The Minimum Spectral Radius of tP_3- or K_5-Saturated Graphs via the Number of 2-Walks

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Submitted: Oct 21, 2023; Accepted: Jan 12, 2025; Published: Feb 28, 2025

Abstract

For a given graph H, a graph G is H-saturated if G does not contain H as a subgraph, but for $e \in E(\overline{G})$, $G+e$ contains H as a subgraph; the spectral saturation number of H, written $sat_\rho(n, H)$, is the minimum value of $\rho(G)$ in an n-vertex H-saturated graph G.

For a vertex $v \in V(G)$, let $l_2(v)$ be the number of 2-walks starting from v. In this paper, when G is an n-vertex tP_3- or K_5-saturated connected graph, for each vertex $v \in V(G)$, we prove the best lower bounds for $l_2(v)$ in terms of n and $d(v)$, implying that $sat_\rho(n, tP_3) = \rho(F)$ and $sat_\rho(n, K_5) = \rho(S_{n,4})$, where F is the 6-vertex graph obtained from K_3 by attaching a pendant vertex to each vertex in K_3 and $S_{n,4}$ is the join of K_3 and $(n-3)K_1$.

Mathematics Subject Classifications: 05C15, 05C50, 15A18

1 Introduction

For undefined terms of graph theory, see West [31]. For basic properties of spectral graph theory, see Brouwer and Haemers [3] or Godsil and Royle [16].

Given a graph H, determining the maximum number of edges in an n-vertex H-free graph, written $ex(n, H)$, has a long history. (See K_3 [22], K_r [30], $K_{s,t}$ [21, 20], C_{2k+1} [26], C_{2k} [2, 25], P_k [8], T_k [7], C_4 [10, 4], $K_{2,t}$ [13], $K_{3,3}$ [4, 14], Q_8 [11], consecutive cycles [8] and three surveys [15, 17, 27].) Thus if an n-vertex graph G has more than $ex(n, H)$ edges, then G must contain H as a subgraph. If we create a copy of H by adding any additional edge to an H-free graph, then it would be an interesting property. Even if

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an n-vertex H-free graph G has less than $ex(n,H)$ edges and if we add any additional edge to G, then the resulting graph may contain a copy of H. In this case, G is called *H-saturated.* More precisely, if G does not contain H as a subgraph but for any edge $e \in E(\overline{G})$, $G+e$ contains H as a subgraph, then we call G H-saturated. Then we may be interested in the minimum number of edges in an n-vertex H-saturated graph, which is called the saturation number of H, written $sat(n,H)$.

Erdős, Hajnal, and Moon [9] proved $sat(n, K_{r+1}) = |E(S_{n,r})|$, which was the first result on saturation number. The graph $S_{n,r}$ is the join of K_{r-1} and $(n-r+1)K_1$.

For a graph G with given number of vertices, the average degree $\frac{2|E(G)|}{|V(G)|}$ and the number of edges give the same information, and we note that the spectral radius of G, written $\rho(G)$, is at least the average degree, where $A(G)$ is the adjacency matrix of G and $\rho(G) = \max_{\{\lambda : A(G)x=\lambda x\}} |\lambda|$. Thus it may be natural to ask what will happen if we replace the average degree with the spectral radius.

Like $sat(n,H)$, we can define the *spectral saturation number* of a graph H, written $sat_\rho(n,H)$, to be the minimum value of $\rho(G)$ in an n-vertex H-saturated graph G.

Nikiforov [24] proved that if G is an n-vertex K_{r+1}-saturated graph, then $\rho(G) \leqslant \rho(T_{n,r})$, where $T_{n,r}$ is the n-vertex r-partite Turán graph; equality holds only when G is $T_{n,r}$. Kim, Kim, Kostochka, and O [18] proved that if G is an n-vertex K_{r+1}-saturated graph, then

$$\rho(G) \geqslant \sqrt{\frac{(n-1)^2(r-1)+(r-1)^2(n-r+1)}{n}} \tag{1}$$

by using the fact $\rho^2(G) \geqslant \frac{1}{n}\sum_{v\in V(G)} d^2(v)$ and by determining the best lower bound for $\sum_{v\in V(G)} d^2(v)$. Note that for $r=2$, equality in (1) holds only when G is $S_{n,2}$ or a Moore graph with diameter 2. However, for $r \geqslant 3$, equality does not hold for any graphs, which means that $sat_\rho(K_{r+1}) > \sqrt{\frac{(n-1)^2(r-1)+(r-1)^2(n-r+1)}{n}}$. For $r=3$, Kim, Kostochka, O, Shi, and Wang [19] proved the sharp lower bound for $\rho(G)$ in an n-vertex K_{r+1}-saturated graph; equality holds only when G is $S_{n,3}$. For $r \geqslant 4$, determining $sat_\rho(n, K_{r+1})$ is still open. In this paper, we determine $sat_\rho(n, K_{r+1})$ when $r=4$.

Note that $\sum_{v\in V(G)} d^2(v) = \sum_{v\in V(G)}\sum_{w\in V(G)}(A^2(G))_{vw} = \sum_{v\in V(G)}\sum_{w\in N(v)} d(w)$. For a vertex $v \in V(G)$, let $l_2(v)$ be the number of 2-walks starting at v in a graph G, where a t-walk is a walk of length t.

Let $P = \{V_1, \ldots, V_s\}$ be a partition of $V(G)$ into s non-empty subsets. The *quotient matrix* Q corresponding to P is the $s \times s$ matrix whose (i,j)-entry is the average number of incident edges in V_j of the vertices in V_i. More precisely, $Q_{i,j} = \frac{|[V_i,V_j]|}{|V_i|}$ if $i \neq j$, and $Q_{i,i} = \frac{2|E(G[V_i])|}{|V_i|}$. A partition P is *equitable* if for each $1 \leqslant i,j \leqslant s$, any vertex $v \in V_i$ has exactly $Q_{i,j}$ neighbors in V_j. In this case, the eigenvalues of the quotient matrix are eigenvalues of G and the spectral radius of the quotient matrix equals the spectral radius of G (see [3, 16] for more details).

Observation 1. *Let Q be a 2×2 equitable quotient matrix of a graph H, and let $p_Q(x)$ be the characteristic polynomial of Q. If G is an n-vertex graph with the spectral radius $\rho(G)$ and $p_Q(\rho(G)) \geqslant 0$, then we have $\rho(G) \geqslant \rho(H)$.*

Since $\{V(K_3), V((n-3)K_1)\}$ and $\{\{v_2, v_3, v_5\}, \{v_1, v_4, v_6\}\}$ (see Figure 2) are equitable partitions of $S_{n,4}$ and F, respectively, the corresponding quotient matrices are

$$Q_1 = \begin{pmatrix} 2 & n-3 \\ 3 & 0 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus the characteristic polynomials are $p_{Q_1}(x) = x^2 - 2x - 3(n-3)$ and $p_{Q_2}(x) = x^2 - 2x - 1$, respectively. If $A(G)\mathbf{x} = \rho(G)\mathbf{x}$, where $\sum_{v \in V(G)} x_v = 1$, then for $i = 1, 2$, we have

$$p_{Q_i}(A(G))\mathbf{x} = p_{Q_i}(\rho(G))\mathbf{x}.$$

Thus we have

$$\begin{aligned} p_{Q_i}(\rho(G)) = p_{Q_i}(\rho(G)) \sum_{v \in V(G)} x_v = \\ \sum_{v \in V(G)} p_{Q_i}(\rho(G)) x_v \\ = \sum_{v \in V(G)} \sum_{w \in V(G)} p_{Q_i}(A(G))_{vw} x_w \\ = \sum_{v \in V(G)} x_v \sum_{u \in V(G)} p_{Q_i}(A(G))_{uv} \\ \geqslant \min_{v \in V(G)} \sum_{u \in V(G)} p_{Q_i}(A(G))_{uv}. \end{aligned}$$

Note that $\sum_{u \in V(G)} A^2(G)_{uv} = l_2(v)$ and $\sum_{u \in V(G)} A(G)_{uv} = d(v)$. Thus for each vertex $v \in V(G)$, if $l_2(v) \geqslant 2d(v) + 3(n-3)$ for $i = 1$, or if $l_2(v) \geqslant 2d(v) + 1$ for $i = 2$, then we have $p_{Q_i}(\rho(G)) \geqslant 0$, which implies that by Observation 1, $\rho(G) \geqslant \rho(S_{n,r})$ or $\rho(G) \geqslant \rho(F)$, respectively.

In Section 4, we prove that if G is an n-vertex K_5-saturated graph, then for each vertex $v \in V(G)$, we have $l_2(v) \geqslant 2d(v) + 3(n-3)$.

Theorem 2. *If G is an n-vertex K_5-saturated graph, then for each vertex $v \in V(H)$, we have*

$$\sum_{w \in N(v)} d(w) \geqslant 2d(v) + 3(n-3).$$

Thus, Theorem 2 implies Theorem 3.

Theorem 3. *If G is an n-vertex K_5-saturated graph, then we have*

$$\rho(G) \geqslant \rho(S_{n,4});$$

equality holds only when G is $S_{n,4}$.

Similarly, we also want to prove that if G is an n-vertex tP_3-saturated graph, then for every vertex $v \in V(G)$, we have $l_2(v) \geqslant 2d(v)+1$. However, depending on n and t, n-vertex tP_3-saturated graphs may be disconnected or just a complete graph. If $t=1$, then each component of G is at most 2 vertices, which implies that $sat_\rho(n, tP_3)=1$. If $2 \leqslant n < 3t$, then $sat_\rho(n, tP_3)=n-1$. We may assume that $t \geqslant 2$ and $n \geqslant 3t$. By Lemma 10, we replace G with a component G_0 of G in the argument above Theorem 3 such that $\rho(G)=\rho(G_0)$ and for each vertex $v \in V(G_0)$, we have $l_2(v) \geqslant 2d(v)+1$. How can we guarantee the existence of such a component in an n-vertex tP_3-saturated graph? In fact, there are some exceptional graphs. For $k \geqslant 2$, let $F_k = K_1 \vee (kK_2)$ and $F_k^+ = K_1 \vee (kK_2 \cup K_1)$. See Figure 1 for F_2 and F_2^+. Let $\mathcal{H} = \{H : H \text{ is isomorphic to } F_k \text{ or } F_k^+ \text{ for some } k \geqslant 2\}$. If an n-vertex tP_3-saturated graph G contains a component G' in $\mathcal{H}$ such that $\rho(G)=\rho(G')$, then there exists a vertex $v \in V(G')$ such that $l_2(v) < 2d(v)+1$. (For all vertices v in any graph in $\mathcal{H}$, we have $l_2(v) \geqslant 2d(v)$.) However, note that $\rho(G)=\rho(G') \geqslant \rho(F_2) > \rho(F)$. Thus we consider n-vertex tP_3-saturated graphs G whose components do not belong to $\mathcal{H}$. Since $t \geqslant 2$ and G is tP_3-saturated, the graph $(t-1)P_3$ is a subgraph of G, which means that some component G_0 on n_0 vertices of G contains P_3 as a subgraph and so $n_0 \geqslant 3$.

Theorem 4. *If any component of an n-vertex tP_3-saturated graph G does not belong to $\mathcal{H}$, then there exists a component G_0 of G such that $\rho(G)=\rho(G_0)$ and for each vertex $v \in V(G_0)$, we have*

$$\sum_{w \in N(v)} d(w) \geqslant 2d(v)+1;$$

equality holds only when G_0 is isomorphic to F, where F is the 6-vertex graph obtained from K_3 by attaching a pendent vertex at each vertex of $V(K_3)$ (see Figure 2).

Thus Theorem 4 implies Theorem 5.

F_2 F_2^+

Figure 1: Two graphs F_2 and F_2^+

Theorem 5. *If G is an n-vertex tP_3-saturated graph with $t \geqslant 2$ and $n \geqslant 3t$, then we have*

$$\rho(G) \geqslant \rho(F);$$

equality holds only when G is the union of $(t-1)F$ and $\lfloor \frac{(n-6t+6)}{2} \rfloor K_2 \cup \eta K_1$, where $\eta = 0$ if n is even or $\eta = 1$ if n is odd.

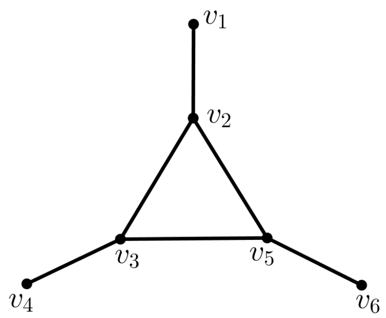

Figure 2: The graph F

Interestingly, the graphs holding equality in the bound of Theorem 5 are from the following conjecture.

Conjecture 6 ([6])**.** Let $t \geqslant 2$ be an integer. For sufficiently large n, $sat(n, tP_3) = \lfloor \frac{n+6t-6}{2} \rfloor$ and $(t-1)F \cup \lfloor \frac{(n-6t+6)}{2} \rfloor K_2 \cup \eta K_1$ is a tP_3-saturated graph with size $sat(n, tP_3)$.

The conjecture was confirmed for $t = 2$ and $n \geqslant 12$, $t = 3$ and $n \geqslant 9$, $t = 4$ and $n \geqslant 3t+2$, and $t = 5$, $n \geqslant 3t+1$ [6, 29, 5]. For $t \geqslant 6$, Conjecture 6 is still open. However, we determine the exact value of $sat_\rho(n, tP_3)$ in this paper (See Theorem 5).

2 Tools

Given a graph G, we denote the minimum degree by $\delta(G)$, the maximum degree by $\Delta(G)$, and the average degree by $d(G)$, respectively. Let $\overline{G}$ denote the complement of a graph G. For a vertex $v \in V(G)$, let $d(v)$ and $N(v)$ denote the degree and neighborhood of v in G, respectively.

For $A \subseteq V(G)$, let $G[A]$ denote the subgraph of G induced by A. For an edge $e \in E(\overline{G})$, $G + e$ is the graph obtained from G by adding e. For a subset $S \subseteq V(G)$, $G - S$ is the graph induced by $V(G) \backslash S$. For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$.

Lemma 7 ([6])**.** *For positive integers n, η, and $t \geqslant 2$, the graph $(t-1)F \cup \lfloor \frac{(n-6t+6)}{2} \rfloor K_2 \cup \eta K_1$ is tP_3-saturated.*

Lemma 8 ([1], Theorem 6.8)**.** *If G is a connected graph and H is a proper subgraph of G, then $\rho(H) < \rho(G)$.*

Lemma 9 ([1], Theorem 6.3 (i))**.** *If G is an n-vertex connected graph with $n \geqslant 2$, then $\rho(G) > 0$ and there is an eigenvector $\mathbf{x} > \mathbf{0}$ corresponding to the eigenvalue $\rho(G)$.*

Lemma 10 ([3], Proposition 1.3.6)**.** *If $G = \bigcup_{i=1}^{s} G_i$, where G_i is a component of G, then the spectrum of G is the union of the spectrum of G_i (and multiplicities are added).*

Lemma 11 ([6])**.** *For an integer $t \geqslant 2$, let G be a tP_3-saturated graph.*

(1) *If $w \in V(G)$ with $d(w) = 2$, then w lies in a triangle.*

(2) *No vertex is adjacent to two vertices of degree* 1.

(3) *G is not a tree.*

Lemma 12 ([5])**.** *For an integer $t \geqslant 2$, let G be a tP_3-saturated graph and $u \in V(G)$.*

(1) *If $d(u) = 1$ and $v \in N(u)$ with $d(v) = 3$, then v lies in a triangle.*

(2) *If $d(u) = 2$ and $v \in V(G)$ with $d(v) = 2$, then $|N(u) \cap N(v)| \leqslant 1$. Additionally, if there exists a vertex $w \in N(u) \cap N(v)$, then $d(w) \geqslant 5$. Furthermore, if $uv \in E(G)$, then $d(w) = n - 1$.*

(3) *If $N(u) = \{v, w\}$, then $d(v) \neq 3$ and $d(w) \neq 3$.*

3 Proof of Theorem 4

Suppose that G is an n-vertex tP_3-saturated graph whose components do not belong to $\mathcal{H}$. To prove Theorem 4, it suffices to show that G contains a component G_0 such that $\rho(G) = \rho(G_0)$ and for each vertex $v \in V(G_0)$, we have $\sum_{w \in N(v)} d(w) \geqslant 2d(v) + 1$. Since $(t-1)P_3$ is a subgraph of G, there exists a component G_0 in G containing P_3 as a subgraph. Let $|V(G_0)| = n_0$. Then $n_0 \geqslant 3$.

Proof of Theorem 4. First, note that equality in the bound holds when $G = F$.

Suppose $n_0 = 3$. Since $n \geqslant 3t \geqslant 6$, there is a vertex $x \in V(G) \setminus V(G_0)$. Let $y \in V(G_0)$. Note that $G + xy$ contains a copy of tP_3, say T. Since $n_0 = 3$, $G - V(G_0) - \{x\}$ contains a copy of $(t-1)P_3$ as a subgraph. Thus G contains a copy of tP_3 as a subgraph, which is a contradiction. If $4 \leqslant n_0 \leqslant 5$, then $G_0 = K_{n_0}$. Otherwise, for an edge $uv \in E(\overline{G_0})$, $G + uv$ must contain a copy of tP_3. Thus there is a copy of $(t-1)P_3$ in $G - V(G_0)$ since $n_0 \leqslant 5$, which implies that G contains a copy of tP_3, a contradiction.

Now, we may assume that $n_0 \geqslant 6$. For $u \in V(G_0)$, we will prove $\sum_{w \in N(u)} d(w) \geqslant 2d(u) + 1$ in the following three cases.

Case 1. $d(u) = 1$.

Let $N(u) = \{w\}$. By Lemma 11 (1), we have $d(w) \neq 2$, which implies $d(w) \geqslant 3$ since $n_0 \geqslant 6$. Then $\sum_{w \in N(u)} d(w) \geqslant 2d(u) + 1$.

Case 2. $d(u) = 2$.

Let $N(u) = \{u_1, u_2\}$. By Lemma 11 (1), we have $u_1u_2 \in E(G_0)$. Since $n_0 \geqslant 6$, we have $d(u_1) \geqslant 3$ or $d(u_2) \geqslant 3$. Then $\sum_{w \in N(u)} d(w) = d(u_1) + d(u_2) \geqslant 5 = 2d(u) + 1$.

Case 3. $d(u) \geqslant 3$.

By Lemma 11 (2), there is at most one vertex with degree 1 in $N(u)$, say x (if it exists). If there exists a vertex $z \in N(u)$ with $d(z) \geqslant 4$, then

$$\sum_{w \in N(u)} d(w) \geqslant 2|N(u) \setminus \{x, z\}| + d(z) + d(x) \geqslant 2(d(u) - 2) + 5 = 2d(u) + 1.$$

Now, we may assume that for any $w \in N(u)$, we have

$$d(w) \leqslant 3. \tag{2}$$

If $d(v) \geqslant 3$ for any vertex $v \in N(u) \setminus \{x\}$, then $\sum_{w \in N(u)} d(w) \geqslant 3|N(u) \setminus \{x\}| + d(x) \geqslant 3(d(u) - 1) + 1 \geqslant 2d(u) + 1$ since $d(u) \geqslant 3$. Thus we may assume that there is a vertex in $N(u)$ with degree 2.

We claim that for any vertex $y \in V(G_0)$ with $|N(y)| = 2$, say $N(y) = \{u, y_1\}$, we have $d(y_1) = 2$. By Lemma 11 (1), y_1 lies in a triangle and $y_1 \in N(u)$. Then by Lemma 12 (3) and Inequality (2), we have $d(y_1) = 2$. Thus by Lemma 12 (2), we have $d(u) = n_0 - 1$. Since $G_0 \notin \mathcal{H}$, there exists a vertex $z \in N(u)$ with $d(z) = 3$ by Inequality (2). Thus we have $|N(z) \cap N(u)| = 2$. Let $N(z) \cap N(u) = \{z_1, z_2\}$. By Lemma 12 (3), we have $d(z_1) \geqslant 3$ and $d(z_2) \geqslant 3$. Thus

$$\begin{aligned}\sum_{w \in N(u)} d(w) &\geqslant 2|N(u) \setminus \{z, z_1, z_2, x\}| + d(z) + d(z_1) + d(z_2) + d(x) \\ &\geqslant 2(d(u) - 4) + 3 + 3 + 3 + 1 \\ &= 2d(u) + 1.\end{aligned}$$

To have equality in the bound means that $\sum_{w \in N(u)} d(w) = 2d(u) + 1$, we have the following cases: In Case 1, we must have $d(w) = 3$, where $N(u) = \{w\}$. In Case 2, we must have $d(u_1) = 2$ and $d(u_2) = 3$ (or the other way). Then by Lemma 12 (2), we have $d(u_2) = n_0 - 1$, which follows that $n_0 = 4$, a contradiction. Thus there is no vertex of degree 2 in G_0. In Case 3, since there is at most one vertex of degree 1 in $N(u)$ and there is no vertex of degree 2 in G_0, we have

$$2d(u) + 1 = \sum_{w \in N(u)} d(w) \geqslant 1 + 3(d(u) - 1) = 3d(u) - 2,$$

which yields $d(u) = 3$. Thus there is a vertex $w_1 \in N(u)$ of degree 1 and $d(w) = 3$ for any $w \in N(u) \setminus \{w_1\}$.

By Lemma 12 (1), u lies in a triangle. Let $N(u) = \{w_1, w_2, w_3\}$. Thus $w_2w_3 \in E(G_0)$ and $d(w_2) = d(w_3) = 3$. Let $N(w_2) = \{u, w_3, w_2'\}$ and $N(w_3) = \{u, w_2, w_3'\}$. By applying $u = w_i$ for any $i \in \{2, 3\}$, we have $d(w_i') = 1$, which gives $G_0 = F$. □

4 Proof of Theorem 3

To prove Theorem 3, we recall the result of Erdős, Hajnal, and Moon [9].

Theorem 13. (Erdős et al. [9]) *If* $2 \leqslant r < n$*, then* $sat(n, K_{r+1}) = (r-1)(n-r+1)+\binom{r-1}{2}$. *The only* n*-vertex* K_{r+1}*-saturated graph with* $sat(n, K_{r+1})$ *edges is the graph* $S_{n,r}$.

Proof of Theorem 3: By Lemma 2, it suffices to show that for each vertex $v \in V(G)$,

$$\sum_{w\in N(v)} d(w) \geqslant 2d(v) + 3(n-3). \tag{3}$$

We consider two cases depending on the property of the closed neighborhood of a vertex.

Case 1. For a vertex $v \in V(G)$*, the graph* $G[N[v]]$ *is* K_5*-saturated.*

Since $G[N[v]]$ is K_5-saturated, $G[N(v)]$ is K_4-saturated. Then we have $|E(G[N(v)])| \geqslant 2d(v)-3$ by Theorem 13. Note that for each $x \notin N[v]$, $G+vx$ contains a copy of K_5 since G is K_5-saturated. Thus we may assume that $|N(x) \cap N(v)| \geqslant 3$, which implies that

$$\begin{aligned}\sum_{w\in N(v)} d(w) &\geqslant d(v) + 2|E(G[N(v)])| + 3(n-d(v)-1) &(4)\\ &\geqslant d(v) + 2(2d(v)-3) + 3(n-d(v)-1) &(5)\\ &= 2d(v) + 3(n-3). &(6)\end{aligned}$$

Equalities in (3) and (5) hold when $|E(G[N(v)])| = 2d(v) - 3$ and $|N(x) \cap N(v)| = 3$ for any $x \in V(G) \setminus N[v]$. Equality in (4) holds when $G[N(v)]$ is isomorphic to $S_{d(v),3}$ by Theorem 13. Note that all triangles of $S_{d(v),3}$ have two common vertices, say y and z. Thus, for any vertex $x \in V(G) \setminus N[v]$, we have $\{y, z\} \subseteq N(x)$, which follows that $d(y) = d(z) = n-1$ and $G[N[y]] = G$ is K_5-saturated. Thus $G[N(y)]$ is isomorphic to $S_{n-1,3}$, which implies that G is isomorphic to $S_{n,4}$.

Case 2. The graph induced by the closed neighborhood $N[v]$ *is not* K_5*-saturated.*

We may assume that G contains at most one vertex of degree $n-1$. Otherwise, there are two vertices u and w with $d(u) = d(w) = n-1$. Then $v \notin \{u, w\}$ and $\{u, w\} \subseteq N(v)$ since $G[N[u]](= G[N[w]] = G)$ is K_5-saturated. Note that $G[N(v)] = S_{d(v),3}$ since G is K_5-free. Thus $G = G[N[v]] = S_{d(v)+1,4}$, which contradicts that $G[N[v]]$ is not K_5-saturated.

Claim 14. $\delta(G) \geqslant 5$.

Proof. If there exists a vertex w with $d(w) = 3$, then every vertex in $V(G) - N[w]$ is adjacent to the three neighbors of w and $G[N(w)] = K_3$ since G is K_5-saturated. Thus each vertex in $N(w)$ has degree $n-1$, which contradicts that G contains at most one vertex of degree $n-1$.

If there exists a vertex w with $d(w) = 4$, then there are two non-adjacent vertices $u, u' \in N(w)$ and $G[N(w)]$ contains a copy of K_3 as a subgraph since G is K_5-saturated. Also note that the copy of K_3 contains the two vertices $x, y \in N(w) \setminus \{u, u'\}$ since $uu' \notin E(G)$. Thus every vertex in $V(G) - N[w]$ must be adjacent to the vertices x and y. If $\{u, u'\} \subseteq N(x) \cap N(y)$, then $d(x) = d(y) = n-1$, a contradiction. Thus

$\{u,u'\} \nsubseteq N(x)\cap N(y)$. We may assume that there is exactly one copy of K_3 in $G[N(w)]$ and it contains u. Then u is adjacent to all the vertices in $V(G)-N[w]$, which implies that $V(G)-N[w]$ is an independent set. Since $\{u,u'\} \nsubseteq N(x)\cap N(y)$ and G is K_5-saturated, $G[N(w)]$ does not contain a copy of K_4 as a subgraph and by adding the edge between u and u', we must have a copy of K_5, which implies that $V(G)-N[w]$ is not independent, a contradiction. □

Now, we partition the vertex set $V(G)-v$ into the following three sets:

$$N_1 = \{x \in N(v) : x \text{ lies in a triangle in } G[N(v)]\},\ N_2 = N(v)-N_1,\ \text{ and }\ N_3 = V(G)-N[v].$$

Note that $N(v) = N_1 \cup N_2$ and $N_3 \neq \varnothing$. For any $u \in N_1$, we have $|N(u)\cap N_1| \geqslant 2$ by the definition of N_1. Suppose that $R'(u)$ is a subset of $N(u)\cap N_1$ with $|R'(u)| = 2$ and $R(u) = R'(u)\cup\{v\}$. Then $R(u) \subseteq N(u)$ and $|R(u)| = 3$ for any $u \in N_1$. For any vertex $w \in N_3$, $G+wv$ contains a copy of K_5 since G is K_5-saturated. We may assume that the vertices in this K_5 are x_1,x_2,x_3,w,v. Thus $\{x_1,x_2,x_3\} \subseteq N_1$ and $G[\{x_1,x_2,x_3,w\}]$ is isomorphic to K_4. For each $w \in N_3$, we choose exactly three such edges x_iw for $i \in \{1,2,3\}$ and put them in M. Then $M \subseteq E(G[N_1,N_3])$ and $|M| = 3|N_3|$. For $u \in N_1$, let $M(u) = \{u' \in N(u)\cap N_3 : u'u \in M\}$ and $S(u) = N(u)-R(u)-M(u)$. Thus $\sum_{u\in N_1}|M(u)| = |M| = 3|N_3|$. Since $\delta(G) \geqslant 5$, we have

$$\sum_{u\in N_2} d(u) \geqslant \sum_{u\in N_2} 5 = 5|N_2|.$$

Note that $|N_1|+|N_2|+|N_3| = n-1$. Now we have

$$\begin{aligned}
\sum_{u\in N(v)} d(u) &= \sum_{u\in N_1}(|R(u)|+|M(u)|+|S(u)|) + \sum_{u\in N_2} d(u)\\
&\geqslant \sum_{u\in N_1}|R(u)| + \sum_{u\in N_1}|M(u)| + \sum_{u\in N_1}|S(u)| + 5|N_2|\\
&= 3|N_1| + 3|N_3| + \sum_{u\in N_1}|S(u)| + 5|N_2|\\
&= 3(|N_1|+|N_2|+|N_3|) + 2|N_2| + \sum_{u\in N_1}|S(u)|\\
&= 3n-3+2|N_2| + \sum_{u\in N_1}|S(u)|. \qquad (7)
\end{aligned}$$

One of our goal is to prove that

$$\sum_{u\in N_1}|S(u)| \geqslant 2|N_1|-6. \qquad (8)$$

Then Inequalities (7) and (8) yield

$$\sum_{u\in N(v)} d(u) \geqslant 3n-3+2|N_2|+2|N_1|-6 = 2d(v)+3(n-3), \qquad (9)$$

as we desired. Another goal is to characterize the structure of the graph when equality in Inequality (9) holds.

Furthermore, we partition the vertex set N_1 into the three subsets A, B_1 and B_0, where

$$A = \{x \in N_1 : |S(x)| \geqslant 2\},\ B_1 = \{x \in N_1 : |S(x)| = 1\}, \text{ and } B_0 = \{x \in N_1 : |S(x)| = 0\}.$$

Then we have $\sum_{u\in N_1} |S(u)| \geqslant 2|A| + |B_1| = 2|N_1| - |B_1| - 2|B_0|$. If $|B_1| + 2|B_0| \leqslant 5$, then $\sum_{u\in N_1} |S(u)| < 2|N_1| - 6$, as we desired. Thus we may assume that $|B_1| + 2|B_0| \geqslant 6$. If $|B_0| = 2$ and $|B_1| = 2$, or $|B_0| = 3$ and $|B_1| = 0$, then $|B_1| + 2|B_0| = 6$ and so $\sum_{u\in N_1} |S(u)|=2|N_1| - 6$. We will complete the proof of Inequality (8) by demonstrating that $\sum_{u\in N_1} |S(u)| < 2|N_1| - 6$ in the following three cases.

Case 2.1. $|B_0| \geqslant 4$ *and* $|B_1| = 0$.

For any vertex $x \in B_0$, we have $|N(x) \cap N_1| = 2$ by the definition of $S(x)$. Since $|B_0| \geqslant 4$, there are two nonadjacent vertices $x_1, x_2 \in B_0$. Then $G + x_1x_2$ contains a copy of K_5. Assume to the contrary that the copy contains a vertex y in N_3. Then x_1 and x_2 are adjacent to y. By the definition of M, we have $x_iy \notin M$ for some $i \in \{1, 2\}$, which means that $|S(x_i)| \neq 0$, a contradiction. Thus we have $V(K_5) \subseteq N[v]$. If the copy of K_5 does not contain v, then there is a copy of K_4 in $G[N(v)]$, which implies that there is a copy of K_5 in $G[N[v]]$, a contradiction. Thus the copy contains v and the other two vertices in $N(v)$, say a and b. Then $\{x_1, x_2\} \subseteq N(a) \cap N(b)$ and $ab \in E(G)$, as shown in Figure 3. We can see $|N(a) \cap N_1| \geqslant |\{b, x_1, x_2\}| = 3$ and $|N(b) \cap N_1| \geqslant |\{a, x_1, x_2\}| = 3$, which means that $\{a, b\} \cap B_0 = \emptyset$. Since $|B_0| \geqslant 4$ and $|N(x_i) \cap N_1| = 2$ for any $x_i \in B_0 \setminus \{x_1\}$, $x_1x_i \notin E(G)$. Similarly, $G + x_1x_i$ contains a copy of K_5 with all five vertices belonging to $N[v]$ as a subgraph. Since $\{x_1, x_i\} \subseteq B_0$, we have $N(x_i) \cap N_1 = \{a, b\}$, as shown in Figure 3. Thus we have $B_0 \subseteq N(a) \cap N(b)$. Now we have

$$S(a) \geqslant |N(a) \cap N(v)| - 2 \geqslant |B_0| + 1 - 2 = |B_0| - 1$$

and

$$S(b) \geqslant |N(b) \cap N(v)| - 2 \geqslant |B_0| + 1 - 2 = |B_0| - 1.$$

Thus we have

$$\begin{aligned}
\sum_{u\in N_1} |S(u)| &= \sum_{u\in A} |S(u)| + \sum_{u\in B_1} |S(u)| + \sum_{u\in B_0} |S(u)| \\
&= \sum_{u\in A} |S(u)| \\
&\geqslant |S(a)| + |S(b)| + 2(|A| - 2) \\
&\geqslant 2(|B_0| - 1) + 2(|A| - 2) \\
&\geqslant 2|N_1| - 6,
\end{aligned}$$

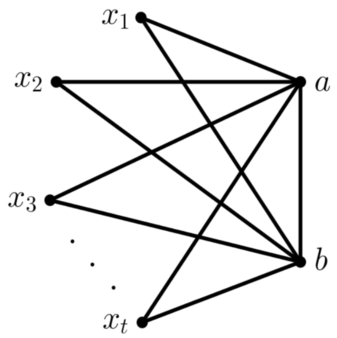

Figure 3: The book graph

saying that Inequality (8) holds. If equality in Inequality (8) holds, then we have

$$N(a) \cap N(v) = B_0 \cup \{b\}, N(b) \cap N(v) = B_0 \cup \{a\}, |S(u)| = 2 \text{ for any } u \in A \setminus \{a, b\},$$

$$\text{and } d(u) = 5 \text{ for any } u \in N_2.$$

In fact, we have $N(v) \neq B_0 \cup \{a, b\}$, all triangles in $G[N(v)]$ have two common vertices a, b, and for any vertex $w \in V(G) \setminus N[v]$, we have $w \in N(a) \cap N(b)$. Those yield that $d(a) = d(b) = n - 1$, which contradicts the fact that there is at most one vertex with degree $n - 1$. Thus there is a vertex $w \in N(v) \setminus B_0$ such that $wx_i \notin E(G)$ for any $x_i \in B_0$. Then $G + wx_i$ contains a copy of K_5 as a subgraph. If the copy contains at least three vertices in $N(v)$, say x_i, w, c, then we have $c \in \{a, b\}$ since $N(x_i) \cap N(v) = \{a, b\}$. Thus we have $w \in N(a) \cup N(b)$, which contradicts $N(a) \cup N(b) = B_0 \cup \{a, b\}$. If the copy contains at most two vertices in $N(v)$, say x_i, w, then it contains three vertices in N_3, say y_1^i, y_2^i, y_3^i. Then we have $\{y_1^i, y_2^i, y_3^i\} \subseteq N(x_i)$, $G[\{y_1^i, y_2^i, y_3^i\}]$ is isomorphic to K_3, and $y_j^i x_i \in M$ for any $j \in \{1, 2, 3\}$ since $x_i \in B_0$. By the definition of M, we have $\{a, b\} \subseteq N(y_1^i) \cap N(y_2^i) \cap N(y_3^i)$ since there is exactly one triangle containing x_i, $x_i ab$. Thus $G[\{a, b, y_1^1, y_2^1, y_3^1\}]$ is isomorphic to K_5, a contradiction. Thus equality in Inequality (8) does not hold.

Case 2.2. $|B_0| = 0$ *and* $|B_1| \geqslant 6$.

By the definition of B_1, for any $u \in B_1$, we have $|N(u) \cap N_1| \leqslant 3$. Since $|B_1| \geqslant 6$, for any $x_1 \in B_1$, there is $x_i \in B_1$ with $i \neq 1$ such that $x_1 x_i \notin E(G)$.

Claim 15. *For any $x_1, x_2 \in B_1$ with $x_1 x_2 \notin E(G)$, there are two triangles $x_1 f_1 g_1$ and $x_2 f_2 g_2$ such that $|\{f_1, g_1\} \cap \{f_2, g_2\}| \geqslant 1$ and $f_1, g_1, f_2, g_2 \in N_1$. Furthermore, if for any two triangles $x_1 f_1 g_1$ and $x_2 f_2 g_2$, we have $|\{f_1, g_1\} \cap \{f_2, g_2\}| \leqslant 1$, then we have $|N(x_i) \cap N(v)| = 2$ for any $i \in \{1, 2\}$.*

Proof. Since G is K_5-saturated, $G + x_1 x_2$ contains a copy of K_5 as a subgraph. If the copy contains at least four vertices in $N(v)$, say $\{x_1, x_2, x_1', x_2'\}$, then $G[\{x_i, x_1', x_2'\}]$ is isomorphic to a triangle for each $i \in \{1, 2\}$, and the two triangles have two common vertices. If the copy contains at most three vertices in $N(v)$, then there are at least two vertices in N_3, say y_1 and y_2. For any vertex y in the copy, if $y \in N_3$, then we have $y \in N(x_1) \cap N(x_2)$. Since $x_1x_2 \notin E(G)$, we have $yx_i \notin M$ for some $i \in [2]$, and there is at most one vertex y such that $yx_i \notin M$ since $|S(x_i)| = 1$. Thus the copy contains at most two vertices in N_3, that is, y_1 and y_2, and three vertices in $N(v)$, say x_1, x_2, w. Then $y_1x_j \notin M$ for some $j \in \{1, 2\}$ and $y_2x_i \notin M$ for some $i \in \{1, 2\}$ by the definition of M. Since $x_1, x_2 \in B_1$, we have $i \neq j$. We may assume that $x_ty_t \notin M$ for any $t \in \{1, 2\}$. Thus $|N(x_t) \cap N(v)| = 2$ for any $t \in \{1, 2\}$. There are two triangles containing $\{x_1, w\}$ and $\{x_2, w\}$, respectively, as claimed. □

If there is a vertex w in N_1 with $B_1 \subseteq N(w)$, then we have $|S(w)| \geqslant |N(w) \cap N_1| - 2 \geqslant |B_1| - 2$. Thus we have

$$\begin{aligned}\sum_{u \in N_1} |S(u)| &= \sum_{u \in A} |S(u)| + \sum_{u \in B_1} |S(u)| + \sum_{u \in B_0} |S(u)| \\ &= \sum_{u \in A} |S(u)| + |B_1| \\ &\geqslant |S(w)| + 2(|A| - 1) + |B_1| \\ &\geqslant |B_1| - 2 + 2|A| - 2 + |B_1| \\ &= 2|N_1| - 4 \\ &> 2|N_1| - 6,\end{aligned}$$

saying that Inequality (8) holds.

In the remaining case, we may assume that there is no vertex w with $B_1 \subseteq N(w)$. For any $x_1 \in B_1$, we have $2 \leqslant |N(x_1) \cap N(v)| \leqslant 3$. Since G is K_5-free, $G[N(v)]$ is K_4-free and so there are at most two triangles in $G[N(v)]$ containing x_1. If there are two triangles containing x_1, then we have $|N(x_1) \cap N(v)| = 3$. By letting $N(x_1) \cap N(v) = \{y_1, y_2, y_3\}$, we have $|E(G[\{y_1, y_2, y_3\}])| = 2$. We may assume that $y_1y_2 \in E(G)$ and $y_1y_3 \in E(G)$. For any vertex $x_2 \in B_1$ with $x_1x_2 \notin E(G)$, Claim 15 guarantees that there is a triangle containing x_2 and one or two vertices of $\{y_1, y_2, y_3\}$. If the triangle contains two vertices in $\{y_1, y_2, y_3\}$, then the two vertices must be $\{y_1, y_2\}$ or $\{y_1, y_3\}$. Note that there is no vertex $w \in N_1$ with $B_1 \subseteq N(w)$. We claim that there exists a vertex x_3 such that there is no triangle containing x_3 and two vertices of $\{y_1, y_2, y_3\}$. Assume to the contrary that for any vertex $x \in B_1 \setminus N(x_1)$, there is a triangle containing x such that the triangle contains $\{y_1, y_2\}$ or $\{y_1, y_3\}$. Then the vertex y_1 is the vertex in N_1 such that $B_1 \subseteq N(y_1)$, a contradiction. By Claim 15, we have $|N(x_1) \cap N(v)| = 2$, which contradicts $|N(x_1) \cap N(v)| = 3$. Thus, for any $x_1 \in B_1$, there is exactly one triangle containing x_1, say $\{x_1, u, w\}$. Since there is no vertex $z \in N_1$ such that $B_1 \subseteq N(z)$, there are two vertices x_2 and x_3 in B_1 such that there is no triangle containing x_j having two common vertices in $\{x_1, u, w\}$ for any $j \in \{2, 3\}$. By Claim 15, we have $|N(x_2) \cap N(v)| = |N(x_3) \cap N(v)| = 2$ and there are two triangles containing $\{x_2, u\}$ and $\{x_3, w\}$, respectively, and the first triangle does not contain w and the second does not contain u. Assume that the third vertex in the triangle containing x_j is x'_j for any $j \in \{2, 3\}$. Then we have $x'_2u \in E(G)$, which implies that $x'_2 \neq x_3$ and $x_2x_3 \notin E(G)$. Since for any vertex $b \in B_1$, there is exactly one triangle containing b, by Claim 15, we have $x'_2 = x'_3$. For any vertex $b \in B_1$, we assert that $b \in N(u) \cup N(w) \cup N(x'_2)$. Since $|N(x_2) \cap N(v)| = 2$, we have $bx_2 \notin E(G)$ for any $b \in B_1 \setminus \{x_1, x_2, x_3\}$. By Claim 15, there is a triangle containing b and at least one vertex in $\{u, x'_2\}$, as asserted. Thus we have

$$|S(u)| + |S(w)| + |S(x'_2)| \geqslant |B_1 \setminus \{x_1, x_2, x_3\}| + 4 * 3 - 2 * 3 = |B_1| + 3,$$

which gives

$$\begin{aligned}\sum_{z \in N_1} |S(z)| &= \sum_{z \in A} |S(z)| + \sum_{z \in B_1} |S(z)| + \sum_{z \in B_0} |S(z)| \\ &= \sum_{z \in A} |S(z)| + |B_1| \\ &\geqslant |S(u)| + |S(w)| + |S(x'_2)| + 2(|A| - 3) + |B_1| \\ &\geqslant |B_1| + 3 + 2|A| - 6 + |B_1| \\ &= 2|N_1| - 3 \\ &> 2|N_1| - 6,\end{aligned}$$

saying Inequality (8) holds.

Case 2.3. $|B_0| \geqslant 1$, $|B_1| \geqslant 1$, *and* $2|B_0| + |B_1| \geqslant 7$, *or* $|B_0| = 1$ *and* $|B_1| = 4$.

Note that for any $x \in B_0$, we have $|N(x) \cap N(v)| = 2$ and there is exactly one triangle containing x; for any $y \in B_1$, we have $|N(y) \cap N(v)| \leqslant 3$. We assert that there are two vertices $x_1 \in B_0$ and $y_1 \in B_1$ such that $x_1 y_1 \notin E(G)$. Assume to the contrary that $G[B_0, B_1]$ is a complete bipartite graph. Then we have $|B_1| \leqslant 2$, $|B_0| \leqslant 3$, and $2|B_0| + |B_1| \geqslant 7$. Since $|N(x) \cap N(v)| = 2$ and there is exactly one triangle containing x for any $x \in B_0$, we have $|B_0| = |N(y_1) \cap B_0| \leqslant 2$. Thus $2|B_0| + |B_1| \leqslant 6$, a contradiction. Thus $x_1 y_1 \notin E(G)$. Suppose that $N(x_1) \cap N(v) = \{u, w\}$. Then $G[\{x_1, u, w\}]$ is isomorphic to a triangle.

Claim 16. *For any* $y \in B_0 \cup B_1$ *with* $x_1 y \notin E(G)$, *we have* $y \in N(u) \cap N(w)$.

Proof. $G + x_1 y$ contains a copy of K_5 as a subgraph. If the copy contains at least four vertices in $N(v)$, then the four vertices must be $\{x_1, y, u, w\}$ and $y \in N(u) \cap N(w)$, as claimed. If the copy contains at most three vertices in $N(v)$, then the other two vertices must be in N_3, say z_1 and z_2. Then we have $\{x_1, y\} \subseteq N(z_1) \cap N(z_2)$. Since $x_1 y \notin E(G)$, we have $z_i x_1 \notin M$ or $z_i y \notin M$ for any $i \in \{1, 2\}$ by the definition of M. Then we have $x_1 z_i \in M$ and $z_i y \notin M$ for any $i \in \{1, 2\}$ and $|S(y)| \geqslant 2$ since $x_1 \in B_0$, which contradicts $y \in B_0 \cup B_1$. □

Since $x_1 y_1 \notin E(G)$, by Claim 16, we have $y_1 \in N(u) \cap N(w)$. Thus there is another vertex $y_2 \in B_0 \cup B_1$ such that $y_2 x_1 \notin E(G)$, else $B_1 \subseteq \{y_1, u, w\}$, $B_0 = \{x_1\}$, and $2|B_0| + |B_1| \leqslant 5$, a contradiction. By Claim 16, we have $y_2 \in N(u) \cap N(w)$ and so

$\{u,w\} \subseteq A$. For any vertex $b \in (B_0 \cup B_1) \setminus \{x_1\}$, we have $bx_1 \notin E(G)$ and so $b \in N(u) \cap N(w)$ by Claim 16. Thus we have

$$|S(u)| \geqslant |B_0| + |B_1| + 1 - 2 = |B_0| + |B_1| - 1$$

and

$$|S(w)| \geqslant |B_0| + |B_1| + 1 - 2 = |B_0| + |B_1| - 1.$$

Observe that

$$\begin{aligned}
\sum_{z \in N_1} |S(z)| &= \sum_{z \in A} |S(z)| + \sum_{z \in B_1} |S(z)| + \sum_{z \in B_0} |S(z)| \\
&= \sum_{z \in A} |S(z)| + |B_1| \\
&\geqslant |S(u)| + |S(w)| + 2(|A| - 2) + |B_1| \\
&\geqslant 2|B_0| + 2|B_1| - 2 + 2|A| - 4 + |B_1| \\
&= 2|N_1| - 6 + |B_1| \\
&> 2|N_1| - 6,
\end{aligned}$$

saying Inequality (8) holds.

Now we characterize the structure of the graph when equality in Inequality (9) holds in Case 2. In fact, equality holds only when $|B_1| + 2|B_0| = 6$, $|B_0| \geqslant 2$, $|S(x)| = 2$ for any $x \in A$, and $d(u) = 5$ for any $u \in N_2$. If there are two nonadjacent vertices x_1, x_2 in B_0 then $G + x_1x_2$ contains a copy of K_5, and the copy does not contain any vertex in N_3. Also there is a vertex $y \in N_3$ such that $y \in N(x_1) \cap N(x_2)$. Then we have $yx_i \notin M$ for some $i \in \{1,2\}$ since $x_1x_2 \notin E(G)$, which contradicts $x_1, x_2 \in B_0$. Thus the copy must contain four vertices in $N(v)$, say x_1, x_2, z_1, z_2 such that $z_1z_2 \in E(G)$ and $\{x_1, x_2\} \subseteq N(z_1) \cap N(z_2)$. Assume to the contrary that $N_2 \neq \emptyset$. Then there is a vertex $w \in N_2$. Since there is no triangle in $G[N(v)]$ containing w, the copy of K_5 in the graph $G + x_iw$ must contain at least two vertices in N_3, say y_1^i, y_2^i. Then $y_1^iy_2^i \in E(G)$. Since $x_i \in B_0$, we have $y_1^ix_i \in M$ and $y_2^ix_i \in M$. Note that there is exactly one triangle containing x_i. We have $\{y_1^i, y_2^i\} \subseteq N(z_1) \cap N(z_2)$ and $G[\{x_i, y_1^i, y_2^i, z_1, z_2\}]$ is isomorphic to K_5, a contradiction. Now assume that $N_2 = \emptyset$. Then $G[N_1]$ is not isomorphic to a book with the two spine vertices z_1, z_2. Otherwise, by the definition of M, we have $d(z_1) = d(z_2) = n - 1$, a contradiction. Since $|S(z_i)| \leqslant 2$ for any $i \in [2]$ and $G[N_1]$ is not isomorphic to a book with the two spine vertices z_1, z_2, there is a vertex $w \in N_1$ such that $wz_i \notin E(G)$ for some $i \in \{1,2\}$. We may assume that $wz_1 \notin E(G)$. Since $wz_1 \notin E(G)$ and $N(x_i) \cap N_1 = \{z_1, z_2\}$, for any $i \in [2]$, $G + wx_i$ contains a copy of K_5, and the copy contains at most three vertices in N_1 and at least two vertices in N_3, saying y_1^i, y_2^i. Since $x_i \in B_0$, we have $y_1^ix_i \in M$ and $y_2^ix_i \in M$. Also we have $\{y_1^1, y_2^1\} \cap \{y_1^2, y_2^2\} = \emptyset$ since $x_1x_2 \notin E(G)$. Thus we have $|N(w) \cap N_3| \geqslant 4$ and $|S(w)| \geqslant 4$, a contradiction.

Now we may assume that $G[B_0]$ is a complete graph. Assume that $G[B_0]$ is a triangle with three vertices x_1, x_2, x_3. For any vertex $w \in N(v) \setminus B_0$ (such a vertex exists since $\delta(G) \geqslant 5$), we have $wx_i \notin E(G)$ for any $i \in \{1,2,3\}$. Since $|N(x_i) \cap N(v)| = 2$, the $G + wx_i$

contains at least three vertices in N_3, say y_1^i, y_2^i, y_3^i. Thus $G[\{y_1^i, y_2^i, y_3^i\}]$ is a triangle. Since there is exactly one triangle containing x_1, we have $\{y_1^1, y_2^1, y_3^1\} \subseteq N(x_2) \cap N(x_3)$, which yields that $G[\{y_1^1, y_2^1, y_3^1, x_1, x_2\}]$ is isomorphic to K_5, a contradiction. Now assume that B_0 has exactly two vertices x_1, x_2. Suppose that the triangle containing x_1, x_2 is x_1x_2u. Since $\delta(G) \geqslant 5$, there is a vertex $w \in N(v)$ such that $wx_i \notin E(G)$ for any $i \in \{1, 2\}$. Note that a copy of K_5 in $G + wx_i$ must contain at most three vertices in $N(v)$ and at least two vertices in N_3, say y_1^i, y_2^i, since $|N(x_i) \cap N(v)| = 2$. Thus we have $y_1^i x_i \in M$, $y_2^i x_i \in M$, and $y_1^i y_2^i \in E(G)$. Since there is exactly one triangle containing x_i, we have $\{y_1^1, y_2^1\} \subseteq N(x_1) \cap N(x_2) \cap N(u)$ by the definition of M. Thus $G[\{x_1, x_2, u, y_1^1, y_2^1\}]$ is isomorphic to K_5, a contradiction. This completes the proof of Theorem 3.

Acknowledgements

Jiangdong Ai was supported by the Fundamental Research Funds for the Central Universities, Pei Liu was supported by China Scholarship Council, Suil O was supported by NRF-2021K2A9A2A11101617, and Junxue Zhang was supported by the China Postdoctoral Science Foundation (No. 2024M764113).

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