A note on uncountably chromatic graphs

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Abstract

We present an elementary construction of an uncountably chromatic graph without uncountable, infinitely connected subgraphs.

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1 Introduction

Erdős and Hajnal asked in 1985 whether every graph of uncountable chromatic number has an infinitely connected, uncountably chromatic subgraph [2]. In 1988 and 2013, P. Komjáth gave consistent negative answers [3, 4]: He first constructed an uncountably chromatic graph without infinitely connected, uncountably chromatic subgraphs; and later an uncountably chromatic graph without any uncountable, infinitely connected subgraph. In 2015, D. Soukup gave the first ZFC construction of such a graph [6]. Soukup even produces an uncountably chromatic graph G in which every uncountable set of vertices contains two points that are connected by only finitely many independent paths in G (to see that this is a stronger failure of connectivity consider an uncountable clique in which every edge has been subdivided once). In this note we present a short, elementary example for Soukup's result.

2 The example

Let $\mathbb{N} = \{1, 2, 3, \dots\}$. For a countable ordinal α, write T^α for the set of all injective sequences $t\colon \alpha \to \mathbb{N}$ that are *co-infinite*, i.e. such that $|\mathbb{N} \setminus \operatorname{im}(t)| = \infty$. Then $T = \bigcup_{\alpha<\omega_1} T^\alpha$ is a well-founded tree when ordered by *extension*, i.e. $t \leqslant t'$ if $t = t' \restriction \operatorname{dom}(t)$. For a sequence $s \in T^{\alpha+1}$ of successor length, let $\operatorname{last}(s) := s(\alpha) \in \mathbb{N}$ be the last value of s; and $s^\star := s \restriction \alpha \in T^\alpha$ its immediate predecessor. Put $\Sigma(T) = \bigcup_{\alpha<\omega_1} T^{\alpha+1}$. For any $t \in T$, let

$$A_t := \{s \leqslant t \colon s \in \Sigma(T),\ \operatorname{last}(s) = \min\left(\operatorname{im}(t) \setminus \operatorname{im}(s^\star)\right)\},$$

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and let $A_t^\star = \{s^\star \colon s \in A_t\}$.

Let $\mathbf{G}$ be the graph with vertex set $V(\mathbf{G}) = T$ and edge set $E(\mathbf{G}) = \{t't \colon t' \in A_t^\star\}$.

Theorem 1. *The graph* $\mathbf{G}$ *is uncountably chromatic yet every uncountable set of vertices in* $\mathbf{G}$ *has two vertices that are connected by only finitely many independent paths in* $\mathbf{G}$.

3 The proof

We first show that every uncountable set of vertices $A \subseteq V(\mathbf{G})$ contains two points which are connected by only finitely many independent paths in $\mathbf{G}$. For $s \in T$ we write $s{\downarrow} := \{t \in T \colon t < s\}$, and note that the definition of A_s implies that for all $s \leqslant u \in T$ we have

$$A_u \cap s{\downarrow} \subseteq A_s \ \text{ and } \ A_u^\star \cap s{\downarrow} \subseteq A_s^\star. \qquad (\star)$$

Since T contains no uncountable chains, the set A contains two vertices t and t' that are incomparable in T. Let $\alpha \in \operatorname{dom}(t)$ be minimal such that $t(\alpha) \neq t'(\alpha)$, and consider $s = t \restriction (\alpha+1)$. Then $(\star)$ implies that every $t - t'$ path meets $A_s^\star$, and since $|A_s^\star| = |A_s| \leqslant \operatorname{last}(s)$ is finite, there are only finitely many independent $t - t'$ paths in $\mathbf{G}$.

It remains to show that $\mathbf{G}$ has chromatic number $\chi(\mathbf{G}) = \aleph_1$. Colouring the elements of each T^α with a new colour shows $\chi(\mathbf{G}) \leqslant \aleph_1$. To see $\chi(\mathbf{G}) \geqslant \aleph_1$, suppose for a contradiction that $c \colon V(\mathbf{G}) \to \mathbb{N}$ is a proper colouring.

For $t \in \Sigma(T)$, we say $t' \in T$ is an *extension of* t if $t' \geqslant t$ and $\operatorname{im}(t') \setminus \operatorname{im}(t) \subseteq \{n \in \mathbb{N} \colon n > \operatorname{last}(t)\}$. We say $t' \in T$ is a *1-extension of* t if it has the stronger property that $t' > t$ and, letting a_1 be the minimal element of $\mathbb{N} \setminus \operatorname{im}(t)$ with $a_1 > \operatorname{last}(t)$, we have $\operatorname{im}(t') \setminus \operatorname{im}(t) \subseteq \{n \in \mathbb{N} \colon n > a_1\}$. In this case we also say that the 1-extension t' *skips* a_1.

Claim 2. *Every* t *in* $\Sigma(T)$ *has an extension* t' *in* $\Sigma(T)$ *such that every* 1*-extension* t'' *of* t' *satisfies* $c(t'') > c(t^\star)$.

Suppose for a contradiction that the claim is false. Then there exists $t_0 \in \Sigma(T)$ such that all its extensions $t' \in \Sigma(T)$ have a 1-extension t'' such that $c(t'') \leqslant c(t_0^\star)$. Then for the extension $t_0' = t_0$ of t_0, there is a 1-extension t_0'' of t_0' skipping $a_1 \in \mathbb{N}$ with $c(t_0'') \leqslant c(t_0^\star)$. Let $t_1' := t_0'' {}^\frown a_1$. Then $t_1' \in \Sigma(T)$ is itself an extension of t_0, so it has a 1-extension t_1'' skipping a_2 with $c(t_1'') \leqslant c(t_0^\star)$. Let $t_2' := t_1'' {}^\frown a_2$. And so on. Now a_{m+1} witnesses that $t_{m+1}' \in A_{t_n''}$, and so $t_m'' \in A_{t_n''}^\star$ whenever $m < n \in \mathbb{N}$. Hence, the vertices $\{t_n'' \colon n \in \mathbb{N}\}$ induce a complete subgraph of $\mathbf{G}$, contradicting that they have been coloured using only colours $\leqslant c(t_0^\star)$. This proves the claim.

We now complete the proof as follows: Fix an arbitrary $t_0 \in \Sigma(T)$. Let $t_0' \in \Sigma(T)$ be an extension of t_0 as in the claim. Let $a_1 < a_2$ be the two smallest elements of $\mathbb{N} \setminus \operatorname{im}(t_0')$ above $\operatorname{last}(t_0')$. Let $t_1 := t_0' {}^\frown a_2$. Let t_1' be an extension of t_1 as in the claim. Let $a_3 < a_4$ be the two smallest elements of $\mathbb{N} \setminus \operatorname{im}(t_1')$ above $\operatorname{last}(t_1')$. Let $t_2 := t_1' {}^\frown a_4$. And so on. Then $\hat{t} = \bigcup_{n \in \mathbb{N}} t_n$ is an injective sequence. Moreover, $a_1, a_3, a_5, \ldots$ witness that $\hat{t}$ is co-infinite, giving $\hat{t} \in T$. But for each $n \in \mathbb{N}$, the sequence $\hat{t}$ is a 1-extension (skipping a_{2n+1}) of the extension t_n' of t_n, so $c(t_n^\star) \leqslant c(\hat{t})$ according to the claim. However, $a_2, a_4, a_6, \ldots$ witness that the vertices $\{t_n^\star \colon n \in \mathbb{N}\}$ induce a complete subgraph of $\mathbf{G}$, a contradiction. □

4 Remarks

(1) In the terminology of [5], the graph $\mathbf{G}$ is a *T-graph of finite adhesion*. The construction of the sets A_t is inspired by an argument from [1].

(2) The graph with vertex set T but edge set $\{t't\colon t' < t,\ t' \in A_t\}$ has countable chromatic number by colouring all $s \in \Sigma(T)$ by colour $\mathrm{last}(s)$, and noticing that $A_t \subset \Sigma(T)$ for all $t \in T$ implies that $T \setminus \Sigma(T)$ is independent.

(3) The following version of the Erdős-Hajnal problem remains open: *Does every uncountably chromatic graph have a countably infinite, infinitely connected subgraph?*

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