Some Bounds on the Threshold Dimension of Graphs

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Abstract

A graph G on n vertices is a *threshold graph* if there exist real numbers $a_1, a_2, \ldots,$ a_n and b such that the zero-one solutions of the linear inequality

$$\sum_{i=1}^{n} a_i x_i \leqslant b$$

are exactly the characteristic vectors of the cliques of G.

The *threshold dimension* of a graph G is the minimum number of threshold graphs whose intersection yields G. We give tight or nearly tight upper bounds for the threshold dimension of a graph in terms of various graph parameters including treewidth, maximum degree, degeneracy, number of vertices, and vertex cover number. We also study threshold dimension of random graphs and graphs with high girth.

Mathematics Subject Classifications: 05C70, 05C62

1 Introduction

All graphs mentioned in this paper are finite, simple, and undirected. Given a graph $G = (V, E)$, we shall use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. For any $v \in V(G)$, we use $N_G(v)$ to denote the neighborhood of v in G, i.e., $N_G(v) = \{u \in V(G) \colon vu \in E(G)\}$. We use $N_G[v]$ to denote $N_G(v) \cup \{v\}$. For any $S \subseteq V(G)$, we shall use $G[S]$ to denote the subgraph induced by the vertex set S in G. We use $G - S$ to denote the graph $G[V(G) \setminus S]$. A subset of vertices in a graph forms a *clique* if each pair of vertices in this subset has an edge between them; if no pair of vertices have an edge between them, then the subset is called an *independent set*.

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A graph G on n vertices is a *threshold graph* if there exist real numbers $a_1, a_2, \ldots, a_n$ and b such that the zero-one solutions of the linear inequality $\sum_{i=1}^{n} a_i x_i \leqslant b$ are the characteristic vectors of the cliques of G. Chvátal and Hammer [7] showed that threshold graphs are exactly the graphs that contain no induced subgraph isomorphic to $2K_2$, P_4 or C_4 (the graph with four vertices and two disjoint edges, the path on four vertices, and the cycle on four vertices, respectively). Thus, the complement of a threshold graph is also a threshold graph, implying that one can replace 'cliques' with 'independent sets' in the definition of a threshold graph. The complete graph on n vertices is a threshold graph with the corresponding linear inequality being $\sum_{i=1}^{n} x_i \leqslant n$. Similarly, the star graph $K_{1,n-1}$ is a threshold graph, as shown by the linear inequality $x_1 + \sum_{i=2}^{n}(n-1)x_i \leqslant n$. For a graph G, the characteristic vectors of the subsets of $V(G)$ correspond to the corners of the n-dimensional hypercube. Thus, a graph G is threshold if and only if there is a hyperplane in $\mathbb{R}^n$ that separates the corners of the n-dimensional hypercube that correspond to the cliques of G from the other corners of the hypercube. Threshold graphs, which find applications in integer programming and set packing problems, were introduced by Chvátal and Hammer [7]. Refer to the book [14] by Golumbic to know more about the different properties of threshold graphs. A more comprehensive study of threshold graphs can be found in the book [19] by Mahadev and Peled.

The following equivalent characterization of threshold graphs (Corollary 1B in [7]) will be useful for us.

Proposition 1 (Chvátal and Hammer [7])**.** *G is a threshold graph if and only if there is a partition of $V(G)$ into an independent set A and a clique B, and an ordering $u_1, u_2, \ldots, u_k$ of A such that $N_G(u_k) \subseteq N_G(u_{k-1}) \subseteq \cdots \subseteq N_G(u_1)$.*

If $G_1, G_2, \ldots, G_k$ are graphs on the same vertex set as G such that $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_k)$, then we say that G is the intersection of $G_1 \cap G_2 \cap \cdots \cap G_k$ and denote this by $G = G_1 \cap G_2 \cap \cdots \cap G_k$. In a similar way, if $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$, then we say that $G = G_1 \cup G_2 \cup \cdots \cup G_k$. Given a class $\mathcal{A}$ of graphs, Kratochvíl and Tuza [18] defined the *$\mathcal{A}$-dimension* of a graph G, denoted as $\dim_{\mathcal{A}}(G)$, to be the minimum integer k such that there exist k graphs in $\mathcal{A}$ whose intersection is G. Let TH denote the class of threshold graphs. Chacko and Francis [4] studied the parameter $\dim_{\mathrm{TH}}(G)$ of a graph G, which in the language of [18], can be called the *threshold dimension* of G.

Definition 2 (Threshold dimension)**.** The threshold dimension of a graph G, denoted by $\dim_{\mathrm{TH}}(G)$, is the smallest integer k for which there exist threshold graphs $G_1, G_2, \ldots, G_k$ such that $G = G_1 \cap G_2 \cap \cdots \cap G_k$.

Note that Chvátal and Hammer [7] use the term "threshold dimension" of a graph G with a slightly different meaning: they define it to be the minimum integer k for which there exist threshold graphs $G_1, G_2, \ldots, G_k$ such that $G = G_1 \cup G_2 \cup \cdots \cup G_k$. We call this the *threshold cover number* of G and denote it by $\mathrm{cov}_{\mathrm{TH}}(G)$. Since the complement of a threshold graph is also a threshold graph, we have the following.

Observation 3. *For a graph G,* $\mathrm{cov}_{\mathrm{TH}}(G) = \dim_{\mathrm{TH}}(\overline{G})$.

Threshold dimension of graphs has close connections to the circuit complexity of certain kinds of Boolean functions (see [7, 12, 15] for more details).

For a graph G, let $\alpha(G)$, $\omega(G)$, and $\chi(G)$ denote the maximum size of an independent set, the maximum size of a clique, and the chromatic number of G, respectively. It was shown in [7] that for every graph G on n vertices, $\mathrm{cov}_{\mathrm{TH}}(G) \leqslant n - \alpha(G)$. In the same paper, the authors also showed that for every positive ϵ, there is a graph G on n vertices such that $\mathrm{cov}_{\mathrm{TH}}(G) > (1-\epsilon)n$. Yannakakis [26] showed that it is NP-complete to recognize graphs having threshold cover number at most k, for all fixed $k \geqslant 3$. Raschle and Simon [20] showed that there is a polynomial time algorithm that recognizes graphs having threshold cover number at most 2. Combining Observation 3 with the results from [7, 20, 26] mentioned above directly yields the following.

Corollary 4.

(a) *For a graph G on n vertices,* $\dim_{\mathrm{TH}}(G) \leqslant n - \omega(G)$.

(b) *For every positive ϵ, there is a graph G on n vertices such that* $\dim_{\mathrm{TH}}(G) > (1-\epsilon)n$.

(c) *For all fixed $k \geqslant 3$, it is NP-complete to recognize graphs having threshold dimension at most k.*

(d) *There is a polynomial-time algorithm that recognizes graphs having threshold dimension at most* 2.

We now give a lower bound on the threshold dimension of a graph.

Proposition 5. *For a graph G,* $\dim_{\mathrm{TH}}(G) \geqslant \min\{\chi(G - C) \colon C \text{ is a clique of } G\}$.

Proof. Suppose that G is a graph and $G_1, G_2, \ldots, G_k$ are threshold graphs such that $G = G_1 \cap G_2 \cap \cdots \cap G_k$. By Proposition 1, we have that for each $i \in [k]$, there is a partition of $V(G_i)$ into an independent set A_i and a clique B_i. It is not difficult to see that $B = B_1 \cap B_2 \cap \cdots \cap B_k$ is a clique of G, and each A_i, for $i \in [k]$, is an independent set of G. Since $V(G) \setminus B = A_1 \cup A_2 \cup \cdots \cup A_k$, we have that $V(G) \setminus B$ is the union of k independent sets of G. Therefore, $\chi(G - B) \leqslant k$. Thus there always exists a clique B in G such that $k \geqslant \chi(G - B)$. This completes the proof. □

Note that the above proposition actually gives a lower bound on $\dim_{\mathrm{SPLIT}}(G)$, where SPLIT is the class of "split graphs" — the graphs whose vertex set can be partitioned into an independent set and a clique — of which the class of threshold graphs is a subclass.

A graph is an *interval graph* if there is a mapping from the set of vertices of the graph to a set of closed intervals on the real line such that two vertices in the graph are adjacent to each other if and only if the intervals they are mapped to have a non-empty intersection. Let INT denote the class of interval graphs. The parameter $\dim_{\mathrm{INT}}(G)$ is more commonly known as the *boxicity* of the graph G and denoted as $\mathrm{box}(G)$. It is known that threshold graphs form a subclass of the class of interval graphs. This implies the following.

Observation 6. *For a graph G,* $\operatorname{box}(G) \leqslant \dim_{\mathrm{TH}}(G)$.

The graph parameter 'boxicity' was introduced by Roberts [21] in 1969 and has been extensively studied in the literature (see $[1,2,5,6,11,17]$). Observation 6 will help us get tight examples to various bounds we prove for threshold dimension in this paper. Chacko and Francis [4] gave the following upper bound for the threshold dimension of a graph G in terms of its boxicity and chromatic number.

Theorem 7 (Theorem 19 in [4])**.** *For a graph G,* $\dim_{\mathrm{TH}}(G) \leqslant \operatorname{box}(G) \cdot \chi(G)$.

We note here that the above upper bound is tight, as shown by the following observation, which also shows that the threshold dimension of a graph cannot be bounded by any function of its boxicity.

Proposition 8. *There is an interval graph G for which* $\dim_{\mathrm{TH}}(G) = \chi(G) = |V(G)|/2$.

Proof. Consider the graph $2K_n$. This graph is clearly an interval graph, and removing any clique from this graph results in a graph that contains a clique of n vertices. Thus by Proposition 5, we have that $\dim_{\mathrm{TH}}(2K_n) \geqslant n = \chi(2K_n)$. Theorem 7 implies $\dim_{\mathrm{TH}}(2K_n) \leqslant n$. □

In this paper, we prove tighter upper bounds for the threshold dimension of a graph that cannot be obtained from Theorem 7 by plugging in known upper bounds for boxicity.

1.1 Our results

For a graph G, we let $\Delta(G)$ denote the maximum degree of a vertex in G. When the graph G under consideration is clear from the context, we sometimes abbreviate $\Delta(G)$ to just Δ. Let $\operatorname{tw}(G)$ denote the treewidth of a graph G. We prove the following results.

1. Chandran and Sivadasan [6] showed that for any graph G, $\operatorname{box}(G) \leqslant \operatorname{tw}(G) + 2$. Using the folklore result that $\chi(G) \leqslant \operatorname{tw}(G) + 1$ (see for example, Lemma 8 in [6]) and Theorem 7, it follows that for any graph G, $\dim_{\mathrm{TH}}(G) \leqslant (\operatorname{tw}(G)+1)(\operatorname{tw}(G)+2)$. Chacko and Francis [4] ask if the threshold dimension of every graph can be bounded by a linear function of its treewidth. In Section 2, we answer this question in the affirmative by showing that $\dim_{\mathrm{TH}}(G) \leqslant 2(\operatorname{tw}(G) + 1)$. We show that this bound is tight up to a multiplicative factor of 2. Co-comparability graphs, AT-free graphs, and chordal graphs are known to have $O(\Delta)$ upper bounds on their treewidth. We thus get an $O(\Delta)$ upper bound to the threshold dimension of such graphs.

2. From Theorem 7, the result of Scott and Wood [23] that the boxicity of any graph is upper bounded by $\Delta \ln^{1+o(1)} \Delta$, and the fact that the chromatic number of any graph is at most $\Delta+1$, we have that for any graph G, $\dim_{\mathrm{TH}}(G) \leqslant \operatorname{box}(G)\cdot(\Delta+1) \leqslant \Delta(\Delta + 1) \ln^{1+o(1)} \Delta$. Thus, it is clear that $\dim_{\mathrm{TH}}(G)$ is bounded in terms of $\Delta(G)$. Let $\dim_{\mathrm{TH}}(\Delta) := \max\{\dim_{\mathrm{TH}}(G)\colon G \text{ is a graph having maximum degree } \Delta\}$. From the discussion above, it is clear that $\dim_{\mathrm{TH}}(\Delta) = O(\Delta^2 \ln^{1+o(1)} \Delta)$. We improve this

in Section 3 by showing that $\dim_{\mathrm{TH}}(\Delta) = O(\Delta \ln^{2+o(1)} \Delta)$. It was shown by Erdős, Kierstead, and Trotter in [9] that there exist graphs G having boxicity $\Omega(\Delta \ln \Delta)$. Using Observation 6, we get $\dim_{\mathrm{TH}}(\Delta) = \Omega(\Delta \ln \Delta)$. Bridging the gap between the upper and lower bounds for $\dim_{\mathrm{TH}}(\Delta)$ would be interesting.

3. Let G be k-degenerate. We show in Section 4 that $\dim_{\mathrm{TH}}(G) \leqslant 10k \ln n$. It was shown in Section 3.1 in [2] that there exist k-degenerate graphs on n vertices with boxicity $\Omega(k \ln n)$. Together with Observation 6, this implies that the upper bound for $\dim_{\mathrm{TH}}(G)$ we prove in Section 4 is tight up to a constant factor. This bound gives some interesting corollaries.

   (a) Let G be a graph that is drawn uniformly at random from the collection of all graphs on n vertices and m edges (this is usually denoted as $G \in \mathcal{G}(n, m)$). If $m \geqslant n/2$ then asymptotically almost surely, $\dim_{\mathrm{TH}}(G) = O(d_{av} \log n)$, where $d_{av} = \frac{2m}{n}$ denotes the average degree of G.

   (b) If G has girth at least $g + 1$, then $\dim_{\mathrm{TH}}(G) = O(n^{\frac{1}{\lfloor g/2 \rfloor}} \ln n)$.

4. In Section 5, we show that the threshold dimension of any graph is upper bounded by its vertex cover number, which implies that for any graph G, $\dim_{\mathrm{TH}}(G) \leqslant n - \max\{\alpha(G), \omega(G)\}$. We show that this bound is tight. As a corollary we show that for every graph G on n vertices, $\dim_{\mathrm{TH}}(G) \leqslant n - \lfloor 0.72 \ln n \rfloor - 1$.

1.2 Preliminaries

The definition and proposition below help us define many different threshold supergraphs of a given graph. This shall be our main tool for proving the upper bounds in this paper.

Definition 9. Given a graph G, an independent set $A = \{u_1, u_2, \ldots, u_t\}$ in G, and a total ordering $\sigma \colon u_1, u_2, \ldots, u_t$ of the vertices of A, we define the threshold supergraph $\tau(G, A, \sigma)$ of G as follows. Let $B = V(G) \backslash A$ and for $v \in B$, let $s(v) = \max\{i \colon u_i \in N_G(v)\}$ if $N_G(v) \cap A \neq \emptyset$ and $s(v) = 0$ otherwise. In $\tau(G, A, \sigma)$, the vertices of A form an independent set and those of B form a clique and each vertex $v \in B$ is adjacent to exactly the vertices $u_1, u_2, \ldots, u_{s(v)}$. Formally,

$$V(\tau(G, A, \sigma)) = V(G)$$

$$E(\tau(G, A, \sigma)) = E(G) \cup \{xy \colon x, y \in B \textit{ and } x \neq y\} \cup \bigcup_{v \in B} \{vu_1, vu_2, \ldots, vu_{s(v)}\}$$

The following proposition follows directly from the above definition and Proposition 1.

Proposition 10. *Given a graph G, an independent set A of G, and an ordering σ of A, the graph $\tau(G, A, \sigma)$ is a threshold graph and G is its subgraph.*

2 Threshold dimension and treewidth

In this section, we show that, for a graph G, $\dim_{\mathrm{TH}}(G) \leqslant 2(\mathrm{tw}(G)+1)$, where $\mathrm{tw}(G)$ denotes the treewidth of G. We set up some notations and discuss some necessary existing results before going into the proof of the main result.

The notion of treewidth was first introduced by Robertson and Seymour in [22].

Definition 11 (Tree decomposition)**.** A *tree decomposition* of a graph $G = (V, E)$ is a pair $(T, \{X_i : i \in V(T)\})$ where T is a tree and for each $i \in V(T)$, X_i is a subset of $V(G)$ (called a "bag"), such that the following conditions are satisfied:

- $\bigcup_{i \in V(T)} X_i = V(G)$.
- For all $uv \in E(G)$, there exists $i \in V(T)$ such that $u, v \in X_i$.
- For all $i, j, k \in V(T)$: if j is on the path in T from i to k, then $X_i \cap X_k \subseteq X_j$.

The *width* of a tree-decomposition $(T, \{X_i : i \in V(T)\})$ is $\max_{i \in V(T)} |X_i| - 1$.

Definition 12 (Treewidth)**.** The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width over all possible tree decompositions of G.

A tree decomposition $(T, \{X_i : i \in V(T)\})$ of a graph G is said to be a *path decomposition* of G if T is a path. The *pathwidth* of G, denoted by $\mathrm{pw}(G)$, is defined as the minimum width over all possible path decompositions of G. The following result by Chacko and Francis connects threshold dimension of a graph with its pathwidth.

Theorem 13 (Theorem 30 in [4])**.** *For every graph G, $\dim_{\mathrm{TH}}(G) \leqslant \mathrm{pw}(G) + 1$.*

Since path decompositions are special cases of tree decompositions, it can be seen that $\mathrm{tw}(G) \leqslant \mathrm{pw}(G)$. Korach and Solel showed that $\mathrm{pw}(G) = O(\log n \cdot \mathrm{tw}(G))$, where $n = |V(G)|$ (Theorem 6 in [16]). We thus have $\dim_{\mathrm{TH}}(G) = O(\log n \cdot \mathrm{tw}(G))$. Chacko and Francis note that for any graph G, $\dim_{\mathrm{TH}}(G) \leqslant (\mathrm{tw}(G)+1)(\mathrm{tw}(G)+2)$ and ask if there is a linear bound on the threshold dimension of a graph in terms of its treewidth. We give an affirmative answer to this question.

Given an ordering σ of the vertices of a graph G and $u, v \in V(G)$, we denote by $u <_\sigma v$ the fact that u appears before v in the ordering.

Let T be a rooted tree. For any $u, v \in V(T)$, u is an *ancestor* of v, and v a *descendant* of u, if u lies on the path from v to the root of T. It follows from this definition that every vertex of T is both an ancestor and descendant of itself. For a rooted tree T, a *preorder traversal* of T is an ordering of $V(T)$ in the order in which a depth-first search algorithm starting from the root may visit the vertices of T. The following is not difficult to see.

Proposition 14. *If π is a preorder traversal of a rooted tree T, then:*

(i) for $u, v \in V(T)$ such that v is a descendant of u, we have $u <_\pi v$, and

(ii) for $u, v, w \in V(T)$ such that $u <_\pi v <_\pi w$, if w is a descendant of u, then v is also a descendant of u.

Let G be a graph and $\mathcal{T} = (T, \{X_i : i \in V(T)\})$ be a tree decomposition of G having width k. We choose an arbitrary vertex r to be the root of T and henceforth consider T to be a rooted tree. Then a function $b\colon V(G) \to V(T)$ is defined as follows: for a vertex $v \in V(G)$, $b(v)$ is the bag containing v in the tree decomposition that is closest to r. Formally, $b(v)$ is the vertex of T such that $v \in X_{b(v)}$ and $v \notin X_i$ for any $i \in V(T)$ that is an ancestor of $b(v)$.

Lemma 15 (Lemma 10 in [6]). *If $uv \in E(G)$, then $b(u)$ is either an ancestor or descendant of $b(v)$ in T.*

Lemma 16 (Lemma 8 in [6]). *There exists a function $\theta\colon V(G) \to \{0, 1, \ldots, k\}$, such that for any $i \in V(T)$ and for any two distinct nodes $u, v \in X_i$, $\theta(u) \neq \theta(v)$.*

Remark. To obtain such a function θ, consider for example an optimal proper vertex coloring of the chordal graph G' that one obtains from G by adding edges between every pair of vertices that appear together in some bag of the tree decomposition. Clearly, $\mathcal{T}$ is a tree decomposition of G' as well. From the fact that every clique in G' has to be contained in some bag of $\mathcal{T}$, and the fact that chordal graphs are perfect, it follows that θ needs to use only $\max\{|X_i| : i \in V(T)\}$ different colors.

In the following, we denote by θ a function of the type that is guaranteed to exist by Lemma 16. The following lemmas from [6] describe some properties of the functions θ and b that we will use later. These are direct corollaries of the definition of θ and that of tree decompositions.

Lemma 17 (Lemma 9 in [6]). *If $uv \in E(G)$ then $\theta(u) \neq \theta(v)$.*

Lemma 18 (Lemma 11 in [6]). *Let $uv \in E(G)$ and let $b(u)$ be an ancestor of $b(v)$. For any vertex $w \in V(G) \setminus \{u\}$, $\theta(w) \neq \theta(u)$ if $b(w)$ is in the path from $b(v)$ to $b(u)$ in T.*

Let π be a preorder traversal of T. Let σ be an ordering of $V(G)$ such that for any two vertices $u, v \in V(G)$, $u <_\sigma v$ if $b(u) <_\pi b(v)$. (In σ, we let the ordering between two vertices $u, v \in V(G)$ such that $b(u) = b(v)$ be arbitrary. Thus, if $u <_\sigma v$, then $b(u) \leqslant_\pi b(v)$.) Let σ^{-1} denote the ordering of $V(G)$ obtained by reversing the ordering σ. Given a set $A \subseteq V(G)$, we denote by $\sigma|_A$ the ordering of vertices of A in the order in which they appear in σ.

For $i \in \{0, 1, \ldots, k\}$, we define $C_i = \{v \in V(G) : \theta(v) = i\}$. From Lemma 17, we know that θ is a proper coloring of G, which implies that C_i is an independent set of G. For each $i \in \{0, 1, \ldots, k\}$, we define two graphs $G_i^1 = \tau(G, C_i, \sigma|_{C_i})$ and $G_i^2 = \tau(G, C_i, \sigma^{-1}|_{C_i})$.

Lemma 19. *Let u, v be distinct vertices in G. Then there do not exist $x_u, y_u \in N_G(u)$ and $x_v, y_v \in N_G(v)$ such that $x_u <_\sigma v <_\sigma y_u$, $x_v <_\sigma u <_\sigma y_v$, $\theta(u) = \theta(x_v)$, and $\theta(v) = \theta(x_u)$.*

Proof. Suppose that there exist $x_u, y_u \in N_G(u)$ and $x_v, y_v \in N_G(v)$ such that $x_u <_\sigma v <_\sigma y_u$, $x_v <_\sigma u <_\sigma y_v$, $\theta(u) = \theta(x_v)$, and $\theta(v) = \theta(x_u)$. Clearly, we have either $u <_\sigma v$ or $v <_\sigma u$. Let us assume without loss of generality that $u <_\sigma v$. Then we have $u <_\sigma v <_\sigma y_u$, which implies that $b(u) \leqslant_\pi b(v) \leqslant_\pi b(y_u)$. Since $uy_u \in E(G)$, we have from Lemma 15 that $b(u)$ is either an ancestor or descendant of $b(y_u)$. As π is a preorder traversal of T, Proposition 14(i) implies that $b(u)$ is an ancestor of $b(y_u)$ in T. As $b(u) \leqslant_\pi b(v) \leqslant_\pi b(y_u)$, it now follows from Proposition 14(ii) that $b(v)$ is a descendant of $b(u)$. Similarly, $x_v <_\sigma u <_\sigma v$ implies that $b(x_v) \leqslant_\pi b(u) \leqslant_\pi b(v)$, and $vx_v \in E(G)$ then implies by Lemma 15, Proposition 14(i) and (ii) that $b(u)$ is a descendant of $b(x_v)$. Now applying Lemma 18 to x_v, u and v, we have that $\theta(x_v) \neq \theta(u)$, which is a contradiction. □

Lemma 20. $G = \bigcap_{0 \leqslant i \leqslant k} (G_i^1 \cap G_i^2)$

Proof. Consider any two distinct vertices u and v of G. Since G_i^1 and G_i^2, for $1 \leqslant i \leqslant k$, are both supergraphs of G by definition, we have that if $uv \in E(G)$, then uv is an edge of both G_i^1 and G_i^2. So in order to prove the lemma, we only need to prove that whenever $uv \notin E(G)$, there exists $i \in \{0, 1, \ldots, k\}$ and $j \in \{1, 2\}$ such that $uv \notin E(G_i^j)$.

Suppose that $uv \notin E(G)$. First, let us consider the case when $\theta(u) = \theta(v) = i$. Since the class C_i is an independent set in G_i^1 and G_i^2, uv is an edge in neither G_i^1 nor G_i^2, and we are done. So let us assume that $\theta(u) \neq \theta(v)$. Let $\theta(u) = i$ and $\theta(v) = j$. We claim that uv is not an edge in one of the graphs G_i^1, G_i^2, G_j^1, or G_j^2. Suppose for the sake of contradiction that $uv \in E(G_i^1) \cap E(G_i^2) \cap E(G_j^1) \cap E(G_j^2)$. Then uv is an edge in each of the graphs $\tau(G, C_i, \sigma|_{C_i})$, $\tau(G, C_i, \sigma^{-1}|_{C_i})$, $\tau(G, C_j, \sigma|_{C_j})$, $\tau(G, C_j, \sigma^{-1}|_{C_j})$. Since $uv \in E(\tau(G, C_i, \sigma|_{C_i}))$, by Definition 9, we have that there exists $y_v \in C_i \cap N_G(v)$ such that $u <_\sigma y_v$. Further, since $uv \in E(\tau(G, C_i, \sigma^{-1}|_{C_i}))$, there exists $x_v \in C_i \cap N_G(v)$ such that $u <_{\sigma^{-1}} x_v$, or in other words, $x_v <_\sigma u$. As $uv \in E(\tau(G, C_j, \sigma|_{C_j}))$ and $uv \in E(\tau(G, C_j, \sigma^{-1}|_{C_j}))$, we can similarly conclude that there exist $x_u, y_u \in C_j \cap N_G(u)$ such that $x_u <_\sigma v <_\sigma y_u$. Since $\theta(x_u) = \theta(v) = j$ and $\theta(x_v) = \theta(u) = i$, we now have a contradiction to Lemma 19. □

From Proposition 10 and Definition 9, it follows that G_i^1 and G_i^2 are both threshold graphs for each $i \in \{0, 1, 2, \ldots, k\}$. Thus by Lemma 20, we get that $\dim_{\mathrm{TH}}(G) \leqslant 2(k+1)$, which leads to the following theorem.

Theorem 21. *For any graph G,* $\dim_{\mathrm{TH}}(G) \leqslant 2(\mathrm{tw}(G) + 1)$.

Tightness of the bound

Note that from Proposition 8, we know that the graph $2K_n$ has threshold dimension n and it is easy to see that the treewidth of this graph is $n - 1$. Thus the upper bound on threshold dimension given by Theorem 21 is tight up to a multiplicative factor of 2. We give below another example showing the same tightness result.

Example 22. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. Let G be a graph defined as $V(G) = A \cup B$ and $E(G) = \{a_i a_j : 1 \leqslant i < j \leqslant n\} \cup \{a_i b_i : 1 \leqslant i \leqslant n\}$. Let H be the complement of the graph G.

We claim that $\dim_{\mathrm{TH}}(H) = n$. To show that $\dim_{\mathrm{TH}}(H) \leqslant n$, it is easy to see that the edges of G can be covered using n threshold graphs (for each $\ell \in \{1, 2, \ldots, n\}$, we can construct a threshold graph having vertex set $V(G)$ and edge set $\{a_ia_j : 1 \leqslant i < j \leqslant n\} \cup \{a_\ell b_\ell\}$; the union of these graphs is G). In order to prove that $\dim_{\mathrm{TH}}(H) \geqslant n$, assume that there is a collection $\mathcal{C}$ of fewer than n threshold graphs whose intersection gives H. Then, for each $i \in [n]$ there must exist a threshold graph in $\mathcal{C}$ in which a_i is non-adjacent to b_i. As $|\mathcal{C}| < n$, this means that there exists a threshold graph in $\mathcal{C}$ in which a_i is non-adjacent to b_i and a_j is non-adjacent to b_j, for some distinct $i, j \in [n]$. This implies the existence of an induced P_4 (the path $a_ib_jb_ia_j$) in this threshold graph, which is a contradiction.

Next, we show that $\mathrm{tw}(H) = n-1$. Since H contains a clique of size n, $\mathrm{tw}(H) \geqslant n-1$. Let $X_0 = \{b_1, \ldots, b_n\}$, $X_i = \{a_i\} \cup (B \setminus \{b_i\})$, for all $i \in [n]$. Let T be the tree having vertex set $\{0, 1, \ldots, n\}$ in which the vertex 0 has degree n and all other vertices have degree 1. Observe that the pair $(T, \{X_i\}_{i\in\{0,1,\ldots,n\}})$ is a tree decomposition of H having width $n-1$. Thus, $\mathrm{tw}(H) \leqslant n-1$. Hence, this example also demonstrates that the bound in Theorem 21 is tight up to a multiplicative factor of 2.

3 Threshold dimension and maximum degree

Let $\dim_{\mathrm{TH}}(\Delta) := \max\{\dim_{\mathrm{TH}}(G) : G \text{ is a graph having maximum degree } \Delta\}$. In this section, we show that $\dim_{\mathrm{TH}}(\Delta) = O(\Delta \ln^{2+o(1)} \Delta)$.

3.1 Definitions, notations, and auxiliary results

Given a graph G and an $S \subseteq V(G)$, recall that we use $G[S]$ to denote the subgraph induced by the vertex set S in G. For any disjoint pair of sets $S, T \subseteq V(G)$, we use $G[S, T]$ to denote the bipartite subgraph of G such that $V(G[S, T]) = S \cup T$ and $E(G[S, T]) = \{uv : u \in S,\ v \in T,\ uv \in E(G)\}$. Let $G^*[S, T]$ denote the graph constructed from $G[S, T]$ by making T a clique. That is, $V(G^*[S, T]) = S \cup T$ and $E(G^*[S, T]) = E(G[S, T]) \cup \{uv : u, v \in T, u \neq v\}$.

We state below the definition of a k-suitable family of permutations that was introduced by Dushnik in [8].

Definition 23 (k-suitable family of permutations)**.** Let n, k be integers such that $n \geqslant k \geqslant 1$. A family of permutations (or linear orders), $\sigma := \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ of $[n]$, is called a k-suitable family of permutations of $[n]$ if for all k-sized subsets A of $[n]$ and an element $x \in A$ there exists a permutation $\sigma_i \in \sigma$ such that x succeeds all the elements $y \in A \setminus \{x\}$ in σ_i; i.e., $y <_{\sigma_i} x$ for all $y \in A \setminus \{x\}$.

The following lemma is due to Spencer [24] though the exact value of k and n are worked out by Scott and Wood in Lemma 5 of [23]. We shall use the same values in our calculations too.

Lemma 24 (Spencer [24])**.** *For every $k \geqslant 2$ and $n \geqslant 10^4$ there is a k-suitable family of permutations of size at most $k2^k \ln\ln n$.*

Lemma 25 (Lemma 12 in [23]). *Let G be a bipartite graph with bipartition $\{A, B\}$, where vertices in A have degree at most Δ and vertices in B have degree at most d. Let r, t, ℓ be positive integers such that*

$$\ell \geqslant e\left(\frac{ed}{r+1}\right)^{1+1/r} \quad \textit{and } t \geqslant \ln(4d\Delta).$$

Then there exist t colorings $c_1, \dots, c_t$ of A, each with ℓ colors, such that for each vertex $v \in B$, for some coloring c_i, each color is assigned to at most r neighbors of v under c_i.

We use Lemmas 24 and 25 to prove the following lemma which is a prerequisite to our proof of Theorem 29.

Lemma 26. *Let G be a bipartite graph with bipartition $\{A, B\}$, where vertices in A have degree at most Δ and vertices in B have degree at most d, for some $2 \leqslant d \leqslant \Delta$. Then,*

$$\dim_{\mathrm{TH}}(G^*[A, B]) \leqslant (81 + o(1)) d \ln(d\Delta) \ln\ln \Delta (2e)^{\sqrt{\ln d}},$$

when $d \to \infty$.

Proof. We follow the proof idea of Lemma 13 in [23]. Let $r = \left\lceil \sqrt{\ln d} \right\rceil$, $\ell = \left\lceil e\left(\frac{ed}{r+1}\right)^{1+1/r} \right\rceil$, and $t = \lceil \ln(4d\Delta) \rceil$. Hence, we know from Lemma 25 that there exist t colorings $c_1, c_2, \dots,$ c_t of A, each with ℓ colors, such that for each vertex $v \in B$, for some coloring c_j, each color is assigned to at most r neighbors of v under c_j. To obtain the threshold dimension of $G^*[A, B]$ we further partition B sequentially into t parts, namely $B_1, B_2, \dots, B_t$, based on the t colorings of A. A vertex $v \in B$ is in B_j if and only if j is the smallest integer such that each color appears on at most r neighbors of v under c_j. For a particular coloring c_j and $1 \leqslant k \leqslant \ell$, we define $A_{j,k}$ as the set containing all the vertices $v \in A$ such that $c_j(v) = k$. Let $G_{j,k}$ be the supergraph of $G^*[A, B]$ obtained from $G^*[A_{j,k}, B_j]$ by adding as universal vertices all the vertices of G that are not in $A_{j,k} \cup B_j$. Let H be the threshold supergraph of $G^*[A, B]$ defined as: $V(H) = A \cup B$, $E(H) = \{uv \colon u \in B,\ v \in V(H) \setminus \{u\}\}$. Then we have the following:

$$G^*[A, B] = H \cap \left(\bigcap_{1 \leqslant j \leqslant t} \bigcap_{1 \leqslant k \leqslant \ell} G_{j,k} \right). \tag{1}$$

Now we are going to calculate $\dim_{\mathrm{TH}}(G_{j,k})$. In order to use the kind of threshold supergraphs defined in Definition 9, we need an ordering of the vertices in $A_{j,k}$, which is an independent set in $G_{j,k}$. Let G' denote the graph with $V(G') = A_{j,k}$ and two vertices $x, y \in A_{j,k}$ are adjacent in G' if and only if they have a common neighbor in B_j. We properly color G' using $r\Delta + 1$ colors as the maximum degree of a vertex in G' is at most $r\Delta$. Let the color classes be $C_1, C_2, \dots, C_{r\Delta+1}$. Then $A_{j,k} = C_1 \uplus C_2 \uplus \cdots \uplus C_{r\Delta+1}$ and in $G_{j,k}$, every vertex in B_j has at most one neighbor in each color class C_i. We determine the ordering of the vertices in $A_{j,k}$ based on an $(r+1)$-suitable family of permutations, $\sigma_1, \sigma_2, \dots, \sigma_p$, of $C_1, C_2, \dots, C_{r\Delta+1}$. From Lemma 24, we can assume that

$p \leqslant (r+1)2^{(r+1)} \ln\ln(r\Delta+1)$. From each σ_a, where $1 \leqslant a \leqslant p$, we construct two linear orderings σ_a^1 and σ_a^2 of $A_{j,k}$ as described below:

$$\sigma_a^1 := \psi_{\sigma_a(1)}, \psi_{\sigma_a(2)}, \ldots, \psi_{\sigma_a(r\Delta+1)} \ ,$$

$$\sigma_a^2 := \psi^{-1}_{\sigma_a(1)}, \psi^{-1}_{\sigma_a(2)}, \ldots, \psi^{-1}_{\sigma_a(r\Delta+1)} \ .$$

In the above, for $1 \leqslant i \leqslant r\Delta+1$, ψ_i denotes an arbitrary ordering of the vertices of C_i and ψ_i^{-1} denotes the reverse of ψ_i. Now that we have total orderings σ_a^1 and σ_a^2 of $A_{j,k}$, we consider the two threshold supergraphs $\tau(G_{j,k}, A_{j,k}, \sigma_a^1)$ and $\tau(G_{j,k}, A_{j,k}, \sigma_a^2)$ of $G_{j,k}$.

Claim 27. $G_{j,k} = \bigcap_{1 \leqslant a \leqslant p} (\tau(G_{j,k}, A_{j,k}, \sigma_a^1) \cap \tau(G_{j,k}, A_{j,k}, \sigma_a^2))$.

Proof. From Definition 9, we know that if $uv \in E(G_{j,k})$, then uv is present in both $\tau(G_{j,k}, A_{j,k}, \sigma_a^1)$ and $\tau(G_{j,k}, A_{j,k}, \sigma_a^2)$, for all $a \in [p]$. Hence we only need to show that if $uv \notin E(G_{j,k})$ then there exists at least one threshold supergraph in the collection where u and v are non-adjacent. If $u, v \in A_{j,k}$ then $uv \notin E(\tau(G_{j,k}, A_{j,k}, \sigma_a^1))$ and $uv \notin E(\tau(G_{j,k}, A_{j,k}, \sigma_a^2))$, for all $a \in [p]$. Without loss of generality, assume $u \in A_{j,k}$ and $v \in B_j$. Let $C \in \{C_1, C_2, \ldots, C_{r\Delta+1}\}$ be the color class containing u. We know from the property of the color classes C_i that v has at most one neighbor in every C_i (in particular, in C). Suppose $|N_{G_{j,k}}(v) \cap C| = 0$. We know that a vertex $v \in B_j$ has at most r neighbors in $A_{j,k}$. Since we have performed $(r+1)$-suitability on the color classes $C_1, C_2, \ldots, C_{r\Delta+1}$, there exists a permutation $\sigma \in \{\sigma_1, \sigma_2, \ldots, \sigma_p\}$ where C succeeds all the color classes that contain a neighbor of v. Thus, u succeeds all the neighbors of v in $A_{j,k}$ in both σ^1 and σ^2. Hence, u and v are non-adjacent in both $\tau(G_{j,k}, A_{j,k}, \sigma^1)$ and $\tau(G_{j,k}, A_{j,k}, \sigma^2)$. Suppose $|N_{G_{j,k}}(v) \cap C| = 1$. Let $\{w\} = N_{G_{j,k}}(v) \cap C$. There exists a permutation $\sigma \in \{\sigma_1, \sigma_2, \ldots, \sigma_p\}$ such that C succeeds all the other color classes that contain a neighbor of v in σ. Then, w succeeds all the other neighbors of v in $A_{j,k}$ in both σ^1 and σ^2. Since u succeeds w in one of σ^1 or σ^2, it follows that u and v are non-adjacent in either $\tau(G_{j,k}, A_{j,k}, \sigma^1)$ or $\tau(G_{j,k}, A_{j,k}, \sigma^2)$. □

Therefore, $\dim_{\mathrm{TH}}(G_{j,k}) \leqslant 2p$. Now from (1) we can write:

$$\dim_{\mathrm{TH}}(G^*[A, B]) \leqslant 1 + 2pt\ell$$

Before substituting the values of p, ℓ, and t in the above inequality, we simplify them below.

$$\begin{aligned}
p &\leqslant (r+1)2^{r+1} \ln\ln(r\Delta+1) \leqslant (r+1)2^{r+1} \ln\ln\left(r\Delta\left(1+\frac{1}{r\Delta}\right)\right) \\
&\leqslant (r+1)2^{r+1} \ln\ln\left(r\Delta \cdot e^{\frac{1}{r\Delta}}\right) = (r+1)2^{r+1} \ln\left(\ln r\Delta + \frac{1}{r\Delta}\right) \\
&\leqslant (r+1)2^{r+1} \ln\left(\ln\Delta\left(1+\frac{r\Delta\ln r+1}{r\Delta\ln\Delta}\right)\right) = (r+1)2^{r+1}\ln\left(\ln\Delta(1+o(1))\right) \\
&= (1+o(1))(r+1)2^{r+1}\ln\ln\Delta
\end{aligned}$$

$$
\begin{aligned}
t &= \lceil \ln(4d\Delta) \rceil \leqslant \ln 4 + \ln(d\Delta) + 1 = \left(1 + \frac{1+\ln 4}{\ln(d\Delta)}\right) \ln(d\Delta) \leqslant (1+o(1))\ln(d\Delta) \\
\ell &= \left\lceil e\left(\frac{ed}{r+1}\right)^{1+\frac{1}{r}} \right\rceil \leqslant e^{2+\frac{1}{r}} \cdot (\frac{d}{r+1})^{1+\frac{1}{r}} + 1 \leqslant e^3 \cdot \left(\frac{d}{r+1}\right)^{1+\frac{1}{r}} + 1 \\
&= (1+o(1))e^3 \left(\frac{d}{r+1}\right)^{1+\frac{1}{r}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\dim_{\mathrm{TH}}(G^*[A,B]) &\leqslant 1 + \Bigg(2 \cdot (1+o(1))(r+1)2^{r+1} \ln\ln\Delta \\
&\qquad \cdot (1+o(1))\ln(d\Delta) \cdot (1+o(1))e^3 \left(\frac{d}{r+1}\right)^{1+\frac{1}{r}}\Bigg) \\
&\leqslant 1 + \left(4e^3(1+o(1))d\ln(d\Delta)\ln\ln\Delta(2^r d^{\frac{1}{r}})\frac{1}{(r+1)^{\frac{1}{r}}}\right) \\
&\leqslant 1 + \left((4e^3+o(1))d\ln(d\Delta)\ln\ln\Delta(2e)^{\sqrt{\ln d}}\right) \\
&\leqslant (81+o(1))d\ln(d\Delta)\ln\ln\Delta(2e)^{\sqrt{\ln d}}. \qquad \square
\end{aligned}
$$

3.2 Proof of the main theorem

We need the following partitioning lemma by Scott and Wood.

Corollary 28 (Corollary 11 in [23]). *For every graph G with maximum degree $\Delta \geqslant 2$ and for all integers $d \geqslant 100\ln\Delta$ and $k \geqslant \frac{3\Delta}{d}$, there is a partition $V_1, \ldots, V_k$ of $V(G)$, such that $|N_G(v) \cap V_i| \leqslant d$ for each $v \in V(G)$ and $i \in [k]$.*

Theorem 29. *For a graph G with maximum degree Δ,*

$$\dim_{\mathrm{TH}}(G) \leqslant (24300+o(1))\Delta\ln^2\Delta\ln\ln\Delta(2e)^{\sqrt{(1+o(1))\ln\ln\Delta}},$$

when $\Delta \to \infty$.

Proof. Let $d = \lceil 100\ln\Delta \rceil$ and $k = \lceil \frac{3\Delta}{d} \rceil$. Using Corollary 28, we get a partition of $V(G)$ into k parts, $V_1, V_2, \ldots, V_k$, such that for any vertex $v \in V(G)$, $|N_G(v) \cap V_i| \leqslant d$, where $1 \leqslant i \leqslant k$. Since the maximum degree of $G[V_i]$ is at most d, we can properly color $G[V_i]$ using $d+1$ colors. Therefore, for all $i \in [k]$, each part V_i can further be partitioned into $d+1$ parts, namely $V_i^1, V_i^2, \ldots, V_i^{d+1}$, where each part is an independent set in G.

Claim 30.

$$G = \bigcap_{1\leqslant i\leqslant k} \bigcap_{1\leqslant j\leqslant d+1} G^*[V_i^j, V(G) \setminus V_i^j].$$

Proof. From the fact that V_i^j is an independent set in G and from the construction of $G^*[V_i^j, V(G) \setminus V_i^j]$, it is clear that $G^*[V_i^j, V(G) \setminus V_i^j]$, for $i \in [k], j \in [d+1]$, is a supergraph of G. Suppose that $uv \notin E(G)$. If $u, v \in V_i^j$ for some $i \in [k]$ and $j \in [d+1]$, then u and v are non-adjacent in $G^*[V_i^j, V(G) \setminus V_i^j]$. Otherwise, $u \in V_i^j$ for some $i \in [k]$ and $j \in [d+1]$, and $v \in V(G) \setminus V_i^j$, in which case u and v are non-adjacent in $G^*[V_i^j, V(G) \setminus V_i^j]$. $\square$

Applying Lemma 26 we can write,

$$\begin{aligned}
\dim_{\mathrm{TH}}(G) &\leqslant k \cdot (1 + o(1))d \cdot (81 + o(1))d \ln(d\Delta) \ln\ln\Delta (2e)^{\sqrt{\ln d}} \\
&\leqslant (243 + o(1))\Delta d \ln\Delta \ln\ln\Delta (2e)^{\sqrt{\ln d}} \quad \Bigg(\text{since } k = \left\lceil \frac{3\Delta}{d} \right\rceil \text{ and} \\
&\qquad\qquad \ln(d\Delta) \leqslant (1 + o(1)) \ln\Delta\Bigg) \\
&\leqslant (24300 + o(1))\Delta \ln^2\Delta \ln\ln\Delta (2e)^{\sqrt{(1+o(1))\ln\ln\Delta}} \\
&\qquad\qquad (\text{since } d = \lceil 100 \ln\Delta \rceil). \qquad \square
\end{aligned}$$

Since

$$(2e)^{\sqrt{(1+o(1))\ln\ln\Delta}} \ln\ln\Delta = (\ln\Delta)^{\frac{\ln(2e)\sqrt{(1+o(1))\ln\ln\Delta}}{\ln\ln\Delta} + \frac{\ln\ln\ln\Delta}{\ln\ln\Delta}} = \ln^{o(1)}\Delta$$

we get the following corollary.

Corollary 31.

$$\dim_{\mathrm{TH}}(\Delta) \in O(\Delta \ln^{2+o(1)} \Delta).$$

4 Threshold dimension and degeneracy

Given a graph G and a positive integer k, an ordering of the vertices of G such that no vertex has more than k neighbors after it is called a *k-degenerate ordering* of G. We say a graph is *k-degenerate* if it has a k-degenerate ordering. The minimum k such that G is k-degenerate is called the *degeneracy* of G. From its definition, it is clear that the degeneracy of a graph is at most its maximum degree. In this section, we derive upper bounds on the threshold dimension of a graph in terms of its degeneracy. The techniques we adopt are mostly inspired by those in [2].

Throughout this section, we shall assume that G is a k-degenerate graph on n vertices with vertex set $\{v_1, v_2, \ldots, v_n\}$ and that $v_1, v_2, \ldots, v_n$ is a k-degenerate ordering of G. Thus, for each $i \in \{1, 2, \ldots, n\}$, $|N_G(v_i) \cap \{v_{i+1}, v_{i+2}, \ldots, v_n\}| \leqslant k$. The vertices in $N_G(v_i) \cap \{v_{i+1}, v_{i+2}, \ldots, v_n\}$ are called the *forward neighbors* of v_i. Let $i < j$ and $v_i v_j \notin E(G)$. A coloring f of the vertices of G is *desirable* for the non-adjacent pair (v_i, v_j) if (i) f is a proper coloring, and (ii) $f(v_j) \neq f(v_t)$, for all neighbors v_t of v_i such that $t > j$.

Lemma 32. *Let G be a k-degenerate graph on n vertices and let $v_1, v_2, \ldots, v_n$ be a k-degenerate ordering of G. Let $r = \lceil \ln n \rceil$. Then there is a collection $\{f_1, \ldots, f_r\}$, where each $f_i \colon V(G) \to [10k]$ is a proper coloring of the vertices of G, such that for every non-adjacent pair (v_i, v_j), where $i < j$, there exists an $\ell \in [r]$ such that f_ℓ is a desirable coloring for the pair (v_i, v_j).*

Proof. We explain a randomized procedure for constructing the coloring f_1 below. Start coloring the vertices from v_n and color them all the way down to v_1 in the following way. Assume we have colored the vertices v_n to v_{i+1} and are about to color v_i. From the set of $10k$ colors, remove the colors that have been assigned to the forward neighbors of v_i. This leaves us with a set of at least $9k$ colors. Uniformly at random, choose one color from this set and assign it to v_i. This completes our description of the construction of the coloring f_1. The procedure ensures that f_1 is a proper coloring. Independently, repeat the above procedure to construct the colorings $f_2, f_3, \ldots, f_r$.

Consider a non-adjacent pair (v_i, v_j), where $i < j$. The probability that f_1 is not a desirable coloring for this pair is equal to the probability that a forward neighbor of v_i that is after v_j in the k-degenerate ordering gets the same color as that of v_j. This probability is at most $k/9k = 1/9$. Let $A_{i,j}$ denote the bad event that none of the colorings $f_1, f_2, \ldots, f_r$ is a desirable coloring for the pair (v_i, v_j). Then, $Pr[A_{i,j}] \leqslant 1/9^r < 1/n^2$. Applying the union bound, $Pr[\bigcup_{v_iv_j \notin E(G),\ i<j} A_{i,j}] \leqslant \sum\limits_{v_iv_j \notin E(G),\ i<j} Pr[A_{i,j}] < \binom{n}{2}\frac{1}{n^2} < 1$. Thus, the statement of the lemma holds with non-zero probability. □

Theorem 33. *Let G be a k-degenerate graph on n vertices. Then,* $\dim_{\mathrm{TH}}(G) \leqslant 10k \ln n$.

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ and let $\sigma \colon v_1, v_2, \ldots, v_n$ be a k-degenerate ordering of G. Let $\{f_1, \ldots, f_{\lceil \ln n \rceil}\}$ be the collection of proper colorings of $V(G)$, where each coloring uses at most $10k$ colors, given by Lemma 32. For each coloring f_a, $a \in [\lceil \ln n \rceil]$, and each color $b \in [10k]$, we construct a threshold supergraph $T_{a,b}$ of G as follows. Let $C_b^a = \{v \in V(G) \colon f_a(v) = b\}$. Since f_a is a proper coloring, C_b^a is an independent set. We define $T_{a,b} := \tau(G, C_b^a, \sigma|_{C_b^a})$ (see Definition 9 and Proposition 10).

We claim that $G = \bigcap_{a \in [\lceil \ln n \rceil],\ b \in [10k]} T_{a,b}$. Since each $T_{a,b}$ is a supergraph of G, all we need to do is show that for every non-adjacent pair (v_i, v_j) in G, where $i < j$, there is a threshold supergraph in our collection that does not contain the edge v_iv_j. Assume f_a is a desirable coloring for (v_i, v_j) and $f_a(v_j) = b$ (Lemma 32 guarantees that such a coloring exists). Then, we claim that $v_iv_j \notin E(T_{a,b})$. If $f_a(v_i) = b$, then $v_iv_j \notin E(T_{a,b})$ as C_b^a is an independent set in $T_{a,b}$. Suppose $f_a(v_i) \neq b$. Since no neighbor u of v_i that is after v_j in the k-degenerate ordering has $f_a(u) = b$, all the neighbors of v_i in C_b^a appear before v_j in the ordering $\sigma|_{C_b^a}$. Thus, $v_iv_j \notin E(T_{a,b})$. This completes the proof of the theorem. □

4.1 Random graphs

By a random graph, we mean a graph that is chosen from some probability distribution on graphs. Please refer [3] for more details on random graphs. The two random graph models that are extensively used in the literature are (i) $\mathcal{G}(n, m)$: where a graph is drawn uniformly at random from the collection of all graphs on n vertices and m edges, and (ii) $\mathcal{G}(n,p)$: where a graph on n vertices is obtained by selecting each possible edge independently with probability p. The following lemma was proved in [2].

Lemma 34 (Lemma 12 in [2])**.** *For a random graph $G \in \mathcal{G}(n,p)$, where $p = \frac{c}{n-1}$ and $1 \leqslant c \leqslant n-1$, $Pr[G$ is $4ec$-degenerate$] \geqslant 1 - \frac{1}{\Omega(n^2)}$.*

Applying Lemma 34 and Theorem 33, we get the following lemma.

Lemma 35. *For a random graph $G \in \mathcal{G}(n,p)$, where $p = \frac{c}{n-1}$ and $1 \leqslant c \leqslant n-1$, $Pr[\dim_{\mathrm{TH}}(G) \in O(c \ln n)] \geqslant 1 - \frac{1}{\Omega(n^2)}$.*

It is known that (see page 35 of [3])

$$P_m(Q) \quad \leqslant \quad 3\sqrt{m} P_p(Q) \tag{2}$$

where (i) Q is a property of graphs of order n, (ii) $P_m(Q)$ is the probability that Property Q is satisfied by a graph $G \in \mathcal{G}(n,m)$, and (iii) $P_p(Q)$ is the probability that Property Q is satisfied by a graph $G \in \mathcal{G}(n,p)$ with $p = \frac{m}{\binom{n}{2}} = \frac{2m/n}{n-1}$. Assume $m \geqslant n/2$. Then, $p = \frac{2m/n}{n-1} \geqslant \frac{1}{n-1}$ and by Lemma 35, $Pr[\dim_{\mathrm{TH}}(G) \notin O(\frac{2m}{n} \ln n)] \leqslant \frac{1}{\Omega(n^2)}$. Applying Equation 2, for a random graph $G \in \mathcal{G}(n,m)$, $m \geqslant n/2$, $Pr[\dim_{\mathrm{TH}}(G) \notin O(\frac{2m}{n} \ln n)] \leqslant \frac{3\sqrt{m}}{\Omega(n^2)} \leqslant \frac{1}{\Omega(n)}$. We thus have the following theorem.

Theorem 36. *For a random graph $G \in \mathcal{G}(n,m)$, $m \geqslant n/2$, $Pr[\dim_{\mathrm{TH}}(G) \in O(\frac{2m}{n} \ln n)] \geqslant 1 - \frac{1}{\Omega(n)}$. In other words, $Pr[\dim_{\mathrm{TH}}(G) \in O(d_{av} \ln n)] \geqslant 1 - \frac{1}{\Omega(n)}$, where d_{av} denotes the average degree of G.*

4.2 Graphs of high girth

The *girth* of a graph is the length of a smallest cycle in it. We assume that if the graph is acyclic, then its girth is ∞. We apply Theorem 33 to prove an upper bound for the threshold dimension of a graph in terms of its girth and the number of vertices. It is known that for any graph G on n vertices having girth at least $g+1$, the average degree is less than $n^{\frac{1}{g/2}} + 1$ when g is even and less than $n^{\frac{1}{(g+1)/2-1}} + 1$ when g is odd (see Section 4.1 of [13]). Thus, any graph G on n vertices having girth at least $g+1$ has average degree less than $n^{\frac{1}{\lfloor g/2 \rfloor}} + 1$.

Let G be any graph on n vertices having girth at least $g+1$. Let H be any subgraph of G on s vertices. Clearly, H has girth at least $g+1$. By the observation above, we have that the average degree of H is less than $s^{\frac{1}{\lfloor g/2 \rfloor}} + 1 \leqslant n^{\frac{1}{\lfloor g/2 \rfloor}} + 1$. Since the minimum degree of a graph is an integer and is at most the average degree, we have that the minimum degree of H is at most $k = \left\lceil n^{\frac{1}{\lfloor g/2 \rfloor}} \right\rceil$. Thus G is k-degenerate. Hence, we get the following corollary to Theorem 33.

Corollary 37. *Let G be a graph on n vertices with girth at least $g+1$. Then, $\dim_{\mathrm{TH}}(G) \leqslant 10 \left\lceil n^{\frac{1}{\lfloor g/2 \rfloor}} \right\rceil \ln n$.*

The bipartite graph G obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$ is known to have boxicity equal to $\lceil \frac{n}{2} \rceil$ (see [25]). From Observation 6 and by applying Corollary 37 with $g = 3$, we have $\frac{n}{2} \leqslant \dim_{\mathrm{TH}}(G) = O(n \ln n)$. This graph shows that the bound given by Corollary 37 is asympotically tight upto a multiplicative factor of $\ln n$; i.e. we cannot expect to get an upper bound of the form $O\left(n^{\frac{1-\epsilon}{\lfloor g/2 \rfloor}}\right)$, for any $\epsilon > 0$, for the threshold dimension of a graph with girth at least $g + 1$.

5 Threshold dimension and vertex cover number

A *vertex cover* of G is a set of vertices $S \subseteq V(G)$ such that for all $e \in E(G)$, at least one endpoint of e is in S. A *minimum vertex cover* of G is a vertex cover of G of the smallest cardinality. We use $\beta(G)$ to denote the cardinality of a minimum vertex cover of G, more commonly referred to as the *vertex cover number* of G. In this section, we prove a tight upper bound for the threshold dimension of a graph in terms of its vertex cover number.

Proposition 38. *For a graph G,* $\dim_{\mathrm{TH}}(G) \leqslant \beta(G)$.

Proof. Let B denote a minimum vertex cover of G, and $b := |B| = \beta(G)$. Then, $A := V(G) \setminus B$ is a maximum independent set in G. Let $B = \{v_1, v_2, \ldots, v_b\}$. For each $i \in [b-1]$, we construct threshold supergraph $G_i := \tau(G, \{v_i\}, \sigma_i)$, where σ_i denotes the trivial ordering of the vertex inside the singleton set $\{v_i\}$. To construct the last threshold supergraph G_b, let π_b be an ordering of the vertices of A where every vertex in $N_G(v_b) \cap A$ appear before every vertex in $A \setminus N_G(v_b)$. We define $G_b := \tau(G, A, \pi_b)$. We claim that $G = \bigcap_{i=1}^{b} G_i$. We know from our construction that every G_i is a supergraph of G. Suppose $xy \notin E(G)$, for some $x, y \in V(G)$. If $x, y \in A$, then $xy \notin E(G_b)$. Assume at least one of x or y belongs to B. If $x = v_i$ or $y = v_i$, for some $i < b$, then $xy \notin E(G_i)$. We are left with the case when $x = v_b$ and $y \in A$ (or vice versa). In this case, it can be verified that $xy \notin E(G_b)$. □

Since $\alpha(G) = |V(G)| - \beta(G)$, by combining Corollary 4(a) with Proposition 38, we get the following theorem.

Theorem 39. *For a graph G on n vertices,* $\dim_{\mathrm{TH}}(G) \leqslant n - \max\{\alpha(G), \omega(G)\}$.

In Ramsey theory, $R(k, k)$ denotes the smallest positive integer n such that every graph on n vertices has either an independent set of size k or a clique of size k. A classical result due to Erdős and Szekeres [10] states that $R(k+1, k+1) \leqslant \binom{2k}{k} \leqslant 4^k$. This implies that if $n \geqslant 4^k$, every graph on n vertices has either an independent set or a clique (or both) of size at least $k + 1$. In other words, every graph on n vertices has a clique or an independent set on $k + 1$ vertices for every $k \leqslant \lfloor \log_4(n) \rfloor$. As $\log_4(n) = \frac{\ln n}{\ln 4} > 0.72 \ln n$, we have that every graph on n vertices contains a clique or independent set containing at least $\lfloor 0.72 \ln n \rfloor + 1$ vertices. This gives us the following corollary.

Corollary 40. *Every graph G on n vertices satisfies* $\dim_{\mathrm{TH}}(G) \leqslant n - \lfloor 0.72 \ln n \rfloor - 1$.

Tightness of the bound in Theorem 39

It can be verified that the graph H on $2n$ vertices having threshold dimension n constructed in Example 22 satisfies $\alpha(H) = \omega(H) = \beta(H) = n$. Hence, the bounds in Theorem 39 and Proposition 38 are tight.

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