The number of edge colorings with small independence number and no monochromatic H

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Abstract

In 1974, Erdős and Rothschild initiated the study of the maximum possible number, known as $F(n,r,k)$, of distinct edge-colorings of a graph on n vertices with r colors which contain no monochromatic copy of K_k. The number $F(n,r,k)$ is not well understood except for a few of non-trivial cases. Recently, Balogh, Liu and Sharifzadeh (2017) introduced an extension of such Erdős-Rothschild problem: given a function $f(n)$ and a graph H, let $RF(n,r,H,f(n))$ be the maximum number of distinct r-edge-colorings that an n-vertex graph with independence number at most $f(n)$ can have without a monochromatic copy of H. In particular, they determined the values of $RF(n,2,K_k,o(n))$ for $k \geqslant 3$ and $RF(n,3,K_3,o(n))$.

Define the *forest arboricity* of H, denoted $arb_f(H)$, as the minimum integer p such that $V(H)$ can be partitioned into $\lceil \frac{p}{2} \rceil$ sets $V_1,\dots,V_{\lceil \frac{p}{2} \rceil}$ such that V_i spans a forest for each $1 \leqslant i \leqslant \lfloor \frac{p}{2} \rfloor$, and the last class $V_{\lceil \frac{p}{2} \rceil}$ spans an independent set if p is odd. In this paper, we mainly obtain the asymptotic values of $RF(n,r,H,o(n))$ for $r \in \{3,4,5\}$, where H is any graph with $arb_f(H)=3$ and chromatic number $\chi(H) \geqslant 3$. As a corollary, we have the asymptotic values of $RF(n,r,H,o(n))$ for $r \in \{3,4,5\}$ when H is an odd cycle, or a book (fan) graph.

Keywords: Erdős-Rothschild problem; Ramsey-Turán number; Regularity lemma

Mathematics Subject Classifications: 05C35

1 Introduction

Ramsey theorem [25] states that for any integers p_1, p_2, there exists a minimum integer, now called Ramsey number $r = r(p_1,p_2)$, such that any red/blue edge-coloring the complete graph K_r contains a red K_{p_1} or a blue K_{p_2}. Motivated by this theorem, Turán [30, 31] proved that the balanced complete $(k-1)$-partite graph on n vertices, so-called

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Turán graph $T_{n,k-1}$, is the unique extremal graph which attains the maximum number of edges among all n-vertex K_k-free graphs.

Given graphs G and H, denote by $F(G,r,H)$ the number of distinct edge colorings of G with r colors which contain no monochromatic copy of H. If $H = K_k$ is a complete graph with k-vertices, then $F(G,r,K_k)$ is written as $F(G,r,k)$. Let

$$F(n,r,H) = \max\{F(G,r,H)|\ G \text{ is a graph on } n \text{ vertices}\}.$$

Let $t_{n,k}$ be the number of edges in $T_{n,k}$. Note that every r-edge-coloring of the Turán graph $T_{n,k-1}$ contains no monochromatic k-clique, we immediately have

$$F(n,r,k) \geqslant r^{t_{n,k-1}}. \tag{1}$$

Erdős and Rothschild [9] conjectured that for sufficiently large n, the above obvious lower bound is optimal for 2-edge-colorings. This was verified for $k = 3$ by Yuster [32]. In 2004, Alon, Balogh, Keevash, and Sudakov [2] settled this conjecture showing that, for all $k \geqslant 3$ and sufficiently large n, the Turán graph $T_{n,k-1}$ maximizes the number of 2-edge-colorings and 3-edge-colorings with no monochromatic copy of K_k among all graphs:

$$F(n,2,k) = 2^{t_{n,k-1}} \quad \text{and} \quad F(n,3,k) = 3^{t_{n,k-1}}. \tag{2}$$

Furthermore, they showed that (2) can not be extended to more than three colors, and indeed for $r \geqslant 4, k \geqslant 3$ and all sufficiently large n, there exists a graph G on n vertices for which $F(G,r,k)$ is larger than $r^{t_{n,k-1}}$ by a factor that is exponential in n^2.

For 4-edge-colorings, we only know that $F(n,4,3)$ and $F(n,4,4)$; Alon, Balogh, Keevash, and Sudakov [2] obtained an asymptotic result; Pikhurko and Yilma [24] obtained the exact result by showing that $T_{n,4}$ and $T_{n,9}$ maximize the number of 4-edge-colorings with no monochromatic K_3 and K_4, respectively. For 5-edge-colorings and 6-edge-colorings, Botler et al. [6] announced the determination of $F(n,5,3)$ and $F(n,6,3)$. For $r = 6$ they proved that $T_{n,8}$ is the unique extremal graph, and also proved a stability result. For $r = 5$, they uncovered new behaviour: for large n there are two infinite families $\{S_{n,\alpha,\beta} : 0 \leqslant \alpha + \beta \leqslant \frac{1}{4}\}$ and $\{T_{n,\alpha,\beta} : 0 \leqslant \alpha, \beta \leqslant \frac{1}{4}\}$ of asymptotically optimal graphs with either 4, 6 or 8 parts, where $S_{n,\alpha,\beta}$ denotes the complete partite graph with parts of size $\frac{n}{4}$, $\frac{n}{4}$, αn, αn, βn, βn, $(1/4-\alpha-\beta)n$, $(1/4-\alpha-\beta)n$ and $T_{n,\alpha,\beta}$ denotes the complete partite graph with parts of size αn, αn, $(1/4-\alpha)n$, $(1/4-\alpha)n$, βn, βn, $(1/4-\beta)n$, $(1/4-\beta)n$. For 7-edge-colorings, Pikhurko and Staden [22] showed that $T_{n,8}$ is also the unique extremal graph, with colorings coming from Hadamard matrices of order 8. We refer the reader to [21, 23] for more recent developments.

As we know, Turán graphs have large independent sets of size linear in n, so it is natural to ask for the maximum number of edges of an n-vertex K_{k+1}-free graph without large independent set. Erdős and Sós [14] initiated the study of such Ramsey-Turán type problems, which have attracted a great deal of attention.

Denote by $RT(n,k,m)$ the Ramsey-Turán function for K_k, i.e., the maximum size of an n-vertex K_k-free graph with independence number at most m. We mainly concern the case when $m = o(n)$, which means that the ratio of the independence number and n tends to 0 as $n \to \infty$. The Ramsey number $r(k,m)$ is the minimum integer N such that any red/blue edge coloring of the complete graph K_N contains either a red K_k or a blue K_m. Clearly, there is no graph G of order N which is K_k-free and $\alpha(G) < m$ if $N \geqslant r(k,m)$, and in this case we let $RT(n,k,m) = 0$. For odd cliques, Erdős and Sós [14] proved that $RT(n, 2p+1, o(n)) = \frac{1}{2}(1-\frac{1}{p})n^2+o(n^2)$ for all $p \geqslant 1$. The problem for even cliques is much harder apart from the trivial case K_2. Erdős and Sós [14] showed that $RT(n,4,o(n)) \leqslant \frac{1}{6}n^2 + o(n^2)$. As an early application of the regularity lemma, Szemerédi [28] showed that $RT(n,4,o(n)) \leqslant \frac{1}{8}n^2 + o(n^2)$. No non-trivial lower bound on $RT(n,4,o(n))$ was known until Bollobás and Erdős [5] provided a matching lower bound using an ingenious geometric construction, now called **BE-graph**, showing that $RT(n,4,o(n)) = \frac{1}{8}n^2+o(n^2)$, i.e., **BE-graph** is a n-vertex K_4-free graph with independence number $o(n)$ and $\frac{1}{8}n^2+o(n^2)$ edges. Finally, Erdős, Hajnal, Sós and Szemerédi [12] proved $RT(n, 2p, o(n)) = \frac{1}{2}(1 - \frac{3}{3p-2})n^2+o(n^2)$ for all $p \geqslant 2$. We refer the reader to the nice survey [26] and its references.

Let us turn our attention to the Ramsey-Turán number for non-complete graphs. Given a forbidden graph H, the Ramsey-Turán number $RT(n,H,o(n))$ for H is defined similarly. An important open problem is to prove a generalization of Erdős-Stone Theorem [16], i.e., $RT(n,H,o(n)) = RT(n,p,o(n))$ for some parameter p that depends only on H. Define the *forest arboricity* of H, denoted by $arb_f(H)$, as the minimum integer p such that $V(H)$ can be partitioned into $\lceil \frac{p}{2} \rceil$ sets $V_1, \ldots, V_{\lceil \frac{p}{2} \rceil}$ such that V_i spans a forest for each $1 \leqslant i \leqslant \lfloor \frac{p}{2} \rfloor$, and $V_{\lceil \frac{p}{2} \rceil}$ spans an independent set if p is odd. Erdős et al. [11] proved that $RT(n,H,o(n)) \leqslant RT(n, arb_f(H), o(n))$, and the inequality is sharp for odd $arb_f(H)$. Denote $\chi(H)$ by the chromatic number of H. Then we have

$$\chi(H) \leqslant arb_f(H) \leqslant 2\chi(H) - 1, \tag{3}$$

where the upper bound holds because a k-partite graph has forest arboricity at most $2k-1$.

Since the Turán graph is extremal in the Erdős-Rothschild problem for $r = 2,3$, it is natural to consider its Ramsey-Turán extension, firstly introduced by Balogh, Liu and Sharifzadeh [4].

Definition 1 (Balogh, Liu and Sharifzadeh [4])**.** Given a function $f(n)$ and a graph H, we define

$$RF(n,r,H,f(n))$$

to be the maximum number of r-edge-colorings that an n-vertex graph with independence number at most $f(n)$ can have without a monochromatic copy of H. If $H = K_k$ is a complete graph with k-vertices, then $RF(n,r,K_k,f(n))$ will be rewritten by $RF(n,r,k,f(n))$.

Clearly, we have that

$$RF(n,r,H,f(n)) \leqslant F(n,r,H). \tag{4}$$

We mainly concern the case when $f(n) = o(n)$. Similarly, since there exists an n-vertex H-free graph with $RT(n,H,o(n))$ edges and independence number $o(n)$, we have that

$$RF(n,r,H,o(n)) \geqslant r^{RT(n,H,o(n))}. \tag{5}$$

Unlike (2), $RF(n,r,k,o(n))$ exhibits rather different behavior than $F(n,r,k)$, even in the 2-edge-coloring case when K_4 is forbidden, as observed by Balogh, Liu and Sharifzadeh [4] as follows. Let G be a graph obtained by putting a copy of $n/2$-vertex K_3-free graph with independence number $o(n)$ in each part of $T_{n,2}$. We can color the edges inside one part red, the edges inside the other part blue, and color all the remaining crossing-edges either red or blue. Clearly, none of these colorings contain monochromatic K_4's, hence $RF(n,2,4,o(n)) \geqslant 2^{\frac{n^2}{4}}$, which is much larger than that obtained from (5) by noting $RT(n,4,o(n)) = (\frac{1}{8}+o(1))n^2$.

In [4], Balogh et al. obtained the values of $RF(n,2,k,o(n))$ for $k \geqslant 3$ and $RF(n,3,3,o(n))$.

Theorem 2 (Balogh, Liu and Sharifzadeh [4])**.** *$RF(n,2,3,o(n)) = 2^{o(n^2)}$. For $t \geqslant 1$ and $i \in [3]$,*

$$RF(n,2,3t+i,o(n)) = 2^{RT(n,4t+i,o(n))+o(n^2)}.$$

Moreover, $RF(n,3,3,o(n)) = 2^{\frac{n^2}{4}+o(n^2)}$.

In this paper, we first determine the asymptotic behavior of $RF(n,r,H,o(n))$ for $r = 3,4,5$, where H is a graph with $arb_f(H) = 3$ and $\chi(H) = 3$.

Theorem 3. *Let H be a graph with $arb_f(H) = 3$. Then*

$$RF(n,r,H,o(n)) \leqslant \begin{cases} (2^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)} & if \quad r=3, \\ (3^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)} & if \quad r=4, \\ (2^{\frac{1}{4}}3^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)} & if \quad r=5. \end{cases}$$

Furthermore, all inequalities are asymptotically best possible if $\chi(H) = 3$.

Let B_k (F_k) be a book (fan) graph, which consists of k copies of K_3 all sharing a common edge (vertex). For $H \in \{C_{2k+1}, B_k, F_k\}$, since $arb_f(H) = \chi(H) = 3$, the following corollary is immediate.

Corollary 4. *For any fixed integer $k \geqslant 1$ and $H \in \{C_{2k+1}, B_k, F_k\}$, we have that*

$$RF(n,r,H,o(n)) = \begin{cases} (2^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)} & if \quad r=3, \\ (3^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)} & if \quad r=4, \\ (2^{\frac{1}{4}}3^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)} & if \quad r=5. \end{cases}$$

We also have the following bounds of $RF(n,r,3,o(n))$ for every fixed $r \geqslant 6$.

Proposition 5. *For every fixed $r \geqslant 6$,*

$$\left(\frac{r-1}{2} - 2\sqrt{(r-1)\log(r-1)}\right)^{(1-\frac{1}{r-1})\left(\binom{n}{2}+o(n^2)\right)} \leqslant RF(n,r,3,o(n))$$
$$\leqslant \min\left\{\left(\frac{r}{2}\right)^{\binom{n}{2}+o(n^2)}, (3^{\frac{r-1}{6}})^{\binom{n}{2}+o(n^2)}\right\}.$$

In particular, if $n \gg r \to \infty$, then $RF(n,r,3,o(n)) = \left(\frac{r}{2}+o(1)\right)^{\binom{n}{2}+o(n^2)}$.

Notation: Let $G=(V,E)$ be a graph with vertex set V and edge set E. We use uv to denote an edge of G. For $X \subseteq V$, $G[X]$ denotes the subgraph of G induced by X. For disjoint $X_1,\dots,X_t \subset V$, $G[X_1,\dots,X_t]$ denotes the subgraph induced by all edges between them. Let $X \sqcup Y$ denote the disjoint union of X and Y. A complete k-partite graph with vertex set $\sqcup_{i=1}^k V_i$, where $|V_i| = n_i$, is denoted by $K_k(n_1,\dots,n_k)$. For each n and k, $T_{n,k}$ denotes the n-vertex Turán graph, which is the n-vertex complete k-partite graph such that each part has size either $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$. Let $[n]=\{1,2,\dots,n\}$, and $[m,n]=\{m,m+1,\dots,n\}$. We always omit the subscripts if there is no confusion from the context.

2 Preliminaries

Let $X,Y \subseteq V(G)$ be disjoint nonempty sets of vertices in a graph G. The density of (X,Y) is $d_G(X,Y)=\frac{e_G(X,Y)}{|X||Y|}$. For $\varepsilon > 0$, the pair (X,Y) is ε-regular in G if for every pair of subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geqslant \varepsilon|X|$ and $|Y'| \geqslant \varepsilon|Y|$ we have $|d_G(X,Y)-d_G(X',Y')| \leqslant \varepsilon$. Additionally, we say that ε-regular pair (X,Y) is (ε,ξ)-regular if $d_G(X,Y) \geqslant \xi$ for some $\xi > 0$. We say a partition $V(G)=\sqcup_{i=0}^m V_i$ of G is equitable with exceptional set V_0 if $|V_i|=|V_j|$ for all distinct i and j in $[m]$. A partition $V(G)=\sqcup_{i=0}^m V_i$ is ε-regular if the following two conditions hold: (i) $|V_1|=|V_2|=\cdots=|V_m|$ and $|V_0| \leqslant \varepsilon|V(G)|$. (ii) all but at most εm^2 pairs (V_i,V_j) with $1 \leqslant i < j \leqslant m$ are ε-regular. For every $A \subseteq V(G)$ and an r-coloring of $E(G)$ with colors $[r]$, let $G_k[A]$ be the k-colored subgraph of G induced by the vertex set A.

Szemerédi regularity lemma [29] is a powerful tool in extremal graph theory. In order to show Theorem 3 and Proposition 5, we will use the following multicolor regularity lemma. See also [20, Theorem 11.9].

Lemma 6 (Komlós and Simonovits [18])**.** *For every $\varepsilon > 0$ and integer r, there exists an M such that for every $n > M$ and every r-coloring of the edges of an n-vertex graph G with colors $[r]$, all monochromatic graphs have a same partition $V(G)=\sqcup_{i=0}^m V_i$ that is ε-regular with exceptional set V_0 and $\frac{1}{\varepsilon} < m < M$.*

Remark. As we know, the regularity lemma is quite flexible. For example, we can start with an arbitrary partition of $V(G)$ instead of the trivial partition in the proof of Lemma 6, in order to obtain a partition that is a refinement of a given partition.

We will also use the following lemma by Balogh, Liu and Sharifzadeh [4, Lemma 3.1] which refines a result of Erdős, Hajnal, Simonovits, Sós and Szemerédi [10, Lemma 2].

Lemma 7 (Balogh, Liu and Sharifzadeh [4])**.** *For every $0 < c < 1$, $r \geqslant 2$, and $\mu \leqslant c^{3\cdot 2^{r-2}-1}$ the following holds. Let G be an n-vertex graph with $\alpha(G) \leqslant \mu n$ and an r-edge-coloring $\varphi : E(G) \to [r]$. Then there exists a partition $V(G) = \sqcup_{i=1}^{r} V_i$ such that for every $k \in [r]$, $\alpha(G_k[V_k]) \leqslant cn$.*

A useful notion associated with a regular partition is a cluster graph. For every $\varepsilon > 0$, positive integer t, and an n-vertex graph $G = (V, E)$, let $V(G) = \sqcup_{i=1}^{m} V_i$ be an ε-regular equitable partition of $V(G)$ with $m \geqslant t$, and $\xi > 0$ is some fixed constant (to be thought of as small, but much large than ε). Let $\mathcal{V} = \{v_1, \ldots, v_m\}$, where the vertex v_i represents the vertex set V_i for all $i \in [m]$. Denote by R the cluster graph (with respect to ε and ξ) with vertex set $\mathcal{V}$, and v_i and v_j are adjacent if the pair (V_i, V_j) is (ε, ξ)-regular.

We now define the weighted cluster graph, $R = (\mathcal{V}, \omega)$ (with respect to ε and ξ), on the vertex set $\mathcal{V}$ as follows. For an ε-regular pair (V_i, V_j), we will define the following:

$$\omega(v_i, v_j) = \begin{cases} 0 & \text{if } d(V_i, V_j) \leqslant \xi \text{ or } (V_i, V_j) \text{ is an irregular pair,} \\ \frac{1}{2} & \text{if } \xi < d(V_i, V_j) \leqslant \frac{1}{2} + \xi, \\ 1 & \text{if } \frac{1}{2} + \xi < d(V_i, V_j). \end{cases}$$

Definition 8. A weighted graph G is an ordered triple (V, E, ω), where $E = \binom{V}{2}$, set of all unordered pairs of vertices, and $\omega : E \to \{0, 1/2, 1\}$. Define $G_{1/2} = (V, E_{1/2})$, where $E_{1/2} = \{e \in E : \omega(e) \geqslant 1/2\}$, and $G_1 = (V, E_1)$, where $E_1 = \{e \in E : \omega(e) = 1\}$. Denote by $e(G) = \Sigma_{e \in E(G)} \omega(e)$. For $Y \subseteq X \subseteq V$ ($Y = \emptyset$ is possible), we call (Y, X) a weighted $(|Y|, |X|)$-clique or weighted complete subgraph of size ℓ if $\binom{Y}{2} \subseteq E_1$ and $\binom{X}{2} \subseteq E_{1/2}$ and $|X| + |Y| = \ell$. Also, let the weighted clique number of G be the size of the largest weighted complete subgraph of G.

The following Lemma is very important for our results, which is due to Erdős, Hajnal, Sós and Szemerédi [12].

Lemma 9 (Erdős, Hajnal, Sós and Szemerédi [12])**.** *Let H be a fixed graph with $arb_f(H) = \ell \geqslant 3$. For every $\xi > 0$, there exist $\delta, \varepsilon > 0$ and n_0 such that for every n-vertex graph G with $n \geqslant n_0$, if its weighted cluster graph $R = (\mathcal{V}, \omega)$ with respect to ε and ξ contains a weighted clique (Y, X) of size ℓ such that $\alpha(G[U_Y]) \leqslant \delta n$ where $U_Y = \sqcup\{V_i \subseteq V(G) : v_i \in Y \subseteq \mathcal{V}\}$, then G contains a copy of H.*

Remark. The above lemma provides an approach for embedding a given graph H to the host graph G. To this end, we usually find a weighted clique (Y, X) of size $\ell = arb_f(H)$ in the weighted cluster graph R of the host graph such that $\alpha(G[U_Y]) \leqslant \delta n$.

3 Proof of Theorem 3

3.1 Lower bounds

$RF(n,3,H,o(n))$: Let G be a graph obtained from the Turán graph $T_{n,2}$ by putting an extremal graph for $RT(\frac{n}{2},H,o(n))$ in each part. Consider the following set of 3-edge-colorings of G. Color the edges inside each part red, and color all the remaining crossing-edges green, or blue. Clearly, there are no monochromatic H since $\chi(H)=3$. Thus $RF(n,3,H,o(n)) \geqslant 2^{\lceil n/2\rceil\lfloor n/2\rfloor} \geqslant \sqrt{2}^{\binom{n}{2}+(n-1)/2}$.

$RF(n,4,H,o(n))$: Let G be a graph obtained from the Turán graph $T_{n,2}$ by putting an extremal graph for $RT(\frac{n}{2},H,o(n))$ in each part. Consider the following set of 4-edge-colorings of G. Color the edges inside each part red, and color all the remaining crossing-edges, black, green, or blue. Clearly, there are no monochromatic H since $\chi(H)=3$. Thus $RF(n,4,H,o(n)) \geqslant 3^{\lceil n/2\rceil\lfloor n/2\rfloor} \geqslant \sqrt{3}^{\binom{n}{2}+(n-1)/2}$.

$RF(n,5,H,o(n))$: Let G be a graph obtained from the Turán graph $T_{n,4}$ by putting an extremal graph for $RT(\frac{n}{4},H,o(n))$ in each part. Let $V_1,\ldots,V_4$ be the classes of the partition and let $\{a,b,c,d,f\}$ be the set of colors. Let $c(1,2)=c(3,4)=\{a,b,c\}$, $c(1,3)=c(2,4)=\{a,b,d\}$, $c(1,4)=c(2,3)=\{c,d\}$. Consider the set of colorings as follows: for $i,j\in[4]$, all edges inside V_i are colored by color f, and every edge between V_i and V_j must have one of the colors belonging to the set $c(i,j)$. Clearly, there are no monochromatic H in any of these colorings since the graph of edges which could be colored a, b, c, d are all bipartite, and $\chi(H)=3$. Therefore, $RF(n,5,H,o(n)) \geqslant (2^{\frac{1}{4}}3^{\frac{1}{2}})^{\binom{n}{2}+\frac{n}{2}-2}$ from a simple calculation. It should be noted that the coloring of the crossing edges of $V_1,\ldots,V_4$ of this construction was first used by Alon, Balogh, Keevash, and Sudakov [2] to prove the lower bound of $F(n,4,3)$. Pikhurko and Yilma [24] showed that the coloring of the crossing edges of $V_1,\ldots,V_4$ of this construction is the unique extremal graph of $F(n,4,3)$.

3.2 Upper bounds

The proofs of the following upper bounds involves the idea of Alon, Balogh, Keevash, and Sudakov in [2].

Let H be a graph with $arb_f(H)=3$. We separate the proof into three parts.

Part (I): $\boldsymbol{RF(n,3,H,o(n)) \leqslant (2^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)}}$

Denote by $|V(H)|=h$. We shall prove that for any $\eta>0$, there exist $\gamma>0$ and $n_0>0$ such that for any $n\geqslant n_0$ the following holds. If G is an n-vertex graph with $\alpha(G)\leqslant\gamma^5 n$, then the number of 3-edge-colorings of G without a monochromatic H is at most $(2^{\frac{1}{2}})^{\binom{n}{2}+\eta n^2}$.

For any sufficiently small $\xi>0$, let δ,ε and M be the constants chosen from Lemma 9 and Lemma 6, respectively. Throughout the proof, we may assume that $0<1/n_0\ll\gamma\ll\delta\ll 1/M\ll\varepsilon\ll\xi\ll\eta<1$. Let G be an n-vertex graph with $n\geqslant n_0$ and $\alpha(G)\leqslant\gamma^5 n$. For any fixed 3-edge-coloring of G, $\varphi: E(G)\to[3]$, we apply Lemma 7 with $r=3$, $c=\gamma$.

Let $\{A_1, A_2, A_3\}$ be the partition such that for $r \in [3]$,

$$\alpha(G_r[A_r]) \leqslant \gamma n. \tag{6}$$

We then apply Lemma 6 to the graph G with coloring φ to obtain a partition $V(G) = \sqcup_{i=1}^m V_i$ by refining the $\{A_1, A_2, A_3\}$-partition which is ε-regular with respect to G_r for every $r \in [3]$, where $M > m \geqslant 1/\varepsilon$. We may assume that $|V_0| = 0$ and $|V_i| = \frac{n}{m}$ for $i \in [m]$ since it does not affect the result. Let R_1, R_2, R_3 be the weighted cluster graphs for colors 1, 2, 3, respectively, on the vertex set $\mathcal{V} = \{v_1, \ldots, v_m\}$, where the vertex v_i represents the vertex set V_i for all $i \in [m]$. Denote $A_\ell^R = \{v_i \in \mathcal{V} : V_i \subseteq A_\ell\}$ for each $\ell \in [3]$, then $\sum_{\ell=1}^3 |A_\ell^R| = m$.

First we bound the number of 3-edge-colorings of G that could give rise to this particular partition and these weighted cluster graphs. By definition, there are at most $m\left(\frac{n}{m}\right)^2 + \varepsilon m^2 \left(\frac{n}{m}\right)^2 \leqslant 2\varepsilon n^2$ edges that either lie within some class of the partition or join a pair of classes that is not regular with respect to some color. Also there are at most $\xi \cdot m^2 \cdot \left(\frac{n}{m}\right)^2$ edges that join a pair of classes in which their color has density smaller than ξ. Altogether, this gives no more than $2\xi n^2$ edges. There are at most $\binom{n^2/2}{2\xi n^2}$ ways to choose this set of edges and they can be colored in at most $3^{2\xi n^2}$ different ways. Now, for any pair $1 \leqslant i \neq j \leqslant m$ consider the remaining edges between V_i and V_j. If $v_i v_j$ is an edge in exactly ℓ of the weighted cluster graphs, where $\ell \in [3]$, then every remaining edge between V_i and V_j has only ℓ possible colors. Clearly $e(V_i, V_j) \leqslant \left(\frac{n}{m}\right)^2$, so there are at most $\ell^{\left(\frac{n}{m}\right)^2}$ ways of coloring these edges. Let e_ℓ denote the number of edges $v_i v_j$ that lie in exactly ℓ of the weighted cluster graphs. Then, by the above discussion, the number of potential 3-edge-colorings of G that could give this vertex partition and these weighted cluster graphs is at most

$$\lambda := \binom{n^2/2}{2\xi n^2} 3^{2\xi n^2} (2^{e_2} 3^{e_3})^{\left(\frac{n}{m}\right)^2}. \tag{7}$$

Let $\Gamma(x) = -x \log_2 x - (1-x)\log_2(1-x)$ be the entropy function, then we may use the well-known estimate $\binom{y}{xy} \leqslant 2^{\Gamma(x)y}$ for $x \in (0,1)$. Thus,

$$\lambda \leqslant 2^{\Gamma(4\xi)\cdot n^2/2} 3^{2\xi n^2} (2^{e_2} 3^{e_3})^{n^2/m^2} \leqslant 3^{(\Gamma(4\xi)+4\xi)\cdot n^2/2} (2^{e_2} 3^{e_3})^{n^2/m^2}.$$

Clearly, $\Gamma(4\xi)$ tends to zero as $\xi \to 0$.

Claim 10. $e_3 = 0$.

Proof. Suppose to the contrary that $v_1 v_2 \in \cap_{i\in[3]} E(R_i) \subseteq E(R_1)$. Without loss of generality, we may assume that $v_1 \in A_1^R$, then $(\{v_1\}, \{v_1, v_2\})$ is a weighted clique in R_1 of size 3 by noting the weight of edge $v_1 v_2$ in R_1 is bigger than 1/2, which together with $\alpha(G_1[A_1]) \leqslant \gamma n$ from (6). Since H is a graph with $arb_f(H) = 3$, it follows that G contains H as a subgraph of color 1 from Lemma 9, a contradiction. □

Now consider the graph F on $\{v_1, \ldots, v_m\}$ where $v_i v_j$ is an edge of F if it is an edge in exactly 2 of the weighted cluster graphs. Clearly, $e(F) = e_2$.

Claim 11. $e_2 = e(F) \leqslant \sum_{\ell=1}^{3} \frac{|A_\ell^R|^2}{4}$.

Proof. For any $v_iv_j \in e(F)$, v_iv_j must lie in $F[A_\ell^R]$ for some $\ell \in [3]$; otherwise, we may assume that $v_iv_j \in F[A_1^R, A_2^R]$ and $v_i \in A_1^R, v_j \in A_2^R$ without loss of generality, it follows from Lemma 9 that $v_iv_j \notin R_\ell$ by noting $\alpha(G_\ell[A_\ell]) \leqslant \gamma n$ from (6) for $\ell \in [2]$, a contradiction since $v_iv_j \in E(F)$.

Clearly, for each $\ell \in [3]$, $F[A_\ell^R]$ is triangle-free. Thus $e(F) = \sum_{\ell=1}^{3} e(F[A_\ell^R]) \leqslant \sum_{\ell=1}^{3} \frac{|A_\ell^R|^2}{4}$, as claimed. □

Claim 12. *For each* $\ell \in [3]$, $e(R_\ell) \leqslant \frac{(m-|A_\ell^R|)^2}{4}$.

Proof. We only show $e(R_1) \leqslant (m - |A_1^R|)^2/4$. Since G contains no H of color 1, each edge of R_1 is not incident to any vertices in A_1^R by noting Lemma 9 and (6). Thus, all edges of color 1 must be contained in $R_1[\cup_{\ell=2}^{3} A_\ell^R]$. Moreover, $R_1[\cup_{\ell=2}^{3} A_\ell^R]$ is triangle-free; otherwise, suppose that $\{v_1, v_2, v_3\}$ forms a triangle in $R_1[\cup_{\ell=2}^{3} A_\ell^R]$, then $(\emptyset, \{v_1, v_2, v_3\})$ is a weighted clique in R_1 of size 3 by noting the weight of edges v_1v_2, v_1v_3, v_2v_3 in R_1 are bigger than 1/2, which together with $\alpha(G_1[U_Y]) = 0 \leqslant \gamma n$ by noting $Y = \emptyset$ and $U_Y = \emptyset$. Thus, G contains H as a subgraph of color 1 from Lemma 9, a contradiction. Recall $\sum_{\ell=1}^{3} |A_\ell^R| = m$, so we have $e(R_1) \leqslant \frac{(m-|A_1^R|)^2}{4}$ as desired. □

By the definition of e_ℓ and $\sum_{\ell=1}^{3} |A_\ell^R| = m$, we have that

$$\sum_{\ell=1}^{3} \ell \cdot e_\ell = \sum_{\ell=1}^{3} e(R_\ell) \overset{Claim\ 12}{\leqslant} \sum_{\ell=1}^{3} \frac{(m - |A_\ell^R|)^2}{4} = \frac{1}{4}m^2 + \frac{1}{4}\sum_{\ell=1}^{3} |A_\ell^R|^2. \tag{8}$$

We now determine the maximum value of $2^{e_2}3^{e_3}$ subject to $\sum_{\ell=1}^{3} \ell \cdot e_\ell \leqslant \frac{1}{4}m^2 + \frac{1}{4}\sum_{\ell=1}^{3} |A_\ell^R|^2$ from (8), and $e_2 = e(F) \leqslant \sum_{\ell=1}^{3} \frac{|A_\ell^R|^2}{4}$ from Claim 11, and $e_3 = 0$ from Claim 10, and $\sum_{\ell=1}^{3} |A_\ell^R| = m$. Indeed, the maximum occurs at $e_1 = e_3 = 0$ and $e_2 = \frac{m^2}{4}$. Hence, there are at most $3^{(\Gamma(4\xi)+4\xi)(n^2/2)} 2^{n^2/4}$ H-free 3-edge-colorings of G under this vertex partition and the corresponding weighted cluster graphs. Note that M is a constant and there are at most M^n partitions of the vertex set of G into at most M parts, and the number of ways to fix an $\{A_1, \dots, A_3\}$-partition of $V(G)$ at most 3^n. Also, for every such partition there are at most $2^{3(M^2/2)}$ choices for weighted cluster graphs R_1, R_2, R_3. Thus for sufficiently large n,

$$RF(n, 3, H, \gamma^{11}n) \leqslant 3^n \cdot M^n \cdot 2^{3(M^2/2)} \cdot 3^{(\Gamma(4\xi)+4\xi)(n^2/2)} 3^{n^2/4} \leqslant (3^{\frac{1}{2}})^{\binom{n}{2}+\eta n^2},$$

as desired. □

Part (II): $\boldsymbol{RF(n, 4, H, o(n)) \leqslant (3^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)}}$

Denote by $V(H) = h$. We shall prove that for any $\eta > 0$, there exist $\gamma > 0$ and $n_0 > 0$ such that for any $n \geqslant n_0$ the following holds. If G is an n-vertex graph with $\alpha(G) \leqslant \gamma^{11}n$, then the number of 4-edge-colorings of G without a monochromatic H is at most $(3^{\frac{1}{2}})^{\binom{n}{2}+\eta n^2}$.

For any sufficiently small $\xi > 0$, let δ, ε and M be the constants chosen from Lemma 9 and Lemma 6, respectively. Throughout the proof, we may assume that $0 < 1/n_0 \ll \gamma \ll \delta \ll 1/M \ll \varepsilon \ll \xi \ll \eta < 1$. Let G be an n-vertex graph with $n \geqslant n_0$ and $\alpha(G) \leqslant \gamma^{11} n$. For any fixed 4-edge-coloring of G, $\varphi : E(G) \to [4]$, we apply Lemma 7 with $r = 4$, $c = \gamma$. Let $\{A_1, \dots, A_4\}$ be the partition such that for $r \in [4]$,

$$\alpha(G_r[A_r]) \leqslant \gamma n. \tag{9}$$

We then apply Lemma 6 to graph G with coloring φ to obtain a partition $V(G) = \sqcup_{i=1}^{m} V_i$ which is ε-regular with respect to G_r for every $r \in [4]$, where $M > m \geqslant 1/\varepsilon$. We may assume that $|V_0| = 0$ and $|V_i| = \frac{n}{m}$ for $i \in [m]$ since it does not affect the result. Note that we may assume the regularity partition $\{V_1, \dots, V_m\}$ refines the $\{A_1, \dots, A_4\}$-partition. Let R_i $(i \in [4])$ be the weighted cluster graphs for colors $1, \dots, 4$, respectively, on the vertex set $\mathcal{V} = \{v_1, \dots, v_m\}$, where the vertex v_i represents the vertex set V_i for all $i \in [m]$. Denote $A_\ell^R = \{v_i \in \mathcal{V} : V_i \subseteq A_\ell\}$ for each $\ell \in [4]$, then $\sum_{\ell=1}^4 |A_\ell^R| = m$. For $\ell \in [4]$, let e_ℓ denote the number of edges $v_i v_j$ that lie in exactly ℓ of the weighted cluster graphs. By a similar argument as **Part (I)**, we obtain the number of potential 4-edge-colorings of G that could give this vertex partition and these weighted cluster graphs is at most

$$\lambda := \binom{n^2/2}{2\xi n^2} 4^{2\xi n^2} (2^{e_2} 3^{e_3} 4^{e_4})^{\left(\frac{n}{m}\right)^2}. \tag{10}$$

Let $\Gamma(x) = -x \log_2 x - (1-x)\log_2(1-x)$ be the entropy function. Similarly, we have

$$\lambda \leqslant 2^{\Gamma(4\xi)\cdot n^2/2} 4^{2\xi n^2} (2^{e_2} 3^{e_3} 4^{e_4})^{n^2/m^2} \leqslant 4^{(\Gamma(4\xi)+4\xi)\cdot n^2/2} (2^{e_2} 3^{e_3} 4^{e_4})^{n^2/m^2}.$$

By a similar argument as Claim 10, we have the following claim.

Claim 13. $e_4 = 0$.

Now consider the graph F on $\{v_1, \dots, v_m\}$ where $v_i v_j$ is an edge of F if it is an edge in exactly 3 of the weighted cluster graphs. Clearly, $e(F) = e_3$.

Claim 14. $e_3 = e(F) \leqslant \sum_{\ell=1}^4 \frac{|A_\ell^R|^2}{4}$.

Proof. For any $v_i v_j \in e(F)$, $v_i v_j$ must lie in $F[A_\ell^R]$ for some $\ell \in [4]$; otherwise, we may assume that $v_i v_j \in F[A_1^R, A_2^R]$ and $v_i \in A_1^R, v_j \in A_2^R$ without loss of generality, it follows from Lemma 9 that $v_i v_j \notin R_\ell$ by noting $\alpha(G_\ell[A_\ell]) \leqslant \gamma n$ from (9) for $\ell \in [2]$, a contradiction since $v_i v_j \in E(F)$.

Clearly, for each $\ell \in [4]$, $F[A_\ell^R]$ is triangle-free. Thus $e(F[A_\ell^R]) \leqslant \frac{|A_\ell^R|^2}{4}$. Therefore, $e(F) = \sum_{\ell=1}^4 e(F[A_\ell^R]) \leqslant \sum_{\ell=1}^4 \frac{|A_\ell^R|^2}{4}$, as desired. □

From a similar proof as Claim 12, we have the following.

Claim 15. *For each* $\ell \in [4]$, $e(R_\ell) \leqslant \frac{(m-|A_\ell^R|)^2}{4}$.

By the definition of e_ℓ and $\sum_{\ell=1}^{4}|A_\ell^R| = m$, we have that

$$\sum_{\ell=1}^{4}\ell\cdot e_\ell = \sum_{\ell=1}^{4}e(R_\ell) \overset{Claim\ 15}{\leqslant} \sum_{\ell=1}^{4}\frac{(m-|A_\ell^R|)^2}{4} = \frac{1}{2}m^2 + \frac{1}{4}\sum_{\ell=1}^{4}|A_\ell^R|^2. \tag{11}$$

We now determine the maximum value of $2^{e_2}3^{e_3}4^{e_4}$ subject to $\sum_{\ell=1}^{4}\ell\cdot e_\ell \leqslant \frac{1}{2}m^2 + \frac{1}{4}\sum_{\ell=1}^{4}|A_\ell^R|^2$ from (11), and $e_4 = 0$ from Claim 13, and $e_3 = e(F) \leqslant \sum_{\ell=1}^{4}\frac{|A_\ell^R|^2}{4}$ from Claim 14, and $\sum_{\ell=1}^{4}|A_\ell^R| = m$. Indeed, the maximum occurs at $e_1 = e_2 = 0$ and $e_3 = \frac{m^2}{4}$. Hence, there are at most $4^{(\Gamma(4\xi)+4\xi)(n^2/2)}3^{n^2/4}$ H-free 4-edge-colorings of G under this vertex partition and the corresponding weighted cluster graphs. Note that M is a constant and there are at most M^n partitions of the vertex set of G into at most M parts, and the number of ways to fix an $\{A_1,\dots,A_4\}$-partition of $V(G)$ at most 4^n. Also, for every such partition there are at most $2^{4(M^2/2)}$ choices for weighted cluster graphs $R_1,\dots,R_4$. Thus for sufficiently large n,

$$RF(n,4,H,\gamma^{11}n) \leqslant 4^n\cdot M^n\cdot 2^{4(M^2/2)}\cdot 4^{(\Gamma(4\xi)+4\xi)(n^2/2)}3^{n^2/4} \leqslant (3^{\frac{1}{2}})^{\binom{n}{2}+\eta n^2},$$

as desired. $\square$

Part (III): $\boldsymbol{RF(n,5,H,o(n)) \leqslant (2^{\frac{1}{4}}3^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)}}$

Denote by $V(H) = h$. We shall prove that for any $\eta > 0$, there exist $\gamma > 0$ and $n_0 > 0$ such that for any $n \geqslant n_0$ the following holds. If G is an n-vertex graph with $\alpha(G) \leqslant \gamma^{23}n$, then the number of 5-edge-colorings of G without a monochromatic H is at most $(2^{\frac{1}{4}}3^{\frac{1}{2}})^{\binom{n}{2}+\eta n^2}$.

For any sufficiently small $\xi > 0$, let δ,ε and M be the constants chosen from Lemma 9 and Lemma 6, respectively. Throughout the proof, we may assume that $0 < 1/n_0 \ll \gamma \ll \delta \ll 1/M \ll \varepsilon \ll \xi \ll \eta < 1$. Let G be an n-vertex graph with $n \geqslant n_0$ and $\alpha(G) \leqslant \gamma^{23}n$. For any fixed 5-edge-coloring of G, $\varphi : E(G) \to [5]$, we apply Lemma 7 with $r = 5$, $c = \gamma$. Let $\{A_1,\dots,A_5\}$ be the partition such that for $r \in [5]$,

$$\alpha(G_r[A_r]) \leqslant \gamma n. \tag{12}$$

We then apply Lemma 6 to graph G with coloring φ to obtain a partition $V(G) = \sqcup_{i=1}^{m}V_i$ which is ε-regular with respect to G_r for every $r \in [5]$, where $M > m \geqslant 1/\varepsilon$. We may assume that $|V_0| = 0$ and $|V_i| = \frac{n}{m}$ for $i \in [m]$ since it does not affect the result. Note that we may assume the regularity partition $\{V_1,\dots,V_m\}$ refines the $\{A_1,\dots,A_5\}$-partition. Let $R_1,\dots,R_5$ be the corresponding weighted cluster graphs on the vertex set $\mathcal{V} = \{v_1,\dots,v_m\}$, where the vertex v_i represents the vertex set V_i for all $i \in [m]$. Denote $A_\ell^R = \{v_i \in \mathcal{V} : V_i \subseteq A_\ell\}$ for each $\ell \in [5]$, then $\sum_{\ell=1}^{5}|A_\ell^R| = m$. For $\ell \in [5]$, let e_ℓ denote the number of edges v_iv_j that lie in exactly ℓ weighted cluster graphs. By a similar argument as **Part (I)**, we obtain the number of potential 5-edge-colorings of G that could give this vertex partition and these weighted cluster graphs is at most

$$\binom{n^2/2}{2\xi n^2}5^{2\xi n^2}(2^{e_2}3^{e_3}4^{e_4}5^{e_5})^{\left(\frac{n}{m}\right)^2} \leqslant 5^{(\Gamma(4\xi)+4\xi)(n^2/2)}(2^{e_2}3^{e_3}4^{e_4}5^{e_5})^{n^2/m^2}. \tag{13}$$

Similar to Claim 10, we have the following claim.

Claim 16. $e_5 = 0$.

Let the graph F be defined on $\{v_1, \dots, v_m\}$ where v_iv_j is an edge of F if it lies in 3 or 4 weighted cluster graphs. Clearly, $e(F) = e_3 + e_4$.

The following fact is clear.

Fact 17. *If we choose three sets of size at least 3 from a 4-element set of colors, then there is a common color in all three.*

Claim 18. $e_3 + e_4 = e(F) \leqslant \frac{m^2}{4}$.

Proof. We only need to show that F is triangle-free. On contrary, we may assume that $\{v_1, v_2, v_3\}$ forms a K_3 without loss of generality. Then, there exist $i, j, k \in [5]$ such that $v_1 \in A_i^R$, $v_2 \in A_j^R$ and $v_3 \in A_k^R$. We have three cases.

Case 1: i, j and k are all distinct.

By symmetry, we may assume that $i = 1$, $j = 2$ and $k = 3$. Then, v_1v_2, v_1v_3, v_2v_3 are in exactly three of the weighted cluster graphs; otherwise, by a similar argument as the proof of Claim 14, there exists a monochromatic weighted clique of size 3, and so G contains a monochromatic H from Lemma 9, a contradiction. Furthermore, $v_1v_2 \in \cap_{\ell\in[5]\setminus\{1,2\}}E(R_\ell)$, $v_1v_3 \in \cap_{\ell\in[5]\setminus\{1,3\}}E(R_\ell)$, and $v_2v_3 \in \cap_{\ell\in[5]\setminus\{2,3\}}E(R_\ell)$. Thus, $(\emptyset, \{v_1, v_2, v_3\})$ forms a weighted clique in R_4 of size 3, and so G contains a H of color 4 from Lemma 9, a contradiction.

Case 2: $i = j \neq k$.

By symmetry, we may assume that $i = j = 1$, $k = 2$. Then, v_1v_3, v_2v_3 are in exactly three of the weighted cluster graphs, and v_1v_2 lies in at least three (3 or 4) weighted cluster graphs. Since $v_1, v_2 \in A_1^R$ and $v_3 \in A_2^R$, $v_1v_2 \notin E(R_1)$ and $v_1v_3, v_2v_3 \in \cap_{\ell\in[5]\setminus\{1,2\}}E(R_\ell)$. Thus, we can use Fact 17 to obtain a monochromatic K_3 in weighted cluster graph, and so G contains a monochromatic H from Lemma 9, a contradiction.

Case 3: $i = j = k$.

By symmetry, we may assume $i = j = k = 1$. Then, v_1v_2, v_1v_3, v_2v_3 lie in at least three (3 or 4) weighted cluster graphs, and $v_1v_2, v_1v_3, v_2v_3 \notin E(R_1)$. Similar to case 2, this will lead to a contradiction.

Therefore, F is triangle-free, and so $e_3 + e_4 = e(F) \leqslant \frac{m^2}{4}$, as desired. □

From a similar proof as Claim 12, we have the following.

Claim 19. *For each $\ell \in [5]$, $e(R_\ell) \leqslant \frac{(m-|A_\ell^R|)^2}{4}$.*

By the definition of e_ℓ and $\sum_{\ell=1}^5 |A_\ell^R| = m$, we have that

$$\sum_{\ell=1}^{5} \ell \cdot e_\ell = \sum_{\ell=1}^{5} e(R_\ell) \overset{\textit{Claim } 19}{\leqslant} \sum_{\ell=1}^{5} \frac{(m-|A_\ell^R|)^2}{4} = \frac{3}{4}m^2 + \frac{1}{4}\sum_{\ell=1}^{5} |A_\ell^R|^2. \tag{14}$$

We now determine the maximum value of $2^{e_2}3^{e_3}4^{e_4}5^{e_5}$ subject to $\sum_{\ell=1}^{5} \ell \cdot e_\ell \leqslant \frac{3}{4}m^2 + \frac{1}{4}\sum_{\ell=1}^{5} |A_\ell^R|^2$ from (14), and $e_5 = 0$ from Claim 16, and $e_3 + e_4 = e(F) \leqslant \frac{m^2}{4}$ from Claim 18, and $\sum_{\ell=1}^{5} |A_\ell^R| = m$. Clearly, we should choose $e_1 = 0$. Setting $x = e_2 + 2e_4$, we only need to maximize $x \log 2 + e_3 \log 3$ subject to $2x + 3e_3 \leqslant m^2$ and $e_3 + e_4 \leqslant \frac{m^2}{4}$. Note that $(\log 3)/3 > (\log 2)/2$, the maximum occurs at $e_3 = \frac{m^2}{4}$, $e_4 = 0$ and $e_2 = \frac{m^2}{8}$. Hence, there are at most $5^{(\Gamma(4\xi)+4\xi)(n^2/2)} 2^{n^2/8} 3^{n^2/4}$ H-free 5-edge-colorings of G under this vertex partition and the corresponding weighted cluster graphs. Note that M is a constant and there are at most M^n partitions of the vertex set of G into at most M parts, and the number of ways to fix an $\{A_1, \ldots, A_5\}$-partition of $V(G)$ at most 5^n. Also, for every such partition there are at most $2^{5(M^2/2)}$ choices for weighted cluster graphs $R_1, \ldots, R_5$. Therefore, for sufficiently large n,

$$RF(n, 5, H, \gamma^{23}n) \leqslant 5^n \cdot M^n \cdot 2^{5(M^2/2)} \cdot 5^{(\Gamma(4\xi)+4\xi)(n^2/2)} 2^{n^2/8} 3^{n^2/4} \leqslant (2^{\frac{1}{4}} 3^{\frac{1}{2}})^{\binom{n}{2} + \eta n^2},$$

as desired. □

4 Proof of Proposition 5

Let us begin with a geometric construction by Erdős and Rogers [13].

Erdős graph (or the **Erdős-Rogers** graph): There are a constant $c > 0$ and n_0 such that for every $n > n_0$ there exists an n-vertex graph G_n satisfying $K_3 \nsubseteq G_n$, and $\alpha(G_n) \leqslant n^{1-c}$.

Remark. There are many constructions on K_3-free n-vertex graphs with independence number $o(n)$. In [1], Alon constructed an n-vertex graph G_n that is K_3-free and $\alpha(G_n) = O(n^{2/3})$. For more constructions, see [3, 19] and the related references therein. Indeed, we can take G_n such that it is K_3-free and $\alpha(G_n) = O(\sqrt{n \log n})$ from the celebrated result of Kim [15]. However, Erdős graph suffices for us.

Recall that $F(G, r, k)$ is the number of the distinct edge-colorings of G with r colors which contains no monochromatic copy of K_k, and

$$F(n, r, k) = \max\{F(G, r, k)|\ G \text{ is a graph on } n \text{ vertices}\}.$$

In 2004, Alon et al. [2] obtained a bound of $F(n, r, 3)$ for all $r \geqslant 6$.

Theorem 20 (Alon, Balogh, Keevash, and Sudakov [2]). *For every fixed $r \geqslant 6$, the following holds.*

$$\left(\frac{r}{2} - 2\sqrt{r \log r}\right)^{(1-\frac{1}{r})(\binom{n}{2}+o(n^2))} \leqslant F(n, r, 3) \leqslant \left(\frac{r}{2}\right)^{\binom{n}{2}+o(n^2)}.$$

For every fixed $r \geqslant 6$, we can obtain a lower bound of $RF(n, r, 3, o(n))$ from Theorem 20. Furthermore, applying a similar idea as the proof of Theorem 3, we can also obtain a non-trivial upper bound of $RF(n, r, 3, o(n))$. We only sketch the proof of Proposition 5 as follows.

Proof sketch of Proposition 5. We apply the construction of [2]. Let H be a graph satisfies the following properties: H is a Turán graph $T_{n,r-1}$ with $(r-1)$-parts $V_1, \dots, V_{r-1}$; for each $p \in [r-1]$, let H_p be a copy of the Turán graph $T_{r-1,2}$ on the set of $r-1$ vertices $R = \{1, \dots, r-1\}$, placed randomly on R; for each fixed pair i, j of distinct members of R, let $S_{ij} = \{p : ij \in E(H_p)\}$ denote the set of all graphs H_p containing the edge ij; all colorings of H in which every edge between V_i and V_j is colored by a color from S_{ij}. From the proof of Theorem 20, such a random coloring on H attains a lower bound of $F(n, r-1, 3)$. Let G be a graph obtained from H by putting a copy of Erdős-Rogers graph in each partite set, and all edges inside each partite set are colored by a new color. Since H contains no monochromatic triangle (in any of the $r-1$ colors) and Erdős-Rogers graph is triangle-free, we have that G contains no monochromatic triangle in any of the r colors. Therefore,

$$RF(n, r, 3, o(n)) \geqslant \left(\frac{r-1}{2} - 2\sqrt{(r-1)\log(r-1)}\right)^{(1-\frac{1}{r-1})(\binom{n}{2}+o(n^2))}.$$

Now we focus on the upper bound. Let G be an extremal graph, $\varphi : E(G) \to [r]$ be a r-edge-coloring with no monochromatic triangle, and let $\{A_1, \dots, A_r\}$ be the partition obtained from Lemma 7 such that $\alpha(G_k[A_k]) = o(n)$ for all $k \in [r]$. Using a similar argument as in Section 3.2, we apply Lemma 6 to G and let $R_1, \dots, R_r$ be the corresponding weighted cluster graphs on the vertex set $\mathcal{V} = \{v_1, \dots, v_m\}$, where the vertex v_i represents the vertex set V_i for all $i \in [m]$. Denote $A_\ell^R = \{v_i \in \mathcal{V} : V_i \subseteq A_\ell\}$ for each $\ell \in [r]$, then $\sum_{\ell=1}^r |A_\ell^R| = m$. For $\ell \in [r]$, let e_ℓ denote the number of pairs (v_i, v_j), $i < j$ that are edges in exactly ℓ of the weighted cluster graphs. Then, the number of the potential r-edge-colorings of G that could give this vertex partition and these weighted cluster graphs is at most

$$\binom{n^2/2}{2\xi n^2} r^{2\xi n^2} \left(\prod_{\ell=2}^r \ell^{e_\ell}\right)^{\left(\frac{n}{m}\right)^2} \leqslant r^{(\Gamma(4\xi)+4\xi)(n^2/2)} \left(\prod_{\ell=2}^r \ell^{e_\ell}\right)^{n^2/m^2}. \tag{15}$$

From a similar proof as Claim 16, Claim 18, and Claim 19, we have $e_r = 0$, $e_{r-2} + e_{r-1} \leqslant \frac{1}{4}m^2$, and $e(R_\ell) \leqslant \frac{(m-|A_\ell^R|)^2}{4}$ for each $\ell \in [r]$.

We now determine the maximum value of $\prod_{\ell=2}^r \ell^{e_\ell}$ subject to $\sum_{\ell=1}^r \ell \cdot e_\ell \leqslant \frac{(r-2)m^2}{4} + \frac{1}{4}\sum_{\ell=1}^r |A_\ell^R|^2$, and $e_r = 0$, and $e_{r-2} + e_{r-1} \leqslant \frac{m^2}{4}$, and $\sum_{\ell=1}^r |A_\ell^R| = m$. Using a similar argument as in Part (II) in Section 3.2, the maximum occurs at $e_3 = \frac{(r-1)m^2}{12}$ and $e_\ell = 0$ for each $\ell \in [r] \setminus \{3\}$ by noting $(\log 3)/3 > (\log 2)/2 = (\log 4)/4 > \cdots > (\log(r-1))/(r-1)$. Hence, there are at most $r^{(\Gamma(4\xi)+4\xi)(n^2/2)} 3^{\frac{(r-1)n^2}{12}}$ triangle-free r-edge-colorings of G under this vertex partition and the corresponding weighted cluster graphs. Note that M is a constant and there are at most M^n partitions of the vertex set of G into at most M parts, and the number of ways to fix an $\{A_1, \dots, A_r\}$-partition of $V(G)$ is at most r^n. Also, for every such partition there are at most $2^{r(M^2/2)}$ choices for weighted cluster graphs $R_1, \ldots, R_r$. Thus for sufficiently large n,

$$RF(n,r,3,o(n)) \leqslant r^n \cdot M^n \cdot 2^{r(M^2/2)} \cdot r^{(\Gamma(4\xi)+4\xi)(n^2/2)} 3^{\frac{(r-1)n^2}{12}} \leqslant (3^{\frac{r-1}{6}})^{\binom{n}{2}+o(n^2)},$$

which together with $RF(n,r,k,o(n)) \leqslant F(n,r,k)$ from (4) and $F(n,r,3) \leqslant \left(\frac{r}{2}\right)^{\binom{n}{2}+o(n^2)}$ from Theorem 20. The proof of Proposition 5 is complete. $\square$

5 Concluding remarks and problems

Pikhurko, Staden and Yilma [23] showed that for every integer $r \geqslant 2$ and $k \geqslant 3$, at least one extremal graph of $F(n,r,k)$ (i.e., n vertices graphs with $F(n,r,k)$ monochromatic K_k-free r-colorings) is complete multipartite. They also made the following conjecture.

Conjecture 21 (Pikhurko, Staden and Yilma [23])**.** For every integer $r \geqslant 2$ and $k \geqslant 3$, every extremal graph of $F(n,r,k)$ is complete multipartite.

It is not easy to construct the extremal graph of $F(n,r,k)$ even for $k=3$. However, similar to the proof of Proposition 5, we can use an extremal graph of $F(n,r,3)$ which is a complete multipartite graph to construct a graph with independence number $o(n)$ and $F(n,r,3)$ $(r+1)$-edge-colorings without a monochromatic copy of K_k, in which each part of the extremal graph is embedded by an Erdős-Rogers graph with the $(r+1)$th color. Together with (4) and Proposition 5, we have

$$F(n,r,3) \leqslant RF(n,r+1,3,o(n)) \leqslant \min\{F(n,r+1,3), (3^{\frac{r}{6}})^{\binom{n}{2}+o(n^2)}\}.$$

Clearly, when $r \geqslant 8$, then $F(n,r+1,3) \leqslant \left(\frac{r+1}{2}\right)^{\binom{n}{2}+o(n^2)} < (3^{\frac{r}{6}})^{\binom{n}{2}+o(n^2)}$.

Note that $F(n,5,3) = (6^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)}$ and $F(n,6,3) = (2^{\frac{3}{4}}3^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)}$ from [6], and $F(n,7,3) = 2^{\frac{7}{4}\binom{n}{2}+o(n^2)}$ from [22]. Together with Proposition 5 we have $(6^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)} \leqslant RF(n,6,3,o(n)) \leqslant (3^{\frac{5}{6}})^{\binom{n}{2}+o(n^2)}$, and $(2^{\frac{3}{4}}3^{\frac{1}{2}})^{\binom{n}{2}+o(n^2)} \leqslant RF(n,7,3,o(n)) \leqslant 3^{\binom{n}{2}+o(n^2)}$ since $(3^{\frac{5}{6}})^{\binom{n}{2}+o(n^2)} < F(n,6,3)$ and $3^{\binom{n}{2}+o(n^2)} < F(n,7,3)$. Note that $6^{\frac{1}{2}} \approx 2.449$ and $3^{\frac{5}{6}} \approx 2.498$, and $2^{\frac{3}{4}}3^{\frac{1}{2}} \approx 2.913$.

Let us conclude with the following problem.

Problem 22. Determine the value of $RF(n,r,3,o(n))$ for $r \geqslant 6$.

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