Braces of perfect matching width two

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Abstract

Perfect matching width is a treewidth-like parameter designed for graphs with perfect matchings. The concept was originally introduced by Norine for the study of non-bipartite Pfaffian graphs. Additionally, perfect matching width appears to be a useful structural tool for investigating matching minors, a specialised version of minors related to perfect matchings. In this paper we lay the groundwork for understanding the interaction of perfect matching width and matching minors by establishing tight connections between the perfect matching width of any matching covered graph G and the perfect matching width of its bricks and braces (a matching theoretic version of blocks) and proving that perfect matching width is almost monotone under the matching minor relation. As an application, we give several characterisations for braces of perfect matching width two, including one that allows for a polynomial time recognition algorithm.

Mathematics Subject Classifications: 05C83, 05C75

1 Introduction

The significance of matching minors, especially for the study of matching theoretic properties of bipartite graphs, was first observed by Little[1] [Lit75], who used bisubdivisions to characterise Pfaffian bipartite graphs. In the pursuit of a description of non-bipartite Pfaffian graphs, Norine introduced a matching theoretic analogue of treewidth, called *perfect matching width* [Nor05], and gave an algorithm that decides whether a graph from a class of bounded perfect matching width is Pfaffian in XP-time. While the general problem of characterising non-bipartite Pfaffian graphs remains open, perfect matching width appears

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[1]Although the word ‘matching minor’ did not appear before the work of Norine and Thomas [Nor05, NT07].

to be a valuable tool for the study of matching minors. For further information on Pfaffian graphs the reader is referred to the surveys of McCuaig and Thomas [McC04, Tho06].

Bipartite and non-bipartite graphs with perfect matchings behave in fundamentally different ways. One particular point of contrast is the matching minor relation. There exists an infinite anti-chain of bricks [NT08], however, for braces, no infinite anti-chain of braces is known for the matching minor relation. This hints at the existence of some more fundamental challenges regarding non-bipartite graphs that have to be overcome for establishing a theory of matching minors. Hence in this paper we focus on bipartite graphs and braces in particular. We mostly consider the class of *matching covered* graphs, that is, graphs in which every edge lies in some perfect matching. Please note that, since matching covered graphs encode all strongly connected digraphs (see for example [RST99, HRW19]), matching covered graphs are already a significant and non-trivial class.

In the following we aim at denoting bipartite graphs with a capital 'B' instead of a 'G' to highlight the parts where we specialise on bipartite graphs. Moreover, we assume every bipartite graph B to come with a two-partition into the *colour classes* V_1 and V_2, where vertices of V_1 are usually depicted as filled vertices and those of V_2 are empty in figures.

We give a short overview of the main results of this paper. In section 2 we consider the following two questions: How does the perfect matching width of a graph relate to the perfect matching width of its matching minors? And, can we determine the perfect matching width of a graph from the perfect matching width of its bricks and braces (its uniquely determined building blocks [Lov87])? In particular we prove the following two theorems.

Theorem 1. *Let G be a matching covered graph and H be a matching minor of G. Then* $\operatorname{pmw}(H) \leqslant 2\operatorname{pmw}(G)$.

Theorem 2. *Let G be a matching covered graph and H be a brick or brace of G maximising* $\operatorname{pmw}(H)$ *over all bricks and braces of G. Then* $\frac{1}{2}\operatorname{pmw}(H) \leqslant \operatorname{pmw}(G) \leqslant \operatorname{pmw}(H)$.

Next we consider the graphs of small perfect matching width. Having established that the maximum perfect matching width among the braces yields an upper bound on the perfect matching width of the graph itself, we start by characterising the braces of width two in section 3. We provide two different characterisations, one in terms of conformal subgraphs of a well-defined class of edge-maximal braces of perfect matching width two, which we refer to as *bipartite ladders*, and the other in form of specific elimination orderings.

Both characterisations allow us to obtain further results. In section 4 we make use of the bipartite ladders to show that, while the perfect matching width of every graph is bounded from above by its treewidth, for every $k \in \mathbb{N}$ there exists a brace of perfect matching width two whose treewidth is at least k. The characterisation via elimination orderings yields a polynomial time algorithm for the construction of a perfect matching decomposition of width two if one exists, which we prove in section 5.

The factor two in Theorem 1 stops us from directly lifting the results for braces to all bipartite graphs of perfect matching width two. However, in section 6, we consider a more restrictive version of perfect matching width, the *M-perfect matching width*, denoted pmw_M, which is known to be parametrically equivalent to perfect matching width [HRW19]. For M-perfect matching width we are able to prove a stronger version of Theorem 1 removing the factor.

This allows us to give a full characterisation of all bipartite graphs B that have a perfect matching M such that $\mathrm{pmw}_M(B) = 2$.

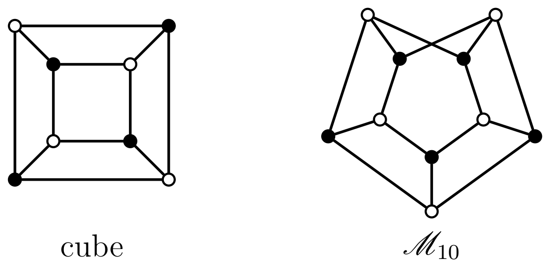

Figure 1: The braces *cube* and the Möbius ladder $\mathscr{M}_{10}$ of order 10.

Theorem 3. *Let B be a bipartite graph with a perfect matching. The following statements are equivalent.*

1. *There exists a perfect matching $M \in \mathcal{M}(B)$ such that* $\mathrm{pmw}_M(B) = 2$*,*
2. *for all $M \in \mathcal{M}(B)$ we have* $\mathrm{pmw}_M(B) = 2$*, and*
3. *B does not contain the cube or $\mathscr{M}_{10}$ as a matching minor (see fig. 1),*

1.1 Preliminaries

All graphs considered in this article are finite and simple, that is, we do not allow for loops or parallel edges. Let G be a graph. A set $M \subseteq E(G)$ of edges is a *matching* if no two edges in M share an endpoint. A matching M is called *perfect* if every vertex of G is an endpoint of some edge of M. We denote the set of perfect matchings of G by $\mathcal{M}(G)$. If G is connected and every edge of G is contained in some perfect matching of G we say that G is *matching covered*. The aim of Matching Theory is to study the structural properties of graphs with perfect matchings, and, within its context a plethora of results revealing a rich structural theory has appeared. For an in-depth exposition of Matching Theory the interested reader is referred to the book by Lovász and Plummer [LP09].

For the study of perfect matchings, there are several adaptions of standard terminology from graph theory to make sure the existence of a perfect matching is preserved. Let G be a graph with a perfect matching M. A set $X \subseteq V(G)$ is *conformal* if $G - X$ has a perfect matching, it is said to be *M-conformal* if M contains a perfect matching of $G - X$. Similarly a subgraph $H \subseteq G$ is *conformal* if $V(H)$ is, and it is *M-conformal* if M contains perfect matchings of $G - V(H)$ and of H. A *bisubdivision* of G is a graph

G' obtained by replacing every edge of G with a path of odd length (possibly one) whose internal vertices are fresh vertices. A *bicontraction* is the operation of contracting both edges incident with a vertex of degree two at the same time. Finally, a *matching minor* is a graph H that can be obtained by a series of bicontractions from a conformal subgraph of G. Note that, if G' is a bisubdivision of G, then G is indeed a matching minor of G'.

A subset of vertices S in a graph G is a *separator*, if $G - S$ is not connected and G is *k-connected* if G does not contain a separator of size less than k. Let G be a graph and $X \subseteq V(G)$. The *edge cut* with *shores* X and $\overline{X} \coloneqq V(G) \setminus X$ is the set $\partial_G(X) \coloneqq \big\{e \in E(G) \mid e \cap X \neq \emptyset,\ e \cap \overline{X} \neq \emptyset\big\}$. The *matching porosity* of $\partial_G(X)$ is the value

$$\mathrm{mp}(\partial_G(X)) \coloneqq \max_{M \in \mathcal{M}(G)} |M \cap \partial_G(X)|.$$

We say that a tree T is *cubic* if every non-leaf vertex of T is of degree three. Moreover, a tree is said to be *subcubic* if the maximum degree of T is three. By $\mathsf{L}(T)$ we denote the set of leaves of any tree, and if $t_1t_2 \in E(T)$ is an edge of T, we denote by T_{t_i} the unique component of $T - t_1t_2$ which contains t_i for any $i \in \{1, 2\}$. Let T be a subcubic tree. We can obtain a cubic tree T' from T by iteratively choosing a degree 2 vertex and contracting one of its two incident edges. The tree T' is, up to isomorphism, uniquely determined by T and we call T' the tree obtained from T by *trimming.* Note that $\mathsf{L}(T) = \mathsf{L}(T')$.

A *perfect matching decomposition* of G is a tuple (T, η) where T is a cubic tree and $\eta\colon \mathsf{L}(T) \to V(G)$ is a bijection. We associate with every edge $t_1t_2 \in E(T)$ a bipartition of $V(G)$ into the sets $\eta(T_{t_i}) \coloneqq \bigcup_{\ell \in \mathsf{L}(T_{t_i})} \{\eta(\ell)\}$ for both $i \in \{1, 2\}$. Moreover, we can associate the edge cut $\partial_G(\eta(T_{t_1})) = \partial_G(\eta(T_{t_2}))$ of G with t_1t_2. As a shorthand, we denote this edge cut by $\partial_G(t_1t_2)$. The *width* of (T, η) is defined as $\max_{e \in E(T)} \mathrm{mp}(\partial_G(e))$, and the *perfect matching width* of G, denoted by $\mathrm{pmw}(G)$, is the minimum width over all perfect matching decompositions of G.

Let M be a perfect matching of a graph G. An *M-decomposition* of G is a perfect matching decomposition (T, η) of G such that for every edge $e = uv \in M$, the leaves t_u and t_v of T with $\eta(t_u) = u$ and $\eta(t_v) = v$ are both adjacent to a vertex t_{uv} of T. Note that a perfect matching decomposition (T, η) of G is an M-decomposition if and only if for every $t_1t_2 \in E(T - \mathsf{L}(T))$ the sets $\eta(T_{t_i})$ are M-conformal. The *M-perfect matching width* of G, denoted by $\mathrm{pmw}_M(G)$, is now the minimum width over all M-decompositions of G. Perfect matching width and M-perfect matching width are parametrically equivalent.

Theorem 4 ([HRW19][2])**.** *Let G be a graph with a perfect matching M, then $\frac{1}{2}\,\mathrm{pmw}_M(G) \leqslant \mathrm{pmw}(G) \leqslant \mathrm{pmw}_M(G)$.*

Tight cuts, bricks and braces

Similar to (undirected) treewidth and directed treewidth being used to study minors and butterfly minors, perfect matching width can be considered a tool for the study of matching minors.

[2]Note that the statement slightly differs from the one in the original paper, it still follows from the proof provided there.

Let $X \subseteq V(G)$ be a set of vertices. The edge cut $\partial_G(X)$ is called *tight* if $\mathrm{mp}(\partial_G(X)) = 1$, it is called *trivial* if $|X| = 1$ or $|\overline{X}| = 1$, we say that X *induces* a tight cut, if $\partial_G(X)$ is tight. Note that the shores of a tight cut are always odd. If X is a shore of a tight cut, we call the operation of identifying X into a single vertex v_X and deleting all resulting loops and parallel edges a *tight cut contraction*. We denote the resulting graph by G_X. Observe that tight cut contractions of a matching covered graph are again matching covered. Let u be the unique vertex of $\overline{X}$ which is incident with the edge of M in $\partial_G(X)$, then we call the perfect matching $M|_{G_X} \coloneqq (M \cap E(G[\overline{X}])) \cup \{uv_X\}$ the *residual* of M in G_X.

A matching covered graph without a non-trivial tight cut is called a *brace* if it is bipartite and a *brick* otherwise. So every matching covered graph either is a brace, a brick or has a non-trivial tight cut and therefore can be decomposed into two smaller matching covered graphs. One can continue with this process of decomposing along tight cuts in the two smaller graphs until there are no more non-trivial tight cuts to be found. This process yields a list of bricks and braces and is known as the *tight cut decomposition* of G. A famous result by Lovász states that this list depends only on G and does not depend on the choices of the tight cuts, thus it is unique for every fixed graph G.

Theorem 5 ([Lov87])**.** *Any two tight cut decomposition procedures of a matching covered graph G yield the same list of bricks and braces.*

We say that two cuts $\partial(S), \partial(T)$ *cross* if all four of the following sets $S \cap T$, $S \cap \overline{T}$, $\overline{S} \cap T$, $\overline{S} \cap \overline{T}$ are non-empty, otherwise $\partial(S)$ and $\partial(T)$ are called *laminar*. Theorem 5 states that all maximal families of pairwise laminar tight cuts of a matching covered graph G induce tight cut decompositions that yield the same list of bricks and braces.

Many matching theoretical concepts can be expressed in terms of matching minors. Especially in bipartite graphs they play a huge role as the following results by Lucchesi et al. illustrate.

Lemma 6 ([LdCM15])**.** *Let G be a bipartite matching covered graph and $\partial(Z)$ a non-trivial tight cut in G. Then the two Z-contractions of G are matching minors of G.*

Corollary 7 ([LdCM15])**.** *Every brace of a bipartite matching covered graph G is a matching minor of G.*

2 Perfect matching width, tight cuts and matching minors

The main objective of this section is to prove Theorems 1 and 2. To this end we start by considering the relation between the parity of vertex sets and the decomposition tree in a perfect matching decomposition. We then consider tight cut contractions and establish that the maximum perfect matching width among the bricks and braces of a matching covered graph yields an upper bound on the perfect matching width of the graph itself. This establishes one of the inequalities in Theorem 2 as it proves that conformal subgraphs cannot have larger perfect matching width than the graph itself. For the remaining part, namely the contractions, we make use of M-decompositions, which cause the factor 2 in Theorems 1 and 2.

2.1 Cubic trees and perfect matching decompositions of even width

The trees of perfect matching decompositions, as for many branch decompositions (see [Vat12] for the general definition), are cubic, or at least subcubic. Just considering the possible structures of the trees themselves can be a useful tool when dealing with this kind of decompositions. We take a closer look at the cubic trees that appear as the trees of perfect matching decompositions.

For a cubic tree T we define the *spine* of T by $\mathrm{spine}(T) \coloneqq T - \mathsf{L}(T)$. The edges in $E(T) \setminus E(\mathrm{spine}(T))$ are called *trivial*. We say that an edge $e \in E(\mathrm{spine}(T))$ is *even*, if the two components of $T - e$ contain an even number of leaves of T each and it is *odd* otherwise. See fig. 2 for an illustration of these definitions.

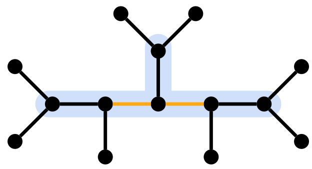

Figure 2: An example for a cubic tree T with its spine and its odd edges.

Note that, if T is the cubic tree underlying a perfect matching decomposition of a graph G, then T has an even number of leaves as G has an even number of vertices. This implies that in T a non-trivial edge e is odd if and only if the two trees of $T - e$ contain an odd number of leaves of T each.

We make the following useful observations on cubic trees with an even number of leaves.

Observation 8. *Let T be a cubic tree with $|\mathsf{L}(T)| = \ell$ even. Then the following statements are true.*

1. $|V(T)| = 2\ell - 2$,
2. $\mathrm{spine}(T)$ *has an even number of vertices,*
3. $\mathrm{spine}(T)$ *has an even number of vertices of degree* 2*, and*
4. $e \in E(\mathrm{spine}(T))$ *is an odd edge of T if and only if the two trees of* $\mathrm{spine}(T) - e$ *contain an even number of vertices each.*

It is easy to see that $\mathrm{spine}(T)$ is a subcubic tree. There is a close correspondence between the occurrence of odd edges in T and vertices of degree 2 in $\mathrm{spine}(T)$.

Lemma 9. *Let T be a cubic tree with an even number of leaves.*

1. *If* $\deg_v(\mathrm{spine}(T)) = 1$*, then v is not incident with an odd edge of T.*
2. *If* $\deg_v(\mathrm{spine}(T)) = 2$*, then v is incident with exactly* 1 *odd edge of T.*

3. *If* $\deg_v(\text{spine}(T)) = 3$*, then* v *is either incident with exactly* 2 *odd edges of* T *or with none.*

Proof. If v is of degree 1 in the spine of T, it is adjacent with exactly two leaves of T and thus, by definition, the unique edge incident with v in spine(T) cannot be odd.

Let v be a vertex of degree 2 in spine(T) and e_1, e_2 the two edges incident with v in the spine. In T itself v is incident with a third edge e_3 whose other endpoint is a leaf of T. Let k_i be the number of leaves of T contained in the component of $T - e_i$ that does not contain v. Then $|\mathsf{L}(T)| = k_1 + k_2 + k_3$ and $k_3 = 1$. Since the total number of leaves is even and k_3 is odd, exactly one of k_1 and k_2 is odd as well. Thus, exactly one of the two edges e_1 and e_2 is an odd edge of T.

At last we consider a degree 3 vertex v in spine(T). Let e_1, e_2, e_3 be the three edges of the spine incident with v and let k_i be the number of leaves of T contained in the component of $T - e_i$. In this case $|\mathsf{L}(T)| = k_1 + k_2 + k_3$ and thus neither all three, nor just one of them can be odd. □

In particular, no vertex in T can be incident with more than two odd edges and every degree 2 vertex of spine(T) is incident with exactly one odd edge of T. Hence, the following corollary holds.

Corollary 10. *Let* T *be a cubic tree with an even number of leaves. Then* spine(T) *is cubic if and only if* T *has no odd edges.*

Moreover, the odd edges of T induce a subforest of spine(T) such that the leaves of this forest are exactly the degree 2 vertices of spine(T). Also, no vertex of spine(T) can be incident with more than two odd edges of T and thus this subforest is actually a collection of paths.

Corollary 11. *Let* T *be a cubic tree with an even number of leaves and* $E_O \subset E(T)$ *the set of odd edges of* T*. Then* $T[E_O]$ *is a collection of pairwise disjoint paths. Moreover, the set of endpoints of these paths is exactly the set of degree* 2 *vertices in* spine(T)*.*

Next, we want to answer the question: How do odd edges interact with the perfect matching width? First, we investigate the influence of the existence of odd edges in the cubic tree of a perfect matching decomposition (T, η) on the parity of the width of (T, η).

Observation 12. *Let* G *be a graph with a perfect matching and* $X \subseteq V(G)$*. Then* $\text{mp}(\partial(X))$ *is odd if and only if* $|X|$ *is odd.*

As an immediate consequence of Observation 12, the tree of every perfect matching decomposition (T, η) of odd width contains an odd edge. Parity plays a huge role in the study of perfect matchings and it can be very useful to control the occurrence of odd edges in a perfect matching decomposition.

2.2 Upper bound by bricks and braces

In order to establish a basic toolkit, in this subsection we present how the perfect matching width of G relates to the perfect matching width of its matching minors. We establish that the bricks and braces of G yield an upper bound on the perfect matching width of G itself. This connection reduces the problem of finding a (close to optimal) perfect matching decomposition of G to finding appropriate decompositions for its bricks and braces.

Proposition 13. *Let G be a matching covered graph. Then*

$$\operatorname{pmw}(G) \leqslant \max_{\substack{H \text{ brick or} \\ \text{brace of } G}} \operatorname{pmw}(H).$$

Proof. Due to Theorem 5 it suffices to show that for every matching covered graph G and every tight cut $\partial(Z)$ holds $\operatorname{pmw}(G) \leqslant \max\{\operatorname{pmw}(G_Z), \operatorname{pmw}(G_{\overline{Z}})\}$.

To this end let (T_i, η_i) be an optimal perfect matching decomposition of G_i, for $i \in \{Z, \overline{Z}\}$ and let ℓ_i be the leaf of T_i such that $\eta_i(\ell_i) = v_j$, where $j \in \{Z, \overline{Z}\} \setminus \{i\}$, that is, v_j is the contraction vertex in G_i.

We construct a perfect matching decomposition (T, η) of G as follows. Let e_i be the edge of T_i that is incident to ℓ_i. First we obtain T from T_Z and $T_{\overline{Z}}$ by identifying the edges e_Z and $e_{\overline{Z}}$ as a new edge e.

$$\eta : \mathsf{L}(T) \to V(G),$$
$$\eta(t) \coloneqq \begin{cases} \eta_Z(t), & \text{if } t \in V(T_Z) \setminus \{\ell_Z\} \\ \eta_{\overline{Z}}(t), & \text{if } t \in V(T_{\overline{Z}}) \setminus \{\ell_{\overline{Z}}\} \end{cases}$$

We claim that the width of (T, δ) is equal to the maximum of the widths of (T_Z, η_Z) and $(T_{\overline{Z}}, \eta_{\overline{Z}})$. As $\partial(Z)$ is a tight cut, $\partial(e)$ has matching porosity 1. Now consider an edge $e' \in E(T) \setminus \{e\}$. Let $i \in \{Z, \overline{Z}\}$ such that $e' \in E(T_i)$ and let $j \in \{Z, \overline{Z}\} \setminus \{i\}$. For every perfect matching $M \in \mathcal{M}(G)$ the $M|_{G_i}$ is a perfect matching of G_i with $|\partial_{G_i}(e') \cap M'| = |\partial_G(e') \cap M|$. Thus $\operatorname{mp}(\partial_G(e')) = \operatorname{mp}(\partial_{G_i}(e'))$.

Therefore, $\operatorname{pmw}(G)$ is at most $\max\{\operatorname{pmw}(G_X), \operatorname{pmw}(G_{\overline{X}})\}$ $\square$

For the study of matching covered graphs of specific perfect matching width it would be helpful to have a notion of obstructions, or at least sources for lower bounds, on the width. Before we continue towards the main result of this section concerning matching minors, we have to discuss conformal subgraphs. These provide a lower bound on the perfect matching width of a graph and therefore are a first step in that direction.

Lemma 14. *Let G be a graph with a perfect matching and $H \subseteq G$ a conformal subgraph of G. Then* $\operatorname{pmw}(H) \leqslant \operatorname{pmw}(G)$.

Proof. Let (T, η) be an optimal perfect matching decomposition of G and

$$L_{\overline{H}} \coloneqq \{\ell \in \mathsf{L}(T) \mid \eta(\ell) \in V(G) \setminus V(H)\}.$$

Then, $T - L_{\overline{H}}$ is a subcubic tree. Now, remove from $T - L_{\overline{H}}$ iteratively all vertices that became leaves and thus are not mapped to any vertex by η obtaining T''. Let T' be the tree obtained from T'' by trimming. We define $\eta' \colon \mathsf{L}(T') \to V(H)$ by restricting η to $\mathsf{L}(T')$ and claim that (T', η') is a perfect matching decomposition of H of width at most $\mathrm{pmw}(G)$.

Now consider an edge $e \in E(T')$ and its corresponding cut $\partial(X')$ in H. Let $M' \in \mathcal{M}(H)$ be a perfect matching in H. By construction, $e \in E(T)$ and thus e corresponds to a cut $\partial(X)$ in G as well. Moreover, $X' \subseteq X$ and $V(H) \setminus X' \subseteq V(G) \setminus X$. Since H is a conformal subgraph of G, there is a perfect matching $M \in \mathcal{M}(G)$ with $M' \subseteq M$ and we obtain $|\partial(X') \cap M'| \leqslant |\partial(X) \cap M| \leqslant \mathrm{pmw}(G)$. Hence, $\mathrm{width}(T', \eta') \geqslant \mathrm{pmw}(G)$. □

We want to reduce the problem of determining the perfect matching width of a matching covered graph to working out the width of its bricks and braces. For now we established that the width of the bricks and braces yields an upper bound and conformal subgraphs provide a lower bound. In order to obtain a lower bound in terms of the bricks and braces, we need to know how tight cuts interact with the perfect matching width.

2.3 *M*-perfect matching width and matching minors

For bipartite matching covered graphs Rabinovich and two of the authors [HRW19] provide a qualitative bound for the perfect matching width of matching minors. In this section we strengthen this to general graphs with perfect matchings. We use the notion of M-perfect matching width (M-pmw), which allows us to restrict ourselves to a specific kind of perfect matching decompositions.

Theorem 4 ([HRW19]**.** *Let G be a graph with a perfect matching M, then $\frac{1}{2}\,\mathrm{pmw}_M(G) \leqslant \mathrm{pmw}(G) \leqslant \mathrm{pmw}_M(G)$.*

We start with tight cut contractions. Given an M-decomposition for G of width k, we want to construct M-decompositions of width at most k for both tight cut contractions of a single tight cut in G. Handling a single tight cut contraction suffices since the M-decompositions we obtain for the two tight cut contractions are M'-perfect matching decompositions again where M' is the restriction of M to the two contractions. This allows us to apply induction and reduce the initial matching covered graph G all the way down to its bricks and braces.

Positioning the contraction vertex Key to obtaining an M-decomposition for a tight cut contraction of G from an M-decomposition (T, η) of G is the decision where in the trimmed version of the decomposition tree to attach a new leaf for the contraction vertex. If there is an edge in T that separates the vast majority of the vertices of one of the tight cut shores from the vertices of the other, this decision is not too complicated to make. But if such an edge does not exists, or in other words (T, η) does not distinguish between the two shores of our tight cut, it is way harder to decide. While Proposition 13 shows that there always exist perfect matching decompositions with edges reflecting the tight cuts, these decompositions are not necessarily optimal and at this point we are not able to provide a bound on the approximation they provide.

Our decision where to position the contraction vertex is based on some implications of Observation 12. If $\partial(Z)$ is a non-trivial tight cut of G, then $|Z|$ is odd and thus for all $X \subseteq V(G)$ the cut $\partial(X)$ of G has exactly one shore that contains an odd number of vertices of Z. If $|X|$ is even, this shore also contains an odd number of vertices of $\overline{Z}$. This observation leads us to the following lemma.

Note that any cut induced by an inner edge of an M-decomposition is even since both shores are M-conformal.

Lemma 15. *Let G be a matching covered graph, X be an even cut, and $\partial(Z)$ a non-trivial tight cut of G as well as v_Z the contraction vertex obtained by the tight cut contraction of Z into the graph G_Z.*

If $|X \cap Z|$ is odd, then $\mathrm{mp}(\partial_{G_Z}((X \setminus Z) \cup \{v_Z\})) \leqslant \mathrm{mp}(\partial_G(X))$.

Proof. Define $X_1 \coloneqq X \cap Z$ and $X_2 \coloneqq \overline{X} \cap \overline{Z}$. Note that the cut $\partial_{G_Z}((X \setminus Z) \cup \{v_Z\})$ in G_Z corresponds to the cut $\partial_G(X_2)$ in G. Now consider a perfect matching M of G. As $|X_1|$ is odd, we obtain

$$\begin{aligned}1 + |M \cap \partial_G(X_2)| &\leqslant |M \cap \partial_G(X_1)| + |M \cap \partial_G(X_2)| \\ &\leqslant |M \cap \partial_G(X)| + |M \cap \partial_G(Z)| = |M \cap \partial_G(X)| + 1.\end{aligned}$$ □

If (T, η) is an M-decomposition of G, then, as we have seen in the proof of Lemma 15, the only cuts whose matching porosity can exceed the width of (T, η) by placing the contraction vertex and "keeping" the rest of the decomposition as it is are those of matching porosity exactly width(T, η).

For each of those cuts we need to indicate which of its two shores contains an odd number of vertices of a tight cut shore. To this end, we define the following orientation of the edges of T. Our definition does not require (T, η) to be an M-decomposition, however, in case it is, we are able to make further observations.

Z-orientations Let G be a matching covered graph, $\partial(Z)$ a non-trivial tight cut of G and (T, η) a perfect matching decomposition of G. We define the *Z-orientation* $\vec{T_Z}$ of T as the orientation of the edges of T, such that for every edge $t_1t_2 \in E(\mathrm{spine}(T))$, $(t_1, t_2) \in E(\vec{T_Z})$ if and only if $|\eta(T_{t_2}) \cap Z|$ is odd. Additionally, every edge $\ell t \in E(T)$, where ℓ is a leaf, is oriented away from ℓ, that is $(\ell, t) \in E(\vec{T_Z})$. Note that the Z-orientation of the edge t_1t_2 is well defined since $|Z|$ is odd (see fig. 3 for an example). If there is a vertex $t \in V(\vec{T_Z})$ such that at least two of its incident edges are outgoing edges, we call t an *inconsistency*.

The idea is that $\vec{T_Z}$ should tell us where to put the contraction vertex in order to obtain a decomposition of the tight cut contraction of G obtained by contracting Z. However, this only works if $\vec{T_Z}$ has no inconsistencies.

In a directed graph a vertex with only incoming edges is called a *sink*. If a Z-orientation does not have any inconsistencies, there exists a unique sink vertex s in $\vec{T_Z}$. We next prove that the Z-orientation of an M-decomposition does not contain any inconsistencies and additionally, s is adjacent to a leaf $t \in V(T)$ and $\eta(t) \in Z$ (see Lemma 16).

So, to obtain a perfect matching decomposition of the tight cut contraction obtained from G by contracting Z into a single vertex v_Z, we now forget all vertices of Z, delete the corresponding leaves from T (except for t) and map t to the contraction vertex v_Z. Finally, we trim this new tree. This not only yields a perfect matching decomposition, the width of this new decomposition is at most the width of the original graph. If our decomposition was an M-decomposition in the first place, we can make even stronger observations (see fig. 3 for an example).

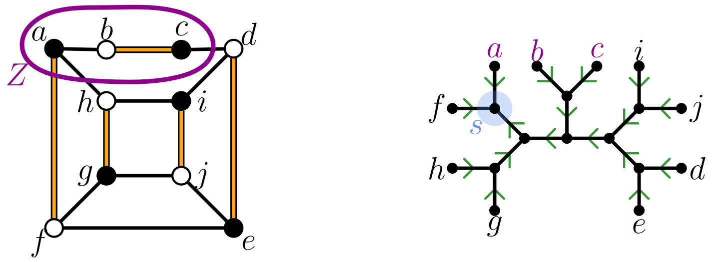

Figure 3: A graph G with the non-trivial tight cut $\partial(Z)$, a perfect matching $M \in \mathcal{M}(G)$, and an M-decomposition (T, η) of width four. The arrows in T are the edges forming $\vec{T}_Z$, note that it is free of inconsistencies and has a unique sink s.

Lemma 16. *Let G be a matching covered graph, $\partial(Z)$ a non-trivial tight cut in G, $M \in \mathcal{M}(G)$, and (T, η) an M-decomposition of G. Then, $\vec{T}_Z$ is free of inconsistencies and has a unique sink that is adjacent to a leaf.*

Proof. As Z defines a non-trivial tight cut, there is a unique edge $xy \in M$ with $x \in Z$ and $y \in \overline{Z}$. All other vertices of Z are matched within Z. In the M-decomposition (T, η) for every $t_1t_2 \in E(T - \mathsf{L}(T))$ the unique subtree T_i, $i \in \{t_1, t_2\}$, with $|\eta(T_i) \cap Z|$ being odd is exactly the one that contains x. Therefore, in $\vec{T}_Z$ every inner edge is oriented towards the subtree that contains x and thus there cannot be an inconsistency as η is a bijection and the tree containing x is well defined for every inner edge.

Moreover, let $t \in V(\vec{T}_Z)$ such that t is adjacent to two leaves ℓ_1 and ℓ_2 where $\eta(\ell_1) = x$. Then, for every vertex $t' \in V(\vec{T}_Z) \setminus \{t, \ell_1, \ell_2\}$ there is a directed path in $\vec{T}_Z$ from t' to t. And by the definition of Z-orientations, $(\ell_1, t), (\ell_2, t) \in E(\vec{T}_Z)$ which implies that t is a sink of $\vec{T}_Z$ and no vertex apart from t can be a sink. $\square$

Note that, in the proof above, $\eta(\ell_2) = y$ and thus, in the decomposition for the tight cut contraction we construct from (T, η), the contraction vertex and y are again siblings. So, if we start out with an M-decomposition of a matching covered graph G, then the Z-orientations of said decomposition behave exactly as intended. This allows us to obtain new decompositions for tight cut contractions of at most the same width and thus yields the following result.

Proposition 17. *Let G be a matching covered graph, $\partial(Z)$ a non-trivial tight cut in G, $M \in \mathcal{M}(G)$, and (T, η) an M-decomposition of G of width k. Moreover, let G_Z be the matching covered graph obtained from G by contracting Z into the vertex v_Z. Then, there is an $M|_{G_Z}$-perfect matching decomposition of G_Z of width at most k.*

Proof. We consider the Z-orientation $\vec{T_Z}$ of T. By Lemma 16, $\vec{T_Z}$ is free of inconsistencies and has a unique sink s. Moreover, as we have seen, s is adjacent to two leaves t_x and t_y of T such that $\eta(t_x) = x \in Z$, $\eta(t_y) = y \in \overline{Z}$ and $xy \in M$ is the unique edge of M in $\partial(Z)$.

We now construct a perfect matching decomposition (T', η') for G_Z. To this end, let $L_Z \coloneqq \{t \in \mathsf{L}(T) \mid \eta(t) \in Z \setminus \{x\}\}$ and T'' be the tree obtained from $T - L_Z$ by repeatedly removing leaves that do not lie in the domain of η. Then, let T' be the cubic tree obtained from T'' by trimming. Then $\mathsf{L}(T') = \mathsf{L}(T) \setminus L_Z$ and for every $t \in \mathsf{L}(T')$ and every inner edge $t_1t_2 \in E(T')$, t is a leaf of the tree T'_i, $i \in \{t_1, t_2\}$, if and only if t is a leaf of the subtree T_i. Therefore, every bipartition of $\mathsf{L}(T)$ induced by an inner edge of T' is also induced by an edge in T.

To obtain η' from η we do not change anything for $\overline{Z}$ and replace x by v_Z. So for all $t \in \mathsf{L}(T')$ let

$$\eta'(t) \coloneqq \begin{cases} v_Z, & \text{if } \eta(t) = x, \text{ and} \\ \eta(t), & \text{otherwise.} \end{cases}$$

The restriction $M|_{G_Z}$ of M to G_Z contains all edges with both endpoints in $\overline{Z}$ and additionally the edge yv_Z, so by construction, (T', η') is an $M|_{G_Z}$-perfect matching decomposition of G_Z.

Now, let $t_1t_2 \in E(T')$ be an inner edge and $\partial_G(t_1t_2)$ the cut induced by t_1t_2 in G via (T, η). Then, $\partial_G(t_1t_2)$ has a unique shore $X \subseteq V(G)$ that contains x and, as (T, η) is an M-decomposition, $|X|$ is even. Moreover, the cut $\partial_{G_Z}(t_1t_2)$ induced by t_1t_2 in G_Z via (T', η') has a shore $X' \coloneqq (X \setminus Z) \cup \{v_Z\}$. As $|X \cap Z|$ is odd by choice of x, Lemma 15 gives us $\mathrm{mp}(\partial_{G_Z}(X')) \leqslant \mathrm{mp}(\partial_G(X)) \leqslant k$ and thus concludes this proof. □

Since Proposition 17 provides an $M|_{G_Z}$-perfect matching decomposition of the tight cut contraction G_Z, we can now choose a new tight cut in G_Z and continue with a new iteration of the tight cut decomposition procedure. So finally, we reach decompositions of the bricks and braces of G with width still bound by the width of the original M-decomposition. By then applying Theorem 4 we obtain the following corollary.

Corollary 18. *Let G be a matching covered graph and H a brick or brace of G. Then $\mathrm{pmw}(H) \leqslant 2\,\mathrm{pmw}(G)$.*

So by iteratively contracting tight cuts we cannot significantly increase the perfect matching width. As bicontractions are a special case of tight cut contractions and by Lemma 14 the width of a conformal subgraph of G is bounded by the width of G itself, we finally obtain our main result on matching minors of this section.

Theorem 1. *Let G be a matching covered graph and H be a matching minor of G. Then $\mathrm{pmw}(H) \leqslant 2\,\mathrm{pmw}(G)$.*

Corollary 18 and Proposition 13 now yield the wanted relation between the perfect matching width of the bricks and braces and the width of the graph itself.

Theorem 2. *Let G be a matching covered graph and H be a brick or brace of G maximising* $\operatorname{pmw}(H)$ *over all bricks and braces of G. Then* $\frac{1}{2}\operatorname{pmw}(H) \leqslant \operatorname{pmw}(G) \leqslant \operatorname{pmw}(H)$.

Moreover, if we consider the M-perfect matching width of a matching covered graph G, we obtain an even stronger result which concludes this section.

Lemma 19. *Let G be a matching covered graph with a perfect matching M and H be a matching minor of G obtained from an M-conformal subgraph of G by a series of bicontractions, or a brick or brace of G. Then* $\operatorname{pmw}_{M|_H}(H) \leqslant \operatorname{pmw}_M(G)$.

3 Braces of Perfect Matching Width 2

The only matching covered graph of perfect matching width one is K_2. Apart from this every perfect matching decomposition of a matching covered graph contains a vertex that is adjacent to two leaves (which, by definition, are mapped to two distinct vertices of G) and, as G is matching covered, there is a perfect matching which does not match these vertices with each other. Therefore, the cut in G induced by the non-leaf edge of said vertex in the decomposition has matching porosity two. So two is a natural lower bound on the perfect matching width of braces. One approach to width parameters can be to investigate the structure of graphs of small width. Since by Proposition 13 the perfect matching width of a graph is bounded from above by the width of its bricks and braces, studying the structure of braces of perfect matching width two appears to be a good starting point towards a better understanding of the parameter itself. We present two possible characterisations of perfect matching width two braces, one in terms of edge-maximal graphs similar to the k-tree characterisation of treewidth k graphs (see [Arn85] for an overview on this topic) and the other one in terms of elimination orderings, which again resembles similar results on treewidth.

3.1 Generalised Tight Cuts

We start by introducing some additional facts about braces and the edge cuts that can be found within a brace. Tight cuts are defined as those edge cuts which contain exactly one edge of every perfect matching of our graph. In a similar way we may define generalised versions of these cuts.

Definition 20 (Generalised Tight Cut)**.** Let G be a matching covered graph, $k \in \mathbb{N}$ a positive integer, and $X \subseteq V(G)$. The edge cut $\partial_G(X)$ is *k-tight* if $|\partial_G(X) \cap M| = k$ for all $M \in \mathcal{M}(G)$. If $\partial_G(X)$ is a k-tight cut, we say that X *induces* a k-tight cut. A k-tight cut is *trivial* if $|X| = k$ or $|\overline{X}| = k$.

In the following we investigate the properties of k-tight cuts in bipartite graphs, and braces in particular, a bit further.

Definition 21 (Minority and Majority)**.** Let B be a bipartite graph, and $X \subseteq V(G)$. If $|X \cap V_1| = |X \cap V_2|$ we say that X is *balanced*, otherwise it is *unbalanced.* Suppose X is unbalanced, then there are $i, j \in \{1, 2\}$, and $k \in \mathbb{N}$ such that $|X \cap V_i| = |X \cap V_j| + k$. In this case we call $X \cap V_i$ the *majority* of X, denoted by $\mathrm{Maj}(X)$, and $X \cap V_j$ is the *minority*, denoted by $\mathrm{Min}(X)$. We say that k is the *imbalance* of X, and in general we set

$$\mathrm{imbalance}(X) \coloneqq \begin{cases} 0, & \text{if } X \text{ is balanced, or} \\ k, & \text{if the imbalance of } X \text{ is } k. \end{cases}$$

Lemma 22. *Let B be a bipartite matching covered graph, $k \in \mathbb{N}$ a positive integer, and $X \subseteq V(B)$ a set of vertices that induces a k-tight cut. Then there exist $k_1, k_2 \in \mathbb{N}$ such that for every perfect matching $M \in \mathcal{M}(B)$ there are exactly k_i vertices of $X \cap V_i$ which are matched by edges of $\partial_B(X) \cap M$ for both $i \in \{1, 2\}$.*

Proof. Let M be some perfect matching of B and for both $i \in \{1, 2\}$, let k_i be the number of vertices in $X \cap V_i$ which are matched by edges of $\partial_B(X) \cap \partial_B(V_i)$. Then every other edge of M either has both or no endpoint in X. Hence there is a number $n \in \mathbb{N}$ of edges of M with both endpoints in X such that $|X| = k_1 + k_2 + 2n$. Now suppose, towards a contradiction, there exist $k'_1, k'_2 \in \mathbb{N}$ together with a perfect matching $M' \in \mathcal{M}(B)$ such that for each $i \in \{1, 2\}$, k'_i is the number of vertices of $X \cap V_i$ that are matched by edges of $\partial_B(X)$, and $k'_1 \neq k_1$, which also implies $k'_2 \neq k_2$. By the same arguments as before, there exists a number n' such that $|X| = k'_1 + k'_2 + 2n'$. Indeed, we have $|X \cap V_i| = k_i + n = k'_i + n'$ for both $i \in \{1, 2\}$. Without loss of generality, let us assume $k'_1 > k_1$. Then $n' < n$ since $k_1 + n = k'_1 + n'$. But since $\partial_B(X)$ is k-tight, we have $k_1 + k_2 = k = k'_1 + k'_2$. Hence

$$|X| = k + 2n > k + 2n' = |X|,$$

which is impossible and thus our claim holds. □

The following can be seen as a generalisation of an observation first made by Lovász (see the proof of Lemma 1.4 in [Lov87]).

Lemma 23. *Let B be a bipartite matching covered graph, $k \in \mathbb{N}$ a positive integer, and $X \subseteq V(G)$ a set of imbalance k. Then $\partial_B(X)$ is k-tight if and only if $N_B(\mathrm{Min}(X)) \subseteq \mathrm{Maj}(X)$.*

Proof. Let us first assume X induces a k-tight cut and suppose there is some edge $e \in \partial_B(X)$ such that e has an endpoint in $\mathrm{Min}(X)$. As G is matching covered there exists $M_e \in \mathcal{M}(B)$ such that $e \in M_e$. Now there are $|\mathrm{Min}(X)| - 1$ many vertices of the minority left which can be matched by M_e to vertices of the majority of X. Hence at least $k + 1$ vertices of $\mathrm{Maj}(X)$ cannot be matched by M_e with vertices inside X. This however means that $|\partial_B(X) \cap M_e| \geqslant k + 2$, contradicting the assumption that X induces a k-tight cut.

For the reverse direction, let us assume $N_B(\mathrm{Min}(X)) \subseteq \mathrm{Maj}(X)$. Then for every $M \in \mathcal{M}(B)$, every vertex of $\mathrm{Min}(X)$ must be matched with a vertex of $\mathrm{Maj}(X)$, therefore leaving exactly $\mathrm{imbalance}(X) = k$ vertices of $\mathrm{Maj}(X)$ which must be matched via edges of $\partial_B(X)$. Therefore $|\partial_B(X) \cap M| = k$ for all $M \in \mathcal{M}(B)$. □

3.2 Spines and Cuts in 2-Extendable Bipartite Graphs

In this subsection we present a few auxiliary results. We start by introducing the notion of k-extendability, which, in the case $k = 2$ is a property equivalent to the absence of non-trivial tight cuts in bipartite matching covered graphs.

Definition 24. Let G be a graph with a perfect matching and $F \subseteq E(G)$ a matching. We say that F is *extendable* if there exists $M \in \mathcal{M}(G)$ such that $F \subseteq M$.

For any positive integer $k \in \mathbb{N}$, G is said to be *k-extendable* if it is connected, has at least $2k + 2$ vertices, and every matching of size k in G is extendable.

This notion provides a characterisation of braces.

Theorem 25 ([LP09])**.** *A bipartite graph B is a brace if and only if it is either isomorphic to C_4, or it is 2-extendable.*

In some sense Theorem 25 generalises to higher values of extendability.

Theorem 26 ([Plu86])**.** *Let B be a bipartite graph and $k \in \mathbb{N}$ a positive integer. The following statements are equivalent.*

1. *B is k-extendable.*
2. *$|V_1| = |V_2|$, and for all non-empty $S \subseteq V_1$ with $|S| \leqslant |V_1| - k$, $|N_B(S)| \geqslant |S| + k$.*
3. *For all sets $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$ with $|S_1| = |S_2| \leqslant k$ the graph $B - S_1 - S_2$ has a perfect matching.*

Additionally, we need the following properties of k-extendable graphs and the well-known Kőnig's Theorem.

Theorem 27 ([Plu80])**.** *Let $k \in \mathbb{N}$ be a positive integer. Then every k-extendable graph is also $(k-1)$-extendable.*

Theorem 28 ([Plu80])**.** *Let $k \in \mathbb{N}$ be a positive integer. Then every k-extendable graph is $(k+1)$-connected.*

In the following we denote the size of a maximum matching in a graph G by $\nu(G)$ and the size of a minimum *vertex cover* by $\tau(G)$, where a vertex cover of G is a set of vertices $S \subseteq V(G)$ such that every edge of G has at least one endpoint in S.

Theorem 29 (Kőnig's Theorem (see for example [LP09]))**.** *If B is a bipartite graph, then $\tau(B) = \nu(B)$.*

Using these we can establish that in bipartite k-extendable graphs no cut of small porosity can have two large shores. To this end we define for every graph G and cut $\partial_G(X)$ in G the graph $G[\partial_G(X)]$ as the subgraph of G induced by all the edges in $\partial_G(X)$.

Lemma 30. *Let $k \in \mathbb{N}$ and B be a bipartite k-extendable graph and $X \subseteq V(B)$, then one of the following holds for every $k' \leqslant k$:*

1. $\nu(B[\partial_B(X)]) > k'$,
2. $|X| \leqslant k'$, *or*
3. $|\overline{X}| \leqslant k'$.

Proof. We assume $\nu(B[\partial_B(X)]) \leqslant k'$. As B is bipartite, the graph $B[\partial_B(X)]$ is as well. By Theorem 29 (Kőnig's Theorem), we thus obtain $\tau(B[\partial_B(X)]) = \nu(B[\partial_B(X)]) \leqslant k'$. So there is a set of vertices S of size at most k' hitting all edges crossing $\partial_B(X)$. This means S is a separator of size at most k' in B separating $X \setminus S$ from $\overline{X} \setminus S$. By Theorem 28, B is $k+1$-connected, therefore $X \subseteq S$ and thus $|X| \leqslant k'$, or $\overline{X} \subseteq S$ and thus $|\overline{X}| \leqslant k'$. □

We now have all necessary tools available to investigate the structure of optimal perfect matching decompositions of 2-extendable bipartite graphs whose perfect matching width is 2.

In this article we are mainly interested in braces, that is 2-extendable graphs. However, we believe that some of the techniques presented here should generalise in some form to the setting of k-extendable bipartite graphs of perfect matching width close to k.

We establish a connection between the matching porosity of a cut and its imbalance in k-extendable bipartite graphs.

Lemma 31. *Let $k \in \mathbb{N}$ be a positive integer, B be a k-extendable and bipartite graph, and $X \subseteq V(G)$ such that $\operatorname{mp}(\partial_B(X)) = k$ and $k+2 \leqslant |X| \leqslant |V(B)| - (k+2)$. Then $\operatorname{imbalance}(X) = k$.*

Proof. We start by observing that every perfect matching of B has at most $\operatorname{Min}(X)$ many edges matching two vertices of X. Thus, the matching porosity of $\partial_B(X)$ yields an upper bound on the imbalance of X, that is, $k = \operatorname{mp}(\partial_B(X)) \geqslant |X| - 2|\operatorname{Min}(X)| = \operatorname{imbalance}(X)$. By Observation 12 and as the imbalance and the size of a vertex set have the same parity, we know that $k \equiv |X| \equiv \operatorname{imbalance}(X) \pmod 2$.

Suppose towards a contradiction that $k' \coloneqq \operatorname{imbalance}(X) \leqslant k-2$. Additionally, we may assume without loss of generality that $\operatorname{Maj}(X) \subseteq V_2$. We split $B[\partial_B(X)]$ into the following two subgraphs.

$$B_1 \coloneqq B[\operatorname{Min}(X) \cup (V_2 \setminus \operatorname{Maj}(X))], \text{ and}$$
$$B_2 \coloneqq B[(V_1 \setminus \operatorname{Min}(X)) \cup \operatorname{Maj}(X)].$$

We have $B_1 \cup B_2 = B[\partial_B(X)]$.

Suppose $\nu(B_1) \geqslant \frac{k-k'}{2} + 1$, then there is a matching F of size $\frac{k-k'}{2} + 1$ in B_1. As $|F| = \frac{k-k'}{2} + 1 \leqslant k$ and B is k-extendable, there is a perfect matching M_F of B with $F \subseteq M_F$. Due to $\operatorname{mp}(\partial_B(X)) = k$, at most $k - (\frac{k-k'}{2} + 1) = \frac{k+k'}{2} - 1$ edges of $M_F \cap \partial_B(X)$

have an endpoint in $\operatorname{Maj}(X)$. Thus we obtain

$$\begin{aligned}|\operatorname{Maj}(X) \setminus V(M_F \cap \partial_B(X))| &\geqslant |\operatorname{Maj}(X)| - (\frac{k+k'}{2} - 1)\\ &= |\operatorname{Maj}(X)| - \frac{k}{2} - \frac{k'}{2} + 1\\ &= |\operatorname{Min}(X)| + k' - \frac{k}{2} - \frac{k'}{2} + 1\\ &> |\operatorname{Min}(X)| - \frac{k-k'}{2} - 1\\ &\geqslant |\operatorname{Min}(X) \setminus V(M_F \cap \partial_B(X))|.\end{aligned}$$

Therefore, $\operatorname{imbalance}(X \setminus V(M_F \cap \partial_B(X))) \geqslant 1$, contradicting M_F being a perfect matching. Thus, $\nu(B_1) \leqslant \frac{k-k'}{2}$. With similar arguments $\nu(B_2) \geqslant \frac{k+k'}{2} + 1$ yields a contradiction. Thus, $\nu(B_2) \leqslant \frac{k+k'}{2}$. It follows that

$$\nu(B[\partial_B(X)]) = \nu(B_1) + \nu(B_2) \leqslant \frac{k-k'}{2} + \frac{k+k'}{2} = k.$$

Together with $k + 2 \leqslant |X| \leqslant |V(B)| - (k + 2)$, this contradicts Lemma 30. Thus, we obtain that $\operatorname{imbalance}(X) = k$. □

Lemma 31 establishes the distribution of the two colours V_1 and V_2 in any set of matching porosity k of sufficient size in a k-extendable brace.

Next, we show that decompositions of 2-extendable bipartite graphs with the special structure of the spine of the spine of the decomposition tree being a path have the following property. The edges of this path, having matching porosity exactly 2, induce shores that have imbalance 2, and the neighbourhood of the minority of these shores is completely contained in the shore. Notice that the above holds, in particular, because both shores are large enough, i.e. have at least three vertices each as a result of taking the spine of the spine. So the shores have a kind of closure property when it comes to the minority: no vertex of the minority has neighbours outside the shore.

Theorem 32. *Let B be a 2-extendable bipartite graph with a perfect matching decomposition (T, η) of width 2 such that* $\operatorname{spine}(\operatorname{spine}(T))$ *is a path. Then, for all $e \in \operatorname{spine}(\operatorname{spine}(T))$ every shore X of $\partial_B(e)$ satisfies*

1. $\operatorname{imbalance}(X) = 2$, *and*
2. $N_B(\operatorname{Min}(X)) \subseteq X$.

Proof. We first consider how the colour classes can be distributed in the shores of two cuts X and Y corresponding to two adjacent edges in the path $\operatorname{spine}(\operatorname{spine}(T))$ such that $X \subseteq Y$. We claim that X and Y differ by exactly one vertex from each colour class.

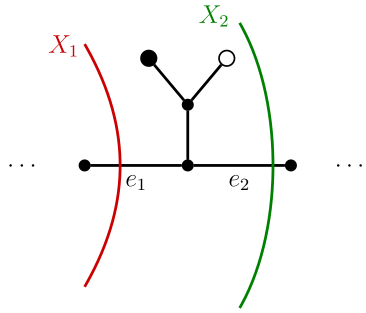

Figure 4: There are exactly two vertices in $X_2 \setminus X_1$ and they come from different colour classes of B.

Claim 33. *If* $|V(B)| \geqslant 8$ *and* e_1, e_2 *are two adjacent edges of* spine(spine(T)) *such that* $\partial_B(e_1)$ *has a shore* X_1 *and* $\partial_B(e_2)$ *has a shore* X_2 *with* $X_1 \subsetneq X_2$ *and* $\mathrm{mp}(\partial_B(X_1)) = \mathrm{mp}(\partial_B(X_2)) = 2$. *Then, we have* $|X_1 \cap V_i| + 1 = |X_2 \cap V_i|$ *for both* $i \in \{1, 2\}$. *See fig. 4 for an illustration.*

Proof. Observation 12 and spine(spine(T)) being a path together imply $|X_2| - |X_1| = 2$. Suppose towards a contradiction that both vertices in $X_2 \setminus X_1$ are from the same colour class, without loss of generality, say V_1, that is, $|X_1 \cap V_1| + 2 = |X_2 \cap V_1|$. Since $e_1, e_2 \in E(\mathrm{spine}(\mathrm{spine}(T)))$ it follows that $|X_i| \geqslant 3$ for both $i \in \{1, 2\}$. This implies, since $\mathrm{mp}(\partial_B(X_i)) = 2$ for both $i \in \{1, 2\}$, by using Observation 12 again, that $|X_i| \geqslant 4$ for both $i \in \{1, 2\}$. Analogously, we obtain $|\overline{X_i}| \geqslant 4$ for both $i \in \{1, 2\}$.

By Lemma 31, we have $\mathrm{imbalance}(X_1) = \mathrm{mp}(\partial_B(X_1)) = 2$, or $|X_1| = 2$ for X_1 and we have $\mathrm{imbalance}(X_2) = \mathrm{mp}(\partial_B(X_2)) = 2$, or $|\overline{X_2}| = 2$. It therefore follows from our discussion above that $\mathrm{imbalance}(X_i) = 2$ for both $i \in \{1, 2\}$

However, this contradicts that the two vertices in $X_2 \setminus X_1$ come from the same colour class. So, we obtain $|X_1 \cap V_i| + 1 = |X_2 \cap V_i|$ for both $i \in \{1, 2\}$. ■

We define $P :=$ spine(spine(T)) with the two endpoints $p_\triangleleft$ and $p_\triangleright$. We order the edges $(e_1, \ldots, e_\ell)$ of P by occurrence along P when traversing it from $p_\triangleleft$ to $p_\triangleright$. Next, we show that P contains two edges that induce a cut with one shore building a star with 3 leaves.

Claim 34. *Let* $e_\triangleleft$ *and* $e_\triangleright$ *be edges of* P *such that for* $\diamond \in \{\triangleleft, \triangleright\}$ *the edge* $e_\diamond$ *is incident with the vertex* $p_\diamond$. *Then the cut* $\partial_B(e_\diamond)$ *has a shore* $X_\diamond$ *of size* 4 *satisfying the following conditions.*

i) $e_\triangleleft = e_\triangleright$ *if and only if* $|V(B)| = 8$,

ii) $X_\triangleleft \cap X_\triangleright = \emptyset$, *and*

iii) $B[X_\diamond]$ *is a star such that its central vertex has no neighbour in* $\overline{X_\diamond}$ *for both* $\diamond \in \{\triangleleft, \triangleright\}$.

Proof. It suffices to show the claim for $e_\triangleleft$. The same for $e_\triangleright$ holds by reversing the ordering of P.

Let $T_\triangleleft$ be the component of $T - e_\triangleleft$ that contains $p_\triangleleft$ and let $X'_\triangleleft \coloneqq \eta(T_\triangleleft)$. As $p_\triangleleft \in V(\text{spine}(\text{spine}(T)))$ it follows that $3 \leqslant |X'_\triangleleft| \leqslant 4$.

Suppose $|X'_\triangleleft| = 3$. Then (T, η) being of width 2 together with Observation 12 imply that $\text{mp}(\partial_B(X'_\triangleleft)) = 1$. Since $|X'_\triangleleft|$ is odd, every perfect matching of B must have at least one edge in $\partial_B(X'_\triangleleft)$. Hence, $\partial_B(X'_\triangleleft)$ is a non-trivial tight cut. This, however, is impossible since B is 2-extendable and, therefore, by Theorem 25, a brace. So we obtain that $|X'_\triangleleft| = 4$ as desired. Notice that this implies item i) from Claim 34 while item ii) is guaranteed by the choice of $e_\triangleleft$ and $e_\triangleright$.

With $|X'_\triangleleft| = 4$ we have, due to Observation 12, that $\text{mp}(\partial_B(X'_\triangleleft)) = 2$. This implies, by Lemma 31, that $\text{imbalance}(X'_\triangleleft) = 2$. So there exists $i \in \{1, 2\}$ such that $X'_\triangleleft$ contains a single vertex of V_i. Let us call this vertex $v_\triangleleft$ (and define $v_\triangleright$ analogously). If $v_\triangleleft$ would have a neighbour $w \in \overline{X'_\triangleleft}$ then, with B being 2-extendable, there would be a perfect matching M of B containing the edge $v_\triangleleft w$. This, however, would mean that $|\partial_B(X'_\triangleleft) \cap M| = 4$ contradicting that $\text{mp}(\partial_B(X'_\triangleleft)) = 2$. Hence, since, by Theorem 28, B is 3-connected, the neighbourhood of $v_\triangleleft$ is exactly the set $X'_\triangleleft \setminus \{v_\triangleleft\}$ and thus, item iii) of Claim 34 also holds. ■

By combining Claim 33 and Claim 34, we now conclude the proof.

If $|V(B)| = 8$, then, by Claim 34, $e_\triangleleft = e_\triangleright$ and this is the only edge of spine(spine(T)) thus, the statement holds.

So, assume that $|V(B)| \geqslant 10$ and $e_\triangleleft \neq e_\triangleright$. We prove the statement for the shores of any edge e_i of P lying between $e_\triangleleft$ and $e_\triangleright$ assuming that the shores of the adjacent edge e_{i-1} fulfil the statement, that is, assuming $\text{imbalance}(X_{e_{i-1}}) = 2$ and $N_B(\text{Min}(X_{e_{i-1}})) \subseteq X_{e_{i-1}}$. Since $e_\triangleleft = e_1$, Claim 34 acts as the base of this inductive argument, so we may assume $i \geqslant 2$.

Let us assume without loss of generality that $\text{Min}(X_{e_i}) \subseteq V_1$. By Claim 33, we know that $\text{imbalance}(X_{e_i}) = \text{imbalance}(X_{e_{i-1}}) = 2$ and there is a unique vertex a in $(X_{e_i} \setminus X_{e_{i-1}}) \cap V_1$. Suppose a has a neighbour b in $\overline{X_{e_i}}$. Then, there is a perfect matching M of B containing ab and $M \cap \partial_B(X_{e_i})$ contains at least 4 edges, because Claim 33 implies $\text{Min}(X_{e_{i-1}}) \subseteq V_1$, a contradiction. Thus, $N_B(\text{Min}(X_{e_i})) \subseteq X_{e_i}$ and we are done. □

3.3 Perfect matching decompositions of width 2

We start by establishing the basic structure of the spine and the spine of the spine in the perfect matching decompositions of bipartite graphs with perfect matching width two have. The first statement, proving that the spine of these decompositions is cubic, even holds for bricks as well.

Lemma 35. *Let G be a brick or brace of perfect matching width two and (T, η) be an optimal perfect matching decomposition. Then,* spine(T) *is cubic.*

Proof. By Corollary 10, it suffices to show that T is free of odd edges. Suppose T has an odd edge t_1t_2, then $X_i \coloneqq \eta(T_{t_i})$ contains an odd number of vertices for $i \in \{1, 2\}$. Then

Observation 12 implies that $\text{mp}(\partial_G(X_1))$ is odd. As the width of (T, η) is 2 and t_1t_2 is an inner edge of T, $|X_1| \geqslant 3$, $|X_2| \geqslant 3$ and $\text{mp}(\partial_G(X_1)) = 1$. Thus $\partial_G(X_1)$ must be a non-trivial tight cut of G contradicting G being a brick or a brace. □

Using our insights on imbalance Lemma 31, we can now prove that there are no degree-3-vertices in the spine of the spine of a width-2-decomposition of a brace. This means that any optimal perfect matching decomposition of a brace B with $\text{pmw}(B) = 2$ has a linear structure.

Proposition 36. *Let B be a brace of perfect matching width two and (T, η) a perfect matching decomposition of minimum width for B. Then,* $\text{spine}(\text{spine}(T))$ *is a path.*

Proof. Suppose there is a vertex $t \in V(\text{spine}(\text{spine}(T)))$ with three neighbours t_1, t_2 and t_3. By Lemma 35, $\text{spine}(T)$ is cubic, and so every t_i is adjacent to exactly two vertices of the spine of T apart from t. Moreover, each of these neighbours again has exactly two neighbours distinct from t_i in T. Let T_i be the component of $T - tt_i$ for $i \in \{1, 2, 3\}$ that does not contain t and let $X_i := \eta(T_i)$. The above observations imply $|X_i| \geqslant 4$ for all $i \in \{1, 2, 3\}$. As T is free of odd edges by Corollary 10 and Lemma 35, $\text{mp}(X_i) = 2$ and so Lemma 31 yields $\text{imbalance}(X_i) = 2$.

Without loss of generality we can assume that two of the three sets have an excess in V_1 while the last one, say X_3, has an excess in V_2. This holds as the case where the excesses of all three sets are of the same colour implies $\text{imbalance}(V(B)) = 6$, a direct contradiction to the existence of a perfect matching in B. However, even under this assumption we still obtain $|V_1| = |V_2| + 2$ and thus, $V(G)$ is not balanced. Since B has a perfect matching, this is impossible, and thus, $\text{spine}(\text{spine}(T))$ cannot have a vertex of degree three. □

This allows us to apply the findings from section 3.2, especially the insights obtained in the proof of Theorem 32 to braces of perfect matching width two, we obtain the structure illustrated in fig. 5.

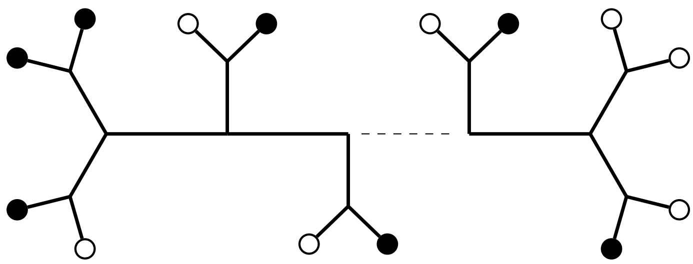

Figure 5: The linear structure of a perfect matching decomposition of width 2 with a claw on each side and two vertices from different colour classes added in each step. The filled vertices in the figure represent the leaves mapped to a vertex in V_1, and the empty vertices represent the leaves mapped to vertices in V_2.

Corollary 37. *Let (T, η) be a perfect matching decomposition of width 2 of a brace B with $2n$ vertices, then*

- spine(spine(T)) *is a path on* $n-2$ *vertices* $t_1, \dots, t_{n-2}$
- *both* $\partial_B(t_1t_2)$ *and* $\partial_B(t_{n-1}t_{n-2})$ *have a shore that is a claw with centre vertices of different colour, and*
- *let* $X_i \coloneqq \eta(T_i)$ *where* T_i *is the component of* $T - t_it_{i+1}$ *that contains* t_1 *for all* $i \in \{1, \dots, n-3\}$*, then* $X_i \setminus X_{i-1}$ *contains exactly two vertices of different colour.*

Of specific interest to us is the direct corollary of Theorem 32.

Corollary 38. *Let* B *be a brace of perfect matching width two,* (T, η) *be an optimal perfect matching decomposition of* G*,* $e \in E(\text{spine}(\text{spine}(T)))$ *and* X *a shore of* $\partial_B(e)$*. Then no vertex of the minority of* X *has a neighbour in* $\overline{X}$*.*

3.4 Elimination Orderings

So, given a perfect matching decomposition (T, η) of width two for a brace B, we know that spine(spine(T)) is a path and each of its endpoints can be identified with a claw in B. Moreover, if the central vertex of such a claw is a vertex of V_1, then spine(spine(T)) induces a linear ordering of V_1 which is uniquely determined by (T, η) except for the order of the last three vertices. Let $a \in V_1$ be any vertex in V_1 and $X_a \subseteq V_1$ be the set of vertices smaller or equal to a in the ordering induced by (T, η), then Corollary 38 together with Lemma 31 implies $|X_a| + 2 = |N_B(X_a)|$. Inspired by this observation, we present a definition for elimination orderings in bipartite matching covered graphs.

Definition 39 (Matching Elimination Width)**.** Let B be a bipartite matching covered graph and $\Lambda(V_i)$ be the set of all linear orderings of V_i for $i \in \{1, 2\}$. Let $\lambda \in \Lambda(V_i)$. For every $v \in V_i$ we define the set of *reachable* vertices in V_{3-i} as

$$\begin{aligned} \text{Reach}[B, \lambda, v] &\coloneqq N_B(\text{Prec}[B, \lambda, v]), \text{ where} \\ \text{Prec}[B, \lambda, v] &\coloneqq \{v' \in V_i \mid \lambda(v') \leqslant \lambda(v)\}. \end{aligned}$$

We also call these the *reachability-set* and the *predecessor-set* respectively. The *width* of such an ordering is given by

$$\text{width}(\lambda) \coloneqq \max_{v \in V_i}(|\text{Reach}[B, \lambda, v]| - |\text{Prec}[B, \lambda, v]|).$$

Now the *matching elimination width* of B (with respect to V_i) is defined as

$$\text{meow}_i(B) \coloneqq \min_{\lambda \in \Lambda(V_i)} \text{width}(\lambda).$$

Please note that by Theorem 26 $|\text{Reach}[B, \lambda, v]| - |\text{Prec}[B, \lambda, v]| \geqslant 0$ for all $\lambda \in \Lambda(V_i)$ and all $v \in V_i$. Moreover, if v is not the largest vertex of λ, then $|\text{Reach}[B, \lambda, v]| - |\text{Prec}[B, \lambda, v]| \geqslant 1$ as B is matching covered. Also note that, in case λ is an ordering whose width with respect to V_1 is some value k, then the ordering λ' obtained by ordering the vertices of V_2 according to their appearance as neighbours of the vertices of V_1, and then reversing this order, is of the same width as λ, but now with respect to V_2. Hence, the choice of $i \in \{1,2\}$ does not influence the width.

What follows is a characterisation of braces of perfect matching width two in terms of their matching elimination width. To be more precise, we show that an ordering of the vertices in V_1 of width two can be used to construct a perfect matching decomposition (T,η) of width two such that spine(spine(T)) is a path. Also, any linear ordering of V_1 obtained from such a path in a perfect matching decomposition (T,η) of width two provides an ordering of V_1 of width two.

Theorem 40. *Let B be a brace on at least 6 vertices. Then $\mathrm{pmw}(B) = 2$ if and only if $\mathrm{meow}_1(B) = 2$.*

Proof. First, let (T,η) be a perfect matching decomposition for B of width two. Then, by Lemma 35, spine(T) is cubic and by Proposition 36, spine(spine(T)) is a path. Let $n := |V_1|$, then $|V(B)| = 2n$ and T has $2n$ leaves. So by Observation 8, spine(T) has $2n-2$ vertices and as spine(T) has a leaf for every two vertices of B, $|\mathsf{L}(\mathrm{spine}(T))| = n$.

Thus, spine(spine(T)) has $n-2$ vertices, let $t_1,\dots,t_{n-2}$ be its vertices ordered by occurrence and t_1 being the endpoint that, by Corollary 37, corresponds to a claw in B whose central vertex is $v_1 \in V_1$. We define a bijective function $\lambda^{-1}\colon \{1,\dots,n\} \to V_1$ whose inverse provides the desired ordering. We set $\lambda^{-1}(1) := v_1$.

For each $i \in \{1,\dots,n-3\}$ let $X_i := \eta(T_i)$ where T_i is the component of $T - t_it_{i+1}$ that contains t_1. By our definition of v_1 and t_1, $X_1 \cap V_1 = \{v_1\}$. Now, consider $i \in \{2,\dots,n-3\}$. Clearly $X_j \subseteq X_i$ for all $j < i$ and by Corollary 37, $X_i \setminus X_{i-1}$ contains exactly two vertices, one being $u_i \in V_2$ and the other one being $v_i \in V_1$. Set $\lambda^{-1}(i) := v_i$. At last, let $\{v_{n-2}, v_{n-1}, v_n\} = \overline{X_{n-3}} \cap V_1$ where the order of these three vertices is chosen arbitrarily and set $\lambda^{-1}(j) := v_j$ for all $j \in \{n-2,n-1,n\}$.

Now, $\lambda = (\lambda^{-1})^{-1}$ is a linear ordering of V_1. Note that $\mathrm{meow}_1(G) \geqslant 2$ due to Theorem 26. Hence, it is only left to show that $\mathrm{width}(\lambda) = 2$.

Let $v \in V_1$ be chosen arbitrarily. If $v \in \{v_{n-2}, v_{n-1}, v_n\}$ we have nothing to show, so suppose $v = v_i$ for some $i \in \{1,\dots,n-3\}$. Then, $\mathrm{Reach}[B,\lambda,v] = X_i \cap V_2$ and $\mathrm{Prec}[B,\lambda,v] = X_i \cap V_1 = \{v_1,\dots,v_i\}$. Lemma 31 yields $\mathrm{imbalance}(X_i) = 2$ and as $\{v_1\}$ is the minority of X_1, we obtain that V_1 contains the minority of X_i from Corollary 37. Therefore, $|\mathrm{Reach}[B,\lambda,v] - \mathrm{Prec}[B,\lambda,v]| = 2$. As i was chosen arbitrarily, $\mathrm{width}(\lambda) = 2$ and thus $\mathrm{meow}_1(G) = 2$.

Second, for the reverse direction, let λ be a linear ordering of V_1 of width two, and let $n := |V_1|$. Since B is a brace, $|\mathrm{Reach}[B,\lambda,v]| - |\mathrm{Prec}[B,\lambda,v]| \geqslant 2$ for all $v \in V_1$ with $\lambda(v) \leqslant n-2$. Let $X_1 := \{v_1\} \cup N_B(v_1)$ and for all $i \in \{1,\dots,n-3\}$ let $X_i := X_{i-1} \cup \{v_i\} \cup N_B(v_i)$ and then let $X_{n-2} := X_{n-3} \cup \{v_{n-2}, v_{n-1}, v_n\} \cup N_B(\{v_{n-2}, v_{n-1}, v_n\})$. We claim that $\mathrm{mp}(\partial_B(X_i)) = 2$ for all $i \in \{1,\dots,n-2\}$ and $|X_j| - |X_{j-1}| = 2$ for all $j \in \{2,\dots,n-2\}$ as well as $|X_1| = |X_{n-2} \setminus X_{n-3}| = 4$.

By construction, for all $i \in \{1,\dots,n-2\}$, $N_B(V_1 \cap X_i) \subseteq X_i$ and thus $\mathrm{mp}(\partial_B(X_i)) = |X_i| - 2|V_1 \cap X_i| = |V_2 \cap X_i| - |V_1 \cap X_i| = 2$, where the last equality follows from the width of λ. Now, consider $j \in \{1,\dots,n-3\}$. By definition, $|X_j \cap V_1| - |X_{j-1} \cap V_1| = 1$ and, as we have seen above, $|V_2 \cap X_j| - |V_1 \cap X_j| = |V_2 \cap X_{j-1}| - |V_1 \cap X_{j-1}|$ hence,

$|X_j \cap V_2| - |X_{j-1} \cap V_2| = 1$ as well. At last, clearly $|X_1| = 4$ by definition and the width of λ. Moreover $|V_2 \cap X_j| - |V_1 \cap X_j| = 2$ and thus $|X_{n-3} \cap V_2| - |X_{n-3} \cap V_1| = 2$ implying $|X_{n-3} \cap V_2| = n-1$, so $|\overline{X_{n-2}}| = 4$.

We now use the X_i to construct a perfect matching decomposition of width two for B. The idea is simple, we introduce a path on $n-2$ vertices $t_1, \ldots, t_{n-2}$ and identify X_i with t_i for all i. We construct a tree T by first introducing two new leaf neighbours for t_1 and t_{n-2} and one new leaf neighbour for each t_j with $j \in \{2, \ldots, n-3\}$ and second, introducing two leaf neighbours again for every leaf added in the first step. This results in the two endpoints of our original path being identified with four new leaves each, while every internal vertex of the path is identified with two leaves of the new tree T. We start creating η by mapping the four leaves identified with t_1 to the vertices of X_1 and the four leaves identified with t_{n-2} to the vertices of $\overline{X_{n-3}}$. By our observations above, for each $j \in \{2, \ldots, n-3\}$, $|X_j| - |X_{j-1}| = 2$ and so for each such j we can map the two leaves of T identified with t_j to the two vertices in $X_j \setminus X_{j-1}$. The result is a perfect matching decomposition (T, η) of B and, since $\mathrm{mp}(\partial_B(X_i)) = 2$ for all $i \in \{1, \ldots, n-2\}$, it is of width two. This completes our proof. □

3.5 Edge-Maximal Braces of Perfect Matching Width Two

Let B be a brace of perfect matching width two and λ a linear ordering of V_1 such that $\mathrm{width}(\lambda) = 2$. Suppose for some $v \in V_1$ there is a $u \in \mathrm{Reach}[B, \lambda, v]$ with $uv \notin E(B)$, then λ is also a width-2-ordering of $B + uv$. Using this observation, we can add edges to our brace until we reach a brace B' such that $\mathrm{meow}_1(B' + uv) > \mathrm{meow}_1(B') = 2$ for every edge uv with $v \in V_1$, $u \in V_2$ and $uv \notin E(B')$.

By following this idea of constructing an edge-maximal brace of perfect matching width two, we obtain a special kind of bipartite graphs. We call a brace $L_n = B$ a *bipartite ladder* of *order* n if $V_1 = \{v_1, \ldots, v_n\}$, $V_2 = \{u_1, \ldots, u_n\}$ and $E(B) = E_1 \cup E_2 \cup E_3$ where

1. $E_1 \coloneqq \{v_i u_j \mid \text{for all } 1 \leqslant j \leqslant i \leqslant n\}$,
2. $E_2 \coloneqq \{v_i u_{i+1} \mid \text{for all } 1 \leqslant i \leqslant n-1\}$, and
3. $E_3 \coloneqq \{v_i u_{i+2} \mid \text{for all } 1 \leqslant i \leqslant n-2\}$.

The graphs L_1, which is a single edge, and L_2, which is isomorphic to C_4, are not too interesting as they are very small. For $n \geqslant 3$ however these graphs grow more complex, see fig. 6 for an illustration of L_3, L_4 and L_5.

The following corollary is a consequence of Theorem 26.

Corollary 41. *Let B be a brace and $v_1 \in V_1$, $v_2 \in V_2$ such that $v_1 v_2 \notin E(B)$. Then $B + v_1 v_2$ is a brace.*

This corollary allows the construction of edge-maximal braces of width 2, which we are aiming for. We conclude this section with a second characterisation of braces with perfect matching width 2, this time in terms of edge-maximal supergraphs.

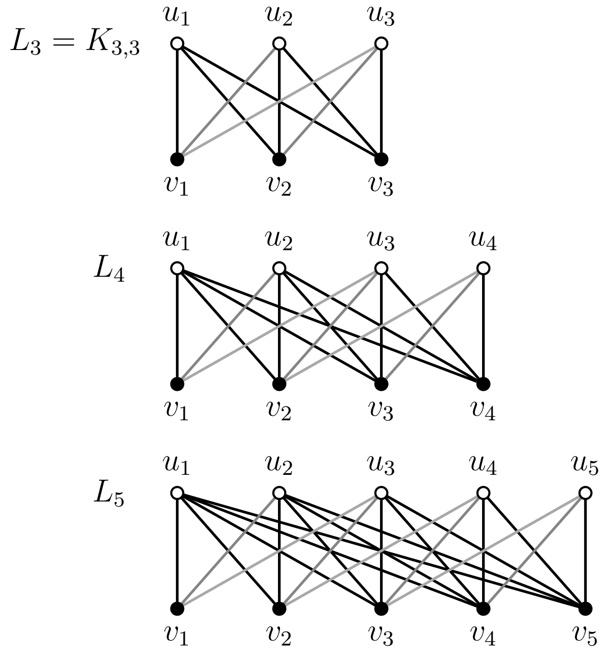

Figure 6: The bipartite ladders of order 3, 4, and 5. Edges of E_1 are black, the ones from E_2 are dark grey and the edges from E_3 are light grey.

Theorem 42. *Let B be brace with $|V_1| = n$. Then,* $\operatorname{pmw}(B) = 2$ *if and only if* $B \subseteq L_n$.

Proof. We start by proving that every conformal subgraph of L_n is of perfect matching width 2 or isomorphic to K_2. To do so, by Lemma 14, it suffices to show $\operatorname{pmw}(L_n) = 2$ for all $n \in \mathbb{N}$ with $n \geqslant 2$. The definition of L_n directly provides an ordering λ of $V_1 = \{v_1, \ldots, v_n\}$ with $\lambda(v_i) = i$. We prove that $\operatorname{width}(\lambda) = 2$. Let $i \in \{1, \ldots, n-3\}$ be arbitrary. By definition, $N_{L_n}(v_i) = \{u_1, \ldots, u_{i+2}\} \subseteq \operatorname{Reach}[L_n, \lambda, v_i]$. Moreover, as $N_{L_n}(v_j) \subseteq N_{L_n}(v_i)$ for all $j \leqslant i$, $N_{L_n}(v_i) = \operatorname{Reach}[L_n, \lambda, v_i]$. Therefore, $|\operatorname{Reach}[L_n, \lambda, v_i]| - |\operatorname{Prec}[L_n, \lambda, v_i]| = 2$ for all $i \in \{1, \ldots, n-2\}$ and thus, $\operatorname{width}(\lambda) = 2$. By Theorem 40, this implies $\operatorname{pmw}(L_n) = 2$, as desired.

Now, let B be a brace of perfect matching width two. Then, there is an ordering λ of V_1 of width two by Theorem 40. Let us number the vertices of V_1 according to λ, so for all $i \in \{1, \ldots, n\}$ let $v_i \coloneqq \lambda^{-1}(i)$. We construct a numbering of the vertices in V_2 as follows. Let $N_B(v_1) = \{u_1, u_2, u_3\}$ be numbered arbitrarily. The size of the neighbourhood of a_1 follows immediately from the width of λ and the fact that B is a brace. Now, as a consequence of Corollary 37, for every $i \in \{1, \ldots, n-2\}$, $\operatorname{Reach}[B, \lambda, v_i] \setminus \operatorname{Reach}[B, \lambda, v_{i-1}]$ contains exactly one vertex, which is in V_2. Let u_{i+2} be this vertex. Now, $N_B(v_i) \subseteq \operatorname{Reach}[B, \lambda, v_i]$ for all $i \in \{1, \ldots, n\}$ and thus B does not contain an edge that does not obey the definition of a bipartite ladder with respect to the orderings of V_1 and V_2 as obtained above. If there are indices $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$ such that $v_iu_j \notin E(B)$, but $j \leqslant i+2$, then we simply add the edge v_iu_j to B. By Corollary 41 $B + v_iu_j$ is still a brace, and by choice of i and j, adding this edge does not change the predecessor- and reachability-sets of any vertices in V_1, hence λ is still an ordering of width two for $G + v_i u_j$. Thus, we can keep adding edges in this fashion until we do not find such a pair of indices any more. In that case, let B' be the newly obtained brace. By construction, B' is isomorphic to L_n and thus B is a conformal subgraph of L_n. □

4 Perfect Matching Width and Treewidth

A natural question for any new width parameter is how it compares to other already-known parameters. We have already seen a way to relate the perfect matching width of bipartite graphs and directed treewidth. However, the graph itself has to be transformed to apply our findings. In the first part of this short section, we discuss the relation between the (undirected) treewidth[3] of G and its perfect matching width. To do this, we use a parameter introduced by Vatshelle [Vat12], which is already known to be equivalent to treewidth but is much closer to perfect matching width in spirit.

Let G be a graph and $X \subseteq V(G)$. We denote by $\mathrm{mm}(\partial_G(X))$ the number $\nu(G[\partial_G(X)])$ which is the maximum number of pairwise disjoint edges in $\partial_G(X)$.

A *maximum matching decomposition* of G is a tuple (T, η) where T is a cubic tree and $\eta\colon \mathsf{L}(T) \to V(G)$ is a bijection.

Recall from the definition of perfect matching width the associated edge cut $\partial_G(e)$ for every edge $e \in E(T)$. The *width* of a maximum matching decomposition (T, η) is defined as the maximum $\mathrm{mm}(\partial_G(e))$ over all $e \in E(T)$, and the *maximum matching width* of G, denoted by $\mathrm{mmw}(G)$, is the minimum width over all maximum matching decompositions of G.

Theorem 43 ([Vat12, JST18]). *Let G be a graph and let $\mathrm{tw}(G)$ denote the treewidth of G. Then $\mathrm{mmw}(G) \leqslant \mathrm{tw}(G) + 1 \leqslant 3\,\mathrm{mmw}(G)$.*

This makes it straightforward to bound the perfect matching width of a graph G with a perfect matching in terms of its treewidth.

Proposition 44. *Let G be a graph with a perfect matching. Then $\mathrm{pmw}(G) \leqslant \mathrm{tw}(G) + 1$.*

Proof. By Theorem 43 we have $\mathrm{mmw}(G) \leqslant \mathrm{tw}(G) + 1$, so there exists a maximum matching decomposition (T, η) for G of width at most $\mathrm{tw}(G) + 1$. Let M be any perfect matching of G and $t_1 t_2 \in E(T)$. Note that $M \cap \partial_G(\eta(T_{t_1}))$ is a matching, hence $|M \cap \partial_G(t_1 t_2)| \leqslant \mathrm{mm}(\partial_G(t_1 t_2)) \leqslant \mathrm{tw}(G) + 1$. Indeed, as M was chosen arbitrarily we have $\mathrm{mp}(\partial_G(t_1 t_2)) \leqslant \mathrm{tw}(G) + 1$ and thus (T, η) is a perfect matching decomposition of G of width at most $\mathrm{tw}(G) + 1$ and our claim follows. □

While treewidth gives us an upper bound on the perfect matching width of G, the reverse is not true in general. With these findings, we close this chapter.

Proposition 45. *For every $k \in \mathbb{N}$ with $k \geqslant 2$ there exists a brace B_k with $\mathrm{pmw}(B_k) = 2$ and $\mathrm{tw}(B_k) \geqslant k$.*

[3]The definition of treewidth can be found in [Vat12] by the interested reader.

Proof. First note that for every $t \in \mathbb{N}$, $\text{tw}(K_{t+1}) = t$. So if we can show that B_k contains K_{k+1} as a minor, we have proven $\text{tw}(B_k) \geqslant k$ since treewidth is monotone under taking minors. Let B_k be the bipartite ladder L_{k+1} of order $k+1$. Then, by Theorem 42, $\text{pmw}(B_k) = 2$ for all k. Now let $k+1 \geqslant 2$ be chosen arbitrarily. We choose the perfect matching $M \coloneqq \{u_i v_i \mid i \in \{1, \dots, k+1\}\}$. By definition of L_{k+1} we know $u_i v_j$ for every $i \in \{1, \dots, k+1\}$ and every $j \in \{i, \dots, k+1\}$, so by contracting all edges in M we obtain a graph on $k+1$ vertices with $\binom{k+1}{2}$ edges. So B_k/M is isomorphic to K_{k+1}, and we are done. □

5 Computing Perfect Matching Decompositions of Width Two

In this section we provide an explicit polynomial time algorithm to compute an optimal perfect matching decomposition for braces of perfect matching width 2. We do so by first finding a matching elimination ordering, as is possible due to Theorem 40. By Corollary 37, such a construction has to start in a degree-3 vertex that builds a claw together with its neighbours. This, in particular, allows us to discard any claw-free graph immediately. Then, the construction proceeds by choosing a new vertex from the same colour class in each step, whose neighbourhood contains at most one vertex that is not already in the neighbourhood of the previously chosen vertices. In the correct decomposition, such a vertex exists by Corollary 38 and Lemma 31. Thus, if there is no such vertex to pick at some point, there are two possible reasons. Either the initial claw was not chosen optimally, then the algorithm starts over with a different claw. Or, the brace has perfect matching width at least 3, this the algorithm concludes when having tried all possible claws.

Lemma 46. *Let $B = (V_1 \cup V_2, E)$ be a brace. Then algorithm 1 computes an ordering λ of width 2 on input V_2 and V_1 if and only if* $\text{pmw}(B) = 2$.

Proof. First, assume algorithm 1 returns an ordering λ for the input V_2 and V_1. Then, we can consider the sets $\text{Prec}[B, \lambda, a]$ and $\text{Reach}[B, \lambda, a]$.

Proving $|\text{Reach}[B, \lambda, \lambda^{-1}(j)]| - |\text{Prec}[B, \lambda, \lambda^{-1}(j)]| \leqslant 2$ for all $j \in \{1, \dots, |V_1|\}$ by induction shows $\text{width}(\lambda) = 2$ as $2 \leqslant \text{width}(\lambda)$ since B is a brace. If $j \in \{1, |V_1|-1, |V_1|\}$, there is nothing to show. So, suppose $2 \leqslant j \leqslant |V_1| - 3$ and let $a \coloneqq \lambda^{-1}(j)$. That is, a is chosen in the iteration for $i = j$ in algorithm 1. Let P_a and U_a be the sets P and U during this step of the algorithm. The set P_a contains all vertices that were previously chosen by algorithm 1 and thus are smaller than a with respect to λ. Hence $\text{Prec}[B, \lambda, a] = P_a \cup \{a\}$ and $\text{Prec}[B, \lambda, \lambda^{-1}(j-1)] = P_a$. With a being chosen at step j, we know $|N(a) \setminus N(P_a)| \leqslant 1$. Therefore,

$$\begin{aligned} |\text{Reach}[B, \lambda, a]| - |\text{Prec}[B, \lambda, a]| =& |N(P_a \cup \{a\})| - |P_a \cup \{a\}| \\ \leqslant& |N(P_a)| + 1 - (|P_a| + 1) \\ \leqslant& |P_a| + 3 - (|P_a| + 1) = 2. \end{aligned}$$

Hence, by Theorem 40, $\text{width}(\lambda) = 2$ and therefore $\text{pmw}(B) = 2$.

Algorithm 1 Compute width-2-ordering

```
procedure ORDER(V_2, V_1)
    λ^{-1} ← ∅
    for all a ∈ V_1 do
        λ^{-1} ← ∅
        if |N(a)| = 3 then
            λ^{-1}(1) ← a
            U ← V_1 \ {a}
            P ← {a}
            for all i ∈ {2, ..., |V_1|} do
                for all a' ∈ U do
                    if |N(a') \ N(P)| ⩽ 1 then
                        λ^{-1}(i) ← a'
                        P ← P ∪ {a'}
                        U ← U \ {a'}
                        break
                if λ^{-1}(i) = ∅ then
                    break
            if λ^{-1}(|V_1|) ≠ ∅ then return λ
    return B is not of perfect matching width 2.
```

Second, assume $\text{pmw}(B) = 2$. By Theorem 40, there exists an ordering σ of V_1 with $\text{width}(\sigma) = 2$. We have already seen that if algorithm 1 returns an ordering λ, it is of width 2. So what remains to show is that the algorithm returns an ordering. Suppose it does not.

Let $a_1 \coloneqq \lambda^{-1}(1)$. Since algorithm 1 only terminates without returning an ordering when it looped through all elements for the choice in algorithm 1, it reaches the point where it chooses a_1. Now, algorithm 1 can choose the next element in algorithm 1, fulfilling the demand in algorithm 1 according to the ordering λ. Since it does not end up returning an ordering, it eventually differs from any optimal ordering and then reaches the point $2 \leqslant k \leqslant |V_1|$ at which there is no element to choose in algorithm 1 fulfilling the demand in algorithm 1. Let $a_1 \dots, a_k$ be elements of V_1 that algorithm 1 ordered this way so far before it gets stuck. Let σ be chosen among all width-2-orderings of V_1 maximising $j \in \{1, \dots, k-1\}$ such that $\sigma^{-1}(i) = a_i$ for all $1 \leqslant i < j$ and $\sigma^{-1}(j) \neq a_j$. We refer to the elements after a_j in σ by $y_h \coloneqq \sigma^{-1}(h)$ for all $h \in \{j+1, \dots, |V_1|\}$. By the definition of the algorithm, $|N(a_j) \setminus N(a_1, \dots, a_{j-1})| \leqslant 1$. Let σ' be the ordering obtained from σ by inserting a_j at the position j instead of its position $j + x$ in σ. So σ' contains the elements of V_1 in the order $a_1, \dots, a_j, y_{j+1}, \dots, y_{j+x-1}, y_{j+x+1}, \dots, y_{|V_1|}$.

Suppose, $\text{width}(\sigma') \geqslant 3$. There is a vertex $y_{h'}$ with $h' \in \{j+1, \dots, j+x-1\}$ such that

$$|\operatorname{Reach}[B, \sigma', y_{h'}]| - |\operatorname{Prec}[B, \sigma', y_{h'}]| \geqslant 3.$$

But $|\operatorname{Prec}[B, \sigma', y_{h'}]| = |\operatorname{Prec}[B, \sigma, y_{h'}]| + 1$ and with

$$|N(a_j) \setminus N(a_1, \ldots, a_{j-1})| \leqslant 1$$

we obtain $|\operatorname{Reach}[B, \sigma', y_{h'}]| \leqslant |\operatorname{Reach}[B, \sigma, y_{h'}]| + 1$. Thus,

$$|\operatorname{Reach}[B, \sigma, y_{h'}]| - |\operatorname{Prec}[B, \sigma, y_{h'}]| \geqslant 3,$$

which contradicts σ to be of width 2. Hence $\operatorname{width}(\sigma') = 2$. However, this is a contradiction to the choice of σ as σ' now coincides on the first j positions with the choice of algorithm 1. Thus, the algorithm does not get stuck once it chose the right claw and therefore, algorithm 1 returns an ordering. □

So algorithm 1 produces an elimination ordering of width 2 if and only if the brace B that was given as input is of perfect matching width 2. This ordering can be translated into a perfect matching decomposition of width 2, as seen in the second part of the proof of Theorem 40. Since all sets necessary for the construction of this decomposition can be computed from the ordering by iterating over edges and vertices of B at most once, this procedure runs in polynomial time and thus, we obtain the following result which concludes this section.

Theorem 47. *Let $B = (V_1 \cup V_2, E)$ be a brace. There is a polynomial time algorithm that computes a perfect matching decomposition of width 2 if and only if* $\operatorname{pmw}(B) = 2$.

6 Bipartite Graphs of M-Perfect Matching Width Two

Section 3 provides a complete characterisation of braces of perfect matching width two. However, we are not able to lift this result to all bipartite matching covered graphs since we do not know whether the braces of a matching covered bipartite graph of perfect matching width two are also of perfect matching width two themselves. To be more precise, for a matching covered bipartite graph B with $\operatorname{pmw}(B) = 2$, the best we know about any brace H of it is $\operatorname{pmw}(H) \in \{2, 3, 4\}$ by Corollary 18. We can however consider the M-perfect matching width instead since here Lemma 19 implies that $\operatorname{pmw}_M(B)$ bounds $\operatorname{pmw}_{M|_H}(H)$. Indeed, since K_2 is the only matching covered graph of M-perfect matching width one, B has M-perfect matching width two if and only if every brace H of B has $M|_H$-perfect matching width two.

In this section, we present a full characterisation of the braces of M-perfect matching width two and, thus, provide a description of all matching covered bipartite graphs that have a perfect matching M such that their M-perfect matching width is 2.

Key to this characterisation is the observation that, given a brace B, $2 \leqslant \operatorname{pmw}(B) \leqslant \operatorname{pmw}_M(B)$ for all $M \in \mathcal{M}(B)$. So, if $\operatorname{pmw}_M(B) = 2$ for some M, then every optimal M-decomposition of B also is an optimal perfect matching decomposition of G. Therefore, we can apply the results from section 3. This immediately implies a rather strict bound on the number of vertices, which in turn narrows down the braces of M-perfect matching width two to exactly two, namely $K_{3,3}$ and C_4.

Proposition 48. *Let B be a brace. Then, the following statements are equivalent.*

1. $\text{pmw}_M(B) = 2$ *for an* $M \in \mathcal{M}(B)$,
2. $\text{pmw}_M(B) = 2$ *for all* $M \in \mathcal{M}(B)$, *and*
3. B *is isomorphic to* C_4 *or* $K_{3,3}$.

Proof. In order to prove this statement, we first deduce item 3 from item 1 and then observe that we can find the same type of decomposition for every $M \in \mathcal{M}(B)$ which then implies item 2.

Let B be a brace and $M \in \mathcal{M}(B)$ such that $\text{pmw}_M(B) = 2$, then $\text{pmw}(B) = 2$ as well. Let (T, η) be an optimal M-decomposition for B, then it also is an optimal perfect matching decomposition of B. Now suppose $|V(B)| \geqslant 8$. Then by Corollary 37, there is an edge $e \in E(\text{spine}(\text{spine}(T)))$ such that $\partial_B(e)$ has a shore X of size 4 that induces a claw in B. In particular, $\text{imbalance}(X) = 2$ and thus X is not M-conformal. This is a contradiction to the definition of M-decompositions as e is an inner edge of T. So $|V(G)| \leqslant 6$. On at most 6 vertices, there are only two braces: C_4 and $K_{3,3}$.

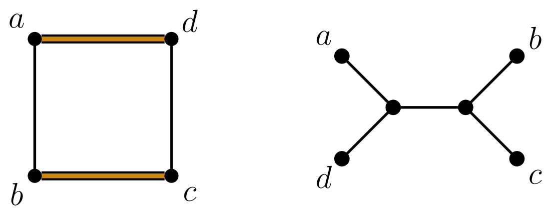

Figure 7: The brace C_4 together with a perfect matching M and an M-decomposition (T, η) of width two.

First, consider C_4. Let $M \in \mathcal{M}(C_4)$ be a perfect matching. Then, $V(C_4) = \{a, b, c, d\}$ and without loss of generality $M = \{ad, bc\}$. As C_4 is a cycle, the only other perfect matching of C_4 is $E(C_4) \setminus M = \{ab, cd\}$. We construct a perfect matching decomposition (T, η) as follows. Take two vertices t_1 and t_2 joined by an edge. We create a cubic tree T by adding two leaves t_i^1 and t_i^2 as new neighbours to each of the t_i for $i \in \{1, 2\}$. Then, let $\eta(t_1^1) \coloneqq a$, $\eta(t_1^2) \coloneqq d$, $\eta(t_2^1) \coloneqq b$ and $\eta(t_2^2) \coloneqq c$ (see fig. 7). Now, (T, η) is an M-decomposition of C_4, and the matching porosity of every cut induced by an edge of T is either one or two. Note that for the other perfect matching of C_4 we just have to adapt the mapping η such that for each $i \in \{1, 2\}$ the leaves t_i^1 and t_i^2 are mapped to the endpoints of the same edge and thus $\text{pmw}_M(C_4) = 2$ for all $M \in \mathcal{M}(C_4)$.

Second consider $K_{3,3}$ and let $V_1 = \{a, b, c\}$ and $V_2 = \{d, e, f\}$ and $M = \{af, be, cd\}$ a perfect matching of $K_{3,3}$. We again construct an M-decomposition (T, η) of our brace. This time, consider a claw on the vertices $\{t, t_1, t_2, t_3\}$ such that t is the central vertex. For each $i \in \{1, 2, 3\}$, we introduce two new neighbours t_i^1 and t_i^2 to t_i, which are the leaves of our cubic tree T. Then let $\eta(t_1^1) \coloneqq a$ and $\eta(t_1^2) \coloneqq f$. For the remaining two edges of M, proceed analogously by choosing an $i \in \{2, 3\}$ for each of the remaining edges

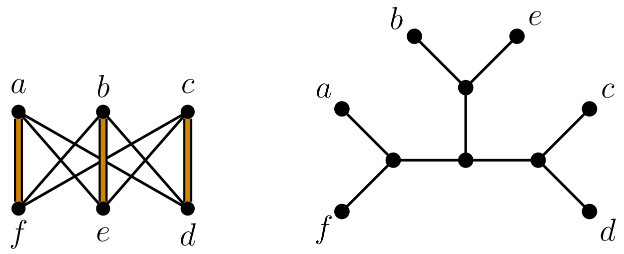

Figure 8: The brace $K_{3,3}$ together with a perfect matching M and an M-decomposition (T, η) of width two.

and then mapping the leaves t_i^1 and t_i^2 to the endpoints of the chosen edge. Now, (T, η) is an M-decomposition of $K_{3,3}$ and for every inner edge e of T the cut induced by e has a shore of size two, hence $\text{width}(T, \eta) = 2$ (see fig. 8 for an illustration). Again, we can adapt the same strategy for every perfect matching $M' \in \mathcal{M}(B)$ and thus $\text{pmw}_M(B) = 2$ for all $M \in \mathcal{M}(B)$.

We have seen that for each of the braces C_4 and $K_{3,3}$, the M-perfect matching width equals two for all perfect matchings M. So, in particular there exists such a matching and thus, item 2 implies item 1 again and the proof is complete. □

With Proposition 48, we are able to deduce a similar theorem for general bipartite matching covered graphs of M-perfect matching width two.

Proposition 49. *Let B be a bipartite matching covered graph. Then, the following statements are equivalent.*

1. $\text{pmw}_M(B) = 2$ *for an* $M \in \mathcal{M}(B)$*,*
2. $\text{pmw}_M(B) = 2$ *for all* $M \in \mathcal{M}(B)$*, and*
3. *Every brace of B is either isomorphic to C_4 or to $K_{3,3}$.*

Proof. If B is a brace, then the statement holds by Proposition 48. Thus, assume that B contains a tight cut.

By [Lov87], it suffices to show that if the statement holds for the two tight cut contractions $B_Z := B/(Z \to v_Z)$, and $B_{\overline{Z}} := B/(\overline{Z} \to v_{\overline{Z}})$ of a bipartite matching covered graph B with tight cut $\partial_B(Z)$, then it also holds for B.

By induction hypothesis, the three statements are equivalent for both B_Z and $B_{\overline{Z}}$.

Assume $\text{pmw}_M(B) = 2$ for an $M \in \mathcal{M}(B)$ (item 1), then by Lemma 19

$$\text{pmw}_{M|_{B_Z}}(B_Z) = \text{pmw}_{M|_{B_{\overline{Z}}}}(B_{\overline{Z}}) = 2$$

and thus, the braces of both B_Z and $B_{\overline{Z}}$ are isomorphic to C_4 or $K_{3,3}$. Since the braces of B are exactly the union of the braces of B_Z and $B_{\overline{Z}}$, item 3 holds for B as well.

Next, assume that item 3 holds for B. Pick any matching $M' \in \mathcal{M}(B)$, then by induction hypothesis $\text{pmw}_{M'|_{B_Z}}(B_Z) = \text{pmw}_{M'|_{B_{\overline{Z}}}}(B_{\overline{Z}}) = 2$. Let $e_Z \in M'|_{B_Z}$ and $e_{\overline{Z}} \in$

$M'|_{B_{\overline{Z}}}$ be the two edges covering v_Z and $v_{\overline{Z}}$ in the respective contractions for the respective reductions of M'. Let u_X be the endpoint of e_X that is not v_X for both $X \in \{Z, \overline{Z}\}$. Moreover, let (T_X, η_X) be an optimal $M'|_X$-decomposition of B_X for both $X \in \{Z, \overline{Z}\}$. In T_Z, there is a vertex t_Z that is adjacent to the two leaves of T_Z that are mapped to v_Z and u_Z, let $t_{\overline{Z}}$ be chosen analogously. Observe, that $M' = ((M'|_{B_Z} \cup M'|_{B_{\overline{Z}}}) \setminus \{e_Z, e_{\overline{Z}}\}) \cup \{u_Z u_{\overline{Z}}\}$. We construct an M'-decomposition (T', η') as follows. Let T'_X be obtained from T_X by deleting the two leaves adjacent to t_X for both $X \in \{Z, \overline{Z}\}$. Then, let T'' be the tree obtained from T'_Z and $T'_{\overline{Z}}$ by identifying t_Z and $t_{\overline{Z}}$, call the new vertex t. At last, let T' be the tree obtained from T'' by adding a new vertex t', the edge tt' and two new leaves t_1 and t_2 adjacent to the new vertex t'. Then, T' is a cubic tree and $|\mathsf{L}(T')| = |V(B)|$. In the next step we define $\eta' : \mathsf{L}(T') \to V(B)$ as follows:

$$\eta'(\ell) := \begin{cases} \eta_Z(\ell), & \text{if } \ell \in \mathsf{L}(T_Z) \setminus \left\{\eta_Z^{-1}(v_Z)\right\}, \\ \eta_{\overline{Z}}(\ell), & \text{if } \ell \in \mathsf{L}(T_{\overline{Z}}) \setminus \left\{\eta_{\overline{Z}}^{-1}(v_{\overline{Z}})\right\}, \\ u_{\overline{Z}}, & \text{if } \ell = t_1, \text{ and} \\ u_Z, & \text{if } \ell = t_2. \end{cases}$$

Now, (T', η') is an M'-decomposition of B. Moreover, let $e \in E(T')$ be an inner edge of T', then either e is an inner edge of T_Z or $T_{\overline{Z}}$ and by construction of T' and the fact that $\partial_B(Z)$ is tight, $\mathrm{mp}(\partial_B(e)) \leqslant 2$, or $e = tt'$. In the latter case, $\partial_B(e)$ has a shore of size two and thus $\mathrm{mp}(\partial_B(e)) = 2$. Therefore, $\mathrm{width}(T', \eta') = 2$ and so M'-pmw$(B) = 2$ for all $M' \in \mathcal{M}(B)$, that is item 2 holds. Since item 2 implies item 1, we are done. ☐

So, in order to recognise a bipartite matching covered graph B of M-perfect matching width two, it suffices to check whether B has a brace not isomorphic to C_4 or $K_{3,3}$. Lovász has shown that the tight cut decomposition of a matching covered graph can be computed in polynomial time (see [Lov87]) and thus, Proposition 49 implies a polynomial recognition algorithm for bipartite matching covered graphs of M-perfect matching width two. Moreover, the proof of Proposition 49 is constructive and can be used to obtain an M-decomposition of width two for any $M \in \mathcal{M}(B)$, given a bipartite matching covered graph B of M-perfect matching width two, from the decompositions of its braces. As these braces are only C_4 and $K_{3,3}$, whose optimal M-decompositions are given in the proof of Proposition 48, we obtain the following corollary.

Corollary 50. *Let B be a bipartite matching covered graph and $M \in \mathcal{M}(B)$. Then, we can compute in polynomial time either an M-decomposition of width two, or a brace of B that is neither isomorphic to C_4, nor to $K_{3,3}$.*

In order to obtain Theorem 3, we consider the family of odd Möbius ladders.

Definition 51 (Odd Möbius ladders)**.** An *odd Möbius ladder* of *order* $k \geqslant 1$ is the graph $\mathscr{M}_{4k+2}$ obtained from the cycle

$$(v_0^1, v_0^2, v_1^1, v_1^2, v_2^1, \ldots, v_{2k-1}^2, v_{2k}^1, v_{2k}^2, v_0^1)$$

by adding the edges $v_i^1 v_{k+i \,(\bmod 2k+1)}^2$, for all $i \in \{0, \ldots, 2k\}$. See fig. 9 for an illustration.

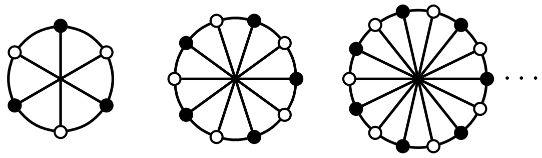

Figure 9: The odd Möbius ladders. They build a chain with respect to the matching minor relation due to Lemma 52.

The odd Möbius ladders build a chain with respect to the matching minor relation.

Lemma 52. *Let $k \in \mathbb{N}$ be a positive integer. Then $\mathscr{M}_{4k+2}$ is a matching minor of $\mathscr{M}_{4k+6}$.*

Proof. Consider $\mathscr{M}_{4k+6}$ and the edge $v_0^1 v_{k+2}^2$. In $\mathscr{M}_{4k+6} - v_0^1 v_{k+2}^2$, the vertices v_0^1 and v_{k+2}^2 are the only two vertices of degree two and thus we may bicontract both of them. As v_0^1 and v_{k+2}^2 come from different colour classes, their neighbourhoods are disjoint, and so the resulting graph, let us call it B, has $4k + 2$ vertices. Moreover, let us denote by u^1 the vertex obtained by identifying v_{2k+2}^2, v_0^1, and v_0^2 and by u^2 the vertex obtained from the identification of v_{k+2}^1, v_{k+2}^2, and v_{k+3}^1. Observe that

$$(u^1, v_1^1, v_1^2, \dots, v_{k+1}^2, u^2, v_{k+3}^2, \dots, v_{2k+2}^1, u^1)$$

is still a Hamilton cycle of B. Indeed, for each $i \in 0, \dots, 2k+2 \setminus \{0, k+2, k+3\}$ we still have the edge $v_i^1 v_{k+i(\mod 2k+3)}$ in B. Additionally, we obtain the edge u^1u^2. By renumbering, it becomes apparent that B is indeed isomorphic to $\mathscr{M}_{4k+2}$. □

We make use of the following two results by McCuaig.

Lemma 53 ([McC04, Lemma 57])**.** *Let B be a bipartite graph with a perfect matching M and a bisubdivision of $K_{3,3}$ as conformal subgraph. Then G has an $\mathscr{M}_{4n+2}$ bisubdivision as M-conformal subgraph for some $n \geqslant 1$.*

Theorem 54 ([McC01, Theorem 29])**.** *Every brace except C_4 has a bisubdivision of $K_{3,3}$ or the cube as conformal subgraph.*

Together, they imply the following statement.

Corollary 55. *Let B be a bipartite matching covered graph. The graph B does not contain the cube or the Möbius ladder $\mathscr{M}_{10}$ as a matching minor if and only if all braces of B are isomorphic to C_4 or $K_{3,3}$.*

Proof. Assume B has a brace that is not isomorphic to C_4 or $K_{3,3}$. Then, by Theorem 54 B contains a bisubdivision of $K_{3,3}$ or the cube as conformal subgraph. If B contains the cube as conformal subgraph, we are done, thus assume B contains $K_{3,3}$ as conformal subgraph. Then, by Lemma 53, B contains an $\mathscr{M}_{4n+2}$ bisubdivision as conformal subgraph for some $n \geqslant 1$. By, Lemma 52 this implies that B contains $\mathscr{M}_{10}$ as matching minor.

For the reverse direction, assume that B contains the cube or $\mathscr{M}_{10}$ as a matching minor. As they are braces, there is a brace J of B containing the cube or $\mathscr{M}_{10}$ as a matching minor. Thus, J has to be of size at least 8 and cannot be isomorphic to C_4 or $K_{3,3}$. □

Proposition 49 and Corollary 55 now immediatly yield Theorem 3.

Theorem 3. *Let B be a bipartite graph with a perfect matching. The following statements are equivalent.*

1. *There exists a perfect matching $M \in \mathcal{M}(B)$ such that* $\text{pmw}_M(B) = 2$*,*
2. *for all $M \in \mathcal{M}(B)$ we have* $\text{pmw}_M(B) = 2$*, and*
3. *B does not contain the cube or $\mathscr{M}_{10}$ as a matching minor (see fig. 1),*

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