# The Degree Threshold for Covering with Connected 3-Graphs with 3 Edges

Yue Ma[a] Xinmin Hou[b,c] Zhi Yin[b]

Submitted: Aug 19, 2023; Accepted: Oct 31, 2024; Published: Feb 28, 2025


**Abstract**

Given two $r$-uniform hypergraphs $F$ and $H$, we say that $H$ has an $F$-covering if every vertex in $H$ is contained in a copy of $F$. Let $c_i(n, F)$ be the least integer such that every $n$-vertex $r$-graph $H$ with $\delta_i(H) > c_i(n, F)$ has an $F$-covering. Falgas-Ravry, Markström and Zhao (*Combin. Probab. Comput.*, 2021) asymptotically determined $c_1(n, K_4^{(3)-})$, where $K_4^{(3)-}$ is obtained by deleting an edge from the complete 3-graph on 4 vertices. Later, Tang, Ma and Hou (*Electron. J. Combin.*, 2023) asymptotically determined $c_1(n, C_6^{(3)})$, where $C_6^{(3)}$ is the linear triangle, i.e. $C_6^{(3)} = ([6], \{123, 345, 561\})$. In this paper, we determine $c_1(n, F_5)$ asymptotically, where $F_5$ is the generalized triangle, i.e. $F_5 = ([5], \{123, 124, 345\})$. We also determine the exact values of $c_1(n, F)$, where $F$ is any connected 3-graph with 3 edges and $F \notin \{K_4^{(3)-}, C_6^{(3)}, F_5\}$.

**Mathematics Subject Classifications:** 05C35, 05C07, 05C65

## 1 Introduction

Given a positive integer $k \geqslant 2$, a *$k$-uniform hypergraph* (or a $k$-graph) $H = (V, E)$ consists of a vertex set $V = V(H)$ and an edge set $E = E(H) \subset \binom{V}{k}$, where $\binom{V}{k}$ denotes the set of all $k$-element subsets of $V$. For simplicity, we write graph for 2-graph. Let $H = (V, E)$ be a simple $k$-graph (with no multiple edges). For any $S \subseteq V(H)$, let $N_H(S) = \{T \subseteq V(H) \backslash S : T \cup S \in E(H)\}$ and the degree $d_H(S) = |N_H(S)|$. For $1 \leqslant i \leqslant k - 1$, the *minimum $i$-degree* of $H$, denoted by $\delta_i(H)$, is the minimum of $d_H(S)$ over all $S \in \binom{V(H)}{i}$. We also call $\delta_1(G)$ the minimum degree of $G$. The *link graph* of a vertex $x$ in $V$, denoted by $H_x$, is a $(k-1)$-graph $H_x = (V(G) \backslash \{x\}, N_H(x))$.

[a]School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing, Jiangsu 210094, China (`yma@njust.edu.cn`).
[b] School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China (`xmhou@ustc.edu.cn`, `yinzhi@mail.ustc.edu.cn`).
[c]Hefei National Laboratory, University of Science and Technology of China, Hefei, Anhui 230088, China.

For $r \geqslant 2$, a *complete r-graph* on $n$ vertices, denoted by $K_n^{(r)}$, is an $r$-graph on $[n]$ with the edge set $\binom{[n]}{r}$. For a vertex set $V$, we also write $K^{(r)}[V]$ for the complete $r$-graph on $V$. We write $K_n$ for $K_n^{(2)}$ and $K[V]$ for $K^{(2)}[V]$ for short. For an $r$-graph $G$ with $U \subset V(G)$, let $G[U] = (U, E(G) \cap E(K^{(r)}[U]))$ and $G - U = G[V(G) \backslash U]$. Also, given two $r$-graphs $G$ and $H$, let $G \cup H$ be the vertex-disjoint union of $G$ and $H$. Let $tH := \bigcup_{i=1}^{t} G_i$ for some $t \geqslant 2$ and $r$-graphs $H, G_1, \ldots, G_t$ if $G_i \cong H$ for $i \in [t]$.

Given a $k$-graph $F$, we say a $k$-graph $H$ has an $F$-covering if each vertex of $H$ is contained in some copy of $F$. For $1 \leqslant i \leqslant k-1$, the *i-degree threshold* for $F$-covering is defined as

$$c_i(n, F) := \max\{\delta_i(G) : G \text{ is a } k\text{-graph on } n \text{ vertices with no } F\text{-covering}\}.$$

We further let the *i-degree F-covering density* be the limit

$$c_i(F) := \lim_{n \to \infty} \frac{c_i(n, F)}{\binom{n-i}{k-i}}.$$

There are two types of extremal problems related to the covering problem. Given a $k$-graph $F$, a $k$-graph $H$ is $F$-free if $H$ does not contain a copy of $F$ as a subgraph. For $0 \leqslant i \leqslant k-1$, define

$$ex_i(n, F) := \max\{\delta_i(G) : G \text{ is } F\text{-free and } |V(G)| = n\}, \text{ and } \pi_i(F) := \frac{ex_i(n, F)}{\binom{n-i}{k-i}},$$

where $\delta_0(G) := |E(G)|$. The quantities $ex_0(n, F)$ and $\pi_0(F)$ are known as the *Turán number* and the *Turán density* of $F$ respectively. For Turán problem on hypergraphs, one can refer to a survey given by Keevash [9].

Given two $k$-graphs $F$ and $H$, an $F$-tiling in $H$ is a spanning subgraph of $H$ which consists of vertex-disjoint copies of $F$. For $1 \leqslant i \leqslant k-1$ and $n \equiv 0 \mod |V(F)|$, define

$$t_i(n, F) := \max\{\delta_i(G) : G \text{ is a } k\text{-graph on } n \text{ vertices with no } F\text{-tiling}\}.$$

The tiling problem in hypergraphs is also widely studied. We recommend a survey given by Zhao [15].

Trivially, for $1 \leqslant i \leqslant k-1$,

$$ex_i(n, F) \leqslant c_i(n, F) \leqslant t_i(n, F),$$

which makes the covering problem an interesting but different extremal problem from Turán problem and the tiling problem.

For a graph $F$, the $F$-covering problem was solved asymptotically in [14] by showing that $c_1(F) = \frac{\chi(F)-2}{\chi(F)-1}$, where $\chi(F)$ is the chromatic number of $F$.

For $r$-uniform hypergraphs with $r \geqslant 3$, there are also some works related, most of them focus on $r = 3$. Here are some exact results for $c_2(n, F)$ and $c_2(F)$ in 3-graphs.

- (Falgas-Ravry, Zhao [6]) For $n > 98$, $c_2(n, K_4^{(3)}) = \lfloor \frac{2n-5}{3} \rfloor$.

- (Yu, Hou, Ma, Liu [13]) $c_2(n, K_4^{(3)-}) = \lfloor \frac{n}{3} \rfloor$ and $c_2(n, K_5^{(3)-}) = \lfloor \frac{2n-2}{3} \rfloor$, where $K_k^{(r)-}$ ($k \geqslant r \geqslant 2$) is an $r$-graph obtained from $K_k^{(r)}$ by deleting an edge.
- (Falgas-Ravry, Zhao [6]) $c_2(C_5^{(3)}) = \frac{1}{2}$, where $C_5^{(3)} = ([5], \{123, 234, 345, 451, 512\})$.

For $c_1(n, F)$ and $c_1(F)$ in 3-graphs, some know results are listed as follows.

- (Falgas-Ravry, Markström, Zhao [5]) $c_1(K_4^{(3)-}) = \frac{\sqrt{13}-1}{6}$.
- (Tang, Ma, Hou [12]) $c_1(C_6^{(3)}) = \frac{3-2\sqrt{2}}{2}$, where $C_6^{(3)} = ([6], \{123, 345, 561\})$.
- (Falgas-Ravry, Markström, Zhao [5]) $\frac{19}{27} \leqslant c_1(K_4^{(3)}) \leqslant \frac{19}{27} + 7.4 \times 10^{-9}$.
- (Falgas-Ravry, Markström, Zhao [5]) $\frac{5}{9} \leqslant c_1(C_5^{(3)}) \leqslant 2 - \sqrt{2}$.
- (Gu, Wang [8]) For $n \geqslant 5$, $\frac{n^2}{9} \leqslant c_1(n, F_5) \leqslant \frac{n^2}{6} + \frac{5}{6}n - 3$, where $F_5 = ([5], \{123, 124, 345\})$.
- (Gu, Wang [8]) For $n \geqslant 8$, $n - 2 \leqslant c_1(n, LP_3) \leqslant n + 4$, where $LP_3 = ([7], \{123, 345, 567\})$.

In this article, we focus on 3-graphs with 3 edges. Let $H$ be a hypergraph. We say $H$ is *connected* if for any pair of vertices $\{u, v\} \subset \binom{V(H)}{2}$, we can find a sequence of edges, say $e_1, e_2, \ldots, e_t \in E(H)$, with $u \in e_1$, $v \in e_t$ and $e_i \cap e_{i+1} \neq \emptyset$ for any $i \in [t-1]$. A maximal connected subgraph for any hypergraph $H$ is called a *component*. Note that a connected hypergraph consists of a unique component.

By a simple enumeration, one can check that there are only 9 kinds of connected 3-graphs with 3 edges. We list all of them in Figure 1.

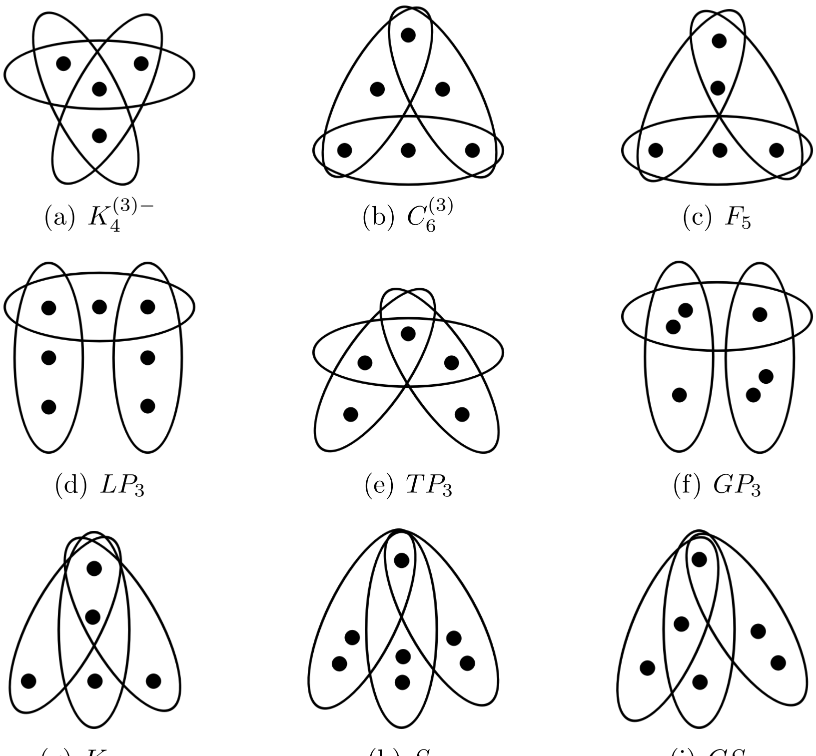
In particular, $K_4^{(3)-}$ and $C_6^{(3)}$ are two examples for connected 3-graphs with 3 edges, whose 1-degree covering densities are already know as mentioned above. Another important example is called a *generalized triangle*, denoted by $F_5$, which is a 3-graph on the vertex set [5] with the edge set $\{123, 124, 345\}$. In 1983, Frankl and Füredi [7] gave the Turán number for $F_5$.

**Theorem 1** ((Frankl, Füredi [7])). *For $n > 3000$, $ex_0(n, F_5) = \lfloor \frac{n}{3} \rfloor \lfloor \frac{n+1}{3} \rfloor \lfloor \frac{n+2}{3} \rfloor$. In particular, $\pi_0(F_5) = \frac{2}{9}$.*

Note that the condition for $n$ in Theorem 1 was later improved to $n > 33$ by Keevash and Mubayi [10]. There are also some other extremal results related to $F_5$, we refer to [1, 2, 3] for example.

To give the extremal construction for Theorem 1, we need some definitions. For two families of sets $\mathcal{A}$ and $\mathcal{B}$, define $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$. For $r \geqslant 2$, a *complete $r$-partite $r$-graph with partition set $V_1, V_2, \ldots, V_r$*, denoted by $K[V_1, V_2, \ldots, V_r]$, is an $r$-graph on $\bigcup_{i=1}^{r} V_i$ with the edge set

$$E(K[V_1, V_2, \ldots, V_r]) = \binom{V_1}{1} \vee \binom{V_2}{1} \vee \cdots \vee \binom{V_r}{1}.$$



(a) $K_4^{(3)-}$ (b) $C_6^{(3)}$ (c) $F_5$

(d) $LP_3$ (e) $TP_3$ (f) $GP_3$

(g) $K_{1,1,3}$ (h) $S_3$ (i) $GS_3$

Figure 1: All possible connected 3-graphs with 3 edges

For an $r$-graph $H$ with $\bigcup_{i=1}^r V_i \subset V(H)$, let $G[V_1, \dots, V_r] = (\bigcup_{i=1}^r V_i, E(H) \cap E(K[V_1, \dots, V_r]))$. If $|V_i| = n_i$ for $i \in [r]$, we write $K_{n_1,\dots,n_r}$ for $K[V_1, \dots, V_r]$. In particular, $K_{1,1} = K_2$ and $K_{1,1,3} = ([5], \{123, 124, 125\})$.

One can check that $K_{\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n+1}{3} \rfloor, \lfloor \frac{n+2}{3} \rfloor}$ on $n$ vertices contains no copy of $F_5$ as its subgraph, which is an extremal construction for Theorem 1. Hence we can easily deduce from Theorem 1 that $\pi_1(F_5) = \frac{2}{9}$. This leads to $c_1(F_5) \geqslant \frac{2}{9}$. In fact, the result of Gu and Wang [8] about $F_5$ implies that $\frac{2}{9} \leqslant c_1(F_5) \leqslant \frac{1}{3}$.

### 1.1 Results

In this paper, we verify the exact value that $c_1(F_5) = \frac{1}{4}$.

**Theorem 2.** *For $n \geqslant 5$, $\frac{1}{8}n^2 - \sqrt{2}n < c_1(n, F_5) < \frac{1}{8}n^2 + \frac{5}{4}n$. In particular, $c_1(F_5) = \frac{1}{4}$.*

For $k \geqslant 1$, a *linear star* with $k$ edges, denoted by $S_k$, is a 3-graph on $[2k+1]$ with edge set $\{123, 145, 167, \dots, 1(2k)(2k+1)\}$. In particular, $S_3 = ([7], \{123, 145, 167\})$.

A *path of length* $k-1$ for some $k \geqslant 2$, denoted by $P_k$, is a graph on $[k]$ whose edge set is $\{12, 23, 34, \dots, (k-1)k\}$. A *cycle of length* $k$ is a graph on $[k]$ with edge set $\{12, 23, \dots, (k-1)k, k1\}$. In 3-graph, however, we have several different definitions for a path. For $k \geqslant 1$, a *linear $k$-path*, denoted by $LP_k$, is a 3-graph on $[2k+1]$ with the edge set $\{123, 345, 567, \dots, (2k-1)2k(2k+1)\}$. In particular, $LP_3 = ([7], \{123, 345, 567\})$.

For $k \geqslant 1$, a *tight $k$-path*, denoted by $TP_k$, is a 3-graph on $[k+2]$ with the edge set $\{123, 234, 345, \ldots, k(k+1)(k+2)\}$. In particular, $TP_3 = ([5], \{123, 234, 345\})$.

There are only two kinds of connected 3-graphs with 3 edges other than $K_4^{(3)-}$, $C_6^{(3)}$, $F_5$, $LP_3$, $TP_3$, $K_{1,1,3}$ and $S_3$. We use $GP_3$ and $GS_3$ to denote them:

$$GP_3 = ([6], \{123, 234, 456\}) \text{ and } GS_3 = ([6], \{123, 124, 156\}).$$

We determine the exact values of $c_1(n, F)$, where $F \in \{LP_3, TP_3, GP_3, K_{1,1,3}, S_3, GS_3\}$ in this paper.

**Theorem 3.** *(1) For $n \geqslant 13$, $c_1(n, LP_3) = n - 2$.*
*(2) For $n \geqslant 8$,*

$$c_1(n, TP_3) = \begin{cases} n-1 & n \equiv 1 \mod 3; \\ n-2 & n \equiv 0, 2 \mod 3. \end{cases}$$

*(3) For $n \geqslant 17$, $c_1(n, GP_3) = n - 2$.*
*(4) For $n \geqslant 9$, $c_1(n, K_{1,1,3}) = n - 1$.*
*(5) For $n \geqslant 11$, $c_1(n, S_3) = n - 1$.*
*(6) For $n \geqslant 13$, $c_1(n, GS_3) = \lfloor \frac{n-1}{2} \rfloor$.*

The rest of the paper is arranged as follows. In Section 2, we prove Theorem 2. In Section 3, we show the other cases in turn and finish the proof of Theorem 3. We give some concluding remarks in Section 4.

## 2 $F_5$: proof of Theorem 2

### 2.1 Lower bound

**Construction 1:** Let $H_1 = (V_1, E_1)$ be a 3-graph with $V_1 = \{u\} \sqcup X \sqcup Y \sqcup Z$, and

$$\begin{aligned} E_1 \quad = \quad & \left(\{\{u\}\} \vee \binom{X}{1} \vee \binom{Y}{1}\right) \cup \left(\binom{Z}{1} \vee \binom{X}{1} \vee \binom{Y}{1}\right) \\ & \cup \left(\binom{X}{1} \vee E_X\right) \cup \left(\binom{Y}{1} \vee E_Y\right) \cup \binom{Z}{3}, \end{aligned}$$

where $|X| = |Y| = \lfloor \frac{\sqrt{2}}{4} n \rfloor - 1$, $E_X \sqcup E_Y = \binom{Z}{2}$ and $||E_X| - |E_Y|| \leqslant 1$.

**Observation 4.** *$\delta_1(H_1) > \frac{1}{8}n^2 - \sqrt{2}n$ and $H_1$ has no $F_5$ covering $u$.*

*Proof.* It is easy to check that $H_1$ has no $F_5$ covering $u$. Let $a = |X| = |Y| = \lfloor \frac{\sqrt{2}}{4} n \rfloor - 1$ and $b = |Z| = n - 1 - 2a$. Since $E_X \sqcup E_Y = \binom{Z}{2}$ and $||E_X| - |E_Y|| \leqslant 1$, $|E_X|, |E_Y| \geqslant \lfloor \frac{1}{2}\binom{b}{2} \rfloor \geqslant \frac{b(b-1)}{4} - \frac{1}{2}$. Note that the case of $n = 5$ is apparently true. For $n \geqslant 6$, we have $a \geqslant 1$ and $b \geqslant 3$. Choose $v \in V(H_1)$.

If $v = u$, then

$$d_{H_1}(v) = a^2 > (\frac{\sqrt{2}}{4} n - 2)^2 = \frac{1}{8}n^2 - \sqrt{2}n + 4 > \frac{1}{8}n^2 - \sqrt{2}n.$$

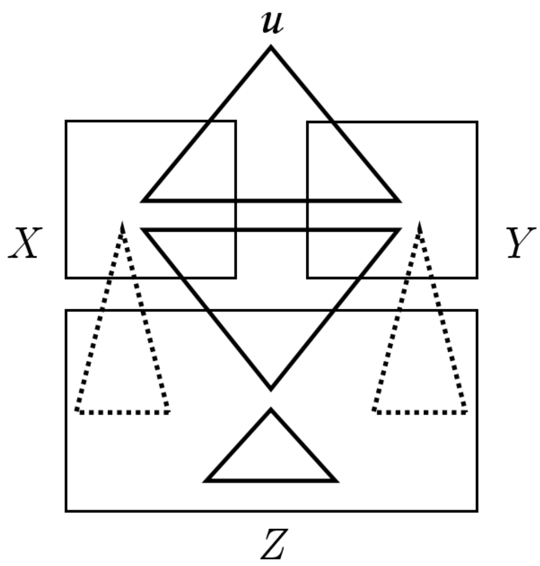



Figure 2: Construction 1

If $v \in X \cup Y$, then

$$\begin{aligned} d_{H_1}(v) &\geqslant a + ab + \frac{b(b-1)}{4} - \frac{1}{2} = \frac{n^2 - 3n}{4} - (a - \frac{3}{2})a \\ &\geqslant \frac{n^2 - 3n}{4} - (\frac{\sqrt{2}}{4}n - \frac{5}{2})(\frac{\sqrt{2}}{4}n - 1) \\ &= \frac{1}{8}n^2 + \frac{7\sqrt{2} - 6}{8}n - \frac{5}{2} > \frac{1}{8}n^2 - \sqrt{2}n \end{aligned}$$

If $v \in Z$, then

$$d_{H_1}(v) > a^2 + \binom{b-1}{2} > a^2 = d_H(u) > \frac{1}{8}n^2 - \sqrt{2}n.$$

Therefore, $\delta_1(H_1) > \frac{1}{8}n^2 - \sqrt{2}n$. □

## 2.2 Upper bound

For any graph $G$, let $\mathcal{E}(G) = \{uv \in \binom{V(G)}{2} : N_G(v) \cap N_G(u) \neq \emptyset\}$ be the graph on $V(G)$ whose edges are all pairs of vertices sharing at least one common neighbor. We have the following result about the number of edges in $\mathcal{E}(G)$.

**Lemma 5.** *For any graph $G$ on $n$ vertices, $|E(\mathcal{E}(G))| \geqslant |E(G)| - \frac{n}{2}$.*

*Proof.* We prove by induction on $n$. Firstly, for $1 \leqslant n \leqslant 3$, the inequality is apparently true. Now let $G$ be a graph on $n \geqslant 4$ vertices and suppose the inequality holds for any graph on less than $n$ vertices. If $G$ is an empty graph, we are done. Otherwise, pick an edge $uv \in E(G)$. By deleting the vertices $u, v$ and all the incidence edges, we get

$$|E(G - \{u, v\})| = |E(G)| - d_G(u) - d_G(v) + 1.$$

On the other hand, the deletion must destroy all the edges incident with one of $u$ and $v$ in $\mathcal{E}(G)$. Note that $u$ (resp. $v$) is incident with all the vertices in $N_G(v) - \{u\}$ (resp. $N_G(u) = \{v\}$) within $\mathcal{E}(G)$. In other words,

$$|E(\mathcal{E}(G - \{u, v\}))| \leqslant |E(\mathcal{E}(G))| - d_G(u) - d_G(v) + 2.$$

Therefore, by induction,

$$\begin{aligned} |E(\mathcal{E}(G))| &\geqslant |E(\mathcal{E}(G - \{u, v\}))| + d_G(u) + d_G(v) - 2 \\ &\geqslant |E(G - \{u, v\})| - \frac{n-2}{2} + d_G(u) + d_G(v) - 2 \\ &= |E(G - \{u, v\})| + d_G(u) + d_G(v) - 1 - \frac{n}{2} \\ &= |E(G)| - \frac{n}{2}. \end{aligned}$$

This completes the proof. □

*Proof of Theorem 2.* It is sufficient to show that every 3-graph $H$ on $n$ vertices with $\delta_1(H) \geqslant \frac{1}{8}n^2 + \frac{5}{4}n$ has an $F_5$-covering.

Suppose on the contrary that there is a 3-graph $H$ on $n$ vertices with $\delta_1(H) \geqslant \frac{1}{8}n^2 + \frac{5}{4}n$ and a vertex $u \in V(H)$ which is not contained in any copy of $F_5$ in $H$. By definition, the link graph $H_u$ contains at least $\delta_1(H)$ edges, so it is not empty. We have the following key claim.

**Claim 6.** *Let $xy \in E(H_u)$ be an edge in $H_u$, then the four sets $E(H_u - \{x, y\})$, $E(H_x - \{u\})$, $E(H_y - \{u\})$ and $E(\mathcal{E}(H_u - \{x, y\}))$ are pairwise disjoint.*

*Proof.* (i) If $E(H_u - \{x, y\}) \cap E(H_x - \{u\}) \neq \emptyset$, we pick a pair $ab$ in it. By definition, $abx, abu, uxy \in E(H)$, which form a copy of $F_5$, a contradiction. The same thing holds for $E(H_u - \{x, y\})$ and $E(H_y - \{u\})$.
(ii) If $E(H_x - \{u\}) \cap E(H_y - \{u\}) \neq \emptyset$, we pick a pair $ab$ in it. Then $abx, aby, xyu$ form a copy of $F_5$, which is a contradiction.
(iii) If $E(H_x - \{u\}) \cap E(\mathcal{E}(H_u - \{x, y\})) \neq \emptyset$, we pick a pair $ab$ in it. By the definition of $\mathcal{E}(H_u - \{x, y\})$, there exists a vertex $c$ with $ac, bc \in E(H_u - \{x, y\})$. Thus, $uca, ucb, abx$ form a copy of $F_5$, a contradiction. The same thing holds for $E(H_y - \{u\})$ and $E(\mathcal{E}(H_u - \{x, y\}))$.
(iv) To complete the proof of this claim, we only need to show that $E(H_u - \{x, y\}) \cap E(\mathcal{E}(H_u - \{x, y\})) = \emptyset$. It is enough to show that there is no triangle in $H_u$. By (i) and (ii), $E(H_u - \{x, y\})$, $E(H_x - \{u\})$ and $E(H_y - \{u\})$ are pairwise disjoint for any $xy \in E(H_u)$. If there is a triangle $\{xy, xz, yz\} \subset E(H_u)$, then it is easy to see that

$E(H_x-\{u\})$, $E(H_y-\{u\})$, $E(H_z-\{u\})$ and $E(H_u-\{x,y,z\})$ are pairwise disjoint since $E(H_u-\{x,y,z\}) = E(H_u-\{x,y\}) \cap E(H_u-\{x,z\}) \cap E(H_u-\{y,z\})$. This means

$$|E(H_u-\{x,y,z\})|+|E(H_x-\{u\})|+|E(H_y-\{u\})|+|E(H_z-\{u\})| \leqslant |\binom{V(H_u)}{2}| = \binom{n-1}{2}.$$

Also, $|E(H_u-\{x,y,z\})| \geqslant \delta_1(H)-(3n-6)$ and $|E(H_w-\{u\})| \geqslant \delta_1(H)-(n-1)$ for $w \in \{x,y,z\}$. This gives $4\delta_1(H)-((3n-6)+3(n-1)) \leqslant \binom{n-1}{2}$, a contradiction by $\delta_1(H) \geqslant \frac{1}{8}n^2+\frac{5}{4}n$. □

Pick an edge $xy \in E(H_u)$. It is easy to check that $|E(H_u-\{x,y\})| \geqslant \delta_1(H)-(2n-3)$ and $|E(H_x-\{u\})| \geqslant \delta_1(H)-(n-1)$. By Lemma 5,

$$|E(\mathcal{E}(H_u-\{x,y\}))| \geqslant |E(H_u-\{x,y\})| - \frac{n-3}{2} \geqslant \delta_1(H) - (\frac{5}{2}n-\frac{9}{2}).$$

By Claim 6, $E(H_u-\{x,y\})$, $E(H_x-\{u\})$, $E(H_y-\{u\})$ and $E(\mathcal{E}(H_u-\{x,y\}))$ are pairwise disjoint. This means

$$|E(H_u-\{x,y\})|+|E(H_x-\{u\})|+|E(H_y-\{u\})|+|E(\mathcal{E}(H_u-\{x,y\}))| \leqslant |\binom{V(H_u)}{2}|.$$

Thus,

$$4\delta_1(H)-(2n-3)-2(n-1)-(\frac{5}{2}n-\frac{9}{2}) \leqslant \binom{n-1}{2},$$

a contradiction by $\delta_1(H) \geqslant \frac{1}{8}n^2+\frac{5}{4}n$. □

## 3 Other cases: proof of Theorem 3

### 3.1 $LP_3$

*Proof of (1).* For the lower bound, we simply consider the following 3-graph $G$ called a *trivial intersecting family* on $V(G)=\{0\}\cup[n-1]$ with edge set $E(G)=\{\{0\}\}\vee\binom{[n-1]}{2}$.

For the upper bound, suppose on the contrary that there is a 3-graph $H$ on $n \geqslant 13$ vertices with $\delta_1(H) \geqslant n-1$ while some vertex $u \in V(H)$ is not contained in any copy of $LP_3$ in $H$.

**Claim 7.** *We can find a copy of $K_{1,2}\cup K_{1,1}$ in the graph $H_u$.*

*Proof.* Note that $H_u$ is a graph on $n-1$ vertices with at least $\delta_1(H) \geqslant n-1$ edges. Pick $v \in V(H_u)$ with the maximum degree $d$ of $H_u$. By Handshaking Lemma, $(n-1)d \geqslant \sum_{x\in V(H_u)} d_{H_u}(x) = 2E(H_u) \geqslant 2(n-1)$. Thus, $d \geqslant 2$, and if $d=2$, then $d_{H_u}(x)=2$ for all the vertices $x \in V(H_u)$. Now suppose $d=2$ and $N_{H_u}(v)=\{x,y\}$. Since $d_{H_u}(x) = d_{H_u}(y)=2$, there are at most $2+2=4$ edges incident with at least one of $x$ and $y$ in $H_u$. Hence we can pick an edge $ab \in E(H_u-\{x,y\})$ since $|E(H_u-\{x,y\})| \geqslant |E(H_u)|-4>0$. Clearly, $v \neq a,b$. This means the three edges $vx, vy, ab$ form a $K_{1,2}\cup K_{1,1}$ in $H_u$. Suppose

$d \geqslant 4$. Pick 4 vertices $w, x, y, z \in N_{H_u}(v)$. Since $d \leqslant |V(H_u)| - 1 = n - 2$, there exists at least $|E(H_u)| - (n-2) \geqslant 1$ edge $ab \in E(H_u - \{v\})$. Note that $v \neq a, b$ and at least two of $w, x, y, z$ are not contained in $\{a, b\}$. Without loss of generality, suppose $w, x \notin \{a, b\}$, then the three edges $vw, vx, ab$ together form a $K_{1,2} \cup K_{1,1}$ in $H_u$. Thus, $d = 3$. Let $N_{H_u}(v) = \{x, y, z\}$. It is easy to see that there are at most $\binom{3}{2} = 3$ edges contained in $\{x, y, z\}$. Note that $|E(H_u - \{v\})| = |E(H_u)| - 3 > 3$. We can pick an edge $ab \in E(H_u - \{v\})$ which is not contained in $\{x, y, z\}$. In other words, $|\{a, b\} \cap \{x, y, z\}| \leqslant 1$, so we can pick two vertices in $\{x, y, z\}$, say $x$ and $y$, which are not in $\{a, b\}$. Hence, the three edges $vx, vy, ab$ form a $K_{1,2} \cup K_{1,1}$ in $H_u$. □

**Claim 8.** *We can find a copy of $K_{1,3} \cup K_{1,2}$ in the graph $H_u$.*

*Proof.* By Claim 7, we choose a set of 5 vertices $\{a, a_1, a_2, b, b_1\} \subset V(H_u)$ with $aa_1, aa_2, bb_1 \in E(H_u)$. We claim that $H_{a_1} - \{u\} \subset \binom{\{a,a_2,b,b_1\}}{2}$. Otherwise, there exists an edge $xy \in E(H_{a_1} - \{u\})$ with $|\{x, y\} \cap \{a, a_2, b, b_1\}| \leqslant 1$. If $\{x, y\} \cap \{a, a_2, b, b_1\} = \emptyset$, then $\{xya_1, a_1au, ubb_1\}$ is a copy of $LP_3$ in $H$, a contradiction. Thus, exactly one of $a, a_2, b, b_1$ is contained in $\{x, y\}$. Without loss of generality, suppose that $x$ is this vertex. If $x = a$, then $\{a_1ya, aa_2u, ubb_1\}$ is a copy of $LP_3$; If $x = a_2$, then $\{a_1ya_2, a_2au, ubb_1\}$ is a copy of $LP_3$; If $x = b$, then $\{a_1yb, bb_1u, uaa_2\}$ is a copy of $LP_3$; If $x = b_1$, then $\{a_1yb_1, b_1bu, uaa_2\}$ is a copy of $LP_3$. Any of the four cases leads to a contradiction. Therefore, $H_{a_1} - \{u\} \subset \binom{\{a,a_2,b,b_1\}}{2}$. In particular, $|E(H_{a_1} - \{u\})| \leqslant \binom{4}{2} = 6$ and then $d_{H_u}(a_1) = d_{H_{a_1}}(u) = |E(H_{a_1})| - |E(H_{a_1} - \{u\})| \geqslant (n-1) - 6 \geqslant 6$. Similarly, $d_{H_u}(a_2) \geqslant 6$. Now, pick three vertices $c_1, c_2, c_3 \in N_{H_u}(a_1) \backslash \{a_2\}$, then we still have at least $6 - 1 - 3 = 2$ vertices $c_4, c_5 \in N_{H_u}(a_2) \backslash \{a_1, c_1, c_2, c_3\}$. This gives 5 edges $a_1c_1, a_1c_2, a_1c_3, a_2c_4, a_2c_5 \in E(H)$, which form a $K_{1,3} \cup K_{1,2}$ in $H_u$. □

Now by Claim 8, we can choose a set of 7 vertices $\{a, a_1, a_2, a_3, b, b_1, b_2\} \subset V(H_u)$ with $aa_1, aa_2, aa_3, bb_1, bb_2 \in E(H_u)$. Similarly as the proof in Claim 8, one can check by simple discussions that, for $i = 1, 2, 3$, $E(H_{a_i} - \{u\}) \subseteq \{ab\}$. This means $|E(H_{a_i} - \{u\})| \leqslant 1$ and then $n - 2 \geqslant d_{H_u}(a_i) = d_{H_{a_i}}(u) = |E(H_{a_i})| - |E(H_{a_i} - \{u\})| \geqslant (n-1) - 1 \geqslant n - 2$ for any $i \in [3]$. This means all the equalities here hold. Hence for $i \in [3]$, $ab \in E(H_{a_i} - \{u\})$ and $a_iv \in E(H_u)$ for any $v \in V(H_u) \backslash \{a_i\}$. In particular, $a_1ab, ua_2a_3 \in E(H)$. Together with the edge $ub_1b \in E(H)$, we get a copy of $LP_3$ in $H$ covering $u$, a contradiction. □

### 3.2 $TP_3$

*Proof of the lower bound of (2).* For $n \equiv 0, 2 \mod 3$, consider the 3-graph $F_{n-2,2}$ on $[n]$ with the edge set $\{\{n-1, n\}\} \vee \binom{[n-2]}{1} \cup \binom{[n-2]}{3}$. one can check that $\delta_1(F_{n-2,2}) = n - 2$ for $n \geqslant 8$ and there is no copy of $TP_3$ containing the vertex $n$ in $F_{n-2,2}$.
For $n \equiv 1 \mod 3$, suppose $n = 3k + 1$ for some integer $k \geqslant 2$. Consider a 3-graph $F$ on the vertex set $\{u\} \cup \bigcup_{i=1}^{k} A_i$ with $|A_i| = 3$ for any $i \in [k]$. The edge set of $F$ is

$$E(F) = \bigcup_{i=1}^{k} \left( \{u\} \vee \binom{A_i}{2} \right) \cup \bigcup_{\{i,j,k\} \in \binom{[k]}{3}} \left( \binom{A_i}{1} \vee \binom{A_j}{1} \vee \binom{A_k}{1} \right).$$

One can also check that $\delta_1(F) = d_F(u) = 3k = n-1$ and $F$ has no copy of $TP_3$ containing $u$. □

*Proof of the upper bound of (2).* Let $g(n)$ be a function with $g(n) = n-1$ for $n \equiv 0, 2 \mod 3$ and $g(n) = n$ for $n \equiv 1 \mod 3$. Suppose on the contrary that there is a 3-graph $H$ on $n \geqslant 8$ vertices with $\delta_1(H) \geqslant g(n)$ and there is a vertex $u \in V(H)$ which is not contained in any copy of $TP_3$ in $H$.

We claim that there is no copy of $P_4$ contained in $H_u$. Otherwise, there must be 4 vertices $x_1, x_2, x_3, x_4 \in V(H_u)$ with $x_1x_2, x_2x_3, x_3x_4 \in E(H_u)$, and we can pick $\{x_1x_2u, x_2ux_3, ux_3x_4\}$ as a copy of $TP_3$ in $H$, a contradiction. This implies that any component of $H_u$ can only be a $K_{1,t}$ for some $t \geqslant 0$ or a $K_3$. Let $n_t$ be the number of components isomorphic to $K_{1,t}$ for any $t \geqslant 0$ and let $m$ be the number of components isomorphic to $K_3$ in $H$. Then $n - 1 = 3m + \sum_{t\geqslant 0}(t+1)n_t$ and

$$d_H(u) = |E(H_u)| = 3m + \sum_{t\geqslant 0} tn_t = n - 1 - \sum_{t\geqslant 0} n_t.$$

If there exists some $i \geqslant 0$ with $n_i \neq 0$, then $d_H(u) \leqslant n-2 < g(n)$, a contradiction. Thus, $n_i = 0$ for any $i \geqslant 0$ and $n = 3m + 1 \equiv 1 \mod 3$. This means $d_H(u) = 3m = n - 1 < n = g(n)$, a contradiction, too. □

### 3.3 $GP_3$

*Proof of the lower bound of (3).* We consider the same 3-graph as mentioned in the proof of (1), i.e, consider a trivial intersecting family $G$ on $V(G) = \{0\} \cup [n-1]$ with edge set $E(G) = \{\{0\}\} \vee \binom{[n-1]}{2}$. Apparently, $\delta_1(G) = n - 2$ and $G$ contains no copy of $GP_3$ covering 0. □

*Proof of the upper bound of (3).* Let $H$ be a 3-graph on $n \geqslant 17$ vertices and $\delta_1(H) \geqslant n-1$. Let $M \subset V(H)$ be the set of all vertices not covered by any copy of $GP_3$ in $H$. Take $u \in M$ with $d_H(u) \leqslant d_H(v)$ for all $v \in M$.

**Claim 9.** *$H_u$ does not contain $K_{1,3} \cup K_{1,1}$ as a subgraph. Moreover, $H_u$ is a 2-regular graph ($d_{H_u}(x) = 2$ for any $x \in V(H_u)$), i.e. $H_u$ is the union of some vertex-disjoint cycles on $n-1$ vertices.*

*Proof.* Suppose on the contrary that there exist $a, a_1, a_2, a_3, b_1, b_2 \in V(H_u)$ with $aa_1$, $aa_2$, $aa_3$, $b_1b_2 \in E(H_u)$. Since $d_H(a_1) \geqslant \delta_1(H) \geqslant n-1 > n-2 \geqslant d_H(\{u, a_1\})$, we can pick $e \in E(H - \{u\})$ be an edge with $a_1 \in e$.

If $e \neq a_1a_2a_3$, then one of $a_2$ and $a_3$, say $a_2$, has $a_2 \notin e$. Then if $a \notin e$, $a_2ua$, $uaa_1$, $e$ form a copy of $GP_3$, a contradiction. Now suppose $e \neq a_1a_2a_3$, then $a \in e$. Then if $e \cap \{b_1, b_2\} = \emptyset$, $e$, $aa_1u$, $ub_1b_2$ form a copy of $GP_3$. Hence we can conclude that $e \in \{a_1a_2a_3, a_1ab_1, a_1ab_2\}$ and $d_{H-\{u\}}(a_1) \leqslant 3$. Similarly, $d_{H-\{u\}}(a_2), d_{H-\{u\}}(a_3) \leqslant 3$.

Recall that $d_H(u) \leqslant d_H(v)$ for all $v \in M$. If $d_{H-\{u\}}(a) = 0$, then all edges containing $a$ must also contain $u$, which means $a \in M$. However, this also implies that $d_H(a) = d_{H_u}(a) < |E(H_u)| = d_H(u)$, where the strict inequality holds since $a \notin b_1b_2 \in E(H_u)$.

This leads to a contradiction by the minimality of $d_H(u)$. Hence, $d_{H-\{u\}}(a) \geqslant 1$. So we can pick $f \in E(H - \{u\})$ with $a \in f$. Thus one of $a_1$, $a_2$, $a_3$, say $a_1$, has $a_1 \notin f$. If $d_{H_u}(a_1) \geqslant 4$, then we can pick $c \in N_{H_u}(a_1)\backslash f$. Then $uca_1$, $ua_1a$, $f$ form a copy of $GP_3$. Thus, $d_{H_u}(a_1) \leqslant 3$. Therefore, $d_H(a_1) = d_{H-\{u\}}(a_1) + d_{H_u}(a_1) \leqslant 3 + 3 = 6 < \delta_1(H)$, a contradiction.

Now $H_u$ is a $K_{1,3} \cup K_{1,1}$-free graph on $n-1$ vertices with at least $n-1$ edges. If $H_u$ does not contain a vertex of degree at least 3, then it is easy to see that $H_u$ must be 2-regular and we are done. Otherwise, pick $v \in V(H_u)$ with at least 3 vertices $v_1, v_2, v_3 \in N_{H_u}(v)$. Clearly, the edges in $H_u$ must incident with $V_0 := \{v, v_1, v_2, v_3\}$ or we get a copy of $K_{1,3} \cup K_{1,1}$. In other words, $N_{H_u}(x) \subset V_0$ for any $x \in V(H_u)\backslash V_0$. Also note that $|E(H_u)| \geqslant n - 1 > \binom{6}{2}$, we have at least $7 - 4 = 3$ vertices, say $x_1$, $x_2$ and $x_3$, other than $v$, $v_1$, $v_2$ and $v_3$ incident with at least one edge in $H_u$. If $x_1v \in E(H_u)$ and some vertex in $V_0\backslash\{v_0\}$, say $v_1$, has $x_2v_1 \in E(H_u)$, then $x_2v_1$, $vv_2$, $vv_3$, $vx_1 \in E(H_u)$ form a copy of $K_{1,3} \cup K_{1,1}$, a contradiction. Thus, if $x_1v \in E(H_u)$, then $x_2v \in E(H_u)$, which then implies that $N_{H_u}(x) \subseteq \{v\}$ for any $x \in V(H_u)\backslash\{v\}$. This gives $|E(H_u)| \leqslant n - 2 < n - 1$, a contradiction. Hence, $N_{H_u}(x) \subset V_1 = \{v_1, v_2, v_3\}$ for any $x \in V(H_u)\backslash V_0$. If there exists some $i \in [3]$ with $|N_{H_u}(v_i) \cap V_0| = 3$, then $v_ix_1$, $v_ix_2$, $v_ix_3 \in E(H_u)$. Thus $v_ix_1$, $v_ix_2$, $v_ix_3$ and $vv_j$ for some $j \neq i$ form a copy of $K_{1,3} \cup K_{1,1}$ in $H_u$, a contradiction. If $|N_{H_u}(v_i) \cap V_0| = 2$, without loss of generality, suppose $v_ix_1$, $v_ix_2 \in E(H_u)$. Since $x_3$ incident with at least one edge in $H_u$, we have $v_jx_3 \in E(H_u)$ for some $j \neq i$. Then $v_ix_1$, $v_ix_2$, $v_iv$ and $v_jx_3$ form a copy of $K_{1,3} \cup K_{1,1}$ in $H_u$, a contradiction. Thus, $|N_{H_u}(v_i)\backslash V_0| \leqslant 1$ for $i \in [3]$, which gives $|E(H_u)| \leqslant \binom{4}{2} + 3 = 9 < n - 1$, a contradiction. □

**Claim 10.** *For any cycle $C \subset H_u$ and edge $e \in E(H - \{u\})$, we have $|V(C) \cap e| \in \{0, 3\}$.*

*Proof.* Suppose $|V(C) \cap e| = 1$ firstly. Let $V(C) = \{c_1, c_2, \ldots, c_\ell\}$, $E(C) = \{c_1c_2, c_2c_3, \ldots, c_{\ell-1}c_l, c_\ell c_1\}$ and let $e = c_1xy$ where $x, y \notin C$. Then $c_1c_2u$, $c_2uc_3$, $c_1xy$ form a copy of $GP_3$ covering $u$, a contradiction. So $|V(C) \cap e| \neq 1$ for any cycle $C \subset H_u$. If $|V(C) \cap e| = 2$, then there must exist another cycle $C'$ with $|V(C') \cap e| = 3 - 2 = 1$, a contradiction. □

Pick a cycle $C_0$ with $V(C_0) = \{c_1, c_2, \ldots, c_\ell\}$ and $E(C_0) = \{c_1c_2, c_2c_3, \ldots, c_{\ell-1}c_\ell, c_\ell c_1\}$.

If $\ell = |V(C_0)| \geqslant 7$, we pick an edge $e$ with $e \cap V(C_0) \neq \emptyset$ (such an edge exists since the degree of vertex in $V(C_0)$ should be more than 2 in $H$ as $\delta_1(H) \geqslant n - 1$). Then $|e \cap V(C_0)| = 3$ by Claim 10. Suppose $e = \{c_i, c_j, c_k\}$ with $1 \leqslant i < j < k \leqslant \ell$. By Pigeonhole Principle, one of $d_1 = j - i$, $d_2 = k - j$, $d_3 = \ell + i - k$, say $d_1$, has $d_1 \geqslant \lceil \ell/3 \rceil \geqslant 3$. This means $j - i \geqslant 3$. Without loss of generality, suppose $i = 1$, so $k > j \geqslant 4$. Then $c_3uc_2$, $uc_2c_1$, $e$ form a copy of $GP_3$ covering $u$.

Therefore, $|V(C_0)| \leqslant 6$. Pick $v \in V(C_0)$. Note that any edge $e$ containing $v$ must have $|e \cap V(C_0)| = 3$, which implies that $d_H(v) \leqslant 2 + \binom{|V(C_0)|-1}{2} \leqslant 12 < n - 1 \leqslant \delta_1(H)$. This is a contradiction. □

### 3.4 $K_{1,1,3}$

*Proof of the lower bound of (4).* Let $W$ be a 3-graph on $[n]$ and let $\mathcal{C} = \{12, 23, \ldots, (n-2)(n-1), (n-1)1\}$. The edge set of $W$ is

$$E(W) = (\{\{n\}\} \vee \mathcal{C}) \cup \left\{ \{i,j,k\} \in \binom{[n-1]}{3} : \binom{\{i,j,k\}}{2} \cap \mathcal{C} = \emptyset \right\}.$$

It is easy to see that $d_W(n) = n-1$ and $d_W(i) = \binom{n-4}{2} - (n-5) + 2 \geqslant n-1$ for $i \in [n-1]$ since $n \geqslant 9$. Hence $\delta_1(W) = n-1$. Also, one can check that there is no copy of $K_{1,1,3}$ covering the vertex $n$. $\square$

*Proof of the upper bound of (4).* Suppose on the contrary that there is a 3-graph $H$ on $n \geqslant 9$ vertices with $\delta_1(H) \geqslant n$ and $u \in V(H)$ is not contained in any copy of $K_{1,1,3}$ in $H$. Then the degree of any vertex in $H_u$ must be at most 2. Otherwise, suppose $d_{H_u}(v) \geqslant 3$ for some $v \in V(H_u)$. Pick $x, y, z \in N_{H_u}(v)$, we get the three edges $uvx, uvy, uvz$ in $H$ which form a $K_{1,1,3}$ in $H$, a contradiction. Thus, $d_{H_u}(v) \leqslant 2$ for any $v \in V(H_u)$. Note that $V(H_u) = n-1$ and $|E(H_u)| \geqslant \delta_1(H) \geqslant n$. By Handshaking Lemma, $2(n-1) \geqslant \sum_{v \in V(H_u)} d_{H_u}(v) = 2|E(H_u)| \geqslant 2n$, a contradiction. $\square$

### 3.5 $S_3$

*Proof of the lower bound of (5).* Let $S$ be a 3-graph on $[n]$ with the edge set

$$E(S) = \left( \{\{n-1\}\} \vee \binom{\{n-2, n-3\}}{1} \vee \binom{[n-4]}{1} \right) \cup \left( \{\{n\}\} \vee \binom{[n-2]}{2} \right).$$

Note that $n \geqslant 11 > 7$. It is easy to check that $d_S(n) = \binom{n-2}{2} > n-1$, $d_S(n-1) = 2(n-4) > n-1$, $d_S(n-2) = d_S(n-3) = 2n-7 > n-1$ and $d_S(i) = n-1$ for $i \in [n-4]$. This means $\delta_1(S) = n-1$. Also, $S$ has no copy of $S_3$ covering the vertex $n-1$. $\square$

Before the proof of the upper bound, we firstly put the famous Tutte-Berge Theorem here.

**Lemma 11** ([4], see also [11]). *A graph $G$ is $(s+1)K_2$-free if and only if there is a set $B \subset V(G)$, such that the vertex sets of all the connected components $G_1, \cdots, G_m$ of $G - B$ have $|V(G_i)| \equiv 1 \mod 2$ ($i \in [m]$), and we have,*

$$|B| + \sum_{i=1}^{m} \frac{|V(G_i)| - 1}{2} \leqslant s \quad \text{and} \quad |B| + \sum_{i=1}^{m} |V(G_i)| = n.$$

*Proof of the upper bound of (5).* Suppose on the contrary that $H$ is a 3-graph on $n \geqslant 11$ vertices with $\delta_1(H) \geqslant n$ and $u \in V(H)$ is not contained in any copy of $S_3$ in $H$. Note that there is no copy of $3K_2$ in $H_u$. Ohterwise, let $\{a_1a_2, b_1b_2, c_1c_2\} \subset H_u$ be a copy of $3K_2$, then$\{ua_1a_2, ub_1b_2, uc_1c_2\}$ is a copy of $S_3$ in $H$, a contradiction. Hence, we can use

Lemma 11 to obtain a set $B \subset V(H_u)$. Then all the components $G_1, \dots G_m$ of $H_u - B$ have $|V(G_i)| \equiv 1 \mod 2$ ($i \in [m]$), and

$$|B| + \sum_{i=1}^{m} \frac{|V(G_i)| - 1}{2} \leqslant 2 \quad \text{and} \quad |B| + \sum_{i=1}^{m} |V(G_i)| = n - 1 \geqslant 10.$$

Without loss of generality, let $|V(G_1)| \geqslant |V(G_2)| \geqslant \cdots \geqslant |V(G_m)|$. Thus $|B| \leqslant 2$. Also, $H_u \subset K[B] \cup K[B, V(H_u) - B] \cup \sum_{i=1}^{m} K[V(G_i)]$.

**Claim 12.** $1 \leqslant |B| \leqslant 2$.

*Proof.* If $|B| = 0$, then $E(H_u) \subset \sum_{i=1}^{m} E(K[V(G_i)])$ and $\sum_{i=1}^{m} \frac{|V(G_i)|-1}{2} \leqslant 2$. Note that $\frac{|V(G_i)|-1}{2}$ is a non-negative integer for any $i \in [m]$. so it is easy to see that either $|V(G_1)| \leqslant 5$ and $|V(G_j)| = 1$ for $j > 1$ or $|V(G_1)|, |V(G_2)| \leqslant 3$ and $|V(G_j)| = 1$ for $j > 2$. This implies $d_H(u) = |E(H_u)| = \sum_{i=1}^{m} |E(K[V(G_i)])| \leqslant 10 < n \leqslant \delta_1(H)$, a contradiction. This gives $1 \leqslant |B| \leqslant 2$. □

**Claim 13.** *For any edge $xy \in E(H_u)$, there is no copy of $2K_2$ in $H_x - \{u, y\}$. Moreover, $|E(H_x - \{u, y\})| \leqslant n - 4$.*

*Proof.* If there exists a set of two disjoint edges $\{a_1a_2, b_1b_2\} \subset E(H_x - \{u, y\})$ as a $2K_2$ in $H_x - \{u, y\}$, then the three edges $xyu, xa_1a_2, xb_1b_2 \in E(H)$ form a copy of $S_3$, a contradiction. Hence, the only non-empty component of $H_x - \{u, y\}$ must be a $K_3$ or a $K_{1,t}$ for some $1 \leqslant t \leqslant n - 4$. This gives $|E(H_x - \{u, y\})| \leqslant n - 4$. □

**Claim 14.** *Let $v \in V(H_u)$ and $d_{H_u}(v) \geqslant 5$. Pick any two vertices $x, y \in N_{H_u}(v)$. If $d_{H_x - \{u\}}(v) \geqslant 1$, then $d_{H_y - \{u\}}(v) \leqslant 1$. Moreover, $\max\{d_{H_u}(x), d_{H_u}(y)\} \geqslant 3$.*

*Proof.* Otherwise, suppose $d_{H_x - \{u\}}(v) \geqslant 1$ and $d_{H_y - \{u\}}(v) \geqslant 2$. then we can pick an edge $va_1 \in H_x - \{u\}$ and another edge $va_2 \in H_y - \{u\}$ with $a_2 \neq a_1$. Since $d_{H_u}(v) \geqslant 5$, we can also pick a vertex $a_3 \in N_{H_u}(v)$ with $a_3 \neq a_1, a_2, x, y$. Then the three edges $va_1x, va_2y, va_3u \in E(H)$ form a copy of $S_3$, a contradiction.

To prove $\max\{d_{H_u}(x), d_{H_u}(y)\} \geqslant 3$, note that $d_H(z) = d_{H_u}(z) + d_{H_z - \{u\}}(v) + |E(H_z - \{u, v\})|$ for $z \in \{x, y\}$. By Claim 13, $|E(H_z - \{u, v\})| \leqslant n - 4$ for $z = x, y$. Hence, for $z \in \{x, y\}$,

$$n \leqslant \delta_1(H) \leqslant d_H(z) \leqslant n - 4 + d_{H_u}(z) + d_{H_z - \{u\}}(v).$$

Now if $d_{H_u}(z) \leqslant 2$ for $z = x, y$, then $n \leqslant n - 2 + d_{H_z - \{u\}}(v)$, which means $d_{H_z - \{u\}}(v) \geqslant 2$ for $z = x, y$. This is impossible by the proof above. □

Now by Claim 12, $1 \leqslant |B| \leqslant 2$.

If $|B| = 1$, let $B = \{v\}$. By $E(H_u) \subset K[B, V(H_u) - B] \cup \sum_{i=1}^{m} K[V(G_i)]$ and $\sum_{i=1}^{m} \frac{|V(G_i)|-1}{2} \leqslant 1$, we have $|V(G_1)| \leqslant 3$, $|V(G_j)| = 1$ for $j > 1$ and $E(H_u) = E(H_u[B, V(H_u) - B]) \cup E(G_1)$. Since $E(H_u) = d_H(u) \geqslant \delta_1(H) \geqslant n$, $d_{H_u}(v) = |E(H_u[B, V(H_u) - B])| \geqslant n - |K[V(G_1)]| = n - 3 > 5 = 2 + 3$. Thus, we can pick two vertices $x, y \in N_{H_u}(v) \backslash V(G_1)$. Then $d_{H_u}(x) = d_{H_u}(y) = 1$, contradicts to $\max\{d_{H_u}(x), d_{H_u}(y)\} \geqslant 3$ by Claim 14.

If $|B| = 2$, let $B = \{v_1, v_2\}$. Similarly, we get $|V(G_j)| = 1$ for any $j \in [m]$ and $E(H_u) = E(H_u[B]) \cup E(H_u[B, V(H_u) - B])$. This means $d_{H_u}(z) \leqslant 2$ for any $x \in V(H_u)\backslash\{v_1, v_2\}$ and $11 \leqslant n \leqslant \delta_1(H) \leqslant |E(H_u)| \leqslant d_{H_u}(v_1) + d_{H_u}(v_2)$. By Pigeonhole Principle, one of $v_1$ and $v_2$, say $v_1$, has $d_{H_u}(v_1) \geqslant \frac{11}{2} > 5$. So we can pick two vertices $x, y \in N_{H_u}(v_1)\backslash\{v_2\}$ and get a contradiction similarly by Claim 14. □

### 3.6 $GS_3$

*Proof of the lower bound of (6).* Consider the graph $F$ with vertex set $\{0\} \cup [n-1]$. Let $B_i = \{2i-1, 2i\} \cap [n-1]$, for $i \in [\lceil \frac{n-1}{2} \rceil]$ and $\mathcal{B} = \{B_i : i \in [\lfloor \frac{n-1}{2} \rfloor]\}$. The edge set of $F$ is

$$E(F) = (\{\{0\}\} \vee \mathcal{B}) \cup \bigcup_{\{i,j,k\} \in \binom{[\lceil \frac{n-1}{2} \rceil]}{3}} \left( \binom{B_i}{1} \vee \binom{B_j}{1} \vee \binom{B_k}{1} \right).$$

Clearly, for $n \geqslant 13 \geqslant 6$, $\delta_1(F) = \lfloor (n-1)/2 \rfloor$, and there is no copy of $GS_3$ covering 0. □

*Proof of the upper bound of (6).* Suppose that $H$ is a 3-graph on $n \geqslant 13$ vertices with $\delta_1(H) \geqslant \lfloor (n-1)/2 \rfloor + 1 \geqslant 7$ and $u$ is a vertex in $H$ not covered by $GS_3$. By averaging, $H_u$ contains at least one vertex $x$ such that $d_{H_u}(x) \geqslant \lceil \frac{2(\lfloor (n-1)/2 \rfloor + 1)}{n-1} \rceil = 2$.

**Claim 15.** *$H_u$ contains no copy of $K_{1,2} \cup K_{1,1}$.*

*Proof.* Assume that $\{x_1x_2, x_2x_3, y_1y_2\}$ is a copy of $K_{1,2} \cup K_{1,1}$ in $H_u$, then $ux_1x_2$, $ux_2x_3$, $uy_1y_2$ form a $GS_3$ covering $u$. □

**Claim 16.** *The only non-empty component of $H_u$ is a star.*

*Proof.* Suppose not and let $x$ be the vertex with maximum degree in $H_u$. Let $N_{H_u}[x] = N_{H_u}(x) \cup \{x\}$. Since $d_{H_u}(x) \geqslant 2$, we have $|N_{H_u}[x]| \geqslant 3$ and any edge in $H_u$ shares at least one vertex in $N_{H_u}[x]$. Otherwise, there would be a copy of $K_{1,2} \cup K_{1,1}$ in $H_u$, which is a contradiction by Claim 15. So we can assume that all edges are incident with $N_{H_u}[x]$. Suppose $N_{H_u}[x] = \{x, y_1, y_2, \ldots, y_d\}$ where $d = d_{H_u}(x) \geqslant 2$.

If $|N_{H_u}[x]| \geqslant 4$, pick an edge $wv$ with $x \notin wv$ (since $H_u$ is not a star), then $wv, xy_i, xy_j$ form a copy of $K_{1,2} \cup K_{1,1}$, where we pick $y_i, y_j \in N_{H_u}[x]\backslash\{w, v\}$. This is a contradiction. If $|N_{H_u}[x]| = 3$, we have $\max\{d_{H_u}(y_1), d_{H_u}(y_2)\} \geqslant \lceil 1 + \frac{|E(H_u)|-2}{2} \rceil \geqslant 4$. Without loss of generality, suppose $d_{H_u}(y_1) \geqslant 4$. We can pick two vertices $z_1$and $z_2$ with $z_1, z_2 \in N_{H_u}(y_1)\backslash\{x, y_2\}$. Then $y_1z_1, y_1z_2, xy_2$ form a copy of $K_{1,2} \cup K_{1,1}$, a contradiction. □

Now we can assume that the only non-empty component of $H_u$ is $K[\{v\}, V_0]$ for some $v \in V(H_u)$ and $V_0 \subset V(H_u)\backslash\{v\}$. Note that $|V_0| = d_H(u) \geqslant \lfloor (n-1)/2 \rfloor + 1 \geqslant 7$. If there exists an edge $e \in E(H - \{u\})$ with $v \in e$, we can pick 2 vertices $v_1.v_2 \in V_0\backslash e$. Hence we get a contradiction since $e$, $uvv_1$, $uvv_2$ form a copy of $GS_3$ covering $u$.

If there is no edge $e \in E(H - \{u\})$ with $v \in e$, then $d_H(\{u, v\}) = d_H(u) = d_H(v) > 0$ and $\delta_1(H - \{u, v\}) \geqslant \delta_1(H) - 1 \geqslant \lfloor (n-1)/2 \rfloor = \lfloor ((n-3)/2 \rfloor + 1$. We now pick $w \in N_H(\{u, v\})$. Note that $|E((H - \{u, v\})_w)| \geqslant \delta_1(H - \{u, v\}) \geqslant \lfloor ((n-3)/2 \rfloor + 1$.

So we can find a vertex $x$ such that $d_{(H-\{u,v\})_w}(x) \geqslant \lceil \frac{2(\lfloor (n-3)/2 \rfloor+1)}{n-3} \rceil = 2$. Pick $x_1, x_2 \in N_{(H-\{u,v\})_w}(x)$, we get a copy of $GS_3$ in $H$ with edge set $\{uvw, wxx_1, wxx_2\}$ covering $u$. □

## 4 Concluding remarks

In this paper, we determine the exact values of $c_1(F_5)$ and $c_1(n, F)$ for $F = LP_3$, $TP_3$, $K_{1,1,3}$, $S_3$, $GP_3$, $GS_3$. These results, together with some known ones, complete the 1-degree thresholds for all possible coverings by a connected 3-graph with 3 edges.

For 3-graphs $F$ with more than 3 edges, however, we have no non-trivial exact results for $c_1(F)$.

For the 2-degree thresholds, one can easily check that: $c_2(n, F)$ is a small constant for any mentioned connected 3-graph $F$ with 3 edges (except for $K_4^{(3)-}$ done by [13]). For example,

- (Tang, Ma and Hou [12]) For $n \geqslant 6$, $c_2(n, C_6^{(3)}) = 1$;
- (Gu, Wang [8]) For $n \geqslant 5$, $c_2(n, F_5) \in \{1, 2\}$ and $c_2(n, F_5) = 2$ if and only if $n \equiv 1 \mod 3$ and $n \geqslant 10$; for $n \geqslant 8$, $c_2(n, LP_3) = 1$; for $n \geqslant 7$, $c_2(n, S_3) \leqslant 1$.

Hence, it seems to be more interesting to consider $c_1(n, F)$ and $c_1(F)$ than $c_2(n, F)$ and $c_2(F)$ for small 3-graphs $F$.


### Acknowledgements

The work was supported by National Natural Science Foundation of China (No.12401455) to Yue Ma; National Natural Science Foundation of China (No.12471336, 12071453) and Innovation Program for Quantum Science and Technology (2021ZD0302902) to Xinmin Hou.


## References


[1] J. Balogh, J.Butterfield, P. Hu, J. Lenz, Mantel's theorem for random hypergraphs, *Random Structures and Algorithms*, 48(4) (2016), 641-654.

[2] J. Balogh, J.Butterfield, P. Hu, J. Lenz, D. Mubayi, On the chromatic thresholds of hypergraphs, *Combin. Probab. Comput.*, 25(2) (2016), 172-212.

[3] J. Balogh, D. Mubayi, Almostly all triangle-free triple systems are tripartite, *Combinatorica*, 32 (2012), 143-169.

[4] C. Berge, Sur le couplage maximum d'un graphe, *C.R. Acad. Sci. Paris Sér. I Math*, 247(1958), 258-259.

[5] V. Falgas-Ravry, K. Markström, Y. Zhao, Triangle-degrees in graphs and tetrahedron coverings in 3-graphs, *Combin. Probab. Comput.*, 30(2) (2021), 175-199.

[6] V. Falgas-Ravry, Y. Zhao, Codegree thresholds for covering 3-uniform hypergraphs, *SIAM J. Discrete Math.*, 30(4) (2016), 1899-1917.

[7] P. Frankl, Z. Füredi, A new generalization of Erdős-Ko-Rado theorem, *Combinatorica*, 3 (1983), 341-349.

[8] R. Gu, S. Wang, The degree and codegree threshold for generalized triangle and some trees covering, arXiv:2307.01647.

[9] P. Keevash, Hypergraph Turán problems, *London Mathematical Society Lecture Note*, 392(1) (2011), 83-140.

[10] P. Keevash, D. Mubayi, Stability results for cancellative hypergraphs, *J. Combin. Theory Ser. B*, 92 (2004), 163-175.

[11] L. Lovász, M.D. Plummer, Matching theory, *Ann. Discrete Math.*, 29(1986).

[12] Y. Tang, Y. Ma, X. Hou, The degree and codegree threshold for linear triangle covering in 3-graphs, *The Electronic Journal of Combinatorics*, 30(4) (2023), #P4.15.

[13] L. Yu, X. Hou, Y. Ma, B. Liu, Exact minimum codegree thresholds for $K_4^-$-covering and $K_5^-$-covering, *The Electronic Journal of Combinatorics*, 27(3) (2020), #P3.22.

[14] C. Zhang, Matchings and tilings in hypergraphs, PhD thesis, Georgia State University, 2016.

[15] Y. Zhao, Recent advances on Dirac-type problems for hypergraphs, In: A. Beveridge, J. Griggs, L. Hogben, G. Musiker, P. Tetali (eds) Recent Trends in Combinatorics, The IMA Volumes in Mathematics and its Applications 159. Springer, New York, 2016.