Minimum Degree Conditions for Hamilton ℓ-Cycles in k-Uniform Hypergraphs

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Submitted: Jun 16, 2023; Accepted: Oct 1, 2024; Published: Feb 28, 2025

Abstract

We show that for $\eta > 0$ and sufficiently large n, every 5-graph on n vertices with $\delta_2(H) \geqslant (91/216+\eta)\binom{n}{3}$ contains a Hamilton 2-cycle. This minimum 2-degree condition is asymptotically best possible. Moreover, we give some related results on minimum d-degree conditions in k-graphs that guarantee the existence of a Hamilton ℓ-cycle when $\ell \leqslant d \leqslant k-1$ and $1 \leqslant \ell < k/2$.

Mathematics Subject Classifications: 05C35, 05C65

1 Introduction

The study of Hamilton cycles is an important topic in graph theory and extremal combinatorics. Dirac's classic result [5] states that every graph whose minimum degree is at least as large as half the size of the vertex set contains a Hamiltonian cycle. In recent years, extending Dirac's theorem to hypergraphs has attracted a great deal of attention. Given $k \geqslant 2$, a *k-uniform hypergraph* H (in short, *k-graph*) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, where every edge is a k-element subset of V. We denote by $e(H) := |E|$ the numbers of edges in H. Given a k-graph $H = (V, E)$ and a vertex set $S \in \binom{V}{d}$, we define $N(S)$ to be the family of $T \in \binom{V}{k-d}$ such that $T \cup S \in E$ and $\deg_H(S) := |N(S)|$. The *minimum d-degree* of H denoted by $\delta_d(H)$ is the minimum of $\deg_H(S)$ over all d-element vertex sets S in H. For $1 \leqslant \ell < k$, we say that a k-graph is an *ℓ-cycle* if there exists a cyclic ordering of its vertices such that every edge consists of k consecutive vertices and two consecutive edges (in the natural order of the edges) share exactly ℓ vertices. If the ordering is linear, we call it an *ℓ-path* and we say the first and last ℓ vertices are the ends of the path. By the *length* of an ℓ-path, we mean the number of edges contained in it. In k-graphs, a $(k-1)$-cycle is often called a *tight cycle* and a

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$(k-1)$-path is often called a *tight path*. A k-graph on n vertices contains a *Hamilton ℓ-cycle* if it contains an ℓ-cycle as a spanning subhypergraph. Note that a Hamilton ℓ-cycle of a k-graph on n vertices contains exactly $n/(k-\ell)$ edges, implying that $(k-\ell) \mid n$. For $(k-\ell) \mid n$ and $1 \leqslant d \leqslant k-1$, we define the *Dirac threshold* $h_d^\ell(k,n)$ to be the smallest integer h such that every n-vertex k-graph H satisfying $\delta_d(H) \geqslant h$ contains a Hamilton ℓ-cycle. Let

$$h_d^\ell(k) := \limsup_{n\to\infty} h_d^\ell(k,n) / \binom{n}{k-d}.$$

Dirac thresholds of hypergraphs were first investigated by Katona and Kierstead [16], who showed that $1/2 \leqslant h_{k-1}^{k-1}(k) \leqslant 1-1/(2k)$ and conjectured $h_{k-1}^{k-1}(k) = 1/2$, which was confirmed by Rödl, Ruciński, and Szemerédi [26, 27], that is, for $\varepsilon > 0$ and large n, $\delta_{k-1}(H) \geqslant n/2 + \varepsilon n$ guarantees a Hamilton tight cycle. When $(k-\ell) \mid k$, a tight cycle on V trivially contains an ℓ-cycle on V. So the asymptotic Dirac threshold $h_{k-1}^\ell(k) = 1/2$ follows as a consequence of $h_{k-1}^{k-1}(k) = 1/2$ and a construction of Markström and Ruciński [22]. When $(k-\ell) \nmid k$, there have been a series of works on the threshold $h_{k-1}^\ell(k)$. We collect these results in the following theorem.

Theorem 1. *[9, 17, 18, 26, 27] For any $k > \ell \geqslant 1$, we have*

$$h_{k-1}^\ell(k) = \begin{cases} 1/2 & (k-\ell) \mid k \\ \frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} & (k-\ell) \nmid k. \end{cases}$$

For sufficiently large n, some exact thresholds $h_{k-1}^\ell(k,n)$ are known: for $k=3$ and $\ell = 2$ [28] and for $k \geqslant 3$ and $1 \leqslant \ell < k/2$ [4, 13]. For $d = k-2$, Buß, Hàn and Schacht [3] showed that $h_2^1(3) = \frac{7}{16}$. Han and Zhao [14] showed the exact result for $h_2^1(3,n)$. Bastos, Mota, Schacht, Schnitzer and Schulenburg [1] determined $h_{k-2}^\ell(k) = 1 - \left(1 - \frac{1}{2(k-\ell)}\right)^2$ for $k \geqslant 4$, $1 \leqslant \ell < k/2$ and got the exact result in [2], which generalizes the previous results for 3-graphs. For tight cycles, which might be considered as the most difficult and interesting case, Polcyn, Reiher, Rödl, Ruciński, Schacht and Szemerédi [23, 25] showed the asymptotic Dirac threshold $h_1^2(3) = h_2^3(4) = 5/9$. Lang and Sanhueza-Matamala [19] proved that $h_{k-2}^{k-1}(k) = 5/9$ for all $k \geqslant 3$ (the same result was also proved independently by Polcyn, Reiher, Rödl and Schülke [24]) and also provided a general upper bound of $1 - 1/(2(k-d))$ for $h_d^{k-1}(k)$, narrowing the gap to the lower bound of $1 - 1/\sqrt{k-d}$ due to Han and Zhao [15]. For $d \leqslant k-3$, much less is known under d-degree conditions. Recently, Hàn, Han and Zhao [8] determined the exact value of $h_d^{k/2}(k,n)$ for any even integer $k \geqslant 6$, integer d such that $k/2 \leqslant d \leqslant k-1$ and sufficiently n. Gan, Han, Sun and Wang [7] determined the following Dirac thresholds for $1 \leqslant \ell < k/2$.

Theorem 2 ([7])**.** *Suppose that $k \geqslant 3$, $k-\ell \leqslant d < 2\ell \leqslant k-1$ such that $2k-2\ell \geqslant (2(2k-2\ell-d)^2+1)(k-d-1)+1$ or suppose that k is odd, $k \geqslant 7, \ell = (k-1)/2$ and $d = k-3$, then*

$$h_d^\ell(k) = 1 - \left(1 - \frac{1}{2(k-\ell)}\right)^{k-d}.$$

The following "space-barrier" construction shows that the minimum degree conditions in the aforementioned results in [1, 3, 7, 17, 18] are best possible, namely,

$$h_d^\ell(k) \geqslant 1 - \left(1 - \frac{1}{2(k-\ell)}\right)^{k-d}. \tag{1}$$

Given $k \geqslant 3$, $1 \leqslant \ell < k/2$, $1 \leqslant d \leqslant k-1$ with $(k-\ell) \mid n$, let $H_{k,\ell} := (V, E)$ be an n-vertex k-graph such that E consists of all k-sets that intersect $A \subseteq V$, where $|A| = \lceil \frac{n}{2(k-\ell)} \rceil - 1$. Note that an ℓ-cycle on n vertices contains $n/(k-\ell)$ edges and each vertex is contained in at most two edges of any ℓ-cycle for $\ell < k/2$. So $H_{k,\ell}$ contains no Hamilton ℓ-cycle and

$$\delta_d(H_{k,\ell}) = \binom{n-d}{k-d} - \binom{n-|A|-d}{k-d} = \left(1 - \left(1 - \frac{1}{2(k-\ell)}\right)^{k-d} - o(1)\right)\binom{n}{k-d}.$$

Letting n tend to infinity, we obtain (1).

We expect more Dirac thresholds to take the value of the space-barrier (as in Theorem 2) for all $\ell < k/2$. However, this is not true for large ℓ. Indeed, as noticed in [15], for $k - \ell = o(\sqrt{k-d})$), we have $h_d^\ell(k) \to 1$ as $(k-d) \to \infty$ regardless of $(k-\ell) \mid k$ or not.

1.1 Our results: $h_2^2(5)$

In this paper we study more thresholds of Hamilton ℓ-cycles and focus on the case $\ell < k/2$. The following theorem gives the Dirac threshold for $k = 5$, $\ell = (k-1)/2 = 2$ and $d = k - 3 = 2$.

Theorem 3.

$$h_2^2(5) = \frac{91}{216}.$$

Our proof in [7] fails for determining $h_2^2(5)$. The key reason is that in [7] we could only prove Theorem 24 for $\alpha < 1/7$ (and one can see later in Section 3 that we need the case $\alpha = 1/6$ in proving Theorem 3). Moreover, we also derive a connecting lemma (Lemma 12) for $d = \ell$ from the Kruskal–Katona theorem – in [7] we use a connecting lemma from [11] which works for $d > \ell$ only. At last, we also need a better absorbing path lemma because the one we established in [7] only works for $d \geqslant k - \ell$ and $\ell < k/2$.

Given two k-graphs F and H, an F*-tiling* in H is a subgraph of H consisting of vertex-disjoint copies of F. The number of copies of F is called the *size* of the F-tiling. When F is a single edge, an F-tiling is known as a *matching*. For $k > b \geqslant 0$, let $Y_{k,b}$ be the k-graph consisting of two edges that intersect in exactly b vertices. For any $0 < \varepsilon < \eta$, $k \geqslant 3$, $1 \leqslant \ell < k/2$ and $1 \leqslant d \leqslant k-1$, let $t_d^\ell(k, n, \varepsilon)$ denote the minimum t such that every k-graph H on n vertices with $\delta_d(H) \geqslant t$ contains a $Y_{k,2\ell}$-tiling covering all but at most εn vertices of H. Let

$$t(k, d, \ell) := \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{t_d^\ell(k, n, \varepsilon)}{\binom{n}{k-d}}.$$

The aforementioned space barrier construction indeed shows that

$$t(k,d,\ell) \geqslant 1 - \left(1 - \frac{1}{2(k-\ell)}\right)^{k-d}. \tag{2}$$

Using the new connecting lemma and absorbing lemma mentioned above, we show the following two results for general Dirac thresholds for ℓ-cycles with $\ell < k/2$.

Theorem 4. *Suppose that $k \geqslant 3$, $\eta > 0$, $\ell < d \leqslant k-1$ and $1 \leqslant \ell < k/2$, then there exists n_0 such that every k-graph H on $n \geqslant n_0$ vertices with $(k-\ell) \mid n$ and*

$$\delta_d(H) \geqslant (\max\{t(k,d,\ell), 1/3\} + \eta)\binom{n}{k-d}$$

contains a Hamilton ℓ-cycle, that is, $h_d^\ell(k) \leqslant \max\{t(k,d,\ell), 1/3\}$.

Theorem 5. *Suppose that $k \geqslant 3$, $\eta > 0$ and $2 \leqslant \ell < k/2$, then there exists n_0 such that every k-graph H on $n \geqslant n_0$ vertices with $(k-\ell) \mid n$ and*

$$\delta_\ell(H) \geqslant \left(\max\left\{(1/2)^{\frac{k-\ell}{\ell}}, t(k,\ell,\ell)\right\} + \eta\right)\binom{n}{k-\ell}$$

contains a Hamilton ℓ-cycle, that is, $h_\ell^\ell(k) \leqslant \max\left\{(1/2)^{\frac{k-\ell}{\ell}}, t(k,\ell,\ell)\right\}$.

We do need the minimum d-degree condition to be at least $(\frac{1}{3} + o(1))\binom{n}{k-d}$ in both theorems above. The reason that we do not see the constant $1/3$ in Theorem 5 is that $t(k,\ell,\ell) > 1 - 1/\sqrt{e} > 0.39$ by (2). Similarly, when $d > \ell$ and $t(k,d,\ell) > 1/3$, we know that $h_d^\ell(k) \leqslant t(k,d,\ell)$, that is, the value of $h_d^\ell(k)$ is determined solely by the tiling threshold.

Corollary 6. *If $k \geqslant 3$, $1 \leqslant \ell < k/2$ and $0.82(k-\ell) \leqslant k-d < k-\ell$, then $h_d^\ell(k) \leqslant t(k,d,\ell)$.*

Proof. We have

$$t(k,d,\ell) \geqslant 1 - \left(1 - \frac{1}{2(k-\ell)}\right)^{k-d} > 1 - e^{-\frac{k-d}{2(k-\ell)}} \geqslant 1 - e^{-0.41} > 1/3,$$

where we used the inequality $(1-1/n)^n < 1/e$ for integer $n > 0$. Since $0.82(k-\ell) \leqslant k-d < k-\ell$, we have $\ell < d \leqslant k-1$. So $h_d^\ell(k) \leqslant t(k,d,\ell)$ by Theorem 4. □

1.2 New proof ideas

Now we briefly talk about our proof ideas. Theorem 4 and Theorem 5 are proved by using the absorbing method, popularized by Rödl, Ruciński, and Szemerédi in [26]. The proof is divided into the following lemmas: the connecting lemma, the absorbing path lemma, the path cover lemma and the reservoir lemma. Roughly speaking, the absorbing path lemma reduces the task of finding a Hamilton ℓ-cycle to the much easier problem of finding an ℓ-cycle covering the majority of vertices. Furthermore, we compute the value of $t(5,2,2)$ (see Theorem 21), which together with Theorem 5 implies Theorem 3.

To prove our absorbing lemma Lemma 13, we combine the swapping-absorbing idea of Reiher, Rödl, Ruciński, Schacht and Szemerédi [25] and the lattice-based absorbing method of the first author [10]. Roughly speaking, by the swapping-absorbing idea, one can build the absorbers in two steps. In the first step, we show that there are many "end absorbers" from each of which we can "free" a $(k-\ell)$-set of vertices. In the second step, it is shown that every $(k-\ell)$-set can be "swapped" with the free vertex set in the end absorbers, namely, one can find two short ℓ-paths that include either of the sets as interior vertices (e.g., in the graph case, the pair (a,b) is a swapper for u and v if both aub and avb form paths of length two). It is easy to see that one can "concatenate" the swappers to form longer swapper chains, and we use the reachability arguments and the lattice-based absorbing method to control the swappings.

The rest of this paper is organized as follows. In Section 2 we give some preparatory results. We use the absorbing method to prove Theorem 4 and Theorem 5 in Section 3. Finally, we give proofs of Theorem 3 in the Appendix.

2 Preliminaries

One important ingredient of our swapping-absorbing method is the following notion of reachability introduced by Lo and Markström [20]. Given a constant $\beta > 0$, an integer $i \geqslant 1$ and a k-graph H on n vertices, we say that two vertices u, v in H are (β, i)-*reachable* if there are at least $\beta n^{(2k-\ell)i-1}$ $((2k-\ell)i-1)$-sets T such that there exist vertex-disjoint ℓ-paths $P_1, \ldots, P_i$ of length two with $V(P_1 \cup \cdots \cup P_i) = T \cup \{u\}$, and vertex-disjoint ℓ-paths $P'_1, \ldots, P'_i$ of length two with $V(P'_1 \cup \cdots \cup P'_i) = T \cup \{v\}$, where P_j and P'_j have the same ends for all $j \in [i]$. Moreover, we call T a reachable set for $\{u, v\}$. Given a vertex set $U \subseteq V(H)$, U is said to be (β, i)-*closed* if every two vertices in U are (β, i)-reachable in H.

The following simple results will be useful.

Fact 7. *Let $1 \leqslant d' \leqslant d < k$ and H be a k-graph on n vertices. If $\delta_d(H) \geqslant x\binom{n-d}{k-d}$ for some $0 \leqslant x \leqslant 1$, then $\delta_{d'}(H) \geqslant x\binom{n-d'}{k-d'}$.*

Proof. Since $\delta_d(H) \geqslant x\binom{n-d}{k-d}$, we get $\delta_{d'}(H) \geqslant \binom{n-d'}{d-d'} x\binom{n-d}{k-d} / \binom{k-d'}{d-d'} \geqslant x\binom{n-d'}{k-d'}$. □

We use a, by now common, hierarchical notation, writing $x \ll y$ to mean that there exists a function f such that whenever $x \leqslant f(y)$, the subsequent statement holds. When multiple constants appear in a hierarchy, they are chosen from right to left.

Proposition 8. *Let $q \geqslant 2$, $1/n \ll \beta \ll \eta, 1/q$, $1 \leqslant \ell, d < k$ and H be a k-graph on n vertices with $\delta_d(H) \geqslant (1/q + \eta)\binom{n}{k-d}$. Then every set of q vertices of $V(H)$ contains two vertices which are $(\beta, 1)$-reachable in H.*

Proof. Take $\beta \ll \varepsilon \ll \eta, 1/q$. By $\delta_d(H) \geqslant (1/q+\eta)\binom{n}{k-d}$ and Fact 7, we get $\delta_1(H) \geqslant (1/q+\eta/2)\binom{n}{k-1}$. For any q-tuple $v_1, v_2, \ldots, v_q$ of $V(H)$, we have $\sum_{i=1}^{q} |N_H(v_i)| \geqslant (1+$

$q\eta/2)\binom{n}{k-1}$. By the pigeonhole principle, there exist v_i, v_j such that $|N_H(v_i) \cap N_H(v_j)| \geqslant \varepsilon n^{k-1}$. Let P^* be a k-graph which is an ℓ-path of length two. Then the link $(k-1)$-graph P^{**} of a vertex of degree two in P^* is $(k-1)$-partite. By the supersaturation result (see [6]), we can find $(2k-\ell-1)!\beta n^{2k-\ell-1}$ copies of P^{**} in $N_H(v_i) \cap N_H(v_j)$. Given any such copy of P^{**} whose vertex set is denoted by T, we get that both $T \cup \{v_i\}$ and $T \cup \{v_j\}$ form a copy of P^*. Overall there are at least $\beta n^{2k-\ell-1}$ choices of $(2k-\ell-1)$-sets for T. So v_i and v_j are $(\beta, 1)$-reachable in H. □

3 Hamilton ℓ-cycles

In this section, we prove Theorem 4 and Theorem 5

3.1 Connecting Lemmas

We first present two versions of the connecting lemma, both of which state that in any sufficiently large k-graph with large minimum d-degree, we can connect any two disjoint ordered ℓ-sets of vertices by a short ℓ-path. When $d > \ell$, we use the following connecting lemma from [11]. The case $d = k-1$ was proved earlier in [18].

Lemma 9 (Connecting lemma, [11], Lemma 4.1)**.** *Suppose that $k \geqslant 3$ and $1 \leqslant \ell < d \leqslant k-1$ such that $(k-\ell) \nmid k$, and that $1/n \ll \beta \ll \mu, 1/k$. Let H be a k-graph on n vertices satisfying $\delta_d(H) \geqslant \mu\binom{n}{k-d}$. Suppose S and T are two disjoint ordered ℓ-sets of $V(H)$, then there exists an ℓ-path P in H with S and T as ends such that P contains at most $8k^5$ vertices.*

For $\ell < k/2$, we derive the following connecting lemma for ℓ-paths from the Kruskal-Katona theorem. For this, we first introduce the notion of (robust) shadow.

Definition 10. Given $\varepsilon \geqslant 0$ and $\ell \geqslant 1$, the *ε-robust ℓ-shadow* of a k-graph H, denoted by $\partial_\varepsilon^\ell(H) \subseteq \binom{V(H)}{k-\ell}$, is the $(k-\ell)$-graph consisting of all $(k-\ell)$-sets lying in more than εn^ℓ edges in H, that is, $\partial_\varepsilon^\ell(H) = \left\{F \in \binom{V(H)}{k-\ell} : \deg_H(F) > \varepsilon n^\ell\right\}$.

Kruskal-Katona theorem studies the size of the (0-robust) shadow of a hypergraph. We state the following handy version by Lovász [21]. The generalized binomial coefficient is defined for any real number t and a positive integer k as:

$$\binom{t}{k} = \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!}.$$

Theorem 11 (Kruskal–Katona theorem, [21])**.** *For all integers $1 \leqslant \ell \leqslant k \leqslant n$ and $t \in \mathbb{R}$, let H be a k-graph with at least $\binom{t}{k}$ edges. Then*

$$|\partial_0^\ell(H)| \geqslant \binom{t}{k-\ell}.$$

Now we give a connecting lemma for minimum ℓ-degree and ℓ-cycles where $1 \leqslant \ell \leqslant k/2$. The cases for $k \geqslant 3$, $1 \leqslant \ell < k/2$ were proved by Buß, Hàn and Schacht [3] and Bastos, Mota, Schacht, Schnitzer and Schulenburg [1].

Lemma 12 (Connecting lemma). *Suppose that $1 \leqslant \ell \leqslant k/2$ and $\eta > 0$. Let H be a k-graph on n vertices satisfying $\delta_\ell(H) \geqslant \left((1/2)^{\frac{k-\ell}{\ell}} + \eta\right)\binom{n}{k-\ell}$. Then for any two disjoint ordered ℓ-sets S and T of $V(H)$, there exists an ℓ-path of length two in H from S to T.*

Proof. Let $1/n < \varepsilon \ll \eta$. Fix two disjoint ordered ℓ-sets S and T of $V(H)$. In order to get the desired ℓ-path, it suffices to find two $(k-\ell)$-sets S_1 and T_1 with $|S_1 \cap T_1| = \ell$ such that $S \cup S_1 \in E(H)$ and $T \cup T_1 \in E(H)$. For $\ell = k/2$, we have $\min\{|N(S)|, |N(T)|\} \geqslant \delta_\ell(H) \geqslant (1/2+\eta)\binom{n}{k-\ell}$. So there exists an ℓ-set $S_1 \in N(S) \cap N(T)$ such that $S \cup S_1 \in E(H)$ and $T \cup S_1 \in E(H)$. Now we suppose $1 \leqslant \ell < k/2$. Consider N_S and N_T as two $(k-\ell)$-graphs on $V(H)$ with the edge sets $N(S)$ and $N(T)$ respectively. So it suffices to show $\partial_\varepsilon^{k-2\ell}(N_S) \cap \partial_\varepsilon^{k-2\ell}(N_T) \neq \emptyset$. Indeed, suppose $D \in \partial_\varepsilon^{k-2\ell}(N_S) \cap \partial_\varepsilon^{k-2\ell}(N_T)$, then it is easy to find $S_1 \in N_S$ and $T_1 \in N_T$ with $S_1 \cap T_1 = D$.

We apply the following procedure iteratively and get a spanning subgraph of N_S, denoted by G, which satisfies that for any ℓ-set X in $V(G)$, either $\deg_G(X) \geqslant \varepsilon n^{k-2\ell}$ or $\deg_G(X) = 0$. If there is an ℓ-set B with $\deg_{N_S}(B) < \varepsilon n^{k-2\ell}$, delete all the edges containing B in N_S. Note that when the process ends, the number of deleted edges is at most $\varepsilon n^{k-2\ell}\binom{n}{\ell}$. By the minimum ℓ-degree condition of H, $e(N_S) = |N(S)| \geqslant \left((1/2)^{\frac{k-\ell}{\ell}} + \eta\right)\binom{n}{k-\ell}$. So as $\varepsilon \ll \eta$,

$$e(G) \geqslant e(N_S) - \varepsilon n^{k-2\ell}\binom{n}{\ell} > \binom{(1/2)^{\frac{1}{\ell}} n + \varepsilon^{\frac{1}{k-\ell}} n}{k-\ell}$$

and $\partial_\varepsilon^{k-2\ell}(G) = \partial_0^{k-2\ell}(G)$. Using Theorem 11 to G, we get $|\partial_0^{k-2\ell}(G)| > \binom{(1/2)^{\frac{1}{\ell}} n + \varepsilon^{\frac{1}{k-\ell}} n}{\ell}$. Thus

$$|\partial_\varepsilon^{k-2\ell}(N_S)| \geqslant |\partial_\varepsilon^{k-2\ell}(G)| = |\partial_0^{k-2\ell}(G)| > \frac{1}{2}\binom{n}{\ell}.$$

Similar arguments show that $|\partial_\varepsilon^{k-2\ell}(N_T)| > \frac{1}{2}\binom{n}{\ell}$. Hence $\partial_\varepsilon^{k-2\ell}(N_S) \cap \partial_\varepsilon^{k-2\ell}(N_T) \neq \emptyset$. $\square$

3.2 Absorbing Path Lemma

Our main contribution of this paper is to prove the following absorbing path lemma, which gives an absorbing ℓ-path P which can absorb a small but arbitrary set of vertices.

Lemma 13 (Absorbing path lemma). *Suppose that $k \geqslant 3$, $1 \leqslant \ell < k/2$ and $1/n \ll \theta \ll \gamma \ll \eta, 1/k$. Let H be a k-graph on n vertices satisfying*

$$\delta_\ell(H) \geqslant \left(\max\{1/3, (1/2)^{\frac{k-\ell}{\ell}}\} + \eta\right)\binom{n}{k-\ell}$$

with $\ell \geqslant 2$, or $\delta_{\ell+1}(H) \geqslant (1/3+\eta)\binom{n}{k-\ell-1}$. Then there exists an ℓ-path P with $|V(P)| \leqslant \gamma n$ such that P can absorb any set $X \subseteq V(H) \setminus V(P)$ with $|X| \leqslant \theta n$, $(k-\ell) \mid |X|$, that is, there exists an ℓ-path Q with the same ordered ends as P, where $V(Q) = V(P) \cup X$.

Definition 14. Let H be a k-graph. Given a set S of $k-\ell$ vertices of H, we call an ordered set an S-*absorber*, if it is a sequence of ℓ-paths $\mathcal{Q} = (P_1, \dots, P_s)$, and there exists another sequence $\mathcal{Q}' = (P'_1, \dots, P'_s)$ of ℓ-paths such that $V(\mathcal{Q}) = V(\mathcal{Q}') \cup S$ and P_i, P'_i have the same ends for each $i \in [s]$.

It is known that if every $(k-\ell)$-set has many absorbers, then known probabilistic arguments will produce an absorbing path. To establish a similar property, we use a variant of the absorbing method originated from [25] and also developed in [10]. The following example illustrates the idea of absorbers. Given a set of $k-\ell$ vertices $\{v_1, \dots, v_{k-\ell}\}$, consider a set of ℓ-paths $P_1, \dots, P_{k-\ell}$ of length two (swappers) and a k-graph A (end-absorber) containing a spanning ℓ-path P_A with $S'_A = \{w_1, \dots, w_{k-\ell}\} \subseteq V(A)$. For $i \in [k-\ell]$, v_i has degree two in P_i and $V(P_i) \setminus \{v_i\} \cup \{w_i\}$ also forms an ℓ-path in which both two edges contain w_i and with the same ends as P_i. Moreover, we require that $A - S'_A$ also contains a spanning ℓ-path with the same ends as P_A. That is, when we absorb $\{v_1, \dots, v_{k-\ell}\}$, v_i will play the role of w_i in P_i for $i \in [k-\ell]$ and $w_1, \dots, w_{k-\ell}$ will be put inside $A - \{w_1, \dots, w_{k-\ell}\}$. (See Figure 1).

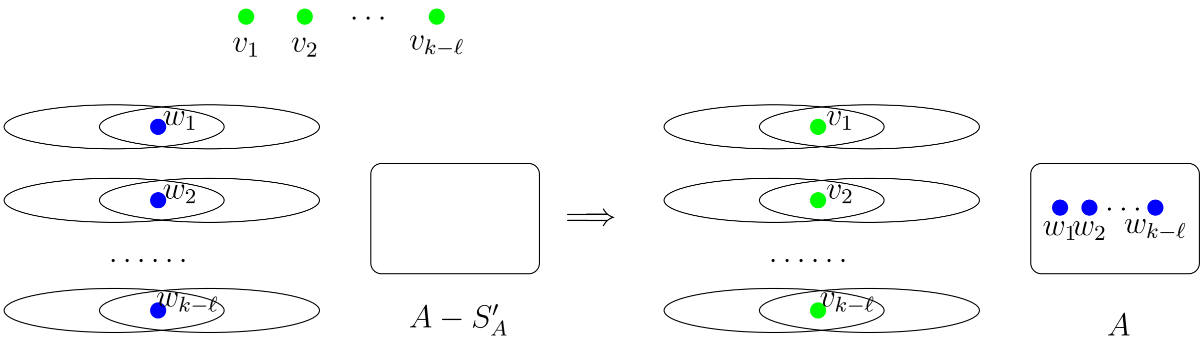

Figure 1: a $\{v_1, \dots, v_{k-\ell}\}$-absorber, where $S'_A = \{w_1, \dots, w_{k-\ell}\}$.

Our actual absorbers are a little bit more complicated, namely, we allow swapper chains of constant length and may concatenate them. The following absorbing path was constructed and used in [18], which we will use as an end-absorber in our proof.

Proposition 15 ([18], Proposition 6.1)**.** *For all integers $k \geqslant 3$ and $1 \leqslant \ell \leqslant k-1$ such that $(k-\ell) \nmid k$, there is a k-partite k-graph $A(k, \ell)$ with the following properties.*

1. $|V(A(k,\ell))| \leqslant k^4$.
2. $V(A(k,\ell)) = S' \cup X$, *where S' and X are disjoint and $|S'| = k - \ell$.*
3. *$A(k,\ell)$ contains an ℓ-path P with vertex set X and ordered ends P^{beg} and P^{end}.*
4. *$A(k,\ell)$ contains an ℓ-path Q with vertex set $S' \cup X$ and ordered ends P^{beg} and P^{end}.*
5. *Each edge of $A(k,\ell)$ contains at most one vertex of S'.*

6. Each vertex class of $A(k, \ell)$ contains at most one vertex of S'.

The lattice-based absorbing method features a partition lemma, which gives a partition of $V(H)$ such that each part is closed. We need the following partition lemma in our context.

Lemma 16 ([11], Lemma 5.4)**.** *Suppose that integers $c, k \geqslant 2$, $1 \leqslant \ell < k$ and $0 < 1/n \ll \beta \ll \beta' \ll \delta \ll 1/c$. Let H be a k-graph on n vertices and every set of $c+1$ vertices in $V(H)$ contains two vertices that are $(\beta', 1)$-reachable in H. Then there exists a partition $\mathcal{P}$ of $V(H)$ into $V_1, V_2, \ldots, V_r, U$ with $r \leqslant c$ such that for any $i \in [r]$, $|V_i| \geqslant (\delta - \beta')n$ and V_i is $(\beta, 2^{c-1})$-closed in H, and $0 \leqslant |U| \leqslant c\delta n$.*

Let $\mathcal{P} = \{V_1, V_2, \ldots, V_r, U\}$ be a vertex partition of H. The *index vector* $i_{\mathcal{P}}(S) \in \mathbb{Z}^r$ of a subset $S \subseteq V$ with respect to $\mathcal{P}$ is the vector whose coordinates are the sizes of the intersections of S with each V_i, $i \in [r]$. We call a vector $\mathbf{i} \in \mathbb{Z}^r$ an s-vector if all its coordinates are non-negative and their sum equals s. Given $\mu > 0$, a k-vector $\mathbf{v}$ is called a μ-robust edge-vector if there are at least μn^k edges e in H satisfying $i_{\mathcal{P}}(e) = \mathbf{v}$. Let $I^{\mu}_{\mathcal{P}}(H)$ be the set of all μ-robust edge-vectors.

Now we are ready to prove Lemma 13. The proof follows the scheme of the absorbing method and uses Lemma 9 and Lemma 12 in the obvious way. The additional work comes from the fact that not all the $(k-\ell)$-sets have many absorbers. To address this we use Lemma 16 to find a partition of $V(H)$ into at most three parts, and classify the $(k-\ell)$-sets that do have many absorbers. Then we show that we can always partition the leftover vertices together with a reserved set R_1 into $(k-\ell)$-sets that have many absorbers in the absorbing path.

Proof of Lemma 13. Suppose we have the constants satisfying the following hierarchy

$$1/n \ll \theta \ll \beta \ll \beta' \ll \mu \ll \delta \ll \gamma \ll \eta, 1/k.$$

Let H be a k-graph on n vertices such that

$$\delta_{\ell+1}(H) \geqslant (1/3 + \eta)\binom{n}{k-\ell-1} \ \text{ or } \ \delta_{\ell}(H) \geqslant \left(\max\left\{1/3, (1/2)^{\frac{k-\ell}{\ell}}\right\} + \eta\right)\binom{n}{k-\ell}.$$

Applying Proposition 8 to H, we get that every triple of vertices in $V(H)$ contains two vertices which are $(\beta, 1)$-reachable in H. Without loss of generality, we suppose that $\delta_d(H) \geqslant (1/3 + \eta)\binom{n}{k-d}$ with $d \geqslant 2$. So by Lemma 16, we get a partition $\mathcal{P}$ of $V(H)$ such that $\mathcal{P} = \{V_1, U\}$ or $\mathcal{P} = \{V_1, V_2, U\}$, where $|V_i| \geqslant (\delta - \beta')n$ and V_i is $(\beta, 2)$-closed in H for $i \in [2]$, and $0 \leqslant |U| \leqslant 2\delta n$. Note that the case $\mathcal{P} = \{V_1, U\}$ is indeed simpler. However, to unify the arguments, when $\mathcal{P} = \{V_1, U\}$ we arbitrarily split V_1 into two sets of equal size and by abusing the notation we call the resulting partition $\mathcal{P} = \{V_1, V_2, U\}$. So we only need to deal with one case, where $\mathcal{P} = \{V_1, V_2, U\}$, $|V_i| \geqslant (\delta - \beta')n$ and V_i is $(\beta, 2)$-closed in H for $i \in [2]$. Suppose $\mathcal{P} = \{V_1, V_2, U\}$ and let $H' := H - U$.

Fact 17. *There exists $1 \leqslant a \leqslant k-1$ such that $(a, k-a) \in I^{\mu}_{\mathcal{P}}(H')$.*

Proof. Let $n' := n - |U|$ and $n_1 := |V_1|$. Then $n' \geqslant (1-2\delta)n$ and $\min\{n_1, n'-n_1\} \geqslant (\delta - \beta')n$. Suppose $(a, k-a) \notin I^{\mu}_{\mathcal{P}}(H')$ for all $1 \leqslant a \leqslant k-1$, that is, $I^{\mu}_{\mathcal{P}}(H') \subseteq \{(0,k),(k,0)\}$. So by $d \geqslant 2$, we have $\sum_{S:i_{\mathcal{P}}(S)=(1,d-1)} \deg_{H'}(S) \leqslant k\mu(n')^k \cdot k^d = k^{d+1}\mu(n')^k$. On the other hand, by the minimum degree condition of H, we have

$$\begin{aligned}\sum_{S:i_{\mathcal{P}}(S)=(1,d-1)} \deg_{H'}(S) &\geqslant n_1\binom{n'-n_1}{d-1}\left(\delta_d(H) - 2\delta n\binom{n}{k-d-1}\right)\\ &\geqslant (\delta-\beta')n\binom{(\delta-\beta')n}{d-1}\left((1/3+\eta)\binom{n}{k-d} - 2\delta n\binom{n}{k-d-1}\right)\\ &> \delta^d/3\binom{n'}{k} > k^{d+1}\mu(n')^k,\end{aligned}$$

which is a contradiction. So there exists $1 \leqslant a \leqslant k-1$ such that $(a, k-a) \in I^{\mu}_{\mathcal{P}}(H')$. $\square$

Suppose $(a, k-a) \in I^{\mu}_{\mathcal{P}}(H')$ as in Fact 17, where $1 \leqslant a \leqslant k-1$. Let

$$m := \begin{cases} a & a \leqslant k/2 \\ a - \ell + 1 & a > k/2. \end{cases}$$

Then $1 \leqslant m \leqslant a$ and $0 \leqslant k - \ell - m \leqslant k - a - 1$. Let $\mathcal{S}$ be the family of all $(k-\ell)$-sets S with $i_{\mathcal{P}}(S) \in \{(m, k-\ell-m), (m-1, k-\ell-m+1)\}$. So for any $S \in \mathcal{S}$, we have

$$|S \cap V_1| \leqslant a \quad \text{and} \quad |S \cap V_2| \leqslant k - a. \tag{3}$$

The following claim says that every $(k-\ell)$-set $S \in \mathcal{S}$ has many absorbers. We postpone its proof until later.

Claim 18. *There exists $b = (4k-2\ell-1)(k-\ell)+r$ for some $r \leqslant k^4$ such that the following holds. For any $(k-\ell)$-set $S \in \mathcal{S}$, H contains $\beta^{k-\ell+1}n^b/2$ S-absorbers, each of which is a b-tuple which spans a family of $2k-2\ell+1$ vertex-disjoint ℓ-paths.*

We select a family $\mathcal{T}$ of b-tuples at random independently from H by including each ordered b-set T with probability $\beta^{k-\ell+2}n^{1-b}$. For a fixed S, let A_S be the set of all members of $\mathcal{T}$ which are S-absorbers. By Claim 18, $\mathbb{E}[|A_S|] \geqslant \beta^{2k-2\ell+3}n/2$. Moreover,

$$\mathbb{E}[|\mathcal{T}|] = \beta^{k-\ell+2}n^{1-b}\binom{n}{b}b! \leqslant \beta^{k-\ell+2}n,$$

$$\mathbb{E}[|(T,T') : T, T' \text{ in } \mathcal{T} \text{are intersecting}|] \leqslant b^2 n^{2b-1}(\beta^{k-\ell+2}n^{1-b})^2 = \beta^{2k-2\ell+4}b^2 n$$

(the corresponding unordered sets intersect). By the Chernoff bound, Markov's inequality and the union bound, we can fix an outcome of our random selection of $\mathcal{T}$ satisfying the following properties:

1. for every $(k-\ell)$-set $S \in \mathcal{S}$, $\mathcal{T}$ contains at least $\beta^{2k-2\ell+3}n/4$ S-absorbers;
2. $|\mathcal{T}| \leqslant 2\beta^{k-\ell+2}n$;

3. there are at most $2\beta^{2k-2\ell+4}b^2n$ overlapping members of $\mathcal{T}$,

We delete one set from each overlapping pairs of members of $\mathcal{T}$. Also delete from $\mathcal{T}$ every member of $\mathcal{T}$ which is not an S-absorber for any $S \in \mathcal{S}$. Then we obtain a family $\mathcal{F}$ of b-tuples such that $|\mathcal{F}| \leqslant 2\beta^{k-\ell+2}n$, each b-tuple is an S-absorber for some $S \in \mathcal{S}$ and for each $S \in \mathcal{S}$, $\mathcal{F}$ contains at least $\beta^{2k-2\ell+3}n/4 - 2\beta^{2k-2\ell+4}b^2n \geqslant \beta^{2k-2\ell+4}n$ S-absorbers. By Claim 18, each S-absorber is a b-tuple which spans a family of $2k-2\ell+1$ vertex-disjoint ℓ-paths. Denote all these ℓ-paths in $\mathcal{F}$ by $\{P_1, P_2, \dots, P_q\}$, where $q \leqslant 2\beta^{k-\ell+2}n(2k-2\ell+1) \leqslant \beta^{k-\ell+1}n$. Let $V(\mathcal{F})$ be the set of all vertices that are covered by members in $\mathcal{F}$.

Now we shall use the minimum d-degree condition of H to greedily construct disjoint edges to cover all vertices in $U \setminus V(\mathcal{F})$ while avoiding the vertices of the paths $P_i, 1 \leqslant i \leqslant q$. Since $\delta_d(H) \geqslant (1/3+\eta)\binom{n}{k-d}$, we have $\delta_1(H) \geqslant (1/3+\eta)\binom{n}{k-1}$. Note that $|U \setminus V(\mathcal{F})| \leqslant |U| \leqslant 2\delta n$ and $|V(\mathcal{F})| \leqslant 2\beta^{k-\ell+2}nb$. For any set $U' \subseteq V(H)$ with $|U'| \leqslant 2\beta^{k-\ell+2}nb+2k\delta n$, $\delta_1(H\setminus U') \geqslant \frac{1}{3}\binom{n}{k-1}$. Thus, we find a matching $M = \{P_{q+1}, \dots, P_{q+h}\}$ of size $h := |U \setminus V(\mathcal{F})| \leqslant 2\delta n$ such that $V(M)$ does not intersect any of the paths $P_1, \dots, P_q$ and each edge of M contains exactly one vertex of $U \setminus V(\mathcal{F})$.

Let P_i^{beg} and P_i^{end} be the ordered ends of P_i for $1 \leqslant i \leqslant q+h$. We see that $\delta_\ell(H\setminus U') \geqslant \left((1/2)^{\frac{k-\ell}{\ell}} + \frac{\eta}{2}\right)\binom{n-|U'|}{k-\ell}$ or $\delta_{\ell+1}(H \setminus U') \geqslant \left(1/3+\frac{\eta}{2}\right)\binom{n-|U'|}{k-\ell-1}$ holds for any vertex set U' with $|U'| \leqslant \gamma n/2$. As $(q+h)(b+8k^5) \leqslant (\beta^{k-\ell+1}n + 2\delta n + 1)(b+8k^5) \leqslant \gamma n/2$, we can use Lemma 12 or Lemma 9 to greedily connect each ordered ℓ-set P_i^{end} to P_{i+1}^{beg} by an ℓ-path P_i' with $|V(P_i')| \leqslant 8k^5$, such that P_i' intersects P_i and P_{i+1} only in the sets P_i^{end} and P_{i+1}^{beg} and does not intersect any other P_j or any previously chosen P_j'. Having found these ℓ-paths, we obtain an ℓ-path P' as $P_1P_1'P_2P_2'\dots P_{q+h-1}P_{q+h-1}'P_{q+h}$ with $|V(P')| \leqslant \gamma n/2$. Note that for any $S \in \mathcal{S}$, P' contains at least $\beta^{2k-2\ell+4}n$ mutually disjoint S-absorbers. Thus P' can greedily absorb a vertex set $W \in V(H) \setminus V(P')$, if $(k-\ell)||W|$, $|W| \leqslant (k-\ell)\beta^{2k-2\ell+4}n$ and there exist nonnegative integers x, y such that $i_{\mathcal{P}}(W) = x(m, k-\ell-m) + y(m-1, k-\ell-m+1)$.

Let $p := \lfloor \theta n \rfloor$. Take $4p$ mutually disjoint vertex sets $S_1, S_2, \dots, S_{2p}, T_1, T_2, \dots, T_{2p}$ from $V(H) \setminus V(P')$ with $i_{\mathcal{P}}(S_i) = (m, k-\ell-m)$ and $i_{\mathcal{P}}(T_i) = (m-1, k-\ell-m+1)$ for $i \in [2p]$. We denote the union of these S_i and T_i, $i \in [2p]$ by R_1. So $R_1 \subseteq V(H) \setminus V(P')$, $|R_1| = 4(k-\ell)p$ and $i_{\mathcal{P}}(R_1) = 2p(m, k-\ell-m) + 2p(m-1, k-\ell-m+1)$. Thus P' can absorb R_1, that is, there exists an ℓ-path P with the same ordered ends as P', where $V(P) = V(P') \cup R_1$. Now we show that P is the desired absorbing ℓ-path. Note that $|V(P)| \leqslant \gamma n$. Fix any set $X \subseteq V(H) \setminus V(P)$ with $|X| \leqslant p$ and $(k-\ell) \mid |X|$ as required by the lemma. Suppose further that $i_{\mathcal{P}}(X) = (t, s) = x(m, k-\ell-m) + y(m-1, k-\ell-m+1)$. Then we have

$$\begin{cases} x = t - \frac{(m-1)(t+s)}{k-\ell} \\ y = \frac{(t+s)m}{k-\ell} - t. \end{cases}$$

Since $t+s = |X|$, $(k-\ell) \mid |X|$ and $m/(k-\ell) \leqslant 1$, we get that x, y are integers and $|x|, |y| \leqslant |X| < 2p$. Thus $i_{\mathcal{P}}(X \cup R_1) = (x+2p)(m, k-\ell-m) + (y+2p)(m-1, k-\ell-m+1)$, where $x+2p > 0, y+2p > 0$ and $|X \cup R_1| \leqslant 4(k-\ell)p + p \leqslant (k-\ell)\beta^{2k-2\ell+4}n$. So P' can absorb $X \cup R_1$, that is, P can absorb X. $\square$

To complete the proof of Lemma 13, it remains to prove Claim 18.

Proof of Claim 18. Let $S := \{v_1, v_2, \ldots, v_{k-\ell}\}$ be a $(k-\ell)$-set in $\mathcal{S}$. Fix a k-partite k-graph $A(k,\ell)$ on $[V_1^A, V_2^A, \ldots, V_k^A]$ satisfying Proposition 15 with $|A(k,\ell)| = r \leqslant k^4$ and $V(A(k,\ell)) = S' \cup X$, where $|S'| = k-\ell$. Without loss of generality, suppose $|S' \cap V_i^A| = 1$ for $i \in [k-\ell]$. Let $b := (4k-2\ell-1)(k-\ell)+r$. Since $(a, k-a) \in I_{\mathcal{P}}^{\mu}(H')$, the number of edges whose index vectors are $(a, k-a)$ is at least μn^k. Since $S \in \mathcal{S}$, by (3), we have $t_1 := |S \cap V_1| \leqslant a$ and $|S \cap V_2| \leqslant k-a$. Since $A(k,\ell)$ is k-partite, by the supersaturation result (see [6]) on the subgraph of H that consists of all edges of index vector $(a, k-a)$, H contains βn^r copies of $A(k,\ell)$ each with $V_1^A, \ldots, V_{t_1}^A \subseteq V_1$ and $V_{t_1+1}^A, \ldots, V_{k-\ell}^A \subseteq V_2$. For such a copy A of $A(k,\ell)$, we denote by S'_A as the set of $k-\ell$ vertices given in the second term of Proposition 15. So $i_{\mathcal{P}}(S'_A) = i_{\mathcal{P}}(S)$ for each such A.

Consider a copy of $A(k,\ell)$ in H which we denote by A. Note that each of V_1, V_2 is $(\beta,2)$-closed in H. Without loss of generality, suppose $S'_A = \{w_1, w_2, \ldots, w_{k-\ell}\}$ such that v_i, w_i are $(\beta,2)$-reachable for $i \in [k-\ell]$ by $i_{\mathcal{P}}(S'_A) = i_{\mathcal{P}}(S)$. By the definition of reachability, for each $i \in [k-\ell]$, there are at least $\beta n^{4k-2\ell-1}$ $(4k-2\ell-1)$-sets T_i such that there exist ℓ-paths $P_i^1, P_i^2, P_i^3, P_i^4$ with $V(P_i^1 \cup P_i^2) = T_i \cup \{v_i\}$ and $V(P_i^3 \cup P_i^4) = T_i \cup \{w_i\}$, where P_i^1 has the same ends as P_i^3, and P_i^2 has the same ends as P_i^4. So there are at least $\beta^{k-\ell+1} n^b$ choices for $A \cup T_1 \cup T_2 \cup \cdots \cup T_{k-\ell}$ as an ordered set. Among them, at most $(k-\ell)n^{b-1}$ of them intersect S and at most $b^2 n^{b-1}$ of them contain repeated vertices. Thus there are at least $\beta^{k-\ell+1} n^b/2$ b-tuples avoiding S such that $A, T_1, T_2, \ldots, T_{k-\ell}$ are pairwise vertex-disjoint.

Now it remains to show that the b-tuple corresponding to $A \cup T_1 \cup T_2 \cup \cdots \cup T_{k-\ell}$ is an S-absorber. Firstly, $T_i \cup \{w_i\}$, $i \in [k-\ell]$ spans two vertex-disjoint ℓ-paths of length two, which together with the spanning ℓ-path in $A \setminus \{w_1, w_2, \ldots, w_{k-\ell}\}$ form a family of $2k-2\ell+1$ ℓ-paths which span $V(A) \cup T_1 \cup T_2 \cup \cdots \cup T_{k-\ell}$. Secondly, $H[T_i \cup \{v_i\}]$ forms two vertex-disjoint ℓ-paths of length two for $i \in [k-\ell]$, which together with the spanning ℓ-path in A gives a family of $2k-2\ell+1$ ℓ-paths which span $S \cup V(A) \cup T_1 \cup T_2 \cup \cdots \cup T_{k-\ell}$ and have the same ends as the family of ℓ-paths above. So the b-tuple corresponding to $A \cup T_1 \cup T_2 \cup \cdots \cup T_{k-\ell}$ is an S-absorber (cf. Figure 1). $\square$

3.3 Proofs of Theorem 4 and Theorem 5

We prove Theorem 4 by following the common approach of absorption (cf. [18, 27]). That is, we decompose the proof in the usual way into the absorbing path lemma, the reservoir lemma, the connecting lemma and the path cover lemma. We will use these lemmas to find an absorbing path and a reservoir firstly, and then we cover the majority of vertices by vertex-disjoint ℓ-paths. We connect up all these ℓ-paths to form an ℓ-cycle. Finally, we absorb the leftover vertices into the absorbing path, thereby completing a Hamilton ℓ-cycle.

We need the following path cover lemma, which states that the vertex set of any sufficiently large k-graph satisfying the minimum degree condition can be covered by a constant number of vertex-disjoint ℓ-paths with a small leftover.

Lemma 19 (Path cover lemma [7], Lemma 3.1)**.** *For all integers $k \geqslant 3$, $1 \leqslant \ell < k/2$ and $1 \leqslant d \leqslant k-1$, suppose $1/n \ll 1/D \ll \varepsilon \ll \mu, 1/k$. Let H be a k-graph on n vertices with $\delta_d(H) \geqslant (t(k,d,\ell)+\mu)\binom{n}{k-d}$. Then there is a family of at most D vertex-disjoint ℓ-paths covering all but at most $4\varepsilon n$ vertices of H.*

We also need the reservoir lemma [18] which guarantees the subhypergraph satisfying the degree condition of the connecting lemma.

Lemma 20 (Reservoir lemma [18], Lemma 8.1)**.** *Suppose that $k \geqslant 2$, $1 \leqslant d \leqslant k-1$ and $1/n \ll \alpha, \mu, 1/k$. Let H be a k-graph on n vertices with $\delta_d(H) \geqslant \mu\binom{n}{k-d}$, and let R be a subset of $V(H)$ of size αn chosen uniformly at random. Then with probability $1-o(1)$, $|N_H(S) \cap \binom{R}{k-d}| \geqslant \mu\binom{\alpha n}{k-d} - n^{k-d-1/3}$ holds for every $S \in \binom{V(H)}{d}$.*

Now we give the proof of Theorem 4 by combining the results as outlined above.

Proof of Theorem 4. Suppose we have the constants satisfying the following hierarchy

$$1/n \ll 1/D \ll \varepsilon \ll \alpha \ll \theta \ll \beta \ll \gamma \ll \eta \ll 1/k$$

and assume that $n \in (k-\ell)\mathbb{N}$. Let $t := \max\{t(k,d,\ell), 1/3\}$. Suppose that H is a k-graph on n vertices such that $\delta_d(H) \geqslant (t+\eta)\binom{n}{k-d}$. Applying Lemma 13, we obtain an ℓ-path P_0 with $|V(P_0)| \leqslant \gamma n$, such that P_0 can absorb any set $S \subseteq V(H) \setminus V(P_0)$ with $|S| \leqslant \theta n$, $(k-\ell) \mid |S|$.

Next let R be a set of αn vertices of $V(H)$ chosen uniformly at random. Applying Lemma 20 to H, we obtain that with probability $1-o(1)$,

$$\left| N_H(S) \cap \binom{R}{k-d} \right| \geqslant (t+\eta/2)\binom{\alpha n}{k-d}$$

for every $S \in \binom{V(H)}{d}$. Since $\mathbb{E}[|R \cap V(P_0)|] = \alpha |V(P_0)|$, by Markov's inequality, with probability at least $1/2$, we have $|R \cap V(P_0)| \leqslant 2\gamma\alpha n$. Then we fix a choice of R which has the two properties above.

Let $V' := V(H) \setminus (R \cup V(P_0))$. Note that $|R \cup V(P_0)| \leqslant \gamma n + \alpha n$. The induced subhypergraph $H' := H[V']$ satisfies

$$\delta_d(H') \geqslant (t+\eta)\binom{n}{k-d} - (\gamma n + \alpha n)\binom{n}{k-d-1} \geqslant (t+\eta/2)\binom{|V'|}{k-d}.$$

Applying Lemma 19 to H', we obtain vertex-disjoint ℓ-paths $P_1, \ldots, P_q$ that together cover all but at most εn vertices of H', where $q \leqslant D$. Denote by X the set of uncovered vertices. Thus $|X| \leqslant \varepsilon n$. We denote the ordered ends of P_i by P_i^{beg} and P_i^{end}, $0 \leqslant i \leqslant q$. Let $P_{q+1}^{beg} := P_0^{beg}$. For $0 \leqslant i \leqslant q$, we now find vertex-disjoint ℓ-paths P_i' by Lemma 9 to connect P_i^{end} and P_{i+1}^{beg}, which actually connects P_i and P_{i+1} such that $V(P_i') \subseteq (R \setminus V(P_0)) \cup P_i^{end} \cup P_{i+1}^{beg}$ and $|V(P_i')| \leqslant 8k^5$. More precisely, suppose that we have chosen such ℓ-paths $P_0', \ldots, P_{i-1}'$. Let $R_i = \left(P_i^{end} \cup P_{i+1}^{beg} \cup R \setminus V(P_0)\right) \setminus \bigcup_{j=0}^{i-1} V(P_j')$. Thus

$$\delta_d(H[R_i]) \geqslant (t+\eta/2)\binom{\alpha n}{k-d} - (8k^5 D + 2\gamma\alpha n)\binom{\alpha n}{k-d-1} \geqslant t\binom{|V(R_i)|}{k-d}$$

and thus we may apply Lemma 9 with t in place of μ to find a desired ℓ-path P_i'.

Let $C := P_0P_0'P_1P_1' \cdots P_qP_q'$ be the ℓ-cycle we have obtained so far and let $R'' := V(H) \setminus V(C)$. Then indeed $R'' = X \cup \left(R \setminus \left(V(P_0) \cup \bigcup_{0\leqslant i\leqslant q} V(P_i')\right)\right)$ and in particular, $|R''| \leqslant (\alpha+\varepsilon)n \leqslant \theta n$. Since $k-\ell$ divides both n and $|V(C)|$, we have $(k-\ell) \mid |R''|$. So we can utilize the absorbing property of P_0 to get an ℓ-path Q_0 with $V(Q_0) = V(P_0) \cup R''$ such that P_0 and Q_0 have the same ordered ends, obtaining a Hamilton ℓ-cycle $C' := Q_0P_0'P_1P_1' \cdots P_qP_q'$ in H. $\square$

Proof of Theorem 5 follows verbatim as the proof of Theorem 4, after replacing Lemma 9 with Lemma 12. Thus we omit it.

Acknowledgements

We would like to express our gratitude to the anonymous reviewers for their valuable comments that greatly improved the presentation of this paper. Jie Han was supported by Natural Science Foundation of China (12371341). Guanghui Wang was supported by Natural Science Foundation of China (12231018) and National Key R&D Program of China (2023YFA1009600).

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A Proof of Theorem 3

In this section we prove Theorem 3. Considering the 5-graph $H_{5,2}$ in Section 1 and Theorem 5, we have

$$\frac{91}{216} \leqslant h_2^2(5) \leqslant \max\left\{(1/2)^{\frac{3}{2}}, t(5,2,2), 1/3\right\}.$$

So it suffices to prove $t(5,2,2) \leqslant \frac{91}{216}$. We will prove the following theorem to determine $t(5,2,2)$.

Theorem 21. *Let $0 < 1/n \ll \varepsilon \ll \eta$. Let H be a 5-graph on n vertices with*

$$\delta_2(H) \geqslant \left(\frac{91}{216} + \eta\right)\binom{n}{3}.$$

Then H contains a $Y_{5,4}$-tiling covering all but at most εn vertices. In particular, $t(5,2,2) = \frac{91}{216}$.

For $p > 0$, fix two k-graphs, F of order p, and H. Let $\mathcal{F}_{F,H} \subseteq \binom{V(H)}{p}$ be the family of p-sets in $V(H)$ that span a copy of F. A *fractional F-tiling* in H is a function $\omega : \mathcal{F}_{F,H} \to [0,1]$ such that for each $v \in V(H)$ we have $\sum_{v\in e\in\mathcal{F}_{F,H}} \omega(e) \leqslant 1$. Then $\sum_{e\in\mathcal{F}_{F,H}} \omega(e)$ is the *size* of ω. Such a fractional F-tiling is called perfect if it has size n/p. Let $0 \leqslant d \leqslant k-1$ and $0 \leqslant s \leqslant n/p$. We denote by $f_d^s(F,n)$ the minimum m so that every n-vertex k-graph H with $\delta_d(H) \geqslant m$ has a fractional F-tiling of size s. In particular, $\delta_0(H) = e(H)$. Gan, Han, Sun and Wang [7] proved the following two lemmas for fractional $Y_{k,b}$-tilings.

Lemma 22 ([7], Lemma 7.2)**.** *Suppose $0 < 1/n \ll \varepsilon \ll \eta$. Let H be a 5-graph on n vertices with $\delta_2(H) \geqslant f_2^{n/6}(Y_{5,4}, n) + \eta\binom{n}{3}$. Then H contains a $Y_{5,4}$-tiling covering all but at most εn vertices.*

Lemma 23 ([7], Lemma 7.1)**.** *For $n \geqslant 5$, we have $f_2^{n/6}(Y_{5,4}, n) \leqslant f_0^{n/6}(Y_{3,2}, n-2)$.*

By Lemma 22, to prove Theorem 21, it suffices to find an almost perfect fractional $Y_{5,4}$-tiling under the minimum 2-degree condition. Then by Lemma 23, we transform this problem to finding a large $Y_{3,2}$-tiling under density condition, so that we can apply the following theorem of Han, Sun and Wang [12].

Theorem 24. *For every $\alpha, \gamma \in (0, 1/4)$ there exists an integer n_0 such that the following holds for every integer $n \geqslant n_0$. Let H be a 3-graph on n vertices such that*

$$e(H) \geqslant \max\left\{\binom{4\alpha n}{3}, \binom{n}{3} - \binom{n-\alpha n}{3}\right\} + \gamma n^3.$$

Then H contains a $Y_{3,2}$-tiling covering more than $4\alpha n$ vertices.

Proof of Theorem 21. Suppose that we have constants such that $0 < 1/n \ll \varepsilon \ll \eta$. Let H be a 5-graph on n vertices such that $\delta_2(H) \geqslant (\frac{91}{216} + \eta)\binom{n}{3}$. By Lemma 23 and Theorem 24 with ε in place of γ and $\alpha = \frac{n}{6(n-2)}$, we have

$$\begin{aligned} f_2^{n/6}(Y_{5,4}, n) &\leqslant f_0^{n/6}(Y_{3,2}, n-2) \\ &\leqslant \max\left\{\binom{\frac{2n^2}{3(n-2)}}{3}, \binom{n-2}{3} - \binom{(n-2)\left(1 - \frac{n}{6(n-2)}\right)}{3}\right\} + \varepsilon(n-2)^3 \\ &\leqslant \left(\frac{91}{216} + \frac{\eta}{2}\right)\binom{n}{3}. \end{aligned}$$

Thus we have $\delta_2(H) \geqslant (\frac{91}{216} + \eta)\binom{n}{3} \geqslant f_2^{n/6}(Y_{5,4}, n) + \frac{\eta}{2}\binom{n}{3}$. Applying Lemma 22 with $\eta/2$ in place of η, we obtain a $Y_{5,4}$-tiling covering all but at most εn vertices.

In particular, by the definition of $t(5,2,2)$, we get $t(5,2,2) \leqslant \frac{91}{216}$. By considering the 5-graph $H_{5,2}$ in Section 1 and Theorem 5, we have $\frac{91}{216} \leqslant h_2^2(5) \leqslant t(5,2,2)$. Thus $t(5,2,2) = \frac{91}{216}$. □