On Set Representation of Bounded Degree Hypergaphs

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Abstract

In their classical paper, Erdős, Goodman and Pósa studied the representation of a graph with vertex set $[n]$ by a family of subsets $S_1, \dots, S_n$ with the property that $\{i, j\}$ is an edge if and only if $S_i \cap S_j \neq \emptyset$. In this note, we consider a similar representation of bounded degree r-uniform hypergraphs and establish some bounds for a corresponding problem.

Mathematics Subject Classifications: 05C62, 05C65, 05D40

1 Introduction

A set S *represents* an r-uniform hypergraph G if there is a family $(S_v)_{v \in V(G)}$ of subsets of S such that for any $\{v_1, \dots, v_r\} \subseteq V(G)$,

$$\{v_1, \dots, v_r\} \in E(G) \iff \left| \bigcap_{i=1}^{r} S_{v_i} \right| \geqslant 1.$$

One can observe that any r-uniform hypergraph can be represented by a finite set and similar to [6], we define the *representation number* of an r-uniform hypergraph G denoted by $\theta(G)$ as the cardinality of the smallest set S that *represents* G.

The study of representing graphs (the case where $r = 2$) can be traced back to the work of Szpilrajn-Marczewski in [9]. In [6], Erdős, Goodman, and Pósa introduced the parameter $\theta(G)$ for 2-graphs, and proved that $\theta(G) \leqslant \lfloor n^2/4 \rfloor$ for any graph G on n vertices.

For graphs G on n vertices whose complement $\overline{G}$ has bounded maximum degree, i.e., $\Delta(\overline{G}) \leqslant \Delta$, Alon [1] proved that $\theta(G) \leqslant c_1 \Delta^2 \log n$. On the other hand, in [5] it was shown that for every $\Delta \geqslant 1$ there are graphs G on n vertices with $\Delta(\overline{G}) \leqslant \Delta$ such that $\theta(G) \geqslant c_2 \frac{\Delta^2}{\log \Delta} \log n$, showing that the upper bound is sharp up to a factor of $\log \Delta$. In [8], these results were extended to r-uniform hypergraphs.

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The related concept of k-representation where k is a positive integer has been studied by a number of authors (for example see [2–4, 7]). For any integer $k > 0$, a set S *k-represents* an r-uniform graph G if there is a family $(S_v)_{v\in V(G)}$ of subsets of S such that for any $\{v_1, \ldots, v_r\} \subseteq V(G)$,

$$\{v_1, \ldots, v_r\} \in E(G) \iff \left|\bigcap_{i=1}^{r} S_{v_i}\right| \geqslant k.$$

The *k-representation number* of an r-uniform hypergraph G, denoted by $\theta_k(G)$, is the cardinality of the smallest set S that *k-represents* G. Note that for $k = 1$, $\theta_1(G) = \theta(G)$ holds.

It may be natural to ask the following question: given a graph G, what is the smallest cardinality of a set S for which there exists a positive integer k such that S k-represents G? In [5] the authors studied this question by defining the parameter

$$\tilde{\theta}(G) := \min_{k\in\mathbb{N}} \theta_k(G),$$

for 2-graphs. In particular, they proved that $\tilde{\theta}(G) \leqslant c_3\Delta^2 \log n$ for any graph G on n vertices with $\Delta(G) \leqslant \Delta$ and that, on the other hand, there exist graphs on n vertices with $\Delta(G) \leqslant \Delta$ and $\tilde{\theta}(G) \geqslant c_4\Delta \log(\frac{n}{2\Delta})$. Here we consider the parameter $\tilde{\theta}(G)$ where G is a bounded degree r-uniform hypergraph.

For a vertex v in $V(G)$ in an r-uniform graph G, let the degree of v, denoted by $d(v)$, be the number of edges that contain v, and further let $\Delta(G)$ be the maximum degree of G. An r-uniform hypergraph G is linear if the intersection of any two edges has size at most 1. We will prove the following theorems.

Theorem 1 (Upper Bound)**.** *For every $r \geqslant 3$, there exists a constant $C_r > 0$ and integers $\Delta_0 = \Delta_0(r), n_0 = n_0(r)$ such that if G is an r-uniform hypergraph on $n \geqslant n_0$ vertices with $\Delta(G) = \Delta \geqslant \Delta_0$, then*

$$\tilde{\theta}(G) \leqslant C_r\Delta^3 \log n. \tag{1}$$

Further, if G is linear, then

$$\tilde{\theta}(G) \leqslant C_r\Delta^{2+\frac{1}{r-1}} \log n. \tag{2}$$

Theorem 2 (Lower Bound)**.** *For every $r \geqslant 3$, there exists an integer $n_0 = n_0(r)$ such that for every $n \geqslant n_0$ and Δ, there exists an r-uniform hypergraph G on n vertices with $\Delta(G) \leqslant \Delta$ such that,*

$$\tilde{\theta}(G) \geqslant \frac{\Delta}{4} \log n. \tag{3}$$

2 Proof of Upper Bound

To prove Theorem 1, we will first decompose the edges of the r-uniform hypergraph G into matchings $M_1, \ldots, M_L$ for some integer L using Lemma 3. We will represent G by a union of L disjoint subsets $S_1, \ldots, S_L$ and a family $(R_e)_{e \in E(G)}$ such that R_e is a subset of S_i whenever e is in M_i. Lemma 4 asserts the existence of such families. We will then assign to each v in $V(G)$, the set S_v which will be the disjoint union of all R_e such that $v \in e$ and show that this forms a k-representation for some $k \in \mathbb{N}$. This is done in Lemma 5. Given an r-uniform graph G, let $\chi'(G)$ denote the chromatic index of G, defined as the smallest integer L such that $E(G)$ can be decomposed into L matchings.

Lemma 3 (Matching Decomposition)**.** *If G is an r-uniform hypergraph on n vertices and $\Delta(G) \leqslant \Delta$, then $\chi'(G) \leqslant L = \Delta \cdot r$.*

Proof. Let $\mathcal{L}$ be the 2-graph such that $V(\mathcal{L}) = E(G)$ and,

$$E(\mathcal{L}) = \{\{e, f\} \subseteq E(G) : e \neq f \text{ and } e \cap f \neq \emptyset\}.$$

Then for any e in $E(G)$, there are at most $(\Delta - 1) \cdot r$ edges f such that $f \neq e$ and $f \cap e \neq \emptyset$. Thus the maximum degree of $\mathcal{L}$ is $(\Delta - 1)r$ and so $\chi(\mathcal{L}) \leqslant (\Delta - 1)r + 1 \leqslant \Delta r$. For a proper coloring of $\mathcal{L}$, with $L = \Delta r$ colors, each color class is an independent set in $\mathcal{L}$ and thus a matching in G. Thus $E(G)$ can be decomposed into matchings $M_1, \ldots, M_L$, each corresponding to a color class. □

In the following lemma, we will use $x = (a \pm b)$ to denote the inequality, $a - b \leqslant x \leqslant a + b$.

Lemma 4. *Let m, ε, p such that $2 \leqslant m \leqslant r$, $0 < \varepsilon < 1$, $0 \leqslant p \leqslant 1$. There exists an integer $n_0 = n_0(r)$ such that if n and t are integers satisfying $n \geqslant n_0$, and*

$$t \geqslant \frac{3(m+1)\log n}{\varepsilon^2 p^m},$$

then there exists a family of subsets $(R_i)_{i \in [n]}$ of a set S of size t, such that

$$\left| \bigcap_{j \in I} R_j \right| = (1 \pm \varepsilon) p^l t \text{ for every } I \in [n]^{(l)}, \tag{4}$$

whenever $1 \leqslant l \leqslant m$.

Proof. Let $n_0 = n_0(r)$ be an integer. Wherever necessary, we will assume n_0 is large enough. Let n, p, ε, t be as given above. Let S be a set of size t and R_i for $i \in [n]$ be random subsets of S with elements chosen independently, each with probability p. Fix $1 \leqslant l \leqslant m$ and let $J \subseteq [n]^{(l)}$, then

$$\mathbb{E}\left[\left| \bigcap_{j \in J} R_j \right| \right] = p^l t.$$

Since the above random variable has a binomial distribution, we have:

$$\mathbb{P}\left(\left|\bigcap_{j\in J} R_j\right| \neq (1\pm\varepsilon)p^l t\right) < 2\exp\left(-\frac{\varepsilon^2 p^l t}{3}\right)$$
$$\leqslant 2\exp\left(-\frac{m+1}{p^{m-l}}\log n\right)$$
$$< 2n^{-(m+1)}.$$

Thus, the probability that

$$\left|\bigcap_{j\in I} R_j\right| = (1\pm\varepsilon)p^l t \text{ for every } J \subseteq [n]^{(l)},$$

whenever $1 \leqslant l \leqslant m$, is at least

$$1 - \sum_{l=1}^{m}\binom{n}{l}2n^{-(m+1)} > 0,$$

for $n \geqslant n_0$ provided n_0 is large enough. □

Lemma 5. *There exists a constant $A > 0$ such that for every integer $r \geqslant 3$, there are positive integers $n_0 = n_0(r)$, and $L_0 = L_0(r)$, such that for every $n \geqslant n_0$ and $L \geqslant L_0$, if G is an r-uniform graph on n vertices with $\chi'(G) \leqslant L$,*

$$\tilde{\theta}(G) \leqslant AL^3 \log n.$$

Moreover, if G is linear, then

$$\tilde{\theta}(G) \leqslant A(r+1)L^{2+\frac{1}{r-1}}\log n.$$

Proof. Fix $r \geqslant 3$. Let $n_0(r), L_0(r)$ be integers that are assumed to be large enough wherever necessary. Let G be any r-uniform hypergraph on n vertices with $\chi'(G) \leqslant L$. Let $E(G)$ be decomposed into matchings $M_1, \dots, M_L$ with $L \geqslant L_0$, that is

$$E(G) = M_1 \sqcup \cdots \sqcup M_L.$$

In what follows, we will give two separate upper bounds for general r-uniform hypergraphs and linear r-uniform hypergraphs. In each of these cases, we will fix parameters m, p and consider pairwise disjoint subsets $\{S_i : i \in [L]\}$, each of size $t = 12(m+1)p^{-m}\log n$, along with families of subsets $(R_e)_{e\in M_i}$ satisfying Eq. (4). The parameter m will allow us to control the size of m-wise intersections for the families $(R_e)_{e\in M_i}$. When G is any r-uniform hypergraph (not necessarily linear), we will choose $m = 2, p = \frac{1}{4L}$, while when G is a linear r-uniform hypergraph, we choose $m = r, p = (\frac{1}{4L})^{\frac{1}{r-1}}$. However, since the analysis for the two cases follow the same steps, we will prove Proposition 6 for a general parameter

$2 \leqslant m \leqslant r$. We will then use it to prove the bounds, considering the cases when G is a general r-uniform hypergraph (not necessarily linear) and when G is linear.

Construction of Representation: Fix an integer m such that $2 \leqslant m \leqslant r$. Let G be an r-uniform hypergraph with the matching decomposition $E(G) = M_1 \sqcup \cdots \sqcup M_L$. Let $\{S_i\}_{i=1}^{L}$ be a collection of pairwise disjoint sets of size t and for each $i \in [L]$, let $(R_e)_{e \in M_i}$ be a family of subsets of S_i satisfying Eq. (4). For any $v \in V(G)$ and $i \in [L]$, let

$$R(v,i) = \begin{cases} R_e & \text{if there exists an } e \in M_i \text{ such that } v \in e, \\ \emptyset & \text{otherwise.} \end{cases} \tag{5}$$

We construct the representation of G as follows. For every $v \in V(G)$, define

$$S_v := \bigcup_{i=1}^{L} R(v,i). \tag{6}$$

Observe that, for any $\{v_1, \ldots, v_r\} \subseteq V(G)$,

$$\left| \bigcap_{j=1}^{r} S_{v_j} \right| = \sum_{i=1}^{L} \left| \bigcap_{j=1}^{r} R(v_j, i) \right|. \tag{7}$$

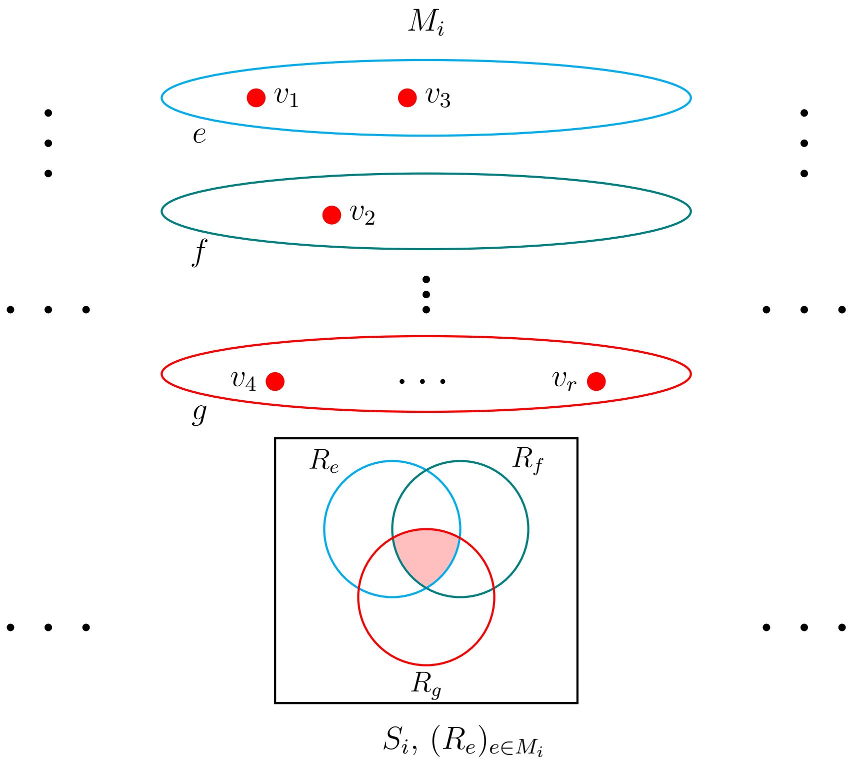


Figure 1: Pairwise disjoint sets S_i with the families $(R_e)_{e \in M_i}$. The shaded area corresponds to $R_e \cap R_f \cap R_g = R(v_1,i) \cap R(v_2,i) \cap \cdots \cap R(v_r,i)$.

If there exists a $k \in \mathbb{N}$ such that

$$\sum_{i=1}^{L} \left| \bigcap_{j=1}^{r} R(v_j, i) \right| \geqslant k \iff \{v_1, \ldots, v_r\} \in E(G), \tag{8}$$

then $S_1 \sqcup \cdots \sqcup S_L$ k-represents G and $\tilde{\theta}(G) \leqslant \theta_k(G) \leqslant |S_1| + \cdots |S_L| = Lt$. We will now find such a k by giving a lower and upper bound on $|R(v_1, i) \cap \cdots \cap R(v_r, i)|$ for each $i \in [L]$ when $\{v_1, \ldots, v_r\}$ is an edge and non-edge respectively.

Bounding the size of intersections $|\boldsymbol{R(v_1, i)} \cap \cdots \cap \boldsymbol{R(v_r, i)}|$: Note that given a fixed $i \in [L]$, the sets $R(v_j, i)$ are not necessarily distinct for distinct j. For example, Fig. 1 depicts the situation when $R(v_1, i) = R(v_3, i) = R_e$. Further, for a fixed $i \in [L]$, since the families $(R_e)_{e \in M_i}$ satisfy Eq. (4), $|R(v_1, i) \cap \cdots \cap R(v_r, i)|$ "shrinks" with the number of distinct $R(v_j, i)$ for $j \in \{1, \ldots, r\}$.
In particular, if $e = \{v_1, \ldots, v_r\}$ is an edge, then it is in some matching M_i, and the sets $R(v_j, i) = R_e$ and the size of the intersection, $|R(v_1, i) \cap \cdots \cap R(v_r, i)|$, is roughly pt. On the other hand, if $\{v_1, \ldots, v_r\}$ is not an edge, then for every matching M_i, there are at least two distinct $R(v_j, i)$ for $j \in \{1, \ldots, r\}$ and $|R(v_1, i) \cap \cdots \cap R(v_r, i)|$ is at most $p^2 t$. Proposition 6 below states a slightly stronger version of this observation. Before we state it, it will be convenient to introduce some notation.
Let $\{v_1, \ldots, v_r\}$ be an r-tuple. Given a matching M_i, let

$$a_i = a_i(\{v_1, \ldots, v_r\}) := \Big| \{ e \in M_i \ : \ e \cap \{v_1, \ldots, v_r\} \neq \emptyset \} \Big|, \tag{9}$$

i.e, a_i is the number of edges in the matching M_i that intersect $\{v_1, \ldots, v_r\}$. Further, let,

$$I_1 = I_1(\{v_1, \ldots, v_r\}) = \{ i \in [L] : \{v_1, \ldots, v_r\} \nsubseteq \cup_{e \in M_i} e \} \text{ and,}$$
$$I_2 = I_2(\{v_1, \ldots, v_r\}) = \{ i \in [L] : \{v_1, \ldots, v_r\} \subseteq \cup_{e \in M_i} e \},$$

i.e., I_1 and I_2 are the sets of those $i \in [L]$ such that the union of the edges in M_i do not and do cover the sets $\{v_1, \ldots, v_r\}$, respectively.

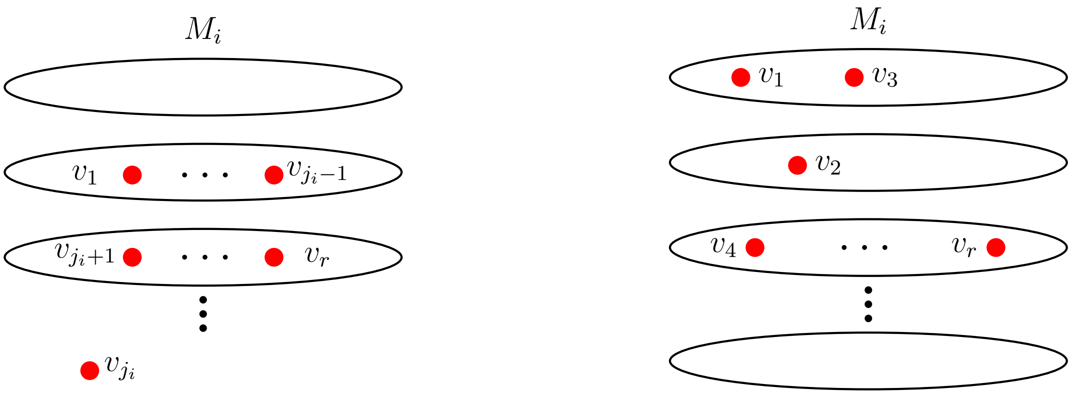
Proposition 6. *For every $i \in [L]$, let $(R_e)_{e \in M_i}$ be a family that satisfies Eq. (4) with a fixed integer m such that $2 \leqslant m \leqslant r$ and for every $v \in V(G)$, let $R(v, i)$ be as given in Eq. (5). Then, for every $\{v_1, \ldots, v_r\} \in E(G)$, there exists $i \in [L]$ such that*

$$\left| \bigcap_{j=1}^{r} R(v_j, i) \right| \geqslant (1 - \varepsilon) pt, \tag{10}$$

and for every $\{v_1, \ldots, v_r\} \notin E(G)$,

- *If $i \in I_1$, then*

$$\left| \bigcap_{j=1}^{r} R(v_j, i) \right| = 0. \tag{11}$$



(a) Case I: $i \in I_1$ (b) Case II: $i \in I_2$

Figure 2

- *If $i \in I_2$, and $a_i \leqslant m$, then*

$$\left| \bigcap_{j=1}^{r} R(v_j, i) \right| \leqslant (1+\varepsilon) p^{a_i} t. \tag{12}$$

Further, $a_i \geqslant 2$.

Proof. If $\{v_1, \ldots, v_r\} \in E(G)$, then there is an $i \in [L]$ such that $\{v_1, \ldots, v_r\} \in M_i$ and hence for all $1 \leqslant j \leqslant r$, $R(v_j, i)$ are identical. Consequently, $a_i = 1$ and

$$\left| \bigcap_{j=1}^{r} R(v_j, i) \right| \geqslant (1-\varepsilon) pt.$$

Next, we fix $\{v_1, \ldots, v_r\} \notin E(G)$, and consider the following cases. Case I (Fig. 2a) considers matchings with isolated vertices, and implies Eq. (11), while Case II (Fig. 2b) considers matchings with no isolated vertices, and implies Eq. (12).

Case I Let $i \in I_1$, i.e., $\{v_1, \ldots, v_r\} \nsubseteq \cup_{e \in M_i} e$. Then there is a v_{j_i} that is not in any edge in M_i (Fig. 2a) and thus $R(v_{j_i}, i) = \emptyset$. Consequently,

$$\left| \bigcap_{j=1}^{r} R(v_j, i) \right| = 0.$$

Case II Let $i \in I_2$, i.e., $\{v_1, \ldots, v_r\} \subseteq \cup_{e \in M_i} e$. Then every $v_j \in \{v_1, \ldots, v_r\}$ is contained in some edge in M_i (Fig. 2b). Since $\{v_1, \ldots, v_r\}$ is not an edge, there are at least two such edges in M_i, and thus $a_i \geqslant 2$. Further, since $(R_e)_{e \in M_i}$ satisfy Eq. (4), whenever $l = a_i \leqslant m$, we have,

$$\left| \bigcap_{j=1}^{r} R(v_j, i) \right| \leqslant (1+\varepsilon) p^{a_i} t. \qquad \square$$

Having described our construction of the representation and computed the bounds on $|R(v_1, i) \cap \cdots \cap R(v_r, i)|$ in Proposition 6, we use Lemma 4 to show that such a construction exists and k-*represents* G for some $k \in \mathbb{N}$.

General Case: First consider the case where G is any r-uniform hypergraph with $E(G) = M_1 \sqcup \cdots \sqcup M_L$. Let $t = \lceil 576L^2 \log n \rceil$, $m = 2$, $p = \frac{1}{4L}$, $\varepsilon = 1/2$ and $k = \lfloor (1-\varepsilon)pt \rfloor$. By Lemma 4, there exists pairwise disjoint sets $\{S_i : i \in [L]\}$, each of size t, and families of subsets $(R_e)_{e \in M_i}$ of S_i, satisfying Eq. (4) with $m = 2$. For every $v \in V(G)$, let S_v be as in Eq. (6), in our construction of the representation.
For every $\{v_1, \dots, v_r\} \in E(G)$, Proposition 6, Eq. (10) implies that,

$$\left| \bigcap_{j=1}^{r} S_{v_j} \right| = \sum_{i=1}^{L} \left| \bigcap_{j=1}^{r} R(v_j, i) \right| \geqslant (1-\varepsilon)pt \geqslant k.$$

On the other hand, since,

$$L(1+\varepsilon)p^2 t = \frac{3}{2} \cdot \frac{1}{16L} t < k.$$

For every $\{v_1, \dots, v_r\} \notin E(G)$, by Eq. (12), we have the upper bound,

$$\left| \bigcap_{j=1}^{r} S_{v_j} \right| = \sum_{i=1}^{L} \left| \bigcap_{j=1}^{r} R(v_j, i) \right| \leqslant \sum_{i=1}^{L} (1+\varepsilon)p^{a_i} t \leqslant L(1+\varepsilon)p^2 t < k.$$

Consequently, G can be k-*represented* by the set $S_1 \cup S_2 \cup \cdots \cup S_L$. This implies that, for $A = 577$, we have

$$\tilde{\theta}(G) \leqslant \theta_k(G) \leqslant Lt \leqslant AL^3 \log n.$$

Linear Case: Let G be a linear r-uniform hypergraph with $E(G) = M_1 \sqcup \cdots \sqcup M_L$.
Let $t = \lceil 384(r+1)L^{\frac{r}{r-1}} \log n \rceil$, $m = r$, $p = \left(\frac{1}{4L}\right)^{\frac{1}{r-1}}$, $\varepsilon = \frac{1}{2}$ and $k = \lfloor (1-\varepsilon)pt \rfloor$. By Lemma 4, there exists pairwise disjoint sets $\{S_i : i \in [L]\}$, each of size t, and families of subsets $(R_e)_{e \in M_i}$ of S_i, satisfying Eq. (4) with $m = r$. For every $v \in V(G)$, let S_v be as in Eq. (6), in our construction of the representation.
For every $\{v_1, \cdots, v_r\} \in E(G)$, in view of Eq. (10),

$$\left| \bigcap_{j=1}^{r} S_{v_j} \right| = \sum_{i=1}^{L} \left| \bigcap_{j=1}^{r} R(v_j, i) \right| \geqslant (1-\varepsilon)pt \geqslant k.$$

Now we consider the case where $\{v_1, \dots, v_r\} \notin E(G)$. Note that, if $i \in I_2 = I_2(\{v_1, \dots, v_r\})$, i.e. the edges of M_i cover $\{v_1, \dots, v_r\}$, then $a_i = r$ if and only if each edge e of M_i satisfies $|e \cap \{v_1, \dots, v_r\}| \leqslant 1$. By linearity of G, there are at most $\binom{r}{2}$ edges that share a pair of vertices with $\{v_1, \dots, v_r\}$ and, consequently, at most $\binom{r}{2}$ matchings M_i with some edge of M_i intersecting $\{v_1, \dots, v_r\}$ in a set of size at least two. Thus, $a_i = r$ for all but at most

$\binom{r}{2}$ matchings in I_2 and, by Proposition 6, $a_i \geqslant 2$ for the remaining matchings. Consequently, for every $\{v_1, \ldots, v_r\} \notin E(G)$,

$$\left|\bigcap_{j=1}^{r} S_{v_j}\right| = \sum_{i=1}^{L} \left|\bigcap_{j=1}^{r} R(v_j, i)\right| < (1+\varepsilon)\left(\left(L - \binom{r}{2}\right) p^r t + \binom{r}{2} p^2 t\right). \tag{13}$$

It remains to show $\left|\cap_{j=1}^{r} S_{v_j}\right| < k$. Indeed, for large enough k and large enough L, the ratio of $\left|\cap_{j=1}^{r} S_{v_j}\right|$ for a non-edge to an edge is,

$$\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \frac{(L - \binom{r}{2})p^r t + \binom{r}{2} p^2 t}{pt} < 3\left(L \cdot \frac{1}{4L} + \frac{\binom{r}{2}}{(4L)^{\frac{1}{r-1}}}\right) < \frac{5}{6}.$$

Thus, for every $\{v_1, \ldots, v_r\} \notin E(G)$, in view of Eq. (13),

$$\left|\bigcap_{j=1}^{r} S_{v_j}\right| = \sum_{i=1}^{L} \left|\bigcap_{j=1}^{r} R(v_j, i)\right| < (1+\varepsilon)\left(\left(L - \binom{r}{2}\right) p^r t + \binom{r}{2} p^2 t\right) < \frac{5}{6}(1-\varepsilon)pt < k.$$

Thus, G can be *k-represented* by the set $S_1 \cup S_2 \cup \cdots \cup S_L$ and for $A = 577$, we have $\tilde{\theta}(G) \leqslant \theta_k(G) \leqslant Lt \leqslant A(r+1)L^{1+\frac{r}{r-1}} \log n = A(r+1)L^{2+\frac{1}{r-1}} \log n$. □

Proof of Theorem 1. Given $r \geqslant 3$ and let $C_r = r^3(r+1)A$, n_0, L_0 be as in Lemma 5. Let $\Delta_0 = \lceil L_0/r \rceil$. For a graph G on $n \geqslant n_0$ vertices with maximum degree $\Delta \geqslant \Delta_0$, by Lemma 3 $\chi'(G) \leqslant L = \Delta r$. Then by Lemma 5,

$$\tilde{\theta}(G) \leqslant A r^3 \Delta^3 \log n < C_r \Delta^3 \log n,$$

and if G is linear,

$$\tilde{\theta}(G) \leqslant A(r+1) r^{2+\frac{1}{r-1}} \Delta^{2+\frac{1}{r-1}} \log n \leqslant C_r \Delta^{2+\frac{1}{r-1}} \log n.$$ □

3 Proof of Lower Bound

The proof of the lower bound extends the approach used in [5] for the case where $r = 2$. Fix $r \geqslant 3$. Whenever necessary, we will assume that n_0 is a large enough integer. Assume that $n \geqslant n_0$. Let $\mathcal{H}^{(r)}(n, \Delta)$ be the collection of r-uniform graphs on the vertex set $[n]$ with bounded degree Δ, and let $\mathcal{M}^{(r)}(n)$ be the collection of all *almost perfect* matchings of r-tuples on $[n]$. Each union of Δ matchings from $\mathcal{M}^{(r)}(n)$ is a graph on $[n]$ with maximum degree $\Delta \leqslant n$, and consequently,

$$|\mathcal{H}^{(r)}(n, \Delta)| \geqslant \binom{|\mathcal{M}^{(r)}(n)|}{\Delta} \geqslant \left(\frac{|\mathcal{M}^{(r)}(n)|}{\Delta}\right)^{\Delta} \geqslant \left(\frac{|\mathcal{M}^{(r)}(n)|}{n}\right)^{\Delta}. \tag{14}$$

Claim 7. *For $r \geqslant 3$ and $n \geqslant n_0$,*

$$|\mathcal{M}^{(r)}(n)| \geqslant \left(\frac{n}{er}\right)^{n/2}$$

Proof. Let $n = qr + s$ where $0 \leqslant s < r$ and $q = \lfloor n/r \rfloor$. We have that $|\mathcal{M}^{(r)}(n)|$ is at least

$$\frac{1}{q!}\binom{n}{r}\binom{n-r}{r}\cdots\binom{r+s}{r} = \frac{1}{q!}\frac{n!}{(r!)^q s!} \geqslant \frac{n!}{(n/r)!(r^r)^{n/r}r!}.$$

We use that $n! \geqslant \sqrt{2\pi n}(n/e)^n$ and consequently, we have, for $n \geqslant n_0(r)$

$$\frac{n!}{(n/r)!(r^r)^{n/r}r!} \geqslant \left(\frac{n}{er}\right)^n \frac{\sqrt{2\pi n}}{r^r}\frac{1}{(n/r)!} \geqslant \left(\frac{n}{er}\right)^n \cdot \frac{1}{(n/r)!} \geqslant \left(\frac{n}{er}\right)^{n/2}.$$

□

Proof of Theorem 2. Given any integer t, there are at most $(2^t)^n$ distinct r-uniform hypergraphs on the vertex set $[n]$ that can be k-represented on the set $[t]$. Consequently, if t is such that $|\mathcal{H}^{(r)}(n,\Delta)| > 2^{tn}$, then there must exist some $G \in \mathcal{H}^{(r)}(n,\Delta)$ that cannot be k-represented by a set of size t for any k, and hence $\tilde{\theta}(G) > t$.
In view of Eq. (14) and Claim 7,

$$\begin{aligned}
\log|\mathcal{H}^{(r)}(n,\Delta)| > \Delta \log\left(\frac{|\mathcal{M}^{(r)}(n)|}{n}\right) &\geqslant \Delta \log\left(\frac{\left(\frac{n}{er}\right)^{n/2}}{n}\right) \\
&= \Delta \cdot \frac{n}{2}\left(\log\left(\frac{n}{n^{2/n}}\right) - \log(er)\right) \\
&\geqslant \Delta \cdot \frac{n}{4}\log n,
\end{aligned}$$

for large enough $n > n_0(r)$. Consequently, for $t = \frac{1}{4}\Delta \log n$, we have that,

$$|\mathcal{H}^{(r)}(n,\Delta)| > 2^{tn}.$$

□

4 Concluding Remarks

In this note we established upper and lower bounds on $\tilde{\theta}(G)$ that differ by a factor of $O(\Delta^2)$, i.e. $\Omega(\Delta \log n) \leqslant \tilde{\theta}(G) \leqslant O(\Delta^3 \log n)$. Closing the gap between these bounds is a problem of interest. Further, since the lower bound in Theorem 2 is nonconstructive it would be interesting to find an explicit construction that matches or improves our lower bound.

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