.11. *[4]* Any graph obtained by splicing two matching covered graphs is also matching covered.

Proposition 2.12. *[4]* Any splicing of two cubic bricks is a cubic brick.

We say that two edge cuts $\partial(X)$ and $\partial(Y)$ *cross* if the four sets $X \cap Y$, $\overline{X} \cap Y$, $X \cap \overline{Y}$ and $\overline{X} \cap \overline{Y}$ are all nonempty.

Lemma 2.13. Let $G_0[A, B]$ be a cubic brace on six or more vertices, $u \in A$ and $v \in B$. And let G_1 and G_2 be two cubic bricks on six or more vertices, $G = (G_0(u) \odot G_1)(v) \odot G_2$. Then G is a brick.

Proof. Obviously, G is 3-connected cubic matching covered graph. Suppose, to the contrary, that $\partial(X)$ is a tight cut of G. By Theorem 8 in [7], $\partial(X)$ is a 3-cut. Let $Y = V(G) \cap V(G_1)$. Then $\partial(Y)$ is a 3-cut of G.

We claim that $\partial(X)$ and $\partial(Y)$ do not cross. Suppose, to the contrary, that $\partial(X)$ and $\partial(Y)$ cross. Interchange X with $\overline{X}$ if necessary, so that $|X \cap Y|$ is odd. Then, $|\overline{X} \cap \overline{Y}|$ is also odd, and $|X \cap \overline{Y}|$ and $|\overline{X} \cap Y|$ are both even and nonempty. Let $C := \partial(X \cap \overline{Y})$ and let $D := \partial(\overline{X} \cap Y)$. Let λ be the set of edges that join $X \cap Y$ to $\overline{X} \cap \overline{Y}$. Then, $|C| + |D| + 2|\lambda| = |\partial(X)| + |\partial(Y)| = 6$. The shores of C and D are nonempty and even, hence C and D are nonempty and even. Thus, at least one of C and D is a 2-cut, a contradiction to the fact that G is 3-connected.

Similarly, $\partial(X)$ and $\partial(Z)$ do not cross, where $Z = V(G) \cap V(G_2)$. Therefore, $\partial(X)$ is a subset of $E(G_0)$, $E(G_1)$ or $E(G_2)$. Thus, either $\partial(X)$ is a nontrivial tight cut of G_0, G_1 or G_2, or $\partial(X) \in \{\partial(Y), \partial(Z)\}$. As the graph G_0 is a brace and the graphs G_1 and G_2 are bricks, we conclude that $\partial(X) \in \{\partial(Y), \partial(Z)\}$.

As G_1 is a brick, the insertion of any triangle produces a brick. Thus, G has a matching, M_1, such that $M_1 \subset E(G_1)$, M_1 contains the three edges of $\partial(Y)$ and M_1 covers all the vertices of Y. Likewise, G has a matching, M_2, such that $M_2 \subset E(G_2)$, $\partial(Z) \subset M_2$ and M_2 covers all the vertices of Z. As G_0 is cubic and has six or more vertices, it does not have multiple edges. Let v_1 and v_2 be two vertices of $G_0 - v$ adjacent to u and let u_1 and u_2 be two vertices of $G_0 - u$ adjacent to v. The graph $G_0 - u_1 - u_2 - v_1 - v_2$ has a perfect matching, M_0, by Theorem 5 in [12]. The set $M_0 \cup M_1 \cup M_2$ is a perfect matching of G that contains three edges in each one of the cuts $\partial(Y)$ and $\partial(Z)$. We deduce that G in fact does not have any nontrivial tight cuts. So G is a brick. $\square$

2.4 Δ-replacements and removable edges

The following proposition can be gotten by the definition of matching covered graphs directly.

Proposition 2.14. Let uv be a (removable) edge of a matching covered graph G with $d_G(v) = 3$. Then no splicing edge in $G\langle v\rangle$ is removable.

For the removability of an edge that is not a splicing edge, we have the following proposition.

Lemma 2.15. Let G_1 and G_2 be matching covered graphs, $u \in V(G_1)$, $v \in V(G_2)$, $d_{G_1}(u) = 3$ and $d_{G_2}(v) = 3$. And let $H = G_1(u) \odot G_2(v)$. For any edge e in G_1 that is not incident with u,
(i). if e is removable in G_1, then e is removable in H; and
(ii). if e is removable in $G_1\langle u\rangle$ and $\partial_H(V(G_1 - u))$ is good in H, then e is removable in H.

Proof. (i). Since $G_1 - e$ is matching covered and $d_{G_1-e}(u) = 3$, (i) follows directly by Proposition 2.11.

(ii). Let $C := \partial(V(G_1) - u)$. For every edge f of the spliced triangle of $G_1\langle u\rangle$, the graph $G_1\langle u\rangle - e$ has a perfect matching that contains the edge f and just one edge in C. Thus, for every edge g in G_2, a perfect matching of G_2 containing the edge g may be extended to a perfect matching of $H - e$.

Suppose that C is good in H and let f be an edge of $G_1 - u - e$. As e is removable in $G_1\langle u\rangle$, $G_1\langle u\rangle - e$ has a perfect matching, say, M, that contains the edge f. If M contains just one edge in C then $M - C$ may be extended to a perfect matching of H, because G_2 is matching covered. Likewise, if M contains the three edges of C then M may be extended to a perfect matching of H, because C is good in H. $\square$

On the other hand, the condition in Lemmas 2.4 and 2.15 (i) is not sufficient. Let's consider the graph G as shown in Figure 3(a). We can see that every edge in $G[X]$ is removable but does not admit the condition of Lemmas 2.4 or 2.15 (i), where X is the set of the three vertices in the central triangle T_3 of G.

The following corollary will be useful for the case when the two vertices are splicing vertices in different splicings.

Corollary 2.16. Assume that G_0, G_1 and G_2 are matching covered graphs, $u_i \in V(G_i)$ $(i = 1, 2)$. Let $v_1, v_2 \in V(G_0)$ and $G = (G_0(v_1) \odot G_1(u_1))(v_2) \odot G_2(u_2)$. For $i = 1$ and 2, let E_i be the set of edges in $E(G_i) \setminus \partial(\{u_i\})$ that are removable in $G_0(v_i) \odot G_i(u_i)$. Then every edge in $E_1 \cup E_2$ is removable in G.

Proof. Note that no edge in E_1 is a splicing edge in the two splicings. By Lemma 2.15 (i), every edge in E_1 is a removable edge in G. Similarly, every edge in E_2 is a removable edge in G, since $(G_0(v_1) \odot G_1(u_1))(v_2) \odot G_2(u_2) = (G_0(v_2) \odot G_2(u_2))(v_1) \odot G_1(u_1)$. So the result follows. □

Proposition 2.17. Let $\partial(X)$ be a 3-cut of a brick G. Then $G^{\Delta}(X)$ is a brick.

Proof. We only need to consider the case when $|\overline{X}| \geqslant 5$. It is clear that $G^{\Delta}(X)$ is 3-connected. We show that $G^{\Delta}(X)$ is bicritical, that is, $G^{\Delta}(X) - \{u, v\}$ has a perfect matching for any two vertices u, v in $G^{\Delta}(X)$.

Let $x_1x_2x_3$ label the replacement-triangle of $G^{\Delta}(X)$, and $\partial_{G^{\Delta}(X)}(\{x_1, x_2, x_3\}) = \{x_1x_1', x_2x_2', x_3x_3'\}$ and $\partial_G(X) = \{x_1'x_1'', x_2'x_2'', x_3'x_3''\}$. Set $w = x_i''$ if $u = x_i$; $w = u$ otherwise. And set $z = x_i''$ if $v = x_i$; $z = v$ otherwise.

Since G is a brick, $G - \{w, z\}$ has a perfect matching, say M. Set $M' = (M \cap G[X]) \cup (M \cap \partial_G(X))$. Since $|X|$ is odd, it can be checked that M' covers all vertices in $G^{\Delta}(X) - \{u, v\}$, or all vertices in $G^{\Delta}(X) - \{u, v\}$ except exactly two vertices in $\{x_1, x_2, x_3\}$. In the latter case, M', together with the edge between the two vertices that are not covered by M', is a perfect matching of $G^{\Delta}(X) - \{u, v\}$ (note that any two vertices in $\{x_1, x_2, x_3\}$ has an edge). □

3 Proof of Theorem 1.1

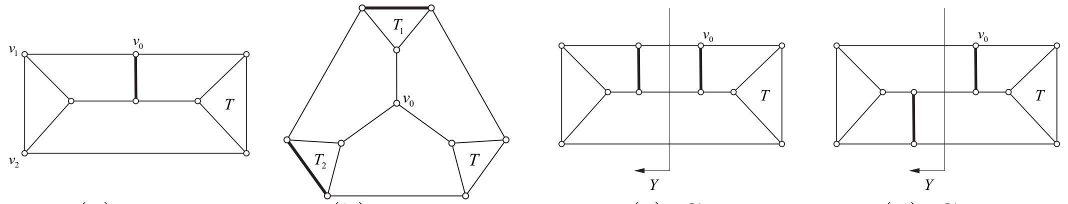
In this section, we consider only cubic graphs. By showing that there exists a 3-cut such that one shore of which contains enough independent removable edges, we will prove the main theorem by induction.

Let $\mathscr{G}$ be the family of K_4 and the cubic graphs obtained from K_4 by a sequence of successive triangle-insertions. It is clear that every graph G in $\mathscr{G}$ is a cubic brick by repeated applications of Proposition 2.12, and except K_4, every vertex in G lies in at most one triangle. Moreover, we have the following proposition.

Proposition 3.1. Let $G \in \mathscr{G} \setminus \{K_4\}$ and let T_0 be a triangle of G. The graph $H := G/V(T_0)$ is in $\mathscr{G}$.

Proof. By induction on the number of vertices of G. If $|V(G)| = 6$ then $H \cong K_4$, hence the result holds. We may thus assume that G has eight or more vertices. Then, G was obtained from a graph $H_1 \in \mathscr{G}$ by the insertion of a triangle, say, T_1, at a vertex t_1. If $T_1 = T_0$ then $H \cong H_1$ and there is nothing more to be proved. Assume then that $T_1 \neq T_0$. As H_1 is a brick, it is 3-connected, hence the set of edges of T_0 still induces a triangle of H_1. Moreover, as T_0 is a triangle of G, the vertex t_1 is not a vertex of T_0. The graph G has eight or more vertices, hence $H_1 \ncong K_4$. By induction, the graph $H_2 := H_1/V(T_0)$ is in $\mathscr{G}$. Let H_0 be the graph obtained from H_2 by the insertion of triangle T_1 at t_1. By definition, the graph H_0 is in $\mathscr{G}$. Moreover, $H_0 \cong H$. The result holds. □

Lemma 3.2. Let $G \in \mathscr{G}$ and let T be triangle of G. If $|V(G)| \geqslant 10$ then G has a 3-cut $C := \partial(X)$ such that (i) $X \cap V(T) = \emptyset$, (ii) $5 \leqslant |X| \leqslant 11$, and (iii) $G[X]$ has a



(a) R_8 (b) tricorn (c) G_1 (d) G_2

Figure 2: The graphs of order 8 and 10 of $\mathscr{G}$; bold lines indicate removable edges not in T.

matching M consisting of two edges both of which are removable in G. Consequently, $|M| \geqslant (|X|+5)/8$.

Proof. Let $n = |V(G)|$. By induction on n. The basis of the inductive hypothesis corresponds to the case in which $n = 10$. The graph R_8 is the only graph of $\mathscr{G}$ on 8 vertices. Up to isomorphism, they are obtained from R_8 by a triangle-insertion at one of the vertices v_0, v_1 and v_2 (see Figure 2(a)). We then obtain the three graphs: the tricorn (Figure 2(b)), G_1 (Figure 2(c)), and G_2 (Figure 2(d)). By Figure 2(b)-(d) and Lemma 2.15, we have the following claim.

Claim 1. *If $\partial(X)$ is a nontrivial 3-cut of G such that $|X| = 7$, then $G[X]$ has a matching consisting of two edges both of which are removable in G.*

The proof of the following auxiliary result is immediate by Corollary 2.5 and Lemma 2.15.

Claim 2. *Let $G' \in \mathscr{G}$ and let X' be a subset of $V(G') \setminus V(T)$. Let $u \in X'$, $G := G'\langle u \rangle$, and $X := X' \cup V(T) \setminus \{u\}$. If $G[X']$ has a matching M' of removable edges of G' then $G[X]$ has a matching M of removable edges of G such that $|M| \geqslant |M'|$.*

If $n = 10$ then the assertion holds, with $X := V(G) \setminus V(T)$. See Figure 2. Note that $|X| = 7$. Suppose that $n \geqslant 12$. Let G' be the graph obtained from G by the contraction of one triangle of G distinct from T, thereby creating the contraction vertex v. By Proposition 3.1, $G' \in \mathscr{G}$. By induction, G' has a subset X' of $V(G') \setminus V(T)$ such that $5 \leqslant |X'| \leqslant 11$ and $G[X']$ has a matching M' consisting of two edges which are removable in G'. If $v \notin X'$ then let $X := X'$; otherwise let X be the set of vertices obtained from X' by the triangle-insertion at v. By Lemma 2.15 and Claim 2, $G[X]$ has a matching consisting of two removable edges of G. Moreover, $|X| = |X'| + 2|X' \cap \{v\}|$. If $v \notin X'$ or if $|X'| \leqslant 9$ then the assertion holds, as $5 \leqslant |X| \leqslant 11$.

We may thus assume that $|X'| = 11$ and $v \in X'$, whereupon $|X| = 13$. Let $H := G^{\Delta}(X)$, and let T_H be the replacement-triangle, that is H is the graph obtained from $G/(\overline{X} \to \overline{x})$ by the triangle-insertion at $\overline{x}$, thereby obtaining the triangle T_H of H. Thus, $|V(H)| = 16$. Suppose that $n > 16$. By induction, $V(H)$ has a subset Y of $V(H) \setminus V(T_H)$ such that $5 \leqslant |Y| \leqslant 11$ and $H[Y]$ has a matching M_H consisting of two removable edges of H. Every edge in $H[Y]$ which is removable in H is also removable in G by Lemma 2.15. Hence, the assertion holds, with $X := Y$ and $M := M_H$.

We may thus assume that $n = 16$. Let G' be obtained from G by repeated applications

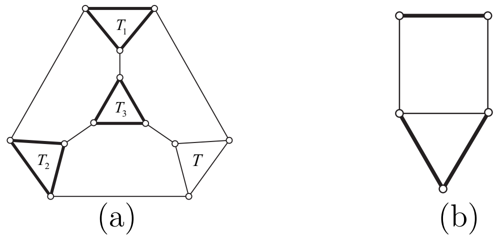


Figure 3: The graphs for Case 2 (the bold lines indicate removable edges).

of three contractions of triangles disjoint with T. Then $|V(G')| = 10$. For $1 \leqslant i \leqslant 3$, we denote by L_i the set of vertices of G' of *level* i, that is the set of vertices resulting from G by the contraction of i triangles. Then $|L_1| \leqslant 3$, $|L_2| \leqslant 1$ and $|L_3| \leqslant 1$.

Case 1 L_3 is not empty.

In that case, L_3 is a singleton, say, v, which was originated from a subset X of $V(G) \setminus V(T)$ having precisely seven vertices. Moreover, as v has degree three, $\partial(X)$ is a 3-cut in G. By Claim 1, every 3-cut $\partial(X)$ such that $|X| = 7$ satisfies the assertion. We may thus assume that L_3 is empty.

Case 2 The graph G' is the tricorn (see Figure 2(b)).

Case 2.1 One of T_1 and T_2 (see Figure 2(b)) is the result of two triangle contractions.

Up to isomorphism, we may assume that T_1 is the result of two triangle contractions. As $L_3 = \emptyset$, the set X which produced T_1 consists of seven vertices. By Claim 1, $G[X]$ has a matching containing two removable edges of G.

Case 2.2 Neither T_1 nor T_2 is the result of two triangle contractions.

In this case, both T_1 and T_2 are the result of at most one triangle contraction. Hence, the vertex v_0 is the result of at least one contraction of a triangle. Let T_3 be the triangle whose contraction produces v_0. We remark that T_3 may contain a contraction vertex. See Figure 3(a).

As L_3 is empty, at least one of T_1 and T_2 is the result of a triangle contraction. Up to isomorphism, we may assume that T_1 is the result of precisely one triangle contraction. In that case, T_1 is the result of the contraction of a vertex set X having only five vertices and such that $G[X]$ contains a matching consisting of two removable edges of G. See Figure 3(b).

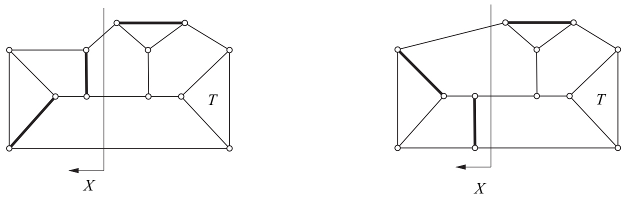
Case 3 $G' = G_1$, the graph in Figure 2(c).

Case 3.1 The vertex set Y, indicated in the Figure 2(c), contains a contraction vertex.

Let X be the vertex set obtained from Y by repeated applications of triangle-insertions on contracted vertices. As $|Y| = 5$, it follows that $7 \leqslant |X| \leqslant 11$. For any 3-cut $\partial(Z)$ such that T and Z are disjoint and $|Z| = 7$, the graph $G[Z]$ has a matching containing two edges which are removable in G by Claim 1. From this and Claim 2, we infer that $G[X]$ contains a matching consisting of two edges which are removable in G.

Case 3.2 The vertex set Y, indicated in the Figure 2(c), does not contain a contraction vertex.

Then, the contracted vertices of G' are the ends of the removable edge of G' not in



(a) G_1' (b) G_2'

Figure 4: The graphs for Case 3 and Case 4 (the bold lines indicate removable edges).

$G[Y]$. As $L_3 = \emptyset$, we deduce that both ends of that edge are contracted vertices of G'. Let G_1' be the result of G' by one triangle-insertion at the vertex v_0. The resulting graph is depicted in Figure 4(a). The graph $G_1'[X]$ has a matching M consisting of two removable edges of G_1'. By Lemma 2.15, $G[X]$ has a matching that consists of two removable edges of G. The assertion holds.

Case 4 $G' = G_2$.

The analysis of this case has several points in common with the analysis of Case 3. In particular, the analysis of the case in which Y contains a contraction vertex of G' is identical to the analysis of Case 3.1.

We may thus assume that Y does not contain any contraction vertex of G'. Again, both ends of the removable edge not in $G[Y]$ are contraction vertices of G'. Let G_2' be the result of G' by one triangle-insertion at the vertex v_0. The resulting graph is depicted in Figure 4(b). As in Case 3.2, the graph $G[X]$ contains a matching consisting of two removable edges of G and $|X| = 5$. The assertion holds.

In each one of the alternatives, we proved the existence of the set X with the asserted properties. $\square$

We say that a cubic brick is *small* if it has at most six vertices. Thus, a cubic brick is small if and only if it is either K_4 or $\overline{C_6}$. Let G be a cubic matching covered graph. Suppose that for some (possibly empty) set S of vertices of G, the graph G is spliced at each vertex in S with a small cubic brick. We then say that the resulting graph is a *decoration* of G.

Lemma 3.3. Let G be a cubic brick and let $C := \partial(X)$ be a (possibly trivial) 3-cut of G such that the C-contraction $H := G/(\overline{X} \to \overline{x})$ is a decoration of an essentially 4-edge-connected graph G' distinct from K_4. Suppose that the contracted vertex $\overline{x}$ of H is also a vertex of G'. Then, $|X| \geqslant 7$ and $G[X]$ has a matching consisting of at least $(|X| + 5)/8$ edges which are removable in G.

Proof. By Corollary 2.10, G' has a perfect matching, say, M', consisting of removable edges of G'. Let e be the edge of M' incident with $\overline{x}$. Let S_1 be the set of vertices v of G' such that G' is spliced at v with a K_4. Let S_2 be the set of vertices v of G' such that G' is spliced at v with a $\overline{C_6}$. Let $S := S_1 \cup S_2$. For $i = 1, 2$, let $s_i := |S_i|$. Let $s := |S|$. Note that $\overline{x} \notin S$. Let $n' := |V(G')|$. Clearly,

$$|X| + 5 = n' + 4 + 2s_1 + 4s_2 \leqslant n' + 4(s + 1). \tag{1}$$

By Theorem 2.6, G' is either a brick or a brace. Suppose that G' is a brick. As G' is an essentially 4-edge-connected brick distinct from K_4, we infer that $|V(G')| \geqslant 8$. Since $|V(H)| \geqslant |V(G')|$, we deduce that $|X| = |V(H)| - 1 \geqslant 7$. By Lemma 2.4, every spliced small cubic brick at a vertex of S contributes with a removable edge of G, at the expense of an edge in $M' - e$. Let M be the matching thereby obtained. Then every edge in M is still removable in G by Lemma 2.15.

Alternatively, suppose that G' is a brace. As G is a brick, it follows that there are two vertices, v_1 and v_2, one in each part of the bipartition of G', such that $\{v_1, v_2\} \subseteq S$. Thus, $|V(H)| \geqslant |V(G')| + 4 \geqslant 10$, hence $|X| = |V(H)| - 1 \geqslant 9$.

The decoration of G' with only the splicings at those two vertices, say G'', has at least two removable edges of G, contributed by the two small cubic bricks, at the expense of the edges of M' incident with v_1 and with v_2. See Figure 5(a). Moreover, G'' is a brick, by Lemma 2.13. Again, the splicing of a small cubic brick at each vertex v in $S \setminus \{v_1, v_2\}$ contributes with a removable edge, at the expense of the edge of M' incident with v. Let M be the matching thereby obtained.

In both alternatives, $|X| \geqslant 7$. Moreover, $|M| \geqslant |M'| - 1 - (s_1 + s_2) + s_1 + s_2 = n'/2 - 1$, hence $|M| \geqslant n'/2 - 1$. We may also disregard the edges of M' and conclude that $|M| \geqslant s_1 + s_2$. In sum,

$$|M| \geqslant \max(n'/2 - 1, s). \tag{2}$$

Case 1. $s + 1 \leqslant n'/2 - 1$.

From (1) and (2), and since $n' \geqslant 6$, $|X| + 5 \leqslant n' + 4(s+1) \leqslant 3n' - 4 < 4n' - 8 \leqslant 8|M|$. The assertion holds.

Case 2. $s + 1 \geqslant n'/2$.

From (2), it follows that

$$|M| \geqslant s \geqslant n'/2 - 1. \tag{3}$$

Case 2.1. $n' \geqslant 8$.

From the hypothesis of the case, and (3), we deduce that $|M| \geqslant 3$. From (1) and (2), and since $n'/2 \leqslant s + 1$, $|X| + 5 \leqslant n' + 4(s + 1) \leqslant 6(s + 1) \leqslant 6|M| + 6 \leqslant 8|M|$, where the last inequality follows from the fact that $|M| \geqslant 3$. The assertion holds.

Case 2.2. $n' \leqslant 6$.

As G' is essentially 4-edge-connected, it follows that $n' = 6$ and $G' \cong K_{3,3}$. Let $t := s_1 + 2s_2$. As indicated in Figure 5(a), $|M| \geqslant s_1 + 2s_2 = t$ by Corollary 2.16. We have assumed that $s + 1 \geqslant n'/2$, hence $t \geqslant s \geqslant 2$. Thus, $|X| + 5 = n' + 2s_1 + 4s_2 + 4 = 2t + 10 < 8t \leqslant 8|M|$. The assertion holds in all cases considered. □

Lemma 3.4. Let G be a cubic brick on $n \geqslant 10$ vertices. Then, G has a (possibly trivial) 3-cut $\partial(X)$ such that $|X| \geqslant 5$ and $G[X]$ has a matching consisting of at least $(|X| + 5)/8$ edges, each of which is removable in G.

Proof. Consider first the case in which G has a 3-cut $\partial(Y)$ such that $|Y| \geqslant 7$ and $G/\overline{Y} \in \mathscr{G}$. Then, $G^{\Delta}(Y)$ is in $\mathscr{G}$ and has at least 10 vertices. By Lemma 3.2, $G^{\Delta}(Y)$ has a 3-cut $\partial(X)$ such that $|X| \geqslant 5$, $X \subset Y$, and $G[X]$ has a matching consisting of at least $(|X| + 5)/8$

(a) (b)

Figure 5: The bold edges represent the removable edges.

edges which are removable in $G^{\Delta}(Y)$. So $G[X]$ has a matching consisting of at least $(|X|+5)/8$ edges which are removable in G by Lemma 2.15.

We may thus assume that if $\partial(Z)$ is a 3-cut such that $G/\overline{Z} \in \mathscr{G}$ then $|Z| \leqslant 5$. In particular, as $n \geqslant 10$, this assumption implies that $G \notin \mathscr{G}$. For every 3-cut C of G, at least one C-contraction of G is not in $\mathscr{G}$. Let X be a minimal set of vertices of G, $|X| > 1$, such that $C := \partial(X)$ is a (possibly trivial) 3-cut of G and the C-contraction $H := G/(\overline{X} \to \overline{x})$ is not in $\mathscr{G}$. We have assumed that for any 3-cut $\partial(Z)$, if $G/\overline{Z}$ is in $\mathscr{G}$ then $|Z| \leqslant 5$. It follows that H is a decoration of an essentially 4-edge-connected graph distinct from K_4. By Lemma 3.3, G has a 3-cut $\partial(X)$ such that $|X| \geqslant 7$ and $G[X]$ has a matching consisting of at least $(|X|+5)/8$ edges, each of which is removable in G. □

Proof of Theorem 1.1. By induction on n, the number of vertices of G. If $n = 8$ then either G is R_8, which has one removable edge, or G is a Möbius ladder, depicted in Figure 5(b), which has a perfect matching consisting solely of removable edges of G. We may thus assume that $n \geqslant 10$.

By Lemma 3.4, G has a 3-cut $C := \partial(X)$ such that $|X| \geqslant 5$ and $G[X]$ has a matching consisting of at least $(|X|+5)/8$ edges which are removable in G. If $|\overline{X}| \leqslant 5$ then M has at least $n/8$ edges.

We may thus assume that $|\overline{X}| \geqslant 7$. Let $H := G^{\Delta}(\overline{X})$. As $|\overline{X}| \geqslant 7$, we have that $|V(H)| \geqslant 10$. As $|X| \geqslant 5$, $|V(H)| < n$. By induction, H has a matching, M_H, consisting of at least $(|\overline{X}|+3)/8$ edges which are removable in H. The edges of C are not removable in H by Proposition 2.14. The replacement-triangle of H contains at most one edge in M_H. By Lemma 2.15, $M' := M_H \cap E(G[\overline{X}])$ is a matching of $G[\overline{X}]$ consisting of at least $|M_H| - 1 = (|\overline{X}| - 5)/8$ edges which are removable in G. Consequently, $M \cup M'$ is a matching of G consisting of at least $n/8$ edges, each of which is removable in G. The result follows. □

It should be noted that the lower bound in Theorem 1.1 is not tight for large $|V(G)|$. We do not know the attainable lower bound of independent removable edges of cubic bricks with any number of vertices.

4 Acknowledgment

The authors are especially grateful to the anonymous referees for many constructive suggestions, which greatly improve the readability and presentation of the paper. This work is supported by NSFC (Nos. 12271235, 12361070) and Fujian Provincial Natural Science Foundation of China (Nos. 2021J01978, 2021J06029).

References

[1] M. H. Carvalho and C. H. C. Little. Matching covered graphs with three removable classes. The Electronic J. of Combin., 21(2): #P2.13, 2014.

[2] M. H. Carvalho, C. L. Lucchesi and U. S. R. Murty. Ear decompositions of matching covered graphs. Combinatorica, 19(2):151-174, 1999.

[3] M. H. Carvalho, C. L. Lucchesi and U. S. R. Murty. On a conjecture of Lovász concerning bricks: II. Bricks of finite characteristic. Journal of Combinatorial Theory, Series B, 85(1):137-180, 2002.

[4] M. H. Carvalho, C. L. Lucchesi and U. S. R. Murty. Graphs with independent perfect matchings. Journal of Graph Theory, 48:19-50, 2005.

[5] M. H. Carvalho, C. L. Lucchesi and U. S. R. Murty. Thin edges in braces. The Electronic J. of Combin., 22(4):#P4.14, 2015.

[6] J. Edmonds, L. Lovász and W. R. Pulleyblank. Brick decompositions and the matching rank of graphs. Combinatorica, 2(3):247-274, 1982.

[7] N. Kothari, M. H. Carvalho, C. L. Lucchesi and C. H. C. Little. On essentially 4-edge-connected cubic bricks. The Electronic J. of Combin., 27(1): #P1.22, 2020.

[8] L. Lovász. Ear decompositions of matching covered graphs. Combinatorica, 2:105-117, 1983.

[9] L. Lovász. Matching structure and the matching lattice. Journal of Combinatorial Theory, Series B, 43(2):187-222, 1987.

[10] L. Lovász and M. D. Plummer. Matching Theory. Number 29 in Annals of Discrete Mathematics. Elsevier Science, 1986.

[11] F. Lu, X. Feng and Y. Wang. b-invariant edges in essentially 4-edge-connected near-bipartite cubic bricks. The Electronic J. of Combin., 27(1): #P1.55, 2020.

[12] W. McCuaig. Brace generation. J. Graph Theory, 38:124-169, 2001.

[13] W. T. Tutte. The factorization of linear graphs. J. London Math. Soc., 22:107-111, 1947.

[14] S. Zhai, C. L. Lucchesi and X. Guo. A lower bound on the number of removable ears of 1-extendable graphs. Discrete Math., 310(5):1123-1126, 2010.