Generalized Crowns in Linear r-Graphs

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Abstract

An r-graph H is a hypergraph consisting of a nonempty set of vertices V and a collection of r-element subsets of V we refer to as the edges of H. An r-graph H is called linear if any two edges of H intersect in at most one vertex. Let F and H be two linear r-graphs. If H contains no copy of F, then H is called F-free. The linear Turán number of F, denoted by $\mathrm{ex}_r^{\mathrm{lin}}(n, F)$, is the maximum number of edges in any F-free n-vertex linear r-graph. The crown $C_{1,3}$ (or E_4) is a linear 3-graph which is obtained from three pairwise disjoint edges by adding one edge that intersects all three of them in one vertex. In 2022, Gyárfás, Ruszinkó and Sárközy initiated the study of $\mathrm{ex}_3^{\mathrm{lin}}(n, F)$ for different choices of an acyclic 3-graph F. They determined the linear Turán numbers for all acyclic linear 3-graphs with at most 4 edges, except the crown. They established lower and upper bounds for $\mathrm{ex}_3^{\mathrm{lin}}(n, C_{1,3})$. In fact, their lower bound on $\mathrm{ex}_3^{\mathrm{lin}}(n, C_{1,3})$ is essentially tight, as was shown in a recent paper by Tang, Wu, Zhang and Zheng. In this paper, we generalize the notion of a crown to linear r-graphs for $r \geqslant 3$, and also generalize the above results to linear r-graphs.

Mathematics Subject Classifications: 05C35, 05C65

1 Introduction

The result presented here is motivated by a number of very recent papers on linear Turán numbers. We extend a result on crown-free linear 3-graphs to linear r-graphs for $r \geqslant 3$. Throughout, we let r be an integer with $r \geqslant 3$.

Let $H = (V, E)$ be an r-graph consisting of a set $V = V(H)$ of vertices and a collection $E = E(H)$ of r-element subsets of V called edges. If any two edges in H intersect in at most one vertex, then H is said to be linear. Let F be a linear r-graph. Then H is called F-free if it contains no copy of F as its subhypergraph. The linear Turán number of F, denoted by $\mathrm{ex}_r^{\mathrm{lin}}(n, F)$, is the maximum number of edges in any F-free linear r-graph on n vertices. More generally, for two linear r-graphs F_1 and F_2, H is called $\{F_1, F_2\}$-free if it

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contains no copy of F_1 or F_2 as its subhypergraph. The linear Turán number of $\{F_1, F_2\}$, denoted by $\mathrm{ex}_r^{\mathrm{lin}}(n, \{F_1, F_2\})$, is the maximum number of edges in any $\{F_1, F_2\}$-free linear r-graph on n vertices.

A linear 3-graph is acyclic if it can be constructed in the following way. We start with one edge. Then at each step we add a new edge intersecting the union of the vertices of the previous edges in at most one vertex. In 2022, Gyárfás, Ruszinkó and Sárközy [5] initiated the study of $\mathrm{ex}_3^{\mathrm{lin}}(n, F)$ for different choices of an acyclic 3-graph F. In [5], they determined the linear Turán numbers of acyclic linear 3-graphs with at most 4 edges, except the crown, for which they gave lower and upper bounds (Theorem 1 below). Here the crown is a linear 3-graph which is obtained from three pairwise disjoint edges on 9 vertices by adding one edge that intersects all three of them in one vertex. In [5], the authors used E_4 to denote a crown, but here we adopt the more intuitive notation $C_{1,3}$ for a crown.

Since the publication of [5], there have appeared several results involving the linear Turán number of some acyclic linear hypergraphs [6, 7, 8]. In the remainder, we focus on results involving $C_{1,3}$, as our aim is to present a natural generalization of these results to linear r-graphs.

In [5], Gyárfás, Ruszinkó and Sárközy obtained the following result.

Theorem 1 ([5])**.**

$$6\left\lfloor \frac{n-3}{4} \right\rfloor + \varepsilon \leqslant \mathrm{ex}_3^{\mathrm{lin}}(n, C_{1,3}) \leqslant 2n,$$

where $\varepsilon = 0$ *if* $n-3 \equiv 0, 1 \pmod 4$, $\varepsilon = 1$ *if* $n-3 \equiv 2 \pmod 4$, *and* $\varepsilon = 3$ *if* $n-3 \equiv 3 \pmod 4$.

Indeed, for the lower bound in Theorem 1, the authors of [5] gave the following construction for obtaining a class of extremal linear $C_{1,3}$-free 3-graphs. We recall this construction for later reference. Start with the graph mK_4 consisting of m disjoint copies of the complete graph on four vertices. The graph mK_4 admits a one-factorization, i.e., a decomposition of the edge set into three edge-disjoint perfect matchings. Each of these matchings corresponds to $2m$ vertex-disjoint pairs of vertices. Add one new vertex for each of the matchings and form $2m$ triples by adding this vertex to each of the $2m$ pairs. Now ignore the edges of the mK_4. This construction consists of $n = 4m+3$ vertices and $6m$ triples, and it is easy to check that the corresponding 3-graph is linear and $C_{1,3}$-free. Thus for $n = 4m + 3$, this construction provides an extremal 3-graph with $6\left\lfloor \frac{n-3}{4} \right\rfloor + \varepsilon$ edges, where ε is defined as in the above theorem. The construction can be adjusted to obtain extremal 3-graphs for the other residue classes modulo 4.

In a later paper [2], Carbonero, Fletcher, Guo, Gyárfás, Wang, and Yan proved that every linear 3-graph with minimum degree 4 contains a crown. The same group of authors conjectured in [1] that $\mathrm{ex}_3^{\mathrm{lin}}(n, C_{1,3}) \sim \frac{3n}{2}$, and proposed some ideas to obtain the exact bounds. After that, Fletcher [4] improved the upper bound to $\mathrm{ex}_3^{\mathrm{lin}}(n, C_{1,3}) \leqslant \frac{5n}{3}$.

Very recently, Tang, Wu, Zhang and Zheng [9] established the following result.

Theorem 2 ([9])**.** *Let G be any $C_{1,3}$-free linear 3-graph on n vertices. Then* $|E(G)| \leqslant \frac{3(n-s)}{2}$, *where s denotes the number of vertices in G with degree at least 6.*

The above result shows that the lower bound in Theorem 1 is essentially tight. Furthermore, the above result, combined with the results in [5], essentially completes the determination of the linear Turán numbers for all acyclic linear 3-graphs with at most 4 edges.

2 Crown-free linear r-graphs

In the remainder, we focus on the following natural generalization of the notion of a crown to linear r-graphs. An r-crown $C_{1,r}$ is a linear r-graph on r^2 vertices and $r+1$ edges obtained from r pairwise disjoint edges by adding one edge that intersects all of them in one vertex. In fact, for our purposes we need a second generalization of the crown to linear r-graphs. We let $C^*_{1,r}$ denote the following linear r-graph on r^2-r+3 vertices and $r+1$ edges. It consists of a set of $r-2$ edges $\{e_1, e_2, \dots, e_{r-2}\}$ that intersect in exactly one vertex v, two additional disjoint edges e_{r-1} and e_r that are also disjoint from $\{e_1, e_2, \dots, e_{r-2}\}$, and one additional edge e intersecting each edge of $\{e_1, e_2, \dots, e_r\}$ in exactly one vertex except for v. Note that both $C_{1,r}$ and $C^*_{1,r}$ are isomorphic to the crown in case $r=3$.

By considering n-vertex $\{C_{1,r}, C^*_{1,r}\}$-free linear r-graphs, we obtain the following generalization of Theorem 2.

Theorem 3. *Fix the integer r such that $r \geqslant 3$. Let G be any $\{C_{1,r}, C^*_{1,r}\}$-free linear r-graph on n vertices, and let s denote the number of vertices with degree at least $(r-1)^2+2$. Then*

$$r(r-1)\left\lfloor \frac{n-r}{(r-1)^2} \right\rfloor \leqslant |E(G)| \leqslant \frac{r(r-2)(n-s)}{r-1}.$$

We postpone the proof for the upper bound to the next section. For the lower bound in Theorem 3, we can use a similar construction as in the description following Theorem 1. We can construct a $\{C_{1,r}, C^*_{1,r}\}$-free linear r-graph on n vertices by using the notion of a transversal design.

Assume that n is a multiple of k for some integer $k \geqslant r-1$. A transversal design $T(n,k)$ is a linear k-graph on n vertices, in which the vertices are partitioned into k sets, each containing $\frac{n}{k}$ vertices, and where each pair of vertices from different sets belongs to exactly one edge on k vertices. Note that $T(n,k)$ is an $\frac{n}{k}$-regular k-partite linear k-graph. It was shown in [3] that $T(k^2,k)$ exists for each $k \geqslant 2$. We use $V_1, V_2, \dots, V_k$ to denote the k vertex sets of the partition, each containing k vertices. We denote these sets by $V_i = \{a_{1,i}, a_{2,i}, \dots, a_{k,i}\}$ for each $1 \leqslant i \leqslant k$. Recall that a perfect matching in a hypergraph $\mathcal{H}$ is a set of disjoint edges whose vertex set union is exactly $V(\mathcal{H})$. There exist k disjoint perfect matchings in $T(k^2,k)$. Fix the integer t such that $1 \leqslant t \leqslant k-1$. For each fixed t,

$$\cup_{i=1}^{k}\{a_{1,i}, a_{2,i+t}, a_{3,i+2t}, \dots, a_{k,i+(k-1)t}\}$$

is a perfect matching, where the second subscript is taken modulo k. This accounts for $k-1$ perfect matchings. The final perfect matching is

$$\cup_{i=1}^{k}\{a_{i,1}, a_{i,2}, a_{i,3}, \dots, a_{i,k}\}.$$

Denote by $T'((r-1)^2, r-1)$ the linear $(r-1)$-graph obtained from $T((r-1)^2, r-1)$ by adding one edge for each set in the partition. Note that for $r = 3$, $T'((r-1)^2, r-1)$ is a K_4. Note that there are r disjoint perfect matchings in $T'((r-1)^2, r-1)$. We next extend m disjoint copies of $T'((r-1)^2, r-1)$ to a $\{C_{1,r}, C^*_{1,r}\}$-free linear r-graph in the same way as we did for $r = 3$ starting with mK_4. Consider a one-factorization of the linear $(r-1)$-graph $mT'((r-1)^2, r-1)$. Each of the r factors (perfect matchings) corresponds to $(r-1)m$ vertex-disjoint $(r-1)$-tuples. Add one new vertex for each of the factors and form $(r-1)m$ edges by adding this vertex to each of the $(r-1)m$ $(r-1)$-tuples. The resulting linear r-graph has $r(r-1)m$ edges and $(r-1)^2m + r$ vertices, and it is $\{C_{1,r}, C^*_{1,r}\}$-free. Let $n = (r-1)^2m + r$. Then the number of edges of the constructed r-graph is at least $r(r-1)\lfloor \frac{n-r}{(r-1)^2} \rfloor$. This proves the lower bound in Theorem 3.

The remainder of this paper is structured in the following way. We present our proof for the upper bound in Theorem 3 in the next section. In the final section, we complete the paper with a short discussion.

3 Proof for the upper bound in Theorem 3

Before we present our proof, we need some additional notation, and we prove a key lemma. Let H be a linear r-graph, let $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_r$ be positive integers, and let $e \in E(H)$. Then we use $D(e) \geqslant \{d_1, d_2, \ldots, d_r\}$ to denote that e can be written as $e = \{u_1, u_2, \ldots, u_r\}$ such that $d(u_i) \geqslant d_i$ for each $i \in [r] = \{1, 2, \ldots, r\}$. Here $d(v)$ denotes the degree of the vertex v, i.e., the number of edges containing the vertex v. We use the shorthand v-edge for an edge containing the vertex v.

Lemma 4. *Fix the integer r such that $r \geqslant 4$. Let G be a $\{C_{1,r}, C^*_{1,r}\}$-free linear r-graph, and let $e \in E(G)$ be such that $D(e) \geqslant \{(r-1)^2+1, (r-1)^2+1, (r-1)^2, \ldots, (r-1)^2\}$. Then*

$$S = \bigcup_{f \in E(G), f \cap e \neq \emptyset} f$$

contains exactly $(r-1)^3 + r$ vertices, and all vertices in S have degree at most $(r-1)^2 + 1$. Moreover,

$$E_S = \{f : f \in E(G), f \cap S \neq \emptyset\}$$

contains at most $r(r-1)^2 + 1$ edges.

Proof. Without loss of generality, suppose $e = \{u_1, u_2, \ldots, u_r\}$ with $d(u_1) \geqslant d(u_2) \geqslant (r-1)^2 + 1$ and $d(u_i) \geqslant (r-1)^2$ for each $3 \leqslant i \leqslant r$. If $d(u_1) \geqslant (r-1)^2 + 2$, we can find a copy of $C_{1,r}$ in the following way. We start with the edge $e = \{u_1, u_2, \ldots, u_r\}$. We can find a u_r-edge $e_1 \neq e$ since $d(u_r) \geqslant (r-1)^2$. By considering i from $r-1$ to 2 one by one, we can find a u_i-edge e_{r-i+1} that does not share a vertex with any edge in $\{e_1, e_2, \ldots, e_{r-i}\}$. Finally, we can choose a u_1-edge e_r that does not share a vertex with $e_1, e_2, \ldots, e_{r-1}$. Hence, we have found a copy of $C_{1,r}$, a contradiction.

Therefore, we have $d(u_1) = d(u_2) = (r-1)^2 + 1$. For $p \in \{u_1, u_2, \ldots, u_r\}$, we use $G(p)$ to denote the set of all vertices outside e that lie on a common edge with p. We first prove the following claim.

Claim 5. $G(u_1) = G(u_2)$.

Proof. Suppose to the contrary that there exists a u_2-edge $e_1 \neq e$ containing some vertex in $V(G) \setminus G(u_1)$. Then there are at most $r-2$ u_1-edges other than e intersecting e_1, so there are at least $(r-2)(r-1)+1$ u_1-edges that are disjoint from e_1. By the degree condition that $d(u_i) \geqslant (r-1)^2$ for each $3 \leqslant i \leqslant r$, we can choose a u_i-edge e_{i-1} for each $3 \leqslant i \leqslant r$ such that e_{i-1} is disjoint from $\{e_1, e_2, \ldots, e_{i-2}\}$, and then choose a u_1-edge e_r that is disjoint from $\{e_1, e_2, \ldots, e_{r-1}\}$. So, in that case $\{e, e_1, e_2, \ldots, e_r\}$ forms a $C_{1,r}$, a contradiction. □

Similarly, we must have $G(u_i) \subset G(u_2)$ for each $3 \leqslant i \leqslant r$. Suppose to the contrary that there exists some $3 \leqslant i \leqslant r$ such that there is a u_i-edge $e_i \neq e$ containing some vertex not in $G(u_2)$. Then there are at most $r-2$ u_2-edges other than e intersecting e_i, so there are at least $(r-2)(r-1)+1$ u_2-edges that are disjoint from e_i. By the degree conditions that $d(u_1) \geqslant (r-1)^2+1$ and $d(u_s) \geqslant (r-1)^2$ for each $3 \leqslant s \leqslant r$, for each s satisfying the conditions $1 \leqslant s \leqslant r, s \neq 2$ and $s \neq i$ we can choose a u_s-edge e_s that is disjoint from $\{e_1, e_3, \ldots, e_{s-1}\}$, and then choose a u_2-edge e_2 that is disjoint from $\{e_1, e_3, \ldots, e_r\}$. So $\{e, e_1, e_2, \ldots, e_r\}$ forms a $C_{1,r}$, a contradiction.

Thus $V(S) \setminus \{u_1, u_2, \ldots, u_r\} = G(u_1) = G(u_2) \supset G(u_i)$ for each $3 \leqslant i \leqslant r$. Denote by F the edge set each edge of which is disjoint from $\{u_1, u_2, \ldots, u_r\}$ and contains at least one vertex of S. If F is empty, then S contains $(r-1)^3+r$ vertices and all vertices in S have degree at most $(r-1)^2+1$ since $G(u_1) = G(u_2) \supset G(u_i)$ for each $3 \leqslant i \leqslant r$. Moreover, E_S contains at most $r(r-1)^2+1$ edges. Hence, it suffices to show that F is empty.

For this purpose, we first construct $r-1$ auxiliary bipartite graphs as follows. Fix an h with $2 \leqslant h \leqslant r$, and let $H_h = (V(H_h) = X_{H_h} \cup Y_{H_h}, E_{H_h})$, where $X_{H_h} = \{f | u_h \in f, f \in E(G), f \neq e\}$, $Y_{H_h} = \{g | u_1 \in g, g \in E(G), g \neq e\}$ and $E_{H_h} = \{\{f, g\} | f \cap g \neq \emptyset\}$. Then H_2 is an $(r-1)$-regular bipartite graph with partition classes of exactly $(r-1)^2$ vertices. For $3 \leqslant h \leqslant r$, H_h is a bipartite graph with one class of exactly $(r-1)^2$ vertices and the other class having at least $(r-1)^2-1$ vertices. Next, we prove two claims on the structure of these bipartite graphs.

Claim 6. *If G is $C_{1,r}$-free, then H_2 must contain a $K_{r-1,r-1}$.*

Proof. By the degree conditions that $d(u_h) \geqslant (r-1)^2$ for each $3 \leqslant h \leqslant r$, we can choose $r-2$ vertex-disjoint edges $e_3, e_4, \ldots, e_r$ satisfying $e_h \neq e$ is a u_h-edge for each $3 \leqslant h \leqslant r$.

Define

$$V_1 = \bigcup_{3 \leqslant h \leqslant r} (V(e_h) \setminus V(e)), W_1 = \{f | f \cap V_1 \neq \emptyset, u_1 \in f\} \text{ and } W_1' = \{g | g \cap V_1 \neq \emptyset, u_2 \in g\}.$$

Then we have $|V_1| = (r-2)(r-1)$, $W_1 \subset Y_{H_2}$ and $W_1' \subset X_{H_2}$. Therefore $|W_1| \leqslant (r-2)(r-1)$ and $|W_2| \leqslant (r-2)(r-1)$. Since $d(u_1) = d(u_2) = (r-1)^2+1$, we have that $|Y_{H_2} \setminus W_1| \geqslant r-1$ and $|X_{H_2} \setminus W_1'| \geqslant r-1$. If there exist a u_1-edge f and a u_2-edge g such that $f \notin W_1$, $g \notin W_1'$ and $f \cap g = \emptyset$, then $e, f, g, e_3, e_4, \ldots, e_r$ forms a $C_{1,r}$, a contradiction. That means there exist two non-adjacent vertices f and g in the graph $H_2 - \{W_1 \cup W_1'\}$.

Hence, if there is no $C_{1,r}$ in G, $H_2 - \{W_1 \cup W_1'\}$ has to be a complete bipartite graph. Since $|Y_{H_2} \setminus W_1| \geqslant r-1$ and $|X_{H_2} \setminus W_1'| \geqslant r-1$, there is a $K_{r-1,r-1}$ in $H_2 - \{W_1 \cup W_1'\}$. Thus H_2 contains a $K_{r-1,r-1}$. □

Claim 7. *If G is $C_{1,r}$-free, then H_h must contain a $K_{r-2,r-1}$ for each $2 \leqslant h \leqslant r$. Furthermore, the partition classes on $r-1$ vertices in these $K_{r-2,r-1}$'s are mutually disjoint.*

Proof. As for the first statement, we already proved it if $h = 2$ by proving there must exist a $K_{r-1,r-1}$ in H_2. Next we will prove it for $3 \leqslant h \leqslant r$.

By Claim 6, we can choose a vertex $e_2 \in V(K_{r-1,r-1}) \subset V(H_2)$ which is also a u_2-edge. Since $d(u_i) \geqslant (r-1)^2$ for each $3 \leqslant i \leqslant r$, we can choose $r-3$ vertex-disjoint edges $e_4, \ldots, e_r$ satisfying $e_t \neq e$ is a u_t-edge and e_t is also disjoint from e_2 for each $4 \leqslant t \leqslant r$.

Define

$$V_2 = \left(\bigcup_{4 \leqslant t \leqslant r} (V(e_t) \setminus V(e)) \right) \cup (V(e_2) \setminus V(e)), W_2 = \{f | f \cap V_2 \neq \emptyset, u_1 \in f\}$$

and

$$W_2' = \{g | g \cap V_2 \neq \emptyset, u_3 \in g\}.$$

Then we have $|V_2| = (r-2)(r-1)$, $W_2 \subset Y_{H_3}$ and $W_2' \subset X_{H_3}$. Therefore $|W_2| \leqslant (r-2)(r-1)$ and $|W_2'| \leqslant (r-2)(r-1)$. Since $d(u_1) = (r-1)^2 + 1$ and $d(u_3) \geqslant (r-1)^2$, we have that $|Y_{H_3} \setminus W_2| \geqslant r-1$ and $|X_{H_3} \setminus W_2'| \geqslant r-2$. If there exist a u_1-edge f and a u_3-edge g such that $f \notin W_2$, $g \notin W_2'$ and $f \cap g = \emptyset$, then $e, f, e_2, g, e_4, e_5, \ldots, e_r$ forms a $C_{1,r}$, a contradiction. That means there exist two non-adjacent vertices f and g in the graph $H_3 - \{W_2 \cup W_2'\}$. Hence, if there is no $C_{1,r}$ in G, $H_3 - \{W_2 \cup W_2'\}$ has to be a complete bipartite graph. Since $|Y_{H_3} \setminus W_2| \geqslant r-1$ and $|X_{H_3} \setminus W_2'| \geqslant r-2$, there is a $K_{r-2,r-1}$ in $H_3 - \{W_2 \cup W_2'\}$. Thus H_3 contains a $K_{r-2,r-1}$.

Note that the $K_{r-2,r-1}$ in H_3 is disjoint from the $K_{r-1,r-1}$ in H_2. And the partition class on $r-1$ vertices in $K_{r-2,r-1}$ consists of u_1-edges. Through a similar process, we can find a $K_{r-2,r-1}$ in H_h for $4 \leqslant h \leqslant r$ such that all of these $K_{r-2,r-1}$'s are pairwise disjoint, all of these $K_{r-2,r-1}$'s are disjoint from the $K_{r-1,r-1}$ in H_2, and the partition class on $r-1$ vertices in the $K_{r-2,r-1}$ consists of u_1-edges for each H_h. □

Let $\{e_1, e_2, \ldots, e_{(r-1)^2}\}$ denote the ordered sequence of all u_1-edges except for e. By Claim 7, we can assume that the $K_{r-2,r-1}$ in H_h contains the $(h-1)$-th $r-1$ u_1-edges of this sequence as one partition class for $2 \leqslant h \leqslant r$. That means one partition class of the $K_{r-2,r-1}$ in H_h is $\{e_{(h-2)(r-1)+1}, e_{(h-2)(r-1)+2}, \ldots, e_{(h-1)(r-1)}\}$ for each $2 \leqslant h \leqslant r$. Denote by U_{h-1} the set of vertices in the $(h-1)$-th $r-1$ u_1-edges of the sequence for $2 \leqslant h \leqslant r$, that is, $U_{h-1} = \cup_{i=1}^{r-1} V(e_{(h-2)(r-1)+i})$. We prove another claim.

Claim 8. *Fix $2 \leqslant i \leqslant r$. Each u_i-edge contains only vertices of one vertex set from $\{U_1, U_2, \ldots, U_{r-1}\}$.*

Proof. By Claim 7 and the above analysis, there must be $r-1$ u_2-edges whose vertices except for u_2 are in U_1, and at least $r-2$ u_h-edges whose vertices except for u_h are in U_{h-1}

for $3 \leqslant h \leqslant r$. Suppose to the contrary that for some $2 \leqslant h \leqslant r$, there exists a u_h-edge f such that $1 \leqslant |f \cap U_i| \leqslant r-2$, $1 \leqslant |f \cap U_j| \leqslant r-2$, and $1 \leqslant i \neq j \leqslant r-1$. We first deal with the case that $|f \cap U_i| = r-2$. Then $|f \cap U_j| = 1$. Let $\{v\} = f \cap U_j$. If f intersects each u_{i+1}-edge in exactly one vertex among U_i, then we can find a $C^*_{1,r}$ in G as follows. At first, we can choose $r-2$ u_{i+1}-edges $\{f_1, f_2, \ldots, f_{r-2}\}$ whose vertices except for u_{i+1} are in U_i. Denote by e' the u_1-edge containing the vertex v. Since $d(u_h) \geqslant (r-1)^2$ and $r \geqslant 4$, we can choose one u_h-edge $f' \neq f$ such that f' is disjoint from $e' \cup \{f_1, f_2, \ldots, f_{r-2}\}$. Note that e' and f' are disjoint, and that they both are disjoint from $\{f_1, f_2, \ldots, f_{r-2}\}$. Also note that f intersects each edge of $\{e', f', f_1, f_2, \ldots, f_{r-2}\}$ in exactly one vertex except for u_{i+1}. Then by the definition of $C^*_{1,r}$ the edges $f, e', f', f_1, f_2, \ldots, f_{r-2}$ form a $C^*_{1,r}$, a contradiction.

Assume that among U_i there exists a u_{i+1}-edge f_1 which is disjoint from f. If $h = i+1$, then there are at least $r-2$ u_h-edges whose vertices except for u_h are in U_i. Without loss of generality, we assume that $r-2$ u_1-edges intersecting f are $e_{(h-2)(r-1)+1}, e_{(h-2)(r-1)+2}, \ldots, e_{(h-1)(r-1)-1}$. We can choose a u_{j+1}-edge g whose vertices except u_{j+1} are in U_j and $v \notin g$. For any t with $t \neq i, j$ and $1 \leqslant t \leqslant r-1$, we choose a u_{t+1}-edge g_t whose vertices except u_{t+1} are in U_t. Hence, $e, e_{(h-1)(r-1)}, g, f$ and $\cup_{1 \leqslant t \neq i,j \leqslant r-1} g_t$ forms a $C_{1,r}$, a contradiction. Next we assume that $h \neq i+1$. Then we can find a $C_{1,r}$ in G as follows. We can find a u_s-edge whose vertices except for u_s are in U_{s-1}, for all s with $2 \leqslant s \leqslant r$ and $s \neq i+1, h$. Then we choose one u_1-edge f_2 whose vertices except for u_1 are in U_{h-1}. Then the edges f, f_1, f_2 together with the $r-2$ u_s-edges form a $C_{1,r}$, a contradiction.

The remaining case is $1 \leqslant |f \cap U_i| \leqslant r-3$ and $1 \leqslant |f \cap U_j| \leqslant r-3$. We can find a u_s-edge whose vertices except for u_s are in U_{s-1} for all s with $2 \leqslant s \leqslant r$ and $s \neq h$. Then we have $r-2$ disjoint edges $f_3, \ldots, f_r$ which are disjoint from f. We choose one u_1-edge f_1 whose vertices except for u_1 are in U_{h-1}. All these edges $e, f, f_1, f_3, f_4, \ldots, f_r$ form a $C_{1,r}$, a contradiction. □

Before we continue with the proof of Lemma 4, we note that the above analysis implies the following about the structure of H_i.

Remark 9. H_2 is the disjoint union of $r-1$ complete bipartite graphs $K_{r-1,r-1}$. Since $d(u_h) \geqslant (r-1)^2$ for each $3 \leqslant h \leqslant r$, H_h is either the disjoint union of $r-1$ complete bipartite graphs $K_{r-1,r-1}$ or the disjoint union of $r-2$ complete bipartite graphs $K_{r-1,r-1}$ and one complete bipartite graph $K_{r-2,r-1}$.

Proof of Remark 9. By the degree condition that $d(u_2) = (r-1)^2 + 1$, together with Claims 5 and 8 and the linearity of H, we have that H_2 is the disjoint union of $r-1$ complete bipartite graphs $K_{r-1,r-1}$. Next we consider the case for each h with $3 \leqslant h \leqslant r$. By the degree condition that $d(u_h) \geqslant (r-1)^2$, together with Claim 8, the argument that $G(u_h) \subset G(u_2) = G(u_1)$ and the linearity of H, H_h is either the disjoint union of $r-1$ complete bipartite graphs $K_{r-1,r-1}$ or the disjoint union of $r-2$ complete bipartite graphs $K_{r-1,r-1}$ and one complete bipartite graph $K_{r-2,r-1}$. □

Now we are ready to prove the statement about F. If F is not an empty set, we let f be an edge of F. There must exist an s with $1 \leqslant s \leqslant r-1$ such that $1 \leqslant |f \cap U_s| \leqslant r$. Let $v \in f \cap U_s$. Denote by g the u_1-edge containing v. As a consequence of Remarks 9, for each $1 \leqslant i \leqslant r-1$ there exist $r-1$ u_2-edges whose vertices except for u_2 are in U_i. Fix h with $3 \leqslant h \leqslant r$. There exists at most one s' with $1 \leqslant s' \leqslant r-1$ such that there exist $r-2$ u_h-edges whose vertices except for u_h are in U_s. For each $1 \leqslant i \neq s' \leqslant r-1$, there exist $r-1$ u_h-edges whose vertices except for u_h are in U_i. Hence, for some t with $1 \leqslant t \neq s \leqslant r-1$, we have that there exist $r-2$ u_t-edges $g_1, g_2, \ldots, g_{r-2}$ with the property that each of them is disjoint from f and each of them intersects g. And there must exist another u_1-edge g' whose vertices except for u_1 are in $U_{t'}$ for some $1 \leqslant t' \neq s \leqslant r-1$ such that g' is disjoint from f. By the definition of $C^*_{1,r}$, we have that $g, f, g', g_1, g_2, \ldots, g_{r-2}$ constitute a $C^*_{1,r}$, a contradiction. This completes the proof of Lemma 4. □

Now we are ready to complete the proof for the upper bound in Theorem 3. By Theorems 1 and 2, the result holds for the case $r = 3$. Next we assume that $r \geqslant 4$. Suppose to the contrary that G is a smallest (in terms of the number of vertices n) $\{C_{1,r}, C^*_{1,r}\}$-free linear r-graph such that G has more than $\frac{r(r-2)(n-s)}{r-1}$ edges. For each $v \in V(G)$, we define $I(v) = 1$ if $d(v) \leqslant (r-1)^2 + 1$, and $I(v) = 0$ otherwise.

We adopt the following useful observation from [9].

$$\sum_{e \in E(G)} \sum_{v \in V(G), v \in e} \frac{I(v)}{d(v)} = \sum_{v \in V(G)} \sum_{e \in E(G), v \in e} \frac{I(v)}{d(v)} = \sum_{v \in V(G)} I(v) = n - s.$$

Since $|E(G)| > \frac{r(r-2)(n-s)}{r-1}$, there must exist an edge $e = \{u_1, u_2, \ldots, u_r\}$ such that

$$\sum_{1 \leqslant i \leqslant r} \frac{I(u_i)}{d(u_i)} < \frac{r-1}{r(r-2)} = \frac{r-1}{(r-1)^2 - 1}. \tag{1}$$

Without loss of generality, we assume $d(u_1) \geqslant d(u_2) \geqslant \ldots \geqslant d(u_r)$. Note that $d(u_r) \geqslant r-1$ and $d(u_2) \geqslant (r-1)^2$, as otherwise (1) would be violated. We can also deduce that $d(u_i) \geqslant (r-i)(r-1) + 2$ for all $3 \leqslant i \leqslant r-1$, as otherwise (1) would be violated. If $d(u_1) \geqslant (r-1)^2 + 2$, then we can easily find a $C_{1,r}$ in the following way. We start with the edge $e = \{u_1, u_2, \ldots, u_r\}$. We can find a u_r-edge $e_1 \neq e$ since $d(u_r) \geqslant r-1 \geqslant 3$. By considering i from $r-1$ to 2 one by one, we can find a u_i-edge e_{r-i+1} that does not share a vertex with any edge in $\{e_1, e_2, \ldots, e_{r-i}\}$. Finally, we can choose a u_1-edge e_r that does not share a vertex with $\{e_1, e_2, \ldots, e_{r-1}\}$, a contradiction. Therefore, we have $d(u_1) \leqslant (r-1)^2 + 1$. By (1), we have $d(u_1) = d(u_2) = (r-1)^2 + 1$ and $d(u_i) \geqslant (r-1)^2$ for each $3 \leqslant i \leqslant r$. Thus, $D(e) \geqslant \{(r-1)^2 + 1, (r-1)^2 + 1, (r-1)^2, \ldots, (r-1)^2\}$.

Now we define S and E_S as in Lemma 4. Let $G - S$ be the linear r-graph obtained by deleting the vertices of S and the edges of E_S. By Lemma 4, $G - S$ has $n' = n - ((r-1)^3 + r)$ vertices and at least $|E(G)| - (r(r-1)^2 + 1)$ edges. Furthermore, the number of vertices in $G - S$ of degree at least $(r-1)^2 + 2$ is exactly s. Therefore, we have

$$|E(G-S)| \geqslant |E(G)| - (r(r-1)^2 + 1) > \frac{r(r-2)(n-s)}{r-1} - (r(r-1)^2 + 1) > \frac{r(r-2)(n'-s)}{r-1},$$

which contradicts the assumption that G is a smallest counterexample.

This completes the proof.

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