A combinatorial interpretation of the noncommutative inverse Kostka matrix

Edward E. Allen Sarah K. Mason

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Abstract

We provide a combinatorial formula for the expansion of immaculate noncommutative symmetric functions into complete homogeneous noncommutative symmetric functions. To do this, we introduce generalizations of Ferrers diagrams that we call GBPR diagrams. A GBPR diagram assigns a color (grey, blue, purple, or red) to each cell of the diagram. We define tunnel hooks on GBPR diagrams; these new objects play a role similar to that of the special rim hooks appearing in the Eğecioğlu-Remmel formula for the symmetric inverse Kostka matrix. We extend this interpretation to skew shapes and fully generalize to define immaculate functions indexed by integer sequences skewed by integer sequences. Finally, as an application of our combinatorial formula, we extend Campbell's results on ribbon decompositions of immaculate functions to a larger class of shapes.

Mathematics Subject Classifications: 05E05, 05E16, 05A05, 05A19

1 Introduction

The ring Sym of symmetric functions on a set of commuting variables consists of all polynomials invariant under the action of the symmetric group. Symmetric functions play an important role in representation theory, combinatorics, and other areas of mathematics and the physical and natural sciences. Bases for Sym are indexed by partitions; two ubiquitous examples are the *Schur functions* s_λ and the *complete homogeneous symmetric functions* h_λ. Schur functions correspond to irreducible representations of the symmetric group, and their multiplication corresponds to the cohomology of the Grassmannian [11, 20, 23].

The inverse Kostka matrix is the transition matrix from the Schur basis of Sym to the complete homogeneous basis. Objects called *special rim hooks* are used by Eğecioğlu and Remmel to construct a combinatorial interpretation of this matrix [10], originating from the Jacobi-Trudi formula.

Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109 (allene@wfu.edu, sarahkmason@gmail.com).

The Hopf algebra NSym of noncommutative symmetric functions is freely generated by a collection of noncommutative algebraically independent generators H_i, one at each positive degree i. The set $\{H_\alpha := H_{\alpha_1}H_{\alpha_2}\cdots H_{\alpha_k}\}$, indexed by compositions α of n, forms a basis for NSym, called the *complete homogeneous basis*. The map χ : NSym $\to$ Sym defined by $\chi(H_\alpha) = h_{\alpha_1}h_{\alpha_2}\cdots h_{\alpha_k}$ (sometimes called the "forgetful map since it "forgets" that the generators don't commute) sends elements of NSym to elements of Sym.

The immaculate basis for NSym is a Schur-like basis that maps to the Schur functions under the forgetful map. This new basis was introduced in [5] through creation operators analogous to those used to construct the Schur functions. It is natural to ask if there is a generalization to NSym of the combinatorial inverse Kostka formula, since there is a Jacobi-Trudi style formulation for the immaculate basis in terms of the complete homogeneous basis for NSym [5].

In this paper, we provide a combinatorial formula for the expansion of the immaculate basis into the complete homogeneous basis for NSym using ribbon-like objects we call *tunnel hooks*. Specifically, we prove the following theorem. This theorem generalizes the Eğecioğlu-Remmel result and in fact provides an alternative method for computing the classical inverse Kostka matrix entries.

Theorem 1. *The decomposition of the immaculate noncommutative symmetric functions into the complete homogeneous noncommutative symmetric functions is given by the following formula.*

$$\mathfrak{S}_\mu = \sum_{\gamma \in THC_\mu} \prod_{r=1}^{k} \epsilon(\mathfrak{h}(r,\tau_r))\ H_{\Delta(\mathfrak{h}(r,\tau_r))}, \tag{1}$$

where $\mu \in \mathbb{Z}^k$, THC_μ denotes the collection of tunnel hook coverings of a diagram of shape μ, and a sign $\epsilon(\mathfrak{h}(r,\tau_r))$ together with an integer value $\Delta(\mathfrak{h}(r,\tau_r))$ are assigned to each tunnel hook $\mathfrak{h}(r,\tau_r)$ in each $\gamma \in THC_\mu$.

Note that the product $\prod_{r=1}^{k} \epsilon(\mathfrak{h}(r,\tau_r))\ H_{\Delta(\mathfrak{h}(r,\tau_r))}$ in Equation (1) is taken in order from $r = 1$ to k so that

$$\prod_{r=1}^{k} \epsilon(\mathfrak{h}(r,\tau_r))\ H_{\Delta(\mathfrak{h}(r,\tau_r))} = \epsilon(\mathfrak{h}(1,\tau_1))\ H_{\Delta(\mathfrak{h}(1,\tau_1))}\cdots\epsilon(\mathfrak{h}(k,\tau_k))\ H_{\Delta(\mathfrak{h}(k,\tau_k))},$$

since the functions $H_{\Delta(\mathfrak{h}(r,\tau_r))}$ do not commute.

Our formula translates, via duality between NSym and the vector space QSym of quasisymmetric functions, to a formula for the expansion of a monomial quasisymmetric function into the dual immaculate basis for QSym, which can then be expanded combinatorially (and positively) into the Young quasisymmetric Schur functions [3]. This expansion provides a potential avenue for attacking questions of Schur positivity, since any symmetric function expanding positivity into the Young quasisymmetric Schur function basis must be Schur positive.

The process of constructing tunnel hook coverings is an iterative process that introduces a skew-shape generalization along the way. This naturally leads to the introduction of a candidate for the *skew immaculate noncommutative symmetric functions* $\mathfrak{S}_{\mu/\lambda}$ and a generalization of Theorem 1 to the skew setting. Although this construction is compatible with skew Schur functions (following [20]) under the forgetful map, this doesn't always align with the Hopf algebraic skewing operator. See Section 5 for details.

Theorem 2. *The decomposition of the skew immaculate noncommutative symmetric functions $\mathfrak{S}_{\mu/\lambda}$ (with $\mu \in \mathbb{Z}^k$ and λ a partition with at most k parts) into the complete homogeneous noncommutative symmetric functions is given by the following formula.*

$$\mathfrak{S}_{\mu/\lambda} = \sum_{\gamma \in THC_{\mu/\nu}} \prod_{r=1}^{k} \epsilon(\mathfrak{h}(r, \tau_r)) \; H_{\Delta(\mathfrak{h}(r,\tau_r))}, \tag{2}$$

where $THC_{\mu/\lambda}$ denotes the collection of tunnel hook coverings of a diagram of shape μ/λ, and a sign $\epsilon(\mathfrak{h}(r, \tau_r))$ together with an integer value $\Delta(\mathfrak{h}(r, \tau_r))$ are assigned to each tunnel hook $\mathfrak{h}(r, \tau_r)$ in each $\gamma \in THC_{\mu/\lambda}$.

As in Theorem 1, the product is taken in order from $r = 1$ to k since the order of the functions matters in NSym.

Loehr and Niese recently published a combinatorial interpretation of the noncommutative inverse Kostka matrix [18]. Their approach uses transitive tournaments and recursively defined sums, which is quite different from our computationally expedient diagrammatic approach.

In addition to the tournament approach, Loehr and Niese also provide a diagrammatic method for computing the decomposition of an immaculate into the complete homogeneous basis when the indexing shape is a partition. Our diagrammatic approach works for all indexing shapes, including all compositions and also all sequences of integers. Our decomposition can be determined directly by looking at the diagram and recording the values of the tunnel hooks. While our formula is not cancellation-free, neither is the Eğecioğlu-Remmel formula. In fact the cancellations are an artifice of the fact that the Jacobi-Trudi formula is not cancellation-free. However, when we restrict to partition shapes, no cancellations appear in our formula.

In Section 2 we review important definitions and properties concerning the rings Sym, NSym, and QSym. In Section 3 we generalize Ferrers diagrams to provide what we call *GBPR diagrams* for sequences of integers skewed by partitions. We also define tunnel hooks and tunnel hook coverings, the main objects involved in our combinatorial formulas. Section 4 contains the proof of Theorem 1. In Section 5 we use determinants of submatrices to generalize Theorem 1 to immaculate functions indexed by sequences skewed by partitions. We then extend this further to sequences skewed by arbitrary sequences and describe the relationship between special rim hooks [10] and tunnel hooks. In Section 6 we apply Theorem 1 to extend results of Campbell [8] regarding the ribbon decompositions of immaculate functions. Section 7 describes several interesting applications and future directions. An abbreviated version of this work appeared as an extended abstract for the conference proceedings of the 35th International Conference on Formal Power Series and Algebraic Combinatorics [4].

2 Foundation and background

We begin by reviewing necessary material to motivate our work. There are a number of excellent sources [5, 8, 12, 20, 23] for further background in this area.

2.1 Classical combinatorial notions

A *sequence* μ of length k (denoted $\ell(\mu)$) consists of a k-tuple of integers $(\mu_1, \mu_2, \ldots, \mu_k)$ where $\mu_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant k$. A *weak composition* is a sequence whose entries are all non-negative integers. A *strong composition* is a weak composition all of whose positive entries appear before any zeros. Often, we simply use the term *composition* to refer to a strong composition. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a composition in which $0 \leqslant \lambda_{i+1} \leqslant \lambda_i$ for $1 \leqslant i \leqslant k-1$. Note that we allow trailing zeros in our partition definition for ease of proofs in this article. We use $\mu \models n$ to denote that μ is a composition of n and $\lambda \vdash n$ to denote that λ is a partition of n. Partitions are often represented by Ferrers diagrams in which (using French notation) there are λ_i boxes (called *cells*) in row i, where the rows are read from the bottom (south) to the top (north).

A *skew shape* is a pair of partitions λ/ν such that $\nu_i \leqslant \lambda_i$ for $1 \leqslant i \leqslant k$. The *diagram* of a skew shape λ/ν, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$, has λ_i cells in row i with the left-most ν_i cells shaded out. A cell in row i and column j is indicated by (i, j). A partition λ is identified with the skew shape $\lambda/\emptyset$, where the empty partition (denoted $\emptyset$) is identified with a sequence of zeros.

Example 3. The diagram of shape

$$\lambda/\nu = (4, 3, 2, 2, 1)/(2, 2, 1, 0, 0)$$

is given by the labelled cells (row, column) in the following figure.

$5,1$			
$4,1$	$4,2$		
	$3,2$		
		$2,3$	
		$1,3$	$1,4$

Note that throughout this paper, we typically use μ and ν to denote sequences (which, depending on the context, includes compositions, weak compositions, and partitions), while λ is used for partitions.

A *semi-standard Young tableau* ($SSYT$) of shape λ/ν (where λ and ν are partitions) is a filling of the non-shaded cells of the diagram with positive integers, so that the entries weakly increase along rows from left to right (west to east) and strictly increase up columns from bottom to top (south to north). The set of all semi-standard Young tableaux of shape λ/ν is denoted by $SSYT(\lambda/\nu)$. The *weight* x^T of a semi-standard Young tableau T is given by

$$x^T = \prod x_i^{\#\text{ of times } i \text{ appears in } T}.$$

In the following definition we suppress the variable set, which can either be finite $\{x_1, x_2, \ldots, x_n\}$ or infinite $\{x_1, x_2, \ldots\}$. In the finite case $SSYT(\lambda/\nu)$ includes all semi-standard Young tableaux with entries in $\{1, 2, \ldots, n\}$. In the infinite case we allow all positive integers to appear as entries in the semi-standard Young tableaux in $SSYT(\lambda/\nu)$.

Definition 4 (**Schur functions**)**.** The *Schur function* $s_{\lambda/\nu}$ is defined by

$$s_{\lambda/\nu} = \sum_{T \in SSYT(\lambda/\nu)} x^T.$$

2.2 The symmetric group and symmetric functions

A permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$, written in one-line notation where S_n denotes the symmetric group, acts on a formal power series $f(x_1, x_2, \ldots, x_n)$ in n variables by

$$\sigma(f(x_1, \ldots, x_n)) = f(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}).$$

The formal power series $f(x_1, x_2, \ldots, x_n)$ in n variables is said to be symmetric if

$$\sigma(f(x_1, x_2, \ldots, x_n)) = f(x_1, x_2, \ldots, x_n)$$

for all $\sigma \in S_n$. The ring of symmetric functions in n variables, denoted by Sym_n, consists of all symmetric formal power series $f(x_1, x_2, \ldots, x_n)$ on commuting variables $x_1, x_2, \ldots, x_n$ with coefficients from a field $\mathbf{K}$. This notion can be extended to symmetric functions in infinitely many variables; a formal power series $f(x_1, x_2, \ldots)$ is in Sym if

$$\sigma(f(x_1, x_2, \ldots)) = f(x_1, x_2, \ldots)$$

for every permutation σ of the positive integers. Bases for Sym are indexed by partitions.

Schur functions form an important basis for Sym since they correspond to characters of irreducible representations of the symmetric group and also to the cohomology of the Grassmannian [11]. Skew Schur functions generalize the Schur functions to pairs of indexing partitions while enjoying many properties similar to those of the Schur functions indexed by partitions [20]. The collection of skew Schur functions do not form a basis for Sym but each skew Schur function expands into a positive sum of Schur functions.

Two other useful bases for Sym are the *complete homogeneous symmetric functions* h_λ and the *elementary symmetric functions* e_λ where $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition. One way to define h_λ is to set $h_n = s_n$ (for $n \in \mathbb{Z}^+$) and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$. Similarly, set $e_n = s_{1^n}$ (for $n \in \mathbb{Z}^+$) and $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. The *Jacobi-Trudi Formula* provides a rule for expanding the Schur functions $s_{\lambda/\nu}$ into the complete homogeneous symmetric functions.

Theorem 5 (Jacobi-Trudi Formula)**.** *[20] Let λ/ν be a skew shape. Then*

$$s_{\lambda/\nu} = \det(h_{\lambda_i - i - (\nu_j - j)})_{i,j}. \tag{3}$$

The inverse Kostka matrix is the transition matrix from the Schur function basis to the complete homogeneous symmetric function basis. Eğecioğlu and Remmel's combinatorial interpretation of the inverse Kostka matrix [10] provides a method for writing Schur functions in terms of complete homogeneous symmetric functions using decompositions of the indexing shapes into collections of cells called *special rim hooks*. A *rim hook tabloid* is constructed by repeated removal of special rim hooks from the diagram of a partition. Each rim hook tabloid has an associated type and a sign, and these are used to determine the coefficients appearing in the expansion of Schur functions into the complete homogeneous symmetric functions. (See Section 5.4 for further details.)

2.3 Quasisymmetric and noncommutative symmetric functions

Quasisymmetric functions generalize symmetric functions. They first appear in the work of Stanley [24] and are formally developed by Gessel [13]. A polynomial (or more generally a bounded degree formal power series on an infinite alphabet $x_1, x_2, \ldots$) f is *quasisymmetric* if the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ in f is equal to the coefficient of $x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}$ in f for any sequence $j_1 < j_2 < \ldots < j_k$ of positive integers and any composition $(\alpha_1, \ldots, \alpha_k)$. Quasisymmetric functions form the terminal object in the category of combinatorial Hopf algebras [1].

Bases for the algebra QSym of quasisymmetric functions over a field $\mathbf{K}$ are indexed by compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. The *monomial quasisymmetric function* M_α is obtained by taking the monomial $x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ and "quasisymmetrizing" it. That is,

$$M_\alpha = \sum_{j_1 < j_2 < \cdots < j_k} x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}.$$

The *fundamental quasisymmetric functions* are a basis for QSym that can be defined in terms of monomial quasisymmetric functions. In particular,

$$F_\alpha = \sum_{\beta \succeq \alpha} M_\beta,$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_j)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, and $\succeq$ is the refinement order. (Recall $\beta \succeq \alpha$ means that there exists a sequence $(g_1, \ldots, g_j)$ with $g_m < g_{m+1}$ for $1 \leqslant m \leqslant j-1$ such that $\beta_i = \alpha_{g_{i-1}+1} + \cdots + \alpha_{g_i}$ for $1 \leqslant i \leqslant j$ with $g_0 = 0$.)

Each Schur function decomposes into a positive sum of fundamental quasisymmetric functions based on a statistic called the *descent set* [25]. Fundamental quasisymmetric functions correspond to characters of irreducible representations of the 0-Hecke algebra [9].

The Hopf algebra NSym is the graded dual of QSym [12]. Recall NSym can be thought of as the free associative algebra $\mathbf{K}\langle H_1, H_2, \ldots \rangle$ generated by algebraically independent, noncommuting complete homogeneneous symmetric functions H_i over a fixed commutative field $\mathbf{K}$ of characteristic zero. Set $H_a := 0$ if a is a negative integer and $H_0 := 1$. Then $H_{(\alpha_1, \alpha_2, \ldots, \alpha_k)} = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_k}$ for $\alpha_i \in \mathbb{Z}$. The collection of complete homogeneous noncommutative symmetric functions indexed by compositions forms a basis for NSym. There is a pairing $\langle \cdot, \cdot \rangle : \mathrm{NSym} \times \mathrm{QSym} \to \mathbf{K}$ defined by $\langle H_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$, where $\delta_{\alpha,\beta}$ is the Kronecker delta. In Section 6, we will work with the *ribbon basis*, which is a basis for NSym dual to the fundamental basis for QSym under this pairing.

2.4 Creation operators and immaculate functions

The Schur function basis for Sym can be defined using creation operators. Let $\mathbf{B}_m : \mathrm{Sym}_n \to \mathrm{Sym}_{m+n}$ be the *Bernstein operator* [27] defined by

$$\mathbf{B}_m := \sum_{i \geqslant 0} (-1)^i h_{m+i} e_i^{\perp},$$

in which $e_i^{\perp}$ is the adjoint operator defined by

$$\langle e_i g, h \rangle = \langle g, e_i^{\perp} h \rangle \quad \text{for all } g, h \in \mathrm{Sym},$$

where $\langle , \rangle$ is the Hall inner product $\langle \cdot, \cdot \rangle$ on Sym defined by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$. The Schur functions are orthonormal under this product, i.e. $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.

For any sequence $\mu \in \mathbb{Z}^k$, the Schur function s_μ can be constructed by repeated application of Bernstein operators [27]; specifically,

$$s_\mu = \mathbf{B}_{\mu_1} \mathbf{B}_{\mu_2} \cdots \mathbf{B}_{\mu_k}(1).$$

These creation operators can be extended to NSym by replacing h_{m+i} with the complete homogeneous non-commutative symmetric function H_{m+i} and replacing $e_i^{\perp}$ with the linear transformation $F_{1^i}^{\perp}$ of NSym that is adjoint to multiplication by the fundamental F_{1^i} in QSym. This construction leads to the following definition.

Definition 6 (noncommutative Bernstein operators [5]). The *noncommutative Bernstein operator* $\mathbb{B}_m$ is given by

$$\mathbb{B}_m = \sum_{i \geqslant 0} (-1)^i H_{m+i} F_{1^i}^{\perp}.$$

The immaculate functions $\mathfrak{S}_\mu$ are then defined analogously to the Schur functions by the following application of creation operators.

Definition 7 (immaculate noncommutative symmetric functions [5]). The *immaculate noncommutative symmetric function* $\mathfrak{S}_\mu$ is defined by

$$\mathfrak{S}_\mu = \mathbb{B}_{\mu_1} \cdots \mathbb{B}_{\mu_k}(1).$$

Immaculate functions correspond to indecomposable modules of the 0-Hecke algebra [6]. The following NSym version of the Jacobi-Trudi Theorem provides a determinantal formula for the expansion of the immaculate functions $\mathfrak{S}_\mu$ into the complete homogeneous basis for NSym.

Theorem 8 ([5]). *For an arbitrary integer sequence μ, let $\mathbb{M}_\mu$ be the matrix given by $(\mathbb{M}_\mu)_{i,j} = H_{\mu_i - i + j}$. Then $\mathfrak{S}_\mu = \mathfrak{det}(\mathbb{M}_\mu)$.*

The above formula for $\mathfrak{S}_\mu$ requires that the determinant be computed using Laplace expansion starting with the top row and then continuing down to the bottom row. We will use $\mathfrak{det}(\mathbb{M}_\mu)$ instead of $\det(\mathbb{M}_\mu)$ to represent this strict expansion process.

Example 9. If $\mu = (-1, 3, 2)$, then

$$\mathbb{M}_\mu = \begin{bmatrix} H_{-1} & H_0 & H_1 \\ H_2 & H_3 & H_4 \\ H_0 & H_1 & H_2 \end{bmatrix}.$$

With $H_a = 0$ if $a \in \mathbb{Z}^{<0}$ and $H_0 = 1$, the decomposition of $\mathfrak{S}_\mu$ into the complete homogeneous basis for NSym is given by

$$\begin{aligned} \mathfrak{S}_\mu &= H_{-1}(H_3H_2 - H_4H_1) - H_0(H_2H_2 - H_4H_0) + H_1(H_2H_1 - H_3H_0) \\ &= -H_{(2,2)} + H_{(4)} + H_{(1,2,1)} - H_{(1,3)}. \end{aligned}$$

3 GBPR diagrams and tunnel cells

Skew diagrams, as traditionally defined, do not extend to the full generality necessary for our work. For our diagrams, we will need to consider shapes μ/ν where μ is an integer sequence (which in particular might contain negative parts) and ν is a partition. Allowing μ to have negative parts requires modifying the typical diagram visualization.

For all of Section 3, let $\mu = (\mu_1, \ldots, \mu_k)$ be an integer sequence and $\nu = (\nu_1, \ldots, \nu_k)$ be a partition. The *Grey-Blue-Purple-Red (GBPR) diagram $D_{\mu/\nu}$* for the skew shape μ/ν is obtained via the following process.

1. Place ν_i grey cells in row i of the diagram (for $1 \leqslant i \leqslant k$), working from bottom to top (to stay consistent with French notation).
2. For each $1 \leqslant i \leqslant k$, there are several cases that impact the placement of red and blue cells into row i.
   (a) If $\mu_i > 0$ and $\nu_i \leqslant \mu_i$, place $\mu_i - \nu_i$ blue cells in row i situated immediately to the right of the grey cells.
   (b) If $\mu_i > 0$ and $\mu_i < \nu_i$, place $\nu_i - \mu_i$ red cells in row i situated immediately to the right of the grey cells.
   (c) If $\mu_i \leqslant 0$, place $|\mu_i| + \nu_i$ red cells in row i situated immediately to the right of the grey cells.

3. Any cell in the first quadrant that is not colored grey, red, or blue is *purple*, but we do not typically color these in illustrations since there are infinitely many purple cells. Purple cells are considered part of the diagram and are available to be claimed (converted to grey or red) as necessary in the following applications. Often, we will draw a purple cell in cell location $(i, \nu_i + 1)$ when the row does not have any blue or red cells to emphasize that the the cell is available for claiming.

Although the GBPR diagram does not necessarily contain exactly μ_i cells in row i, the value of μ_i can be determined from the number of grey, blue, and red cells in the following way.

Lemma 10. *If row i of a GBPR diagram has a_i grey cells, b_i blue cells, and c_i red cells, then $a_i + b_i - c_i = \mu_i$.*

Proof. To see this, consider each of the cases in Step 2 of the GBPR diagram construction. In all cases, $a_i = \nu_i$.

In case (a), $b_i = \mu_i - \nu_i$ and $c_i = 0$, so

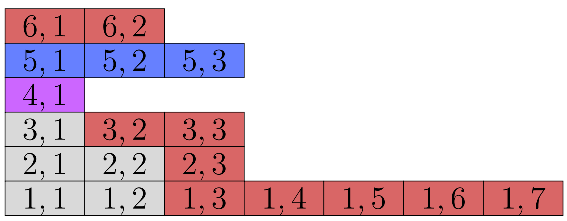
$$a_i + b_i - c_i = \nu_i + \mu_i - \nu_i - 0 = \mu_i.$$

In cases (b) and (c), $b_i = 0$ and $c_i = \nu_i - \mu_i$, so

$$a_i + b_i - c_i = \nu_i + 0 - (\nu_i - \mu_i) = \mu_i. \quad \square$$

Let $C_{\mu/\nu}$ denote the collection of red and blue cells in a diagram $D_{\mu/\nu}$.

Example 11. The following is the diagram $D_{\mu/\nu}$ where $\mu = (-3, 1, -1, 0, 3, -2)$ and $\nu = (2, 2, 1, 0, 0, 0)$ with the corresponding location indicated in each cell, respectively.



Note

$$C_\nu = \{(1,1), (1,2), (2,1), (2,2), (3,1)\}$$

and

$$C_{\mu/\nu} = \{(1,3), (1,4), (1,5), (1,6), (1,7), (2,3), (3,2), (3,3), (5,1), \\ (5,2), (5,3), (6,1), (6,2)\}.$$

Definition 12 (adjacent, connected, and diagonally adjoining cells). Two cells (p, q) and (s, t) are said to be adjacent if and only if $|p - s| + |q - t| = 1$. We say that a collection C of cells is *connected* if and only if for any cells $c, d \in C$ there is a sequence $c = u_1, u_2, \ldots, u_j = d$ where $u_i \in C$ and u_i and u_{i+1} are adjacent for $1 \leqslant i \leqslant j - 1$. A set consisting of just one cell is also considered to be connected. Cells (p, q) and (s, t) are *diagonally adjoining* if both $s = p + 1$ and $t = q + 1$ or both $p = s + 1$ and $q = t + 1$.

It is often convenient to construct diagrams for shapes obtained via prefix removal. In particular, at times we encounter a pair of sequences $\mu \in \mathbb{Z}^k$ and $\nu \in \mathbb{Z}^k$ with $\nu_r \geqslant \nu_{r+1} \geqslant \ldots \geqslant \nu_k \geqslant 0$. If the first $r-1$ parts of ν are not weakly decreasing, we can remove them and consider parts r through k of μ and ν, which allows us to still construct a GBPR diagram for the remaining rows. The following definition makes this precise. We introduce a superscript $(r-1)$ (rather than r) because we will be constructing objects indexed by r directly on the partial diagram with the first $r-1$ rows removed.

Definition 13 (**partial diagrams $\mathbf{D^{(r-1)}_{\mu/\nu^{(r-1)}}}$**)**.** Let $r \in \mathbb{Z}^{>0}, \mu \in \mathbb{Z}^k$, and $\nu^{(r-1)} \in \mathbb{Z}^k$ such that $\nu^{(r-1)}_r \geqslant \nu^{(r-1)}_{r+1} \geqslant \ldots \geqslant \nu^{(r-1)}_k \geqslant 0$. The *partial diagram* $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ is obtained by constructing the GBPR diagram $D_{(\mu_r,\mu_{r+1},\ldots,\mu_k)/(\nu^{(r-1)}_r,\nu^{(r-1)}_{r+1},\ldots,\nu^{(r-1)}_k)}$ and then shifting the resulting diagram up by $r-1$ rows, so that the first nonempty row is in row r of the plane.

The row labels for a shifted diagram correspond to their placement in the xy-plane. Note that any GBPR diagram $D_{\mu/\nu}$ can be considered as a partial diagram by setting $r = 1$ and $\nu^{(r-1)} = \nu^{(0)} = \nu$; that is,

$$D_{\mu/\nu} = D^{(0)}_{\mu/\nu^{(0)}}.$$

Example 14. With $D_{\mu/\nu}$ as in in Example 11, the partial diagram $D^{(2)}_{\mu/\nu^{(2)}}$ (with $\nu^{(2)} = \nu$ and $r = 3$) is depicted below.

$6,1$	$6,2$	
$5,1$	$5,2$	$5,3$
$4,1$		
$3,1$	$3,2$	$3,3$

Intuitively, a boundary cell is any cell in the partial diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ outside of $\nu^{(r-1)}$ that is either horizontally or vertically adjacent to at least one cell in $\nu^{(r-1)}$, or diagonally adjoining a cell in $\nu^{(r-1)}$ (as in Definition 12). Any red or blue cell in row r of the plane in the partial diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ is also a boundary cell. Boundary cells will be used to construct tunnel hooks that will play a role similar to the role of *rim hooks* in the combinatorial interpretation of the symmetric inverse Kostka matrix [10].

Definition 15 (**boundary and tunnel cells**)**.** Let $\mu \in \mathbb{Z}^k$ and let r be a positive integer such that $1 \leqslant r \leqslant k$. Let $\nu^{(r-1)} \in \mathbb{Z}^k$ such that $\nu^{(r-1)}_r \geqslant \nu^{(r-1)}_{r+1} \geqslant \cdots \geqslant \nu^{(r-1)}_k \geqslant 0$. A cell in location (p, q) (with $r \leqslant p \leqslant k$) is a *boundary cell* of $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ if and only if

$$\begin{cases} \nu^{(r-1)}_p + 1 \leqslant q \leqslant \nu^{(r-1)}_{p-1} + 1 & \text{if } p > r \\ \nu^{(r-1)}_r + 1 \leqslant q \leqslant \max\{\nu^{(r-1)}_r + 1, a_r + b_r + c_r\} & \text{if } p = r, \end{cases}$$

where a_r, b_r and c_r are the number of grey cells, blue cells and red cells in row r, respectively. A *tunnel cell* (p, q) of $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ is a boundary cell such that $q = \nu^{(r-1)}_p + 1$.

Note that boundary cells are not necessarily cells of $C_{\mu/\nu^{(r-1)}}$; boundary cells may be red, blue, or purple. The inequality $\nu_r^{(r-1)} + 1 \leqslant q \leqslant \max\{\nu_r^{(r-1)} + 1, a_r + b_r + c_r\}$ for row r forces the cell $(r, \nu_r^{(r-1)} + 1)$ as well as all red or blue cells in row r to be boundary cells. Also note that a cell (p, q) with $p < r$ cannot be a boundary cell of $D^{(r-1)}_{\mu/\nu^{(r-1)}}$.

Let $\mathcal{B}^{(r-1)}_{\mu/\nu^{(r-1)}}$ and $\mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$ denote the collections of boundary cells and tunnel cells of $D^{(r-1)}_{\mu/\nu^{(r-1)}}$, respectively. Let $\mathcal{N}^{(r-1)}_{\mu/\nu^{(r-1)}}$ be the set difference

$$\mathcal{N}^{(r-1)}_{\mu/\nu^{(r-1)}} := \mathcal{B}^{(r-1)}_{\mu/\nu^{(r-1)}} \setminus \mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}. \tag{4}$$

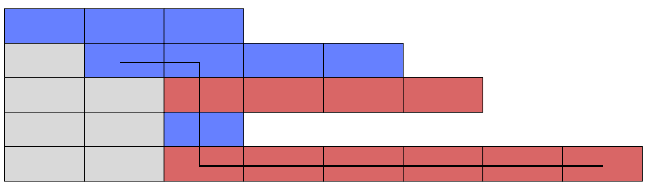
We are now ready to define our combinatorial objects called *tunnel hooks*. A tunnel hook originating in row r of $\mathcal{D}_\mu$ will be constructed on the partial diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$. Therefore in general the superscripts on our partial diagrams will typically be $r-1$ (as above) whereas the tunnel hooks will be indexed by r.

Definition 16 (tunnel hooks, terminal cells, signs, coverings). Let $\tau = (p, q) \in \mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$ be a tunnel cell. A *tunnel hook* on $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ is a collection $\mathfrak{h}(r, \tau)$ consisting of all boundary cells in rows r through p. The cell τ (the farthest northwest cell of $\mathfrak{h}(r, \tau)$) is called the *terminal cell* of $\mathfrak{h}(r, \tau)$ and we say this tunnel hook *terminates* in row p. The *sign* of tunnel hook $\mathfrak{h}(r, \tau)$, denoted by $\epsilon(\mathfrak{h}(r, \tau))$, equals $(-1)^{p-r}$. If cell $\eta \in \mathfrak{h}(r, \tau)$ then we say that $\mathfrak{h}(r, \tau)$ *covers* η. The southeasternmost cell in $\mathfrak{h}(r, \tau_r)$ is called the *initial cell* and we say that $\mathfrak{h}(r, \tau_r)$ *starts* in row r.

There are a number of hook/strip objects in the literature such as skew hooks, ribbons, and border strips [20], and the rim hooks [10] appearing in the combinatorial interpretation of the inverse Kostka matrix in Sym. In order to generalize these combinatorial objects to the NSym setting, tunnel hooks need to "tunnel" into the diagram instead of remaining on the rim. See Section 5.4 for a detailed comparison of rim hooks and tunnel hooks.

Example 17. Let $\mu = (5, 4, -4, 3, -2, 5, 3)$, $\nu^{(2)} = (5, 4, 2, 2, 2, 1, 0)$, and $r - 1 = 2$. Since $(\nu_3^{(2)}, \nu_4^{(2)}, \ldots, \nu_7^{(2)})$ is a partition, we have the following GBPR diagram $D^{(2)}_{\mu/\nu^{(2)}}$ in which we have selected a tunnel hook

$$\mathfrak{h}(3, (6, 2)) = \{(3, 8), (3, 7), (3, 6), (3, 5), (3, 4), (3, 3),$$
$$(4, 3), (5, 3), (6, 3), (6, 2)\}.$$



The boundary cells in the above diagram are

$$\mathcal{B}^{(2)}_{\mu/\nu^{(2)}} = \{(3, 3), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8),$$

$$(4,3),(5,3),(6,2),(6,3),(7,1),(7,2)\},$$

the tunnel cells are

$$\mathcal{T}^{(2)}_{\mu/\nu^{(2)}} = \{(3,3),(4,3),(5,3),(6,2),(7,1)\},$$

and

$$\mathcal{N}^{(2)}_{\mu/\nu^{(2)}} = \{(3,4),(3,5),(3,6),(3,7),(3,8),(6,3),(7,2)\}.$$

The following lemmas will be useful in later proofs.

Lemma 18. *If $(p,q) \in \mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$, then $(p+1,q+1) \notin \mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$.*

Proof. Let $(p,q) \in \mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$. Then $\nu^{(r)}_p + 1 \leqslant q$ by Definition 15. Adding 1 to both sides implies that $\nu^{(r)}_p + 2 \leqslant q+1$. Then $\nu^{(r)}_p + 1 < \nu^{(r)}_p + 2 \leqslant q+1$ contradicts the inequality $q+1 \leqslant \nu^{(r)}_p + 1$ necessary for $(p+1,q+1) \in \mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$. Therefore, $(p+1,q+1) \notin \mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$ and the proof is complete. ☐

Note that Lemma 18 implies the set of all boundary cells does not contain any 2×2 rectangles. Another consequence of Lemma 18 is that for any cell (p,q) contained in a tunnel hook in the diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$, the cell $(p+1,q+1)$ is not a boundary cell in $D^{(r-1)}_{\mu/\nu^{(r-1)}}$. The following lemma shows that every tunnel hook is connected.

Lemma 19. *Let $\mathfrak{h}(r,\tau)$ be a tunnel hook in $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ terminating at cell $\tau = (p,q)$. Then $\mathfrak{h}(r,\tau)$ is connected.*

Proof. For $r < i < p$, the boundary cells Y_i and Y_{i+1} of rows i and $i+1$ are, respectively,

$$Y_i = \{(i, \nu^{(r-1)}_i + 1), (i, \nu^{(r-1)}_i + 2), \ldots, (i, \nu^{(r-1)}_{i-1} + 1)\}$$

and

$$Y_{i+1} = \{(i+1, \nu^{(r-1)}_{i+1} + 1), (i+1, \nu^{(r-1)}_{i+1} + 2), \ldots, (i+1, \nu^{(r-1)}_{i} + 1)\}.$$

Each of these collections is connected. Since $(i, \nu^{(r-1)}_i + 1)$ and $(i+1, \nu^{(r-1)}_i + 1)$ are connected, the union $Y_i \cup Y_{i+1}$ is connected, and hence the collection $\mathfrak{h}(r,\tau)$ is connected as long as the collection Y_r of boundary cells in row r is connected to the boundary cells Y_{r+1} in row $r+1$. Since $(r, \nu^{(r-1)}_r + 1) \in Y_r$ and $(r+1, \nu^{(r-1)}_r + 1) \in Y_{r+1}$, the union $Y_r \cup Y_{r+1}$ is connected and the proof is complete. ☐

Definition 20 (bank and taxi indices). Given a row i of $D^{(r-1)}_{\mu/\nu^{(r-1)}}$, define

$$bank_{\mu/\nu^{(r-1)}}(i) = (\text{\# of blue cells in row } i) - (\text{\# of red cells in row } i).$$

With $\mathfrak{h}(r,\tau)$ a tunnel hook in the partial diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ terminating at cell $\tau=(p,q)$, set

$$\Delta(\mathfrak{h}(r,\tau)) = bank_{\mu/\nu^{(r-1)}}(r) + (\nu_r^{(r-1)} + 1 - q) + (p-r). \tag{5}$$

Note that $(\nu_r^{(r-1)} + 1 - q) + (p-r)$ is the *taxicab* (or *Manhattan*) distance from the cell $(r, \nu_r^{(r-1)} + 1)$ to the cell $\tau = (p,q)$ [16]. Therefore, set

$$taxi(\mathfrak{h}(r,\tau)) = (\nu_r^{(r-1)} + 1 - q) + (p-r). \tag{6}$$

In Equation (5), since $\nu_r^{(r-1)} \geqslant \nu_{r+1}^{(r-1)} \geqslant \cdots \geqslant \nu_k^{(r-1)} \geqslant 0$, the cell $(p,q) \in \mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$ will be weakly west of $(r, \nu_r^{(r-1)} + 1)$.

Example 21. Consider the partial diagram $D^{(2)}_{\mu/\nu^{(2)}}$ depicted in Example 17. The tunnel hook shown has $\Delta(\mathfrak{h}(3,(6,2))) = -6+4 = -2$, since there are six red cells in row 3 and $taxi(\mathfrak{h}(3,(6,2))) = 4$. Similarly (but not shown), $\Delta(\mathfrak{h}(3,(7,1))) = -6+6 = 0$, $\Delta(\mathfrak{h}(3,(5,3))) = -6+2 = -4$, $\Delta(\mathfrak{h}(3,(4,3))) = -6+1 = -5$, and $\Delta(\mathfrak{h}(3,(3,3))) = -6+0 = -6$.

The proof of the following lemma is immediate since there is only one tunnel cell in each row and moving up a row strictly increases the value of $\Delta(\mathfrak{h}(r,\tau))$.

Lemma 22. *Given a partial diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ with μ a sequence and ν a partition, for any fixed $j \in \mathbb{Z}$, there is at most one tunnel cell τ such that $j = \Delta(\mathfrak{h}(r,\tau))$.*

The following iterative procedure provides a method for constructing a *tunnel hook covering* (THC) of the diagram $D_{\mu/\nu^{(0)}}$.

Procedure 23 (tunnel hook covering construction). *Consider a sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathbb{Z}^k$ and a partition $\nu^{(0)} = (\nu_1^{(0)}, \nu_2^{(0)}, \ldots, \nu_k^{(0)})$.*

1. *Construct the partial GBPR diagram $D^{(0)}_{\mu/\nu^{(0)}}$ of shape $\mu/\nu^{(0)}$.*
2. *Repeat the following steps, once for each value of r from 1 to k.*
   (a) *Choose a tunnel hook $\mathfrak{h}(r,\tau_r)$ in $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ and set*
   $$\Delta_r = \Delta(\mathfrak{h}(r,\tau_r)).$$
   (b) *For each $1 \leqslant i \leqslant k$, let $\eta_i^{(r)}$ be the number of cells in row i of $\mathfrak{h}(r,\tau_r)$ and let $\nu^{(r)}$ be the sequence defined for $1 \leqslant i \leqslant k$ by*
   $$\nu_i^{(r)} = \nu_i^{(r-1)} + \eta_i^{(r)}.$$
   (c) *Construct the partial GBPR diagram $D^{(r)}_{\mu/\nu^{(r)}}$.*

3. *Let γ denote the resulting tunnel hook covering and define the* sequence associated to the tunnel hook covering γ *by* $\Delta(\gamma) = (\Delta_1, \ldots, \Delta_k)$.

Step 2b appends the cells covered by $\mathfrak{h}(r,\tau)$ to $\nu^{(r-1)}$. Although $\nu^{(r)}$ is not necessarily a partition, we will prove in Lemma 24 that $(\nu^{(r)}_{r+1}, \nu^{(r)}_{r+2}, \ldots, \nu^{(r)}_k)$ is a partition, as is necessary in order to construct the partial diagram $D^{(r)}_{\mu/\nu^{(r)}}$.

In the diagram $D^{(r)}_{\mu/\nu^{(r)}}$ constructed in Step 2c, all the cells in $\nu^{(r)}$ become grey. Additional red cells are appended to the remaining rows in Step 2c. One additional red cell is appended to row i for every purple cell covered by $\mathfrak{h}(r,\tau)$ in row i and two additional red cells are appended to row i for every red cell covered by $\mathfrak{h}(r,\tau)$ in row i.

Lemma 24. *Let $\mu \in \mathbb{Z}^k$ be an integer sequence and $\nu^{(r)}$ be a sequence of nonnegative integers produced during Step 2b of Procedure 23. Then $(\nu^{(r)}_{r+1}, \nu^{(r)}_{r+2}, \ldots, \nu^{(r)}_k)$ is a partition.*

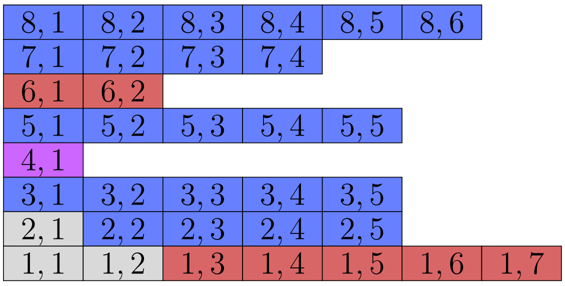
Proof. First note that $\nu^{(0)}$ is assumed to be a partition, possibly the empty partition. Therefore the base case is true. Assume that $(\nu^{(j)}_{j+1}, \nu^{(j)}_{j+2}, \ldots, \nu^{(j)}_k)$ is a partition for $0 \leqslant j \leqslant r-1$ and let $\nu^{(r)}$ be a sequence of nonnegative integers produced during Step 2b of Procedure 23 for the partial GBPR diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$. Let $\mathfrak{h}(r,\tau_r)$ be the tunnel hook constructed during this process. We will prove that $\nu^{(r)}_i \geqslant \nu^{(r)}_{i+1}$ for $r+1 \leqslant i < k$.

First note that since $(\nu^{(r-1)}_r, \nu^{(r-1)}_{r+1}, \ldots, \nu^{(r-1)}_k)$ is a partition by assumption, $\nu^{(r-1)}_i \geqslant \nu^{(r-1)}_{i+1}$ for $r \leqslant i < k$. Assume $r < i < k$ and notice this means we don't need to consider cells from row r, which eliminates the need to consider the second inequality in the boundary cell definition (Definition 15). If $\mathfrak{h}(r,\tau_r)$ does not include cells from row $i+1$, then $\nu^{(r)}_{i+1} = \nu^{(r-1)}_{i+1} \leqslant \nu^{(r-1)}_i \leqslant \nu^{(r)}_i$ so that $\nu^{(r)}_{i+1} \leqslant \nu^{(r)}_i$, as desired.

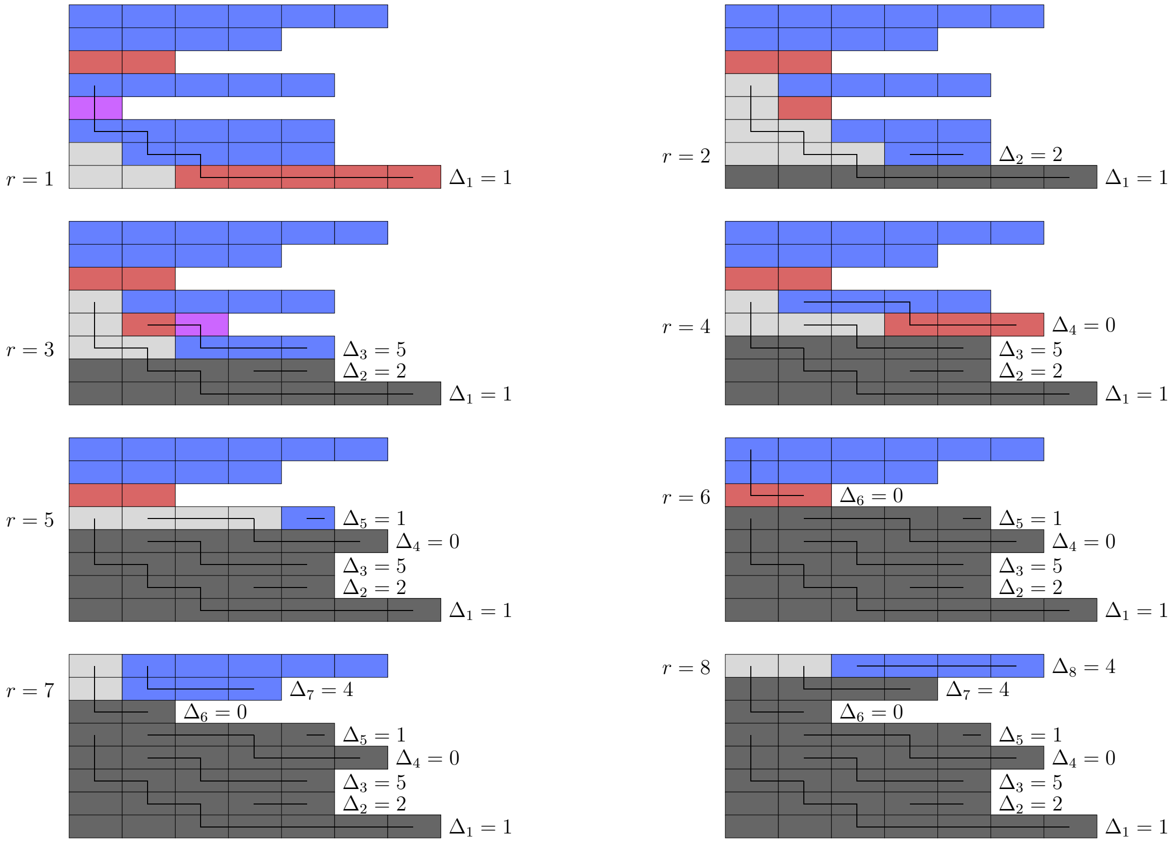
If $\mathfrak{h}(r,\tau_r)$ includes cells from row $i+1$, it must also include cells from row i. Recall the definition of tunnel hooks states that every boundary cell from an included row must be contained in $\mathfrak{h}(r,\tau_r)$. The largest column index for a boundary cell in row $i+1$ is $\nu^{(r-1)}_{i+1-1} + 1 = \nu^{(r-1)}_i + 1$, whereas the largest column index for a boundary cell in row i is $\nu^{(r-1)}_{i-1} + 1$. Since $\nu^{(r-1)}_i + 1 \leqslant \nu^{(r-1)}_{i-1} + 1$, the largest column index for a boundary cell in row $i+1$ is weakly smaller than the largest column index for a boundary cell in row i. Therefore, $\nu^{(r)}_{i+1} \leqslant \nu^{(r)}_i$, and the proof is complete. ☐

Example 25. We describe a tunnel hook covering of $D_{\mu/\nu^{(0)}}$ for the shape $\mu/\nu^{(0)} = (-3, 5, 5, 0, 5, -2, 4, 6)/(2, 1)$. First, we give the GBPR diagram of $D^{(0)}_{\mu/\nu^{(0)}}$. We then provide a series of diagrams detailing the construction of the individual tunnel hooks of a particular tunnel hook covering, summarizing this construction in a table. Finally, we illustrate this particular THC on the GBPR diagram itself.

The GBPR diagram of $D^{(0)}_{\mu/\nu^{(0)}}$ is shown below, with each cell labelled by its corresponding row and column.



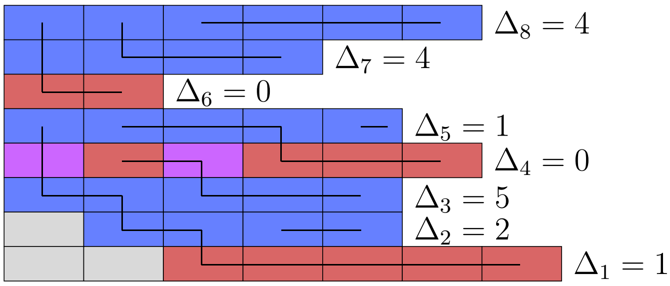
The following series of diagrams illustrates the construction of a particular tunnel hook covering. The dark grey indicates the rows that are removed to create the partial diagrams.



The following table records the process of decomposing the GBPR diagram $D_{\mu/\nu^{(0)}}$ into tunnel hooks. Each row r indicates the situation before the r^{th} tunnel hook is placed. Here τ_r is the tunnel cell at which the tunnel hook beginning in the r^{th} row of the partial diagram terminates. Notice that although $\nu^{(r-1)}$ is not always a partition, the last $k-r+1$ terms of $\nu^{(r-1)}$ do form a partition, as emphasized in boldface in the table below. Therefore $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ is a valid partial GBPR diagram.

r	$(\mu_r, \mu_{r+1}, \ldots, \mu_k)$	$\nu^{(r-1)}$	τ_r	$\Delta(\mathfrak{h}(r,\tau_r))$
1	$(-3,5,5,0,5,-2,4,6)$	$(\mathbf{2},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$	$(5,1)$	$-5+6=1$
2	$(5,5,0,5,-2,4,6)$	$(7,\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0})$	$(2,4)$	$2+0=2$
3	$(5,0,5,-2,4,6)$	$(7,5,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0})$	$(4,2)$	$3+2=5$
4	$(0,5,-2,4,6)$	$(7,5,5,\mathbf{3},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0})$	$(5,2)$	$-3+3=0$
5	$(5,-2,4,6)$	$(7,5,5,6,\mathbf{4},\mathbf{0},\mathbf{0},\mathbf{0})$	$(5,5)$	$1+0=1$
6	$(-2,4,6)$	$(7,5,5,6,5,\mathbf{0},\mathbf{0},\mathbf{0})$	$(8,1)$	$-2+2=0$
7	$(4,6)$	$(7,5,5,6,5,2,\mathbf{1},\mathbf{1})$	$(8,2)$	$3+1=4$
8	(6)	$(7,5,5,6,5,2,4,\mathbf{2})$	$(8,3)$	$4+0=4$

Next, we illustrate γ, the final THC of $D^{(0)}_{\mu/\nu^{(0)}}$ resulting from the construction above. In this diagram, we depict all the tunnel hooks at once. We omit the step of converting the colors of the cells to grey as they are covered by tunnel hooks, so that their color as they are covered by a tunnel hook is retained.



Therefore $\Delta(\gamma) = (1,2,5,0,1,0,4,4)$.

4 Theorem 1 proof and application to QSym

In this section, we prove Theorem 1. Recall Theorem 8 [5] states that

$$\mathfrak{S}_\mu = \mathfrak{det}(\mathbb{M}_\mu) = \sum_{\sigma \in S_k} \epsilon(\sigma) \prod_{i=1}^{k} (\mathbb{M}_\mu)_{i,\sigma_i}.$$

Theorem 1 provides a combinatorial interpretation of this decomposition of immaculate functions into complete homogeneous functions using diagram fillings called tunnel hook coverings. The main idea behind our proof is a natural bijection between tunnel hook coverings and permutations (Proposition 31). Furthermore, each tunnel hook is associated to a number which we will show is equal to the subscript of the corresponding complete homogeneous function appearing in the matrix $\mathbb{M}_\mu$ (Lemma 37). We complete the proof by attaching signs to tunnel hooks and showing that the product of the signs of the tunnel hooks in a tunnel hook covering is equal to the sign of the corresponding permutation. We close the section with an application to QSym, obtaining a combinatorial formula for the expansion of monomial quasisymmetric functions in terms of dual immaculates as a corollary to Theorem 1.

4.1 A procedure for selecting tunnel cells to construct a tunnel hook covering

The first lemma necessary for the proof of Theorem 1 provides foundational insight into how tunnel cells are related to the diagonals parallel to the line $y = x$. Let

$$\mathcal{L}_j = \{(p,q) | p - q + 1 = j\} = \{(j+m, 1+m) | m \in \mathbb{Z}^{\geqslant 0}\} \tag{7}$$

be the collection of cells in the j^{th} diagonal of the first quadrant of the plane, for $1 \leqslant j \leqslant k$. These diagonals (whose properties are described in the following lemma) will correspond to the entries in the permutations used when computing the determinant of the matrix $\mathbb{M}_\mu$. Recall the definition of boundary cells $\mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$, tunnel cells $\mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$, and $\mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$ found in and immediately following Definition 15.

Lemma 26. *Let $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ be a partial GBPR diagram for $\mu \in \mathbb{Z}^k$ and $\nu^{(r-1)} \in \mathbb{Z}^k$ such that $\nu^{(r-1)}_r \geqslant \nu^{(r-1)}_{r+1} \geqslant \cdots \geqslant \nu^{(r-1)}_k \geqslant 0$. Assume $r < k$. Suppose $\tau_r, \omega \in \mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$, $\tau_r \neq \omega$, and $\xi \in \mathcal{N}^{(r-1)}_{\mu/\nu^{(r-1)}}$. Furthermore, suppose $\tau_r = (p_1, q_1)$, $\omega = (p_2, q_2)$, and $\xi = (p_3, q_3)$. Finally, let $\mathfrak{h}(r, \tau_r)$ be a tunnel hook in the diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$. Then*

A. $(p_1 + 1, q_1 + 1) \in \mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$.

B. If $\mathfrak{h}(r, \tau_r)$ does not cover ω then $\omega \in \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$.

C. If $\mathfrak{h}(r, \tau_r)$ covers ω then $(p_2 + 1, q_2 + 1) \in \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$.

D. If $\mathfrak{h}(r, \tau_r)$ does not cover ξ then $\xi \in \mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$.

E. If $\mathfrak{h}(r, \tau_r)$ covers ξ then either $(p_3 + 1, q_3 + 1) \in \mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$ or $p_3 = k$ in which case $(p_3 + 1, q_3 + 1) \notin D^{(r)}_{\mu/\nu^{(r)}}$.

Proof of part A. First note that $p_1, p_2, p_3 \geqslant r$ since the diagram begins in row r. Since $\tau_r = (p_1, q_1)$ is the terminal cell of $\mathfrak{h}(r, \tau_r)$, we have $\nu^{(r-1)}_{p_1} = q_1 - 1$, $\nu^{(r)}_{p_1} \geqslant q_1$, and $\nu^{(r-1)}_{p_1+1} = \nu^{(r)}_{p_1+1}$. Also, the fact that $(\nu^{(r-1)}_r, \nu^{(r-1)}_{r+1}, \ldots, \nu^{(r-1)}_k)$ is a partition (Lemma 24) implies

$$\nu^{(r)}_{p_1+1} = \nu^{(r-1)}_{p_1+1} \leqslant \nu^{(r-1)}_{p_1} = q_1 - 1.$$

Thus,

$$\nu^{(r)}_{p_1+1} + 1 \leqslant q_1 < q_1 + 1 \leqslant \nu^{(r)}_{p_1} + 1,$$

so that $(p_1 + 1, q_1 + 1) \in \mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$. However, $(p_1 + 1, q_1 + 1) \notin \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$ since $\nu^{(r)}_{p_1+1} \leqslant q_1 - 1$ implies $\nu^{(r)}_{p_1+1} + 1 \neq q_1 + 1$. Therefore, $(p_1 + 1, q_1 + 1) \in \mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$, recalling that $\mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$ is the set difference $\mathcal{B}^{(r)}_{\mu/\nu^{(r)}} \setminus \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$. □

Proof of part B. If $\mathfrak{h}(r, \tau_r)$ does not cover $\omega = (p_2, q_2)$, then $p_2 > p_1$ by the tunnel hook definition. Therefore, $q_2 = \nu^{(r-1)}_{p_2} + 1 = \nu^{(r)}_{p_2} + 1$ so that $\omega \in \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$. □

Proof of part C. If $\mathfrak{h}(r,\tau_r)$ covers $\omega = (p_2, q_2) \in \mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$, then $p_2 < p_1$ since $\tau_r \neq \omega$ and at most one cell of each row of $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ belongs to $\mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$. Therefore, $\mathfrak{h}(r,\tau_r)$ includes all boundary cells from row p_2+1. The tunnel hook $\mathfrak{h}(r,\tau_r)$ does not cover (p_2+1, q_2+1) since $(p_2+1, q_2+1) \notin \mathcal{B}^{(r-1)}_{\mu/\nu^{(r-1)}}$ by Lemma 18. Since $\omega \in \mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$, we have $\nu^{(r-1)}_{p_2} = q_2 - 1$. This means $(p_2, q_2 - 1) \notin \mathfrak{h}(r,\tau_r)$, so that all cells in row p_2 covered by $\mathfrak{h}(r,\tau_r)$ lie weakly to the right of (p_2, q_2). The fact that $q_2 + 1 > \nu^{(r-1)}_{p_2}$ implies the cells in row p_2+1 covered by $\mathfrak{h}(r,\tau_r)$ must lie strictly to the left of (p_2+1, q_2+1). Since $\mathfrak{h}(r,\tau_r)$ is connected by Lemma 19 and includes cells in row p_2+1 (since $p_2 + 1 \leqslant p_1$), the cell (p_2+1, q_2) must be covered by $\mathfrak{h}(r,\tau_r)$. Since (p_2+1, q_2) is covered by $\mathfrak{h}(r,\tau_r)$ but (p_2+1, q_2+1) is not covered by $\mathfrak{h}(r,\tau_r)$, we have $q_2 + 1 = \nu^{(r)}_{p_2+1} + 1$ and $(p_2+1, q_2+1) \in \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$. ☐

Proof of part D. Let $\xi = (p_3, q_3)$ be a boundary cell which is not a tunnel cell in $D^{(r-1)}_{\mu/\nu^{(r-1)}}$; i.e., $\xi \in \mathcal{N}^{(r-1)}_{\mu/\nu^{(r-1)}}$. Assume $\mathfrak{h}(r,\tau_r)$ does not cover ξ. Therefore, $p_1 < p_3$ and $\nu^{(r)}_{p_3} = \nu^{(r-1)}_{p_3}$. So ξ is a boundary cell in $D^{(r)}_{\mu/\nu^{(r)}}$ but not a tunnel cell in $D^{(r)}_{\mu/\nu^{(r)}}$ since otherwise ξ would be a tunnel cell in $D^{(r-1)}_{\mu/\nu^{(r-1)}}$. Therefore, $\xi \in \mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$. ☐

Proof of part E. Let $\xi = (p_3, q_3)$ be a boundary cell which is not a tunnel cell; i.e., $\xi \in \mathcal{N}^{(r-1)}_{\mu/\nu^{(r-1)}}$, and assume $\mathfrak{h}(r,\tau_r)$ covers ξ. Assume $p_3 < k$; otherwise $(p_3+1, q_3+1) \notin D^{(r)}_{\mu/\nu^{(r)}}$. We first prove $(p_3+1, q_3+1) \in \mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$ and then show $(p_3+1, q_3+1) \notin \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$.

The assumption that $\mathfrak{h}(r,\tau_r)$ covers ξ implies $q_3 \leqslant \nu^{(r)}_{p_3}$. Adding 1 to each side of this inequality gives

$$q_3 + 1 \leqslant \nu^{(r)}_{p_3} + 1. \tag{8}$$

Since $(p_3, q_3) \in \mathcal{B}^{(r-1)}_{\mu/\nu^{(r-1)}}$, Lemma 18 implies that $(p_3+1, q_3+1) \notin \mathcal{B}^{(r-1)}_{\mu/\nu^{(r-1)}}$. Thus, $(p_3 + 1, q_3 + 1)$ is not covered by $\mathfrak{h}(r,\tau_r)$, since $\mathfrak{h}(r,\tau_r)$ only covers boundary cells. Therefore, $\nu^{(r)}_{p_3+1} < q_3 + 1$ and thus

$$\nu^{(r)}_{p_3+1} + 1 \leqslant q_3 + 1. \tag{9}$$

Together Equations (8) and (9) imply that

$$\nu^{(r)}_{p_3+1} + 1 \leqslant q_3 + 1 \leqslant \nu^{(r)}_{p_3} + 1,$$

meaning $(p_3+1, q_3+1) \in \mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$.

Now we show $(p_3+1, q_3+1) \notin \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$, by proving $q_3 + 1 \neq \nu^{(r)}_{p_3+1} + 1$. Since $(p_3, q_3) \notin \mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$, the fact that (p_3, q_3) is covered by $\mathfrak{h}(r,\tau_r)$ also implies that $(p_3, q_3 - 1)$ is covered by $\mathfrak{h}(r,\tau_r)$ by the tunnel hook definition. So $(p_3, q_3 - 1) \in \mathcal{B}^{(r-1)}_{\mu/\nu^{(r-1)}}$. Therefore, we have $(p_3+1, q_3) \notin \mathcal{B}^{(r-1)}_{\mu/\nu^{(r-1)}}$ by Lemma 18. Then, $\nu^{(r)}_{p_3+1} < q_3$, so $\nu^{(r)}_{p_3+1} + 1 < q_3 + 1$. Therefore, $(p_3+1, q_3+1) \notin \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$. So $(p_3+1, q_3+1) \in \mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$, as desired. ☐

The following lemma states that a boundary cell in $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ remains a boundary cell in $D^{(r)}_{\mu/\nu^{(r)}}$ if it is not covered by $\mathfrak{h}(r,\tau_r)$.

Lemma 27. *Suppose that $h(r,\tau_r)$ is a tunnel hook in the partial GBPR diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ and consider the partial GBPR diagram $D^{(r)}_{\mu/\nu^{(r)}}$ resulting from applying Step 2 of Procedure 23. If $(p,q)\in\mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$, then either $(p,q)\in\mathcal{B}^{(r-1)}_{\mu/\nu^{(r-1)}}$ or $(p-1,q-1)\in\mathfrak{h}(r,\tau_r)$.*

Proof. Suppose $(p,q)\in\mathcal{B}^{(r)}_{\mu/\nu^{(r)}}$. Thus, from Definition 15, we have $\nu^{(r)}_p+1\leqslant q\leqslant\nu^{(r)}_{p-1}+1$. Assume $(p-1,q-1)\notin\mathfrak{h}(r,\tau_r)$, since otherwise we are done. Since $q\leqslant\nu^{(r)}_{p-1}+1$, we have $q-1\leqslant\nu^{(r)}_{p-1}$. Since $(p-1,q-1)\notin\mathfrak{h}(r,\tau_r)$ but $q-1\leqslant\nu^{(r)}_{p-1}$, we must have $q-1\leqslant\nu^{(r-1)}_{p-1}$. So $q\leqslant\nu^{(r-1)}_{p-1}+1$. Also, $\nu^{(r-1)}_p\leqslant\nu^{(r)}_p$, so $\nu^{(r-1)}_p+1\leqslant\nu^{(r)}_p+1\leqslant q$. Therefore $\nu^{(r-1)}_p+1\leqslant q$. Putting these together implies

$$\nu^{(r-1)}_p+1\leqslant q\leqslant\nu^{(r-1)}_{p-1}+1.$$

Therefore, by definition, $(p,q)\in\mathcal{B}^{(r-1)}_{\mu/\nu^{(r-1)}}$. $\square$

Lemma 26 implies an algorithm for identifying the cells in $\mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$ available to become terminal cells at each step r in the construction of a THC. Along the way, we uncover a permutation associated with each THC. This algorithm to identify tunnel cells and produce the associated permutation is described below. The subscripts on the $\mathbb{T}$ emphasize the fact that each new collection of cells depends on the choice of cells in the previous step.

Procedure 28 (associated permutation construction). *Let $\mu\in\mathbb{Z}^k$ and set $\nu=\nu^{(0)}=\emptyset$. The following algorithm constructs a sequence of cells (which we will see are terminal cells for a tunnel hook covering) and also produces a permutation associated to each choice of cells.*

1. *Let $\mathbb{T}_{\mu/\nu^{(0)}}=\{(1,1),(2,1),\ldots,(k,1)\}$ be the collection of all cells in the leftmost column of the GBPR diagram D_μ.*

2. *Select a cell $\tau_1=(p_1,1)$ from $\mathbb{T}_{\mu/\nu^{(0)}}$ and set*

   $$\mathbb{T}_{\mu/\nu^{(1)}}=\{(2,2),(3,2),\ldots,(p_1,2),(p_1+1,1),\ldots,(k,1)\};$$

   $\mathbb{T}_{\mu/\nu^{(1)}}$ is the set constructed from $\mathbb{T}_{\mu/\nu^{(0)}}$ by removing $(p_1,1)$ and adding $(1,1)$ to each cell in $\mathbb{T}_{\mu/\nu^{(0)}}$ situated in a row lower than row p_1.

3. *Let $\sigma_1=p_1-1+1=p_1$. Note that $\mathcal{L}_{\sigma_1}$ (see Equation (7)) is the diagonal containing $(p_1,1)$.*

4. *Repeat the following steps, once for each value of r from 2 to k.*

(a) *Select a cell* $\tau_r = (p_r, q_r)$ *from* $\mathbb{T}_{\mu/\nu^{(r-1)}}$.

(b) *Construct* $\mathbb{T}_{\mu/\nu^{(r)}}$ *from* $\mathbb{T}_{\mu/\nu^{(r-1)}}$ *by removing* τ_r *and adding* $(1,1)$ *to each of cell in* $\mathbb{T}_{\mu/\nu^{(r-1)}}$ *from a row lower than row* p_r.

(c) *Let* $\sigma_r = p_r - q_r + 1$. *Note that* $\mathcal{L}_{\sigma_r}$ *is the diagonal containing* (p_r, q_r).

5. *Set* $\sigma = \sigma_1\sigma_2\cdots\sigma_k$.

Lemma 29. *The cells in the set* $\mathbb{T}_{\mu/\nu^{(r-1)}}$ *are precisely the tunnel cells in* $\mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$ *for* $1 \leqslant r \leqslant k$. *Furthermore, the resulting sequence* $\sigma = \sigma_1\cdots\sigma_k$ *is a permutation in* S_k.

Proof. Let $\mathbb{T}_{\mu/\nu^{(0)}} = \{(1,1),(2,1),\ldots,(k,1)\}$ be the collection of cells in the leftmost column of $D^{(0)}_{\mu/\nu^{(0)}}$. At the beginning of Step $r = 1$ of the procedure, it is clear that $\mathbb{T}_{\mu/\nu^{(0)}} = \mathcal{T}^{(0)}_{\mu/\nu^{(0)}}$. Note that each cell of $\mathcal{T}^{(0)}_{\mu/\nu^{(0)}}$ belongs to a unique diagonal $\mathcal{L}_j$ for $1 \leqslant j \leqslant k$.

Let $\tau_1 = (p,1) \in \mathcal{L}_p$ be the cell chosen as terminal cell in Step 1 of Procedure 28. Then Lemma 26 (A) implies $(p+1,2) \in \mathcal{N}^{(1)}_{\mu/\nu^{(1)}}$. Furthermore, by repeated application of Lemma 26 (D) and (E), no elements of $\mathcal{L}_p$ other than $(p,1)$ can be in any of the collections $\mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$ for any $1 \leqslant r \leqslant k-1$.

Consider an arbitrary cell $(s,1)$ with $s \neq p$. By Lemma 26 (C), if $\mathfrak{h}(1,\tau_1)$ covers the cell $(s,1) \in \mathcal{T}^{(0)}_{\mu/\nu^{(0)}}$, then $(s+1,2) \in \mathcal{T}^{(1)}_{\mu/\nu^{(1)}}$. If $\mathfrak{h}(1,\tau_1)$ does not cover the cell $(s,1)$, then Lemma 26 (B) implies $(s,1) \in \mathcal{T}^{(1)}_{\mu/\nu^{(1)}}$. This proves the containment

$$\mathbb{T}_{\mu/\nu^{(1)}} = \{(2,2),(3,2),\ldots,(p,2),(p+1,1),\ldots,(k,1)\} \subseteq \mathcal{T}^{(1)}_{\mu/\nu^{(1)}}.$$

Since there are exactly $k-1$ cells in $\mathcal{T}^{(1)}_{\mu/\nu^{(1)}}$, the fact that there are $k-1$ cells in $\mathbb{T}_{\mu/\nu^{(1)}}$ implies that $\mathbb{T}_{\mu/\nu^{(1)}} = \mathcal{T}^{(1)}_{\mu/\nu^{(1)}}$.

This argument can be repeated to show that each collection $\mathbb{T}_{\mu/\nu^{(r)}}$ is equal to the set $\mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$ and includes at most one cell from $\mathcal{L}_j$ for each $1 \leqslant j \leqslant k$, since adding $(1,1)$ to a cell does not change the diagonal in which it lies.

The construction of the cells in $\mathbb{T}_{\mu/\nu^{(r)}}$ from those in $\mathbb{T}_{\mu/\nu^{(r-1)}}$ removes a cell in row r or higher and increases the row value for each cell lower than the removed cell. Therefore by induction, the cells in $\mathbb{T}_{\mu/\nu^{(r)}}$ all lie in rows strictly higher than row r.

Finally, repeated application of Lemma 26 (A, D, E) implies that if $\mathcal{L}_{\sigma_r}$ is the diagonal containing the cell τ_r removed during Step r, then no cell from diagonal $\mathcal{L}_{\sigma_r}$ can appear in $\mathcal{T}^{(i)}_{\mu/\nu^{(i)}}$ for $i > r$. Therefore each of the k diagonals $\mathcal{L}_{\sigma_1}, \mathcal{L}_{\sigma_2}, \ldots, \mathcal{L}_{\sigma_k}$ removed during the k steps of the procedure is distinct, and satisfy $1 \leqslant \sigma_j \leqslant k$ for all j. This implies that σ is indeed a permutation in S_k. $\square$

Lemma 29 in fact proves that Procedure 28 is equivalent to the THC construction procedure starting with $\nu = \emptyset$, since selecting a tunnel hook starting in row r can be done by simply selecting its terminal cell. Every possible terminal cell for a tunnel hook starting in row r (for the diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ created from the selection of the first $r-1$ tunnel hooks) is included in $\mathbb{T}_{\mu/\nu^{(r-1)}}$.

The following example demonstrates this construction.

Example 30. Let μ be a composition of length $k = 10$ and consider step $i = 6$ in the tunnel hook covering construction. Assume the first 5 tunnel hooks have been constructed and $\sigma_i \in \{2, 3, 6, 7, 9\}$ for $1 \leqslant i \leqslant 5$. Then the tunnel cells in $\mathbb{T}_{\mu/\nu^{(5)}}$ are in diagonals $\mathcal{L}_1$, $\mathcal{L}_4$, $\mathcal{L}_5$, $\mathcal{L}_8$, and $\mathcal{L}_{10}$. If a cell is in $\mathcal{L}_j$, any cell northwest of this cell will be in a higher diagonal, which means that the tunnel cell in row 6 must be in $\mathcal{L}_1$, the tunnel cell in row 7 must be in $\mathcal{L}_4$, etc. Therefore,

$$\mathbb{T}_{\mu/\nu^{(5)}} = \{(6, 6), (7, 4), (8, 4), (9, 2), (10, 1)\}.$$

If, for example, the cell $(8, 4)$ is selected as the terminal cell for the next tunnel hook, then the new collection of tunnel cells will become

$$\mathbb{T}_{\mu/\nu^{(6)}} = \{(7, 7), (8, 5), (9, 2), (10, 1)\},$$

since $(1, 1)$ is added to the first two entries of $\mathbb{T}_{\mu/\nu^{(5)}}$ and $(8, 4)$ is removed.

If, instead, the cell $(9, 2)$ is selected from $\mathbb{T}_{\mu/\nu^{(5)}}$ for the next terminal cell, the new collection of tunnel cells will become

$$\mathbb{T}_{\mu/\nu^{(6)}} = \{(7, 7), (8, 5), (9, 5), (10, 1)\},$$

since $(1, 1)$ is added to the first three entries of $\mathbb{T}_{\mu/\nu^{(5)}}$ and $(9, 2)$ is removed.

The following proposition, which states that there is a bijection between tunnel hook coverings and permutations, will be used in the proof of our combinatorial interpretation of the NSym Jacobi-Trudi determinant.

Proposition 31. *Let $\mu = (\mu_1, \dots, \mu_k)$ be a sequence. There is a bijection between tunnel hook coverings of the GBPR diagram for μ and permutations $\sigma \in S_k$. In particular, if the tunnel hook covering has terminal cells $\{\tau_1, \tau_2, \cdots, \tau_k\}$, where $\tau_i = (p_i, q_i)$, then the bijection sends this tunnel hook covering to the permutation σ such that $\sigma_i = p_i - q_i + 1$ for all $1 \leqslant i \leqslant k$.*

Proof. We first show how to determine the permutation from the tunnel hook covering of the GBPR diagram. Let

$$\gamma = \{\mathfrak{h}(1, \tau_1), \mathfrak{h}(2, \tau_2), \dots, \mathfrak{h}(k, \tau_k)\}$$

be a tunnel hook covering (THC) of the GBPR diagram for μ, where $\tau_i = (p_i, q_i)$ for $1 \leqslant i \leqslant k$. Let $\sigma(\gamma)$ be the permutation which maps i to $p_i - q_i + 1$. This is precisely the permutation produced by Procedure 28. Lemma 29 guarantees that σ is indeed a permutation in S_k.

To recover the THC from the permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_k$, it is enough to determine the terminal cells $\{\tau_1, \tau_2, \ldots, \tau_k\}$. To this end, set $\tau_r = (\sigma_r + m_r, 1 + m_r)$, where

$$m_r = \#\{i \in \{1, \ \ldots, k\} | i < r \text{ and } \sigma_i > \sigma_r\}.$$

We show that these cells are indeed the terminal cells for a THC by proving inductively that $\tau_r \in \mathbb{T}_{\mu/\nu^{(r-1)}}$ for $1 \leqslant r \leqslant k$.

For the base case, note that $\tau_1 = (\sigma_1, 1)$, since there is no value $i \in \{1, \ldots, k\}$ such that $i < 1$. Since $(\sigma_1, 1) \in \mathbb{T}_{\mu/\nu^{(0)}}$, the base case is satisfied.

Next assume that $\tau_j \in \mathbb{T}_{\mu/\nu^{(j-1)}}$ for $1 \leqslant j < r$. The cell in diagonal $\mathcal{L}_{\sigma_r}$ of $\mathbb{T}_{\mu/\nu^{(0)}}$ is $(\sigma_r, 1)$. With each choice of τ_j (for $j < r$) selected during Procedure 28, this cell either remains the same (if $\sigma_j < \sigma_r$) or is augmented by $(1, 1)$ (if $\sigma_j > \sigma_r$). But this implies that $\tau_r \in \mathbb{T}_{\mu/\nu^{(r-1)}}$ since m_r counts the number of times (in steps 1 through $r - 1$) the coordinates of the terminal cell in diagonal $\mathcal{L}_{\sigma_r}$ are increased by $(1, 1)$.

Thus, the cells $\{\tau_1, \tau_2, \ldots, \tau_k\}$ are indeed tunnel cells and produce precisely the tunnel hook covering associated to σ from Procedure 28. Therefore, the map described is a bijection between tunnel hook coverings of D_μ and permutations in S_k, as desired. $\square$

Example 32. Let $\sigma = 4731625 \in S_7$. Then $\tau_1 = (4, 1)$. Since $m_2 = 0$, we have $\tau_2 = (\sigma_2 + 0, 1 + 0) = (7, 1)$. For σ_3, we have $m_3 = 2$ since both σ_1 and σ_2 are greater than $\sigma_3 = 3$. So $\tau_3 = (3 + 2, 1 + 2) = (5, 3)$. Continuing this process produces the overall collection

$$\{(4, 1), (7, 1), (5, 3), (4, 4), (7, 2), (6, 5), (7, 3)\}$$

of terminal cells for the tunnel hook covering corresponding to σ. Note that these terminal cells are in diagonals $\mathcal{L}_4, \mathcal{L}_7, \mathcal{L}_3, \mathcal{L}_1, \mathcal{L}_6, \mathcal{L}_2, \mathcal{L}_5$, respectively.

The bijection between THC's and permutations allows us to go one step further in giving a combinatorial interpretation for the partitions $(\nu^{(r)}_{r+1}, \nu^{(r)}_{r+2}, \ldots, \nu^{(r)}_k)$ appearing in the partial GBPR diagrams $D^{(r)}_{\mu/\nu^{(r)}}$ associated to $\sigma = \sigma_1\sigma_2\cdots\sigma_k$. In the following, assume $\sigma \in S_k$ and $m < k$. Let $U_m(\sigma)$ be the set $\{\sigma_{m+1}, \sigma_{m+2}, \cdots, \sigma_k\}$ and set $U_{m,j}(\sigma)$ to be the j^{th} smallest element of $U_m(\sigma)$.

Example 33. If $\sigma = 437926581$ then $U_4(\sigma) = \{1, 2, 5, 6, 8\}$ and $U_{4,3}(\sigma) = 5$.

Note that the set $U_m(\sigma)$ is the collection of diagonals $\mathcal{L}_j$ remaining after the first m tunnel hooks for the THC associated to σ have been constructed.

Lemma 34. *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in S_k$, assume $\nu^{(0)} = \emptyset$, and let*

$$D_\mu = D^{(0)}_{\mu/\nu^{(0)}}, D^{(1)}_{\mu/\nu^{(1)}}, \ldots, D^{(k)}_{\mu/\nu^{(k)}}$$

be the sequence of partial GBPR diagrams produced during the construction of the corresponding tunnel hook covering of diagram D_μ. Then $\nu^{(r)}_j$ (for $j > r$) equals the number of entries in the set $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ that are greater than $U_{r,j-r}(\sigma)$.

Proof. The tunnel hook covering construction algorithm (Procedure 28) begins with the collection of cells in the leftmost column, one on each diagonal $\mathcal{L}_a$. At each stage of the algorithm, one diagonal is removed and each tunnel cell in a diagonal below the removed diagonal is augmented by $(1, 1)$. This means that after r iterations of the tunnel hook covering construction, the tunnel hook with initial cell in row $r + 1$ and terminal cell in row $j \geqslant r + 1$ terminates in the $(j - r)^{th}$ smallest remaining diagonal $\mathcal{L}_b$, where $b = U_{r,j-r}(\sigma)$. The coordinates of this tunnel cell are (p_{r+1}, q_{r+1}), where q_{r+1} is one more than the number of tunnel hooks terminating in a diagonal greater than b. But since (p_{r+1}, q_{r+1}) is a tunnel cell, we have $\nu_j^{(r)} = q_{r+1} - 1$, so $\nu_j^{(r)}$ equals the number of tunnel hooks terminating in a diagonal greater than $U_{r,j-r}(\sigma)$. Since σ_i equals the diagonal in which the i^{th} tunnel hook terminates, $\nu_j^{(r)}$ equals the number of entries in $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ greater than $U_{r,j-r}(\sigma)$. ☐

Recall that the *Lehmer code* [17] $L(\sigma)$ of a permutation $\sigma \in S_k$ is given by

$$L(\sigma) = (L(\sigma_1), L(\sigma_2), \ldots, L(\sigma_k)),$$

where $L(\sigma_i) = \#\{j > i | \sigma_j < \sigma_i\}$. Note, if $d := \sum_{i=1}^{n-1} L(\sigma_i)$ counts the total number of inversions of σ, then the sign of σ is given by $\epsilon(\sigma) = (-1)^d$.

The following lemma provides a method for counting the number of rows covered by a tunnel hook. This will be useful in constructing the sign of a tunnel hook covering.

Lemma 35. *Let*

$$\mathfrak{h}(1, \tau_1), \mathfrak{h}(2, \tau_2), \ldots, \mathfrak{h}(k, \tau_k),$$

be a THC of D_μ with corresponding permutation

$$\sigma = \sigma_1\sigma_2 \cdots \sigma_k \in S_k.$$

The number of rows covered by $\mathfrak{h}(r, \tau_r)$ is equal to $L(\sigma_r) + 1$.

Proof. The tunnel hook $\mathfrak{h}(r, (p_r, q_r))$ begins in row r and travels through each tunnel cell situated in a diagonal smaller than σ_r, which means the number of rows covered by $\mathfrak{h}(r, (p_r, q_r))$ equals one plus the number of remaining diagonals in $\mathbb{T}_{\mu/\nu^{(r-1)}}$ smaller than σ_r. But this is equal to one plus the number of entries in the set $\{\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_k\}$ that are less than σ_r. But this is precisely equal to $L(\sigma_r) + 1$, as desired. ☐

Notice that the terminal cells for the tunnel hooks are not dependent on μ. However, the tunnel hooks themselves still vary based on μ due to the cells in the lowest row of each tunnel hook as well as the new red cells that are potentially introduced, depending on μ.

Example 36. Let $\sigma = 4731625 \in S_7$. Then $L(\sigma) = (3, 5, 2, 0, 2, 0, 0)$. Recall from Example 32 that the terminal cells in the THC corresponding to σ are

$$\{(4, 1), (7, 1), (5, 3), (4, 4), (7, 2), (6, 5), (7, 3)\}.$$

Consider, for example, the tunnel hook $\mathfrak{h}(3, (5, 3))$. This tunnel hook starts in row 3 and ends in row 5, so it covers 3 rows, which equals $L(\sigma_3) + 1$.

Our next step in the proof of Theorem 1 is to show that the lengths of the tunnel hooks equal the subscripts of the corresponding entries in $\mathbb{M}_\mu$.

Lemma 37. *Let μ be a sequence of length k and let $\mathbb{M}_\mu$ be defined as above. Assume the first $r-1$ tunnel hooks have been constructed and let j be any positive integer between 1 and k such that none of these $r-1$ tunnel hooks have terminal cell in $\mathcal{L}_j$. Then*

$$(\mathbb{M}_\mu)_{r,j}(=H_{\mu_r-r+j}) = H_{\Delta(\mathfrak{h}(r,\tau))}, \tag{10}$$

where τ is the unique cell in the diagonal $\mathcal{L}_j$ that is also in $\mathbb{T}_{\mu/\nu^{(r-1)}}$.

Proof. First note that there exists a unique s such that $(j+s, 1+s) = \tau \in \mathbb{T}_{\mu/\nu^{(r-1)}}$ by the arguments given in the proof of Lemma 29. By the definition of *taxi* (Equation (6)), we have

$$taxi(\mathfrak{h}(r,\tau)) = (\nu_r^{(r-1)} + 1 - (1+s)) + (j+s-r) = \nu_r^{(r-1)} - r + j.$$

Next, recognize that $\mu_r - \nu_r^{(r-1)} = b_r - c_r = bank(r)$, where the first equality is due to Lemma 10 (since $\nu_r^{(r-1)} = a_r$) and the second is the definition of *bank* (see Definition 20). Therefore,

$$bank(r) + taxi(\mathfrak{h}(r,\tau)) = \mu_r - \nu_r^{(r-1)} + \nu_r^{(r-1)} - r + j = \mu_r - r + j,$$

which is exactly what is necessary since $\Delta(\mathfrak{h}(r,\tau)) = bank(r) + taxi(\mathfrak{h}(r,\tau))$. □

4.2 Proof of Theorem 1

We now have all the pieces we need to complete the proof of Theorem 1.

Proof of Theorem 1. Recall [5] that

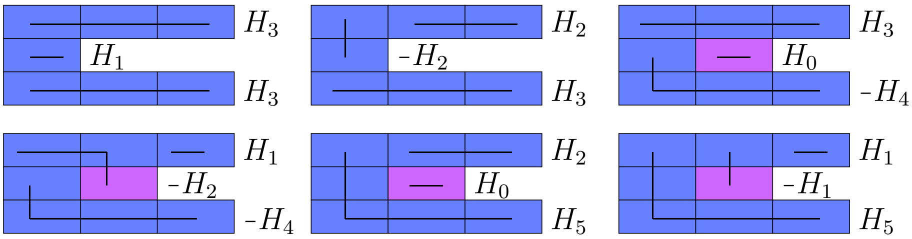
$$\mathfrak{S}_\mu = \mathfrak{det}(\mathbb{M}_\mu) = \sum_{\sigma\in S_k} \epsilon(\sigma) \prod_{i=1}^{n} (\mathbb{M}_\mu)_{i,\sigma_i}.$$

Proposition 31 gives a bijection between permutations and tunnel hook coverings. In Lemma 37 we show that the subscripts on the entries of the matrix $\mathbb{M}_\mu$ are equal to the Δ values for the corresponding tunnel hooks. All that remains is to show that $\epsilon(\sigma)$ equals the product of the signs of the tunnel hooks. Recall that the sign of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ is given by $(-1)^d$, where $d = \sum_i L(\sigma_i)$.

Lemma 35 shows that the sign of each individual tunnel hook $h(r,\tau_i)$ is in fact $(-1)^{L(\sigma)_i}$, since the sign of an individual hook $\mathfrak{h}(i,\tau)$ is one less than the number of rows covered by $\mathfrak{h}(i,\tau)$. Thus, the product of the signs of the tunnel hooks gives the sign of the permutation corresponding to that tunnel hook covering. □

In Examples 38, 39, and 40, we give the complete collection of tunnel hook coverings for shapes $(3,1,3), (3,0,3)$, and $(3,-1,3)$ to provide the expansions of $\mathfrak{S}_{(3,1,3)}$, $\mathfrak{S}_{(3,0,3)}$, and $\mathfrak{S}_{(3,-1,3)}$ into complete homogeneous noncommutative symmetric functions.

Example 38. Below is the complete homogeneous expansion of $\mathfrak{S}_{(3,1,3)}$.



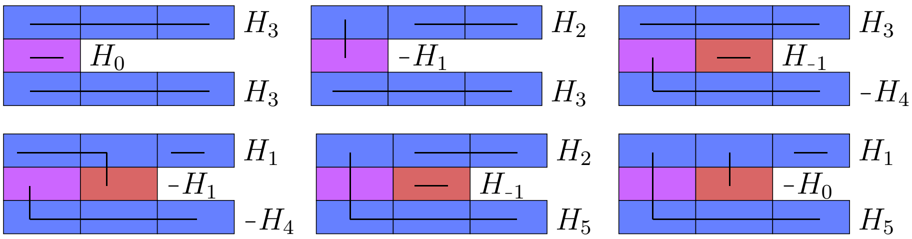
Putting this together produces the expansion

$$\mathfrak{S}_{(3,1,3)} = H_{(3,1,3)} - H_{(3,2,2)} - H_{(4,3)} + H_{(4,2,1)} + H_{(5,2)} - H_{(5,1,1)}.$$

Example 38 does not include red cells in any of the tunnel hook coverings, since no parts are negative and it is not possible in this example for a tunnel hook to cover a purple cell at any point other than in its originating row. Note that it is, however, possible to introduce a red cell in the decomposition of an immaculate function whose indexing composition has all positive parts; consider for example the tunnel hook decomposition of $\mathfrak{S}_{(1,2,1)}$ (not shown), which includes the term $H_3H_2H_{-1}$.

The next example illustrates a situation in which it is possible to introduce a red cell, despite the fact that the indexing composition does not contain any negative parts.

Example 39. The complete homogeneous expansion of $\mathfrak{S}_{(3,0,3)}$ is depicted below.

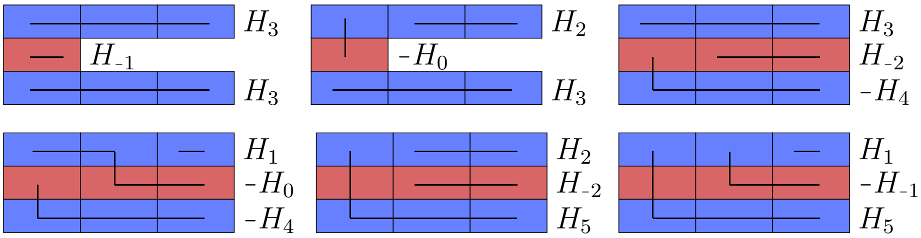


Including signs and recalling that $H_j = 0$ if $j \in \mathbb{Z}^-$, we see that

$$\mathfrak{S}_{(3,0,3)} = H_{(3,3)} - H_{(3,1,2)} + H_{(4,1,1)} - H_{(5,1)}.$$

Finally, Example 40 depicts the decomposition of $\mathfrak{S}_{(3,-1,3)}$ into tunnel hooks. In this situation, even though we begin with a red cell, two of the six tunnel hook coverings result in nonnegative indices.

Example 40. Below is the complete homogeneous expansion of $\mathfrak{S}_{(3,-1,3)}$.



Including signs and recalling that $H_j = 0$ if $j \in \mathbb{Z}^-$, we see that

$$\mathfrak{S}_{(3,-1,3)} = -H_{(3,2)} + H_{(4,1)}.$$

4.3 Quasisymmetric Functions

The Hopf algebra QSym of quasisymmetric functions is dual to the algebra NSym, satisfying the pairing

$$\langle \cdot, \cdot \rangle : \mathrm{NSym} \times \mathrm{QSym} \to \mathbf{K},$$

defined by setting

$$\langle H_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}.$$

Let $\mathcal{C}$ be the set of all compositions. Any pair of bases $\{X_\alpha\}_{\alpha\in\mathcal{C}}$ in NSym and $\{Y_\beta\}_{\beta\in\mathcal{C}}$ in QSym satisfying $\langle X_\alpha, Y_\beta \rangle = \delta_{\alpha,\beta}$ are said to be *dual* to one another. Hence, the complete homogeneous basis for NSym is dual to the monomial basis for QSym.

The *ribbon Schur basis* $\{R_\alpha\}_{\alpha\in\mathcal{C}}$ for NSym can be defined in terms of the complete homogeneous basis by

$$R_\alpha = \sum_{\alpha \preceq \beta} (-1)^{\ell(\alpha)-\ell(\beta)} H_\beta,$$

where $\preceq$ is the refinement ordering on compositions. The ribbon Schur basis for NSym is dual to the fundamental basis for QSym [12, 13].

The basis in QSym dual to the immaculate basis is called the *dual immaculate quasisymmetric function basis* [5]. Elements of this basis are denoted by $\mathfrak{S}^\star_\alpha$ and expand positively in the monomial, fundamental, and Young quasisymmetric Schur bases [19, 3]. The following combinatorial formula for the expansion of the monomial basis for QSym into the dual immaculate basis is an immediate corollary of Theorem 1, due to duality. Intuitively, to expand M_α into dual immaculates, the shapes being covered may vary but the values associated to the tunnel hook coverings must be equal to the composition α.

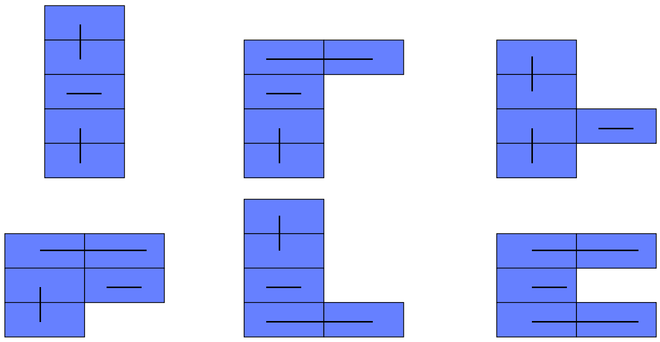
Corollary 41. *The expansion of the monomial quasisymmetric functions into the dual immaculate quasisymmetric function basis is given by the following formula.*

$$M_\alpha = \sum_{\mu \models |\alpha|} \sum_{\substack{\gamma \in THC_\mu, \\ flat(\Delta(\gamma))=\alpha}} \epsilon(\gamma) \mathfrak{S}^\star_\mu, \tag{11}$$

where μ is a composition of $|\alpha|$, THC_μ denotes the collection of tunnel hook coverings of a diagram of shape μ, $\mathfrak{h}(r,\tau_r) \in \gamma$, and the sign $\epsilon(\gamma) = \prod_{r=1}^{\ell(\mu)} \epsilon(\mathfrak{h}(r,\tau_r))$ and composition $\Delta(\gamma)$ associated to each tunnel hook covering γ are as described above.

Note that this formula is not cancellation-free in general. For example, when $\alpha = (1,3,2)$, the shape $(1,2,1,2)$ appears twice: once with associated sequence $(1,3,0,2)$ and once with associated sequence $(1,3,2,0)$. These two tunnel hook coverings have opposite signs and therefore cancel each other out. The following example, however, contains no cancellations and therefore includes every tunnel hook covering whose flat is equal to $(2,1,2)$. We do not draw the tunnel hooks with Δ value zero since for this example all are simply a single purple cell in their originating row. (For instance, the first diagram also has a tunnel hook in cell $(2,2)$ and another in cell $(5,2)$.)

Example 42. Let $\alpha = (2,1,2)$. The following are all THC γ such that $\Delta(\gamma) = (2,1,2)$.



Therefore, $M_{212} = \mathfrak{S}^\star_{11111} - \mathfrak{S}^\star_{1112} + \mathfrak{S}^\star_{1211} - \mathfrak{S}^\star_{122} - \mathfrak{S}^\star_{2111} + \mathfrak{S}^\star_{212}$.

5 Immaculate functions indexed by skew shapes

We now extend Theorem 8 to introduce a definition of skew immaculate functions, just as the Jacobi-Trudi formula can be used to define skew Schur functions in terms of the complete homogeneous symmetric functions.

Definition 43 (skew immaculate functions). Let $\mu, \nu \in \mathbb{Z}^k$ be sequences of integers. Recalling Theorem 5, we define $(\mathbb{M}_{\mu/\nu})_{i,j} = H_{(\mu_i - i)-(\nu_j - j)}$ and

$$\mathfrak{S}_{\mu/\nu} = \mathfrak{det}(\mathbb{M}_{\mu/\nu}). \tag{12}$$

where the determinant $\mathfrak{det}$ is expanded using Laplace expansion starting in the top row and continuing sequentially to the bottom row.

For example, if $\mu = (2,5,3)$ and $\nu = (1,3,0)$, then

$$\mathbb{M}_{\mu/\nu} = \begin{bmatrix} H_1 & H_0 & H_4 \\ H_3 & H_2 & H_6 \\ H_0 & H_{-1} & H_3 \end{bmatrix}.$$

Recalling that $H_a = 0$ if $a \in \mathbb{Z}^-$ and $H_0 = 1$, the resulting decomposition of $\mathfrak{S}_{\mu/\nu}$ into the complete homogeneous basis for NSym is therefore

$$\mathfrak{S}_{\mu/\nu} = H_{(1,2,3)} - H_{(3,3)} + H_{(6)} - H_{(4,2)}.$$

Notice that the skew Schur function $s_{(2,5,3)/(1,3)}$ becomes $s_{(4,3,3)/(2,2)}$ under the well-known *straightening algorithm* that states that for any integer sequences λ', λ'' and any integers a and b,

$$s_{(\lambda',a,b,\lambda'')} = -s_{(\lambda',b-1,a+1,\lambda'')}.$$

This property arises by swapping rows in the Jacobi-Trudi determinant for Schur functions. Note that this relationship does not generally apply to immaculate functions due to the noncommutativity of the complete homogeneous noncommutative symmetric functions. For example, $\mathfrak{S}_{(a,b)} = H_{(a,b)} - H_{(a+1,b-1)}$ while $\mathfrak{S}_{(b-1,a+1)} = H_{(b-1,a+1)} - H_{(b,a)}$ so that $\mathfrak{S}_{(a,b)} \neq -\mathfrak{S}_{(b-1,a+1)}$.

Decomposing the Schur function from our example into the complete homogeneous symmetric functions produces the expansion

$$s_{(4,3,3)/(2,2)} = h_{(3,2,1)} - h_{(3,3)} + h_{(6)} - h_{(4,2)}.$$

This is exactly the decomposition obtained by applying the forgetful map to the expansion produced by our construction in NSym. This is true in general since the starting matrices are identical and the only difference between the H expansion of $\mathfrak{S}_{\mu/\nu}$ and the h expansion of $s_{\mu/\nu}$ is in the commutativity of the indices. The indices for the H basis expansions do not commute whereas the indices for the h basis expansions do commute.

It is important to note that the Hopf algebra operation $\rightharpoonup$ for skewing elements of NSym by elements of QSym (often referred to as the transpose of multiplication [22]), another natural candidate for skew immaculates, does not always coincide with this construction. For example, skewing $\mathfrak{S}_{(2,5,3)}$ by $\mathfrak{S}^{\star}_{(1,3)}$ produces the H decomposition

$$\mathfrak{S}^{\star}_{(1,3)} \rightharpoonup \mathfrak{S}_{(2,5,3)} = \mathfrak{S}_{(1,2,3)} + \mathfrak{S}_{(1,3,2)} + \mathfrak{S}_{(1,4,1)} + \mathfrak{S}_{(1,5)} + \mathfrak{S}_{(2,4)} \neq \mathfrak{S}_{(2,5,3)/(1,3)}.$$

One open problem is to classify the pairs of compositions for which the two skew candidates coincide.

Theorem 2 states that when we restrict ν to be a partition (regardless of whether it is contained inside μ), we can still apply tunnel hook coverings to generate the determinantal decomposition combinatorially. That is,

$$\mathfrak{S}_{\mu/\nu} = \sum_{\gamma \in THC_{\mu/\nu}} \prod_{\mathfrak{h}(r,\tau_r) \in \gamma} \epsilon(\mathfrak{h}(r,\tau_r)) \; H_{\Delta(\mathfrak{h}(r,\tau_r))},$$

where $THC_{\mu/\nu}$ denotes the set of all tunnel hook coverings of the GBPR diagram $D_{\mu/\nu}$.

Section 5.1 describes how to utilize submatrices to produce the homogeneous function expansion of an immaculate function indexed by a skew shape. In Section 5.2, we apply these submatrices to prove Theorem 2. In Section 5.3, we discuss how to extend this approach to the situation in which ν is an arbitrary sequence. Section 5.4 explains how to recover the decomposition of a Schur function into the complete homogeneous symmetric functions in Sym.

5.1 Submatrices and immaculate functions indexed by skew shapes

Recall that the Jacobi-Trudi formula stated in Theorem 8 [5] requires the determinant of $\mathbb{M}_\mu$ be computed expanding using Laplace expansion row by row starting with the first row; the order in which the expansion occurs is important since the complete homogeneous noncommutative symmetric functions are not commutative. Recalling this convention, we have $\mathfrak{S}_\mu = \mathfrak{Det}(\mathbb{M}_\mu)$ where $(\mathbb{M}_\mu)_{i,j} = H_{\mu_i+j-i}$ and $1 \leqslant i, j \leqslant n$. Using the permutation expansion of the determinant, we have

$$\begin{aligned}\mathfrak{S}_\mu &= \sum_{\sigma \in S_k} \epsilon(\sigma)(\mathbb{M}_\mu)_{1,\sigma_1}(\mathbb{M}_\mu)_{2,\sigma_2} \cdots (\mathbb{M}_\mu)_{k,\sigma_k} \\ &= \sum_{\sigma \in S_k} \epsilon(\sigma) H_{\mu_1+\sigma_1-1} H_{\mu_2+\sigma_2-2} \cdots H_{\mu_k+\sigma_k-k},\end{aligned} \tag{13}$$

where $\epsilon(\sigma)$ is the sign of σ. Due to properties of the Jacobi-Trudi matrix, it is possible to describe the submatrices of $\mathbb{M}_\mu$ in terms of permutations. Recall that with $\sigma \in S_k$ and $m < k$, the set $U_m(\sigma)$ is the collection $\{\sigma_{m+1}, \sigma_{m+2}, \cdots, \sigma_k\}$ and $U_{m,j}(\sigma)$ is the j^{th} smallest element of $U_m(\sigma)$.

Let $\mathbb{M}(i|j)$ be the submatrix obtained from the matrix $\mathbb{M}$ by deleting row i and column j. With this notation, the submatrix obtained by deleting row i and column σ_i for $1 \leqslant i \leqslant m$ is

$$\mathbb{M}_\mu^{(m,\sigma)} := (\mathbb{M}_\mu)(1|\sigma_1)(2|\sigma_2)\dots(m|\sigma_m).$$

Notice that $U_m(\sigma)$ is the set of columns that were not deleted in the construction of $\mathbb{M}_\mu^{(m,\sigma)}$.

In the following proposition, we use notation $\tilde{\mu}$ and $\tilde{\nu}$ to denote the last $k-m$ parts of μ and $\nu^{(m)}$, respectively. Although this process of removing the first m parts relies on m, we suppress the m in our notation for clarity of exposition.

Proposition 44. *Let $\mu = (\mu_1, \dots, \mu_k) \in \mathbb{Z}^k$ and let $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ be a permutation in S_k. Let $\tilde{\mu} = (\mu_{m+1}, \mu_{m+2}, \dots, \mu_k)$ be the sequence obtained by deleting the first m parts of μ and let $\tilde{\nu} = (\nu_{m+1}^{(m)}, \nu_{m+2}^{(m)}, \dots, \nu_k^{(m)})$ be the partition consisting of the last $k-m$ parts of the sequence obtained during the construction of the first m tunnel hooks corresponding to the first m entries in the permutation σ. Then*

$$(\mathbb{M}_\mu^{(m,\sigma)})_{i,j} = H_{(\tilde{\mu}_i - i)-(\tilde{\nu}_j - j)}.$$

Proof. We use the Laplace expansion for computing the Jacobi-Trudi determinant. After expanding through the first m rows following the ordering given by the permutation $\sigma \in S_k$, the entries of the resulting submatrix $\mathbb{M}_\mu^{(m,\sigma)}$ are given by the formula

$$(\mathbb{M}_\mu^{(m,\sigma)})_{i,j} = H_{\mu_{m+i}-(m+i)+U_{m,j}(\sigma)}, \tag{14}$$

with $1 \leqslant i, j \leqslant k-m$. For $1 \leqslant j \leqslant k-m$, we set

$$\zeta_j^{(m,\sigma)} = m + j - U_{m,j}(\sigma).$$

There are $k-(U_{m,j}(\sigma))$ entries larger than $U_{m,j}(\sigma)$ in $\sigma \in S_k$ with exactly $k-m-j$ of them in $\{\sigma_{m+1}, \ldots, \sigma_k\}$. Therefore, the number of entries larger than $U_{m,j}(\sigma)$ in $\{\sigma_1, \ldots, \sigma_m\}$ equals

$$k - (U_{m,j}(\sigma)) - (k-m-j) = m + j - (U_{m,j}(\sigma)) = \zeta_j^{(m,\sigma)}.$$

Substituting m for r and $j+m$ for j in Lemma 34 implies $\nu_{m+j}^{(m)} = \zeta_j^{(m,\sigma)}$. Then $U_{m,j}(\sigma) = m + j - \nu_{m+j}^{(m)}$. Equation (14) now produces

$$(\mathbb{M}_\mu^{(m,\sigma)})_{i,j} = H_{\mu_{m+i}-(m+i)+(m+j)-\nu_{m+j}^{(m)}} = H_{(\mu_{m+i}-i)-(\nu_{m+j}^{(m)}-j)},$$

so that

$$(\mathbb{M}_\mu^{(m,\sigma)})_{i,j} = H_{(\tilde{\mu}_i - i)-(\tilde{\nu}_j - j)},$$

as desired. □

Proposition 44 together with Equation (13) imply that the skew immaculate $\mathfrak{S}_{\tilde{\mu}/\tilde{\nu}}$ is equal to the determinant of the submatrix $\mathbb{M}_\mu^{(m,\sigma)}$. This provides a way to start with a sequence μ and produce a skew immaculate $\mathfrak{S}_{\tilde{\mu}/\tilde{\nu}}$. In the next section, we discuss how to start with the skew shape, modify it to create a starting un-skewed sequence, and then apply the tunnel hook covering techniques to construct its H-decomposition combinatorially.

5.2 Proof of Theorem 2

In the following, λ^T denotes the transpose of the partition λ, obtained by reflecting λ across the main diagonal (southwest to northeast).

Proof of Theorem 2. Let $\mu \in \mathbb{Z}^k$ and let λ be a partition with ℓ nonzero parts, where $\ell \leqslant k$. If $\ell < k$, set $\lambda_i = 0$ for $\ell < i \leqslant k$. Prepend λ_1 rows of length λ_1 to the front of μ to obtain $\underline{\mu} = ((\lambda_1)^{\lambda_1}, \mu_1, \ldots, \mu_k)$. (See Example 45.) Applying Theorem 1 to the composition $\underline{\mu}$ produces a collection of tunnel hook coverings whose signed weights produce the decomposition of $\mathfrak{S}_{\underline{\mu}}$ into complete homogeneous noncommutative symmetric functions. Consider the tunnel hook coverings of $\underline{\mu}$ whose first λ_1 tunnel hooks are the collection

$$\{\mathfrak{h}(1,\tau_1), \mathfrak{h}(2,\tau_2), \ldots, \mathfrak{h}(\lambda_1, \tau_{\lambda_1})\},$$

where $\tau_j = (\lambda_1 + (\lambda^T)_j, j)$.

To see that $(\lambda_1 + (\lambda^T)_j, j) \in \mathbb{T}_{\underline{\mu}/\nu^{(j-1)}}$ for each $1 \leqslant j \leqslant \lambda_1$, first note that $\tau_1 = (\lambda_1 + (\lambda^T)_1, 1)$ is in $\mathbb{T}_{\mu/\nu^{(0)}}$, since every cell in the leftmost column of $D_{\mu/\nu^{(0)}}$ is in $\mathbb{T}_{\mu/\nu^{(0)}}$. The tunnel hook $\mathfrak{h}(1,\tau_1)$ is shaped like the letter L; it is comprised precisely of the cells

$$\{(1,1), (1,2), \ldots, (1,\lambda_1), (2,1), (3,1), \ldots, (\lambda_1 + (\lambda^T)_1, 1)\}.$$

In particular, the cell $(\lambda_1 + (\lambda^T)_2, 1)$ is contained in $\mathfrak{h}(1,\tau_1)$ since $(\lambda^T)_2 \leqslant (\lambda^T)_1$. But then $(\lambda_1 + (\lambda^T)_2 - 1, 1) \in \mathfrak{h}(1,\tau_1)$ and is not the terminal cell for $\mathfrak{h}(1,\tau_1)$ since $(\lambda_1 + (\lambda^T)_2 - 1, 1)$ lies immediately below $(\lambda_1 + (\lambda^T)_2, 1)$ and $\mathfrak{h}(1,\tau_1)$ is L-shaped. Step 4b of Procedure 28 implies that $(\lambda_1+(\lambda^T)_2, 2) \in \mathbb{T}_{\mu/\lambda^{(1)}}$ since $(\lambda_1+(\lambda)^T_2, 2)$ is obtained from $(\lambda_1+(\lambda^T)_2-1, 1)$ by adding $(1, 1)$.

The tunnel hook $\mathfrak{h}(2, \tau_2)$ is then also shaped like the letter L, consisting of the cells

$$\{(2,2), (2,3), \ldots, (2, \lambda_1), (3,2), (4,2), \ldots, (\lambda_1 + (\lambda^T)_2, 2)\}.$$

We can yet again use the fact that λ^T is a partition to conclude that $(\lambda_1 + (\lambda^T)_3, 2) \in \mathfrak{h}(2, \tau_2)$. This implies that $(\lambda_1 + (\lambda^T)_3, 3) \in \mathbb{T}_{\mu/\nu^{(2)}}$. Repeating this argument shows that the cells $\{\tau_1, \tau_2, \ldots, \tau_{\lambda_1}\}$ are indeed terminal cells, and so the collection

$$\{\mathfrak{h}(1, \tau_1), \mathfrak{h}(2, \tau_2), \ldots, \mathfrak{h}(\lambda_1, \tau_{\lambda_1})\}$$

is indeed a valid tunnel hook covering.

Next, consider the partial GBPR diagram obtained after λ_1 iterations of Procedure 23 (Step 2). The first λ_1 rows of $\underline{\mu}$ have been removed and the remaining nonzero rows of $\nu^{(\lambda_1)}$ are equal to the rows of λ, since their columns are the rows of λ^T. Therefore, removing the first λ_1 rows entirely from this diagram produces precisely the GBPR diagram $D^{(0)}_{\mu/\lambda}$.

Finally, we prove that the determinant $\mathfrak{det}\left(\mathbb{M}_{\mu/\lambda}\right)$ is equal to the determinant of the submatrix of $\mathbb{M}_{\underline{\mu}}$ obtained by removing rows 1 through λ_1 and columns $\{\lambda_1 + (\lambda^T)_1, \lambda_1 + (\lambda^T)_2 - 1, \ldots, \lambda_1 + (\lambda^T)_{\lambda_1} - \lambda_1 + 1\}$. Let σ be the permutation whose first λ_1 parts are $(\lambda_1 + (\lambda^T)_1, \lambda_1 + (\lambda^T)_2 - 1, \ldots, \lambda_1 + (\lambda^T)_{\lambda_1} - \lambda_1 + 1)$. These parts are obtained by listing the diagonals containing the terminal cells; that is, cell $(\lambda_1 + (\lambda^T)_j, j)$ lies in diagonal $\mathcal{L}_{\lambda_1+(\lambda^T)_j-j+1}$. Proposition 44 implies that

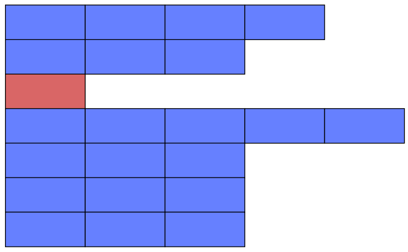
$$\mathfrak{det}(\mathbb{M}^{(\lambda_1,\sigma)}_{\underline{\mu}}) = \mathfrak{det}\left((H_{(\mu_i-i)-(\lambda_j-j)})_{i,j}\right) = \mathfrak{S}_{\mu/\lambda},$$

as desired, since $\lambda_j = \nu^{(\lambda_1)}_{\lambda_1+j}$, as observed above. □

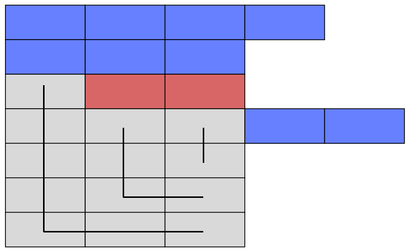
Note that there are other choices for the prefix of μ that yield the same result. This particular choice is taken so that none of the first λ_1 rows of the matrix $\mathbb{M}_{\underline{\mu}}$ contains the entry H_0. In particular, observe that for the tunnel hooks given in the proof of Theorem 2, $\Delta(\mathfrak{h}(j, \tau_j)) > 0$ for $1 \leqslant j \leqslant \lambda_1$. To see this, first note that $\mathfrak{h}(j, \tau_j)$ contains $\lambda_1 - j + 1$ blue cells in row j. This means $bank(j) = \lambda_1 - j + 1 > 0$. Since $\Delta(\mathfrak{h}(j, \tau_j)) \geqslant bank(j)$, we have $\Delta(\mathfrak{h}(j, \tau_j)) > 0$ for $1 \leqslant j \leqslant \lambda_1$.

This allows us to determine the H-decomposition of $\mathfrak{S}_{\mu/\lambda}$ by first computing the H-decomposition of $\mathfrak{S}_{\underline{\mu}}$ and then identifying the terms in this decomposition beginning with $H_{2\lambda_1-1+(\lambda^T)_1} H_{2\lambda_1-3+(\lambda^T)_2} \cdots H_{2\lambda_1-(2\lambda_1-1)+(\lambda^T)_{\lambda_1}}$ (and deleting all other terms). Removing this initial product from each of the identified terms and then summing the resulting terms produces the H-decomposition of $\mathfrak{S}_{\mu/\lambda}$.

Example 45. Let $\mu = (5, -1, 3, 4)$ and $\lambda = (3, 1, 0, 0)$. Then $\ell = 2$ and $\lambda^T = (2, 1, 1, 0)$. Therefore, $\underline{\mu} = (3, 3, 3, 5, -1, 3, 4)$ and the initial GBPR diagram for $\underline{\mu}$ is



The first three (since $\lambda_1 = 3$) tunnel hooks terminate at the tunnel cells $\{(5,1),(4,2),(4,3)\}$ (since $\lambda_1 + (\lambda^T)_1 = 3+2 = 5$, $\lambda_1 + (\lambda^T)_2 = 3+1 = 4$, and $\lambda_1 + (\lambda^T)_3 = 3+1 = 4$), producing the partial tunnel hook covering



The skew immaculate $\mathfrak{S}_{(5,-1,3,4)/(3,1,0,0)}$ can be obtained by selecting the terms appearing in $\mathfrak{S}_{(3,3,3,5,-1,3,4)}$ whose first three terms are $H_7H_4H_2$.

Notice that the submatrix $\mathbb{M}_{\underline{\mu}}(1|5)(2|3)(3|2)$ of the matrix

$$\mathbb{M}_{\underline{\mu}} = \begin{bmatrix} H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 \\ H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 \\ H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\ H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 \\ H_{-5} & H_{-4} & H_{-3} & H_{-2} & H_{-1} & H_0 & H_1 \\ H_{-2} & H_{-1} & H_0 & H_1 & H_2 & H_3 & H_4 \\ H_{-2} & H_{-1} & H_0 & H_1 & H_2 & H_3 & H_4 \end{bmatrix}$$

(obtained by deleting the first through third rows and columns $5,3$, and 2) is given by

$$\begin{bmatrix} H_2 & H_5 & H_7 & H_8 \\ H_{-5} & H_{-2} & H_0 & H_1 \\ H_{-2} & H_1 & H_3 & H_4 \\ H_{-2} & H_1 & H_3 & H_4 \end{bmatrix}$$

which is precisely the matrix defined by $(\mathbb{M}_{\mu/\lambda})_{i,j} = H_{(\mu_i - i)-(\lambda_j - j)}$

5.3 Skewing by non-partition shapes

We now apply NSym analogues of well-known Sym constructions to produce the homogeneous expansion of an immaculate function skewed by a non-partition shape.

First assume that λ is an integer sequence, all of whose parts are nonnegative. As in the analogous Sym situation, if we set

$$\hat{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \lambda_{p+1} - 1, \lambda_p + 1, \lambda_{p+2}, \ldots, \lambda_k)$$

and $(\mathbb{M}_{\mu/\hat{\lambda}})_{i,j} = H_{(\mu_i - i)-(\hat{\lambda}_j - j)}$, then $(\mathbb{M}_{\mu/\lambda})_{i,j} = (\mathbb{M}_{\mu/\hat{\lambda}})_{i,j}$ for $j \notin \{p, p+1\}$. Furthermore, since $\hat{\lambda}_{p+1} = \lambda_p + 1$ and $\hat{\lambda}_p = \lambda_{p+1} - 1$, we have

$$(\mathbb{M}_{\mu/\lambda})_{i,p} = H_{(\mu_i - i)-(\lambda_p - p)} = H_{(\mu_i - i)-(\hat{\lambda}_{p+1} - p - 1)} = (\mathbb{M}_{\mu/\hat{\lambda}})_{i,p+1}$$

and

$$(\mathbb{M}_{\mu/\lambda})_{i,p+1} = H_{(\mu_i - i)-(\lambda_{p+1} - (p+1))} = H_{(\mu_i - i)-(\hat{\lambda}_p + 1 - (p+1))} = (\mathbb{M}_{\mu/\hat{\lambda}})_{i,p}$$

for all $1 \leqslant i \leqslant k$. Thus, $\mathfrak{S}_{\mu/\lambda} = -\mathfrak{S}_{\mu/\hat{\lambda}}$, since $\mathbb{M}_{\mu/\hat{\lambda}}$ is obtained from $\mathbb{M}_{\mu/\lambda}$ by swapping columns p and $p+1$.

Note that if $\lambda_{i+1} = \lambda_i + 1$, for some $1 \leqslant i < k$, then columns i and $i+1$ of $\mathbb{M}_{\mu/\lambda}$ are identical. Thus the row by row expansion of $\mathfrak{Det}(\mathbb{M}_{\mu/\lambda})$ starting in the first row and continuing down yields that $\mathfrak{S}_{\mu/\lambda} = \mathfrak{Det}(\mathbb{M}_{\mu/\lambda}) = 0$.

When taken together, the results of the two previous paragraphs imply that, without loss of generality, we may assume that the entries of λ are weakly decreasing. If the entries of λ are not weakly decreasing apply the straightening operator $\hat{\lambda}$ (adjusting the sign each time) until the result is a partition (in which case Theorem 2 applies) or a sequence containing a one-step increase (in which case the skew immaculate function is zero).

Finally, if any terms of λ are negative, let λ_j be the smallest part of λ. Add $-\lambda_j$ to every part of μ and every part of λ; call the resulting sequences $aug_{\lambda_j}(\mu)$ and $aug_{\lambda_j}(\lambda)$ respectively. Then $\mathbb{M}_{aug_{\lambda_j}(\mu)/aug_{\lambda_j}(\lambda)} = \mathbb{M}_{\mu/\lambda}$, so

$$\mathfrak{S}_{\mu/\lambda} = \mathfrak{S}_{aug_{\lambda_j}(\mu)/aug_{\lambda_j}(\lambda)} \tag{15}$$

and we can apply the above techniques to $\mathfrak{S}_{aug_{\lambda_j}(\mu)/aug_{\lambda_j}(\lambda)}$ to find the decomposition of $\mathfrak{S}_{\mu/\lambda}$ into the complete homogeneous noncommutative symmetric functions.

Example 46. Let $\mu = (2, -5, 0, 1)$ and $\lambda = (2, -3, 1, 6)$. To compute the expansion of $\mathfrak{S}_{\mu/\lambda}$ into the complete homogeneous noncommutative symmetric functions combinatorially, we must apply several steps before finding the tunnel hook coverings.

1. Since λ includes negative parts (with smallest part equal to -3), add 3 to every part of λ and every part of μ to get

$$aug_3(\mu)/aug_3(\lambda) = (5, -2, 3, 4)/(5, 0, 4, 9).$$

2. Since $aug_3(\lambda)$ is not a partition, apply the straightening operator to get

$$\mathfrak{S}_{\mu/\lambda} = -\mathfrak{S}_{(5,-2,3,4)/(5,0,8,5)} = \mathfrak{S}_{(5,-2,3,4)/(5,7,1,5)}$$
$$= -\mathfrak{S}_{(5,-2,3,4)/(6,6,1,5)} = \mathfrak{S}_{(5,-2,3,4)/(6,6,4,2)}.$$

The following corollary (an immediate consequence of Proposition 44 and Proposition 31) provides a way to expand the immaculates $\mathfrak{S}_\mu$ in terms of skew immaculates. A *linear permutation* (or *partial permutation*) of a k-element set is an ordered arrangement of an m-element subset of a k-element set. (Note that in the literature, these are also sometimes referred to as partial permutations or k-permutations.)

Let $A_{k,m}$ be the set of all linear permutations of an m-element subset of a k-element set. Define the *sign* $\epsilon(\pi)$ of a linear permutation $\pi = \pi_1 \cdots \pi_m$ to be $(-1)^\delta$, where δ is the number of pairs a, b such that either $a = \pi_i > \pi_j = b$ with $i < j$ or $a = \pi_i > q$ with q a positive integer not equal to π_j for any j.

Corollary 47. *Let $\mu \in \mathbb{Z}^k$ and m be a fixed integer such that $1 \leqslant m \leqslant k$. If $\tilde{\mu}$ is the sequence obtained by deleting the first m parts of μ, then*

$$\mathfrak{S}_\mu = \sum_{\pi \in A_{k,m}} \epsilon(\pi) \left(\prod_{1 \leqslant i \leqslant m} H_{\mu_i - i + \pi_i} \right) \mathfrak{S}_{\tilde{\mu}/\nu^{(m)}},$$

where $\nu^{(m)}$ is the sequence obtained from the construction of the tunnel hooks corresponding to π in the diagram D_μ.

For example, when $\mu = (4, 3, 3, 2)$ and $m = 2$, we have the following decomposition of $\mathfrak{S}_{(4,3,3,2)}$.

$$\begin{aligned}
\mathfrak{S}_{(4,3,3,2)} = {} & H_{(4,3)}\mathfrak{S}_{(3,2)} - H_{(4,4)}\mathfrak{S}_{(3,2)/(1,0)} + H_{(4,5)}\mathfrak{S}_{(3,2)/(1,1)} \\
& - H_{(5,2)}\mathfrak{S}_{(3,2)} + H_{(5,4)}\mathfrak{S}_{(3,2)/(2,0)} - H_{(5,5)}\mathfrak{S}_{(3,2)/(2,1)} \\
& + H_{(6,2)}\mathfrak{S}_{(3,2)/(1,0)} - H_{(6,3)}\mathfrak{S}_{(3,2)/(2,0)} + H_{(6,5)}\mathfrak{S}_{(3,2)/(2,2)} \\
& - H_{(7,2)}\mathfrak{S}_{(3,2)/(1,1)} + H_{(7,3)}\mathfrak{S}_{(3,2)/(2,1)} - H_{(7,4)}\mathfrak{S}_{(3,2)/(2,2)}
\end{aligned}$$

Note again that applying the forgetful map to this expansion produces the expansion of a Schur function in terms of skew Schur functions with complete homogeneous symmetric functions as coefficients. Thus we have the following Schur function decomposition.

$$\begin{aligned}
s_{(4,3,3,2)} = {} & h_{(4,3)}s_{(3,2)} - h_{(4,4)}s_{(3,2)/(1)} + h_{(4,5)}s_{(3,2)/(1,1)} \\
& - h_{(5,2)}s_{(3,2)} + h_{(5,4)}s_{(3,2)/(2)} - h_{(5,5)}s_{(3,2)/(2,1)} \\
& + h_{(6,2)}s_{(3,2)/(1)} - h_{(6,3)}s_{(3,2)/(2)} + h_{(6,5)}s_{(3,2)/(2,2)} \\
& - h_{(7,2)}s_{(3,2)/(1,1)} + h_{(7,3)}s_{(3,2)/(2,1)} - h_{(7,4)}s_{(3,2)/(2,2)}
\end{aligned}$$

5.4 Recovering the Schur function decomposition

The forgetful map applied to immaculate functions produces the Schur functions, and the forgetful map applied to the H basis for NSym produces the h basis for Sym. Therefore, applying the forgetful map to the decomposition of $\mathfrak{S}_{\mu/\lambda}$ into the H basis (where μ is a partition and λ is a partition such that $\lambda_i \leqslant \mu_i$ for all i) produces the decomposition of the skew Schur function $s_{\mu/\lambda}$ into the h basis.

Our approach, therefore, provides an alternative to the special rim hooks appearing in the Eğecioğlu-Remmel [10] combinatorial interpretation of the inverse Kostka matrix. While tunnel hooks and special rim hooks share some similarities, they are not simply shifts or translates of one another.

Rim hooks are collections of cells in the diagram of a partition satisfying the following properties.

1. Rim hooks consist of cells on the northeastern rim of the diagram.

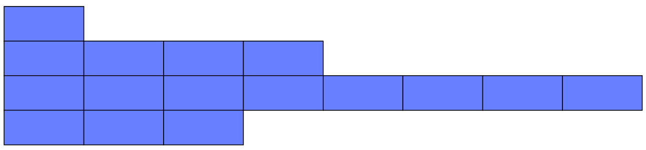
2. The cells in a rim hook are connected.

3. A rim hook contains no 2×2 squares.

4. A rim hook is *special* if it includes a cell in the leftmost column.

If one attempts to apply rim hooks to immaculates indexed by shapes which are not partitions, one must sacrifice either property (1) or (2). Example 48 illustrates this obstruction, since there must be a covering of the shape $(3, 8, 4, 1)$ by special rim hooks whose lengths are $6, 0, 8, 2$ respectively. In order to have a special rim hook of length 8 beginning in the third row, there cannot also be a connected special rim hook of length 6 beginning in the top row.

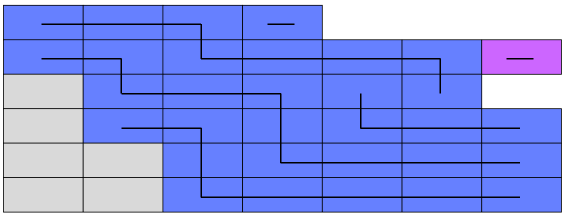
Example 48. Let $\mu = (3, 8, 4, 1)$ and consider the term $H_{(6,0,8,2)}$ in the expansion of $\mathfrak{S}_{(3,8,4,1)}$. To fill the shape $(3, 8, 4, 1)$ with special rim hooks, a special rim hook of length 6 starting in the top row is necessary, but this is not compatible with a special rim hook of length 8 starting in the third row from the top.



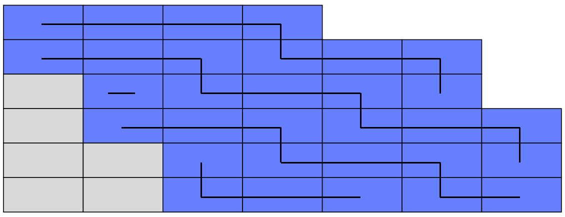
If one relaxes the rules for special rim hooks to try to address this concern, other lengths (such as an initial rim hook of length 7) become available which are not legal options since they do not appear as indices in the H-decomposition of $\mathfrak{S}_{(6,0,8,2)}$. In fact, Loehr and Niese point out that their special rim hook method (which solely applies to immaculates indexed by partitions) does not compute the content of a diagram simply by listing the lengths in some predetermined order [18].

In order to generalize the Eğecioğlu-Remmel decomposition idea to composition shapes (for NSym), we need objects that "tunnel" into the interior of the diagram. Our tunnel hooks satisfy properties (2) and (3) above, as well as a variant of (1) stating that tunnel hooks are comprised of cells on the South-Western border of the diagram. See Example 49 to compare a tunnel hook covering of skew shape $(7, 7, 7, 6, 6, 4)/(2, 2, 1, 1)$ and the corresponding special rim hook tableau.

Example 49. Let $\mu = (7, 7, 7, 6, 6, 4)$ and $\lambda = (2, 2, 1, 1)$. The tunnel hook covering that produces the term $H_{(8,10,4,8,0,1)}$ is shown.



The corresponding special rim hook tableau is shown below. Note that this produces the term $h_{(10,8,8,4,1,0)}$ due to the commutativity of Sym. There are no other rim hook tableaux of this shape corresponding to $h_{(10,8,8,4,1,0)}$.



Our tunnel hook approach also allows us to generalize to ribbon decompositions of certain immaculate functions, improving upon results of Campbell [8].

6 Ribbon decompositions of immaculate functions

The ribbon basis for NSym is another collection of Schur-like functions in NSym. There are a number of excellent sources for background on ribbons in Sym and NSym [12, 20]. Our work in this section extends results of Campbell [8]. We take the following formula as our definition for the ribbon functions.

Definition 50 (ribbon functions). The *ribbon function* in NSym indexed by a composition α is given by the formula

$$R_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} H_\beta,$$

where $\succeq$ is the refinement order on compositions and where $\ell(\alpha)$ is the length (i.e. number of parts) of α.

The set of all ribbon functions indexed by compositions form a basis for NSym. Here for $a \in \mathbb{Z}^{\leqslant 0}$, set $R_a = H_a = 0$ and $R_0 = H_0 = 1$. We will make use of the formula that converts a complete homogeneous function in NSym indexed by a composition into the ribbon basis for NSym [12]. That is,

$$H_\alpha = \sum_{\beta \succeq \alpha} R_\beta.$$

The ribbon basis for NSym is dual to the fundamental basis for QSym. That is, when $\langle \cdot, \cdot \rangle$ is the pairing defined by $\langle H_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$, we have

$$\langle R_\alpha, F_\beta \rangle = \delta_{\alpha,\beta}.$$

Multiplication in the H basis is fairly straightforward; simply concatenate the indexing compositions. Multiplication in the ribbon basis requires two steps, which are described below.

Definition 51 (concatenation, near concatenation and ribbon multiplication). [12] Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_j)$ be integer sequences. Then the concatenation of α and β, denoted by $\alpha \cdot \beta$, is given by

$$\alpha \cdot \beta = (\alpha_1, \alpha_2, \cdots, \alpha_k, \beta_1, \beta_2, \cdots, \beta_j).$$

Define *near concatenation* $\odot$ by

$$\alpha \odot \beta = (\alpha_1, \alpha_2, \cdots, \alpha_k + \beta_1, \beta_2, \cdots, \beta_j).$$

The product of two ribbons R_α and R_β, where α and β are compositions, is given by

$$R_\alpha R_\beta = R_{\alpha \cdot \beta} + R_{\alpha \odot \beta}. \tag{16}$$

We now describe a method for expanding the *refinement* order to weak compositions, by first defining a coarsening of a sequence (generalizing the notion of coarsening compositions).

Definition 52 (refinement ordering and coursening). Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a weak composition. Given a subset $S = \{i_1, i_2, \ldots i_t\}$ of $[k-1]$, the *coarsening* $\Theta(\alpha, S)$ *of* α *with respect to* S is given by

$$\Theta(\alpha, S) = (\alpha_1 \star_1 \alpha_2 \star_2 \alpha_3 \star_3 \cdots \star_{k-2} \alpha_{k-1} \star_{k-1} \alpha_k),$$

where

$$\star_i = \begin{cases} + & \text{if } i \in S. \\ , & \text{otherwise.} \end{cases} \tag{17}$$

The *refinement* ordering on weak compositions is given by $\beta \succeq \alpha$ if and only if β is a *coarsening* of α. If $\beta \succeq \alpha$, we say that α is a *refinement* of β.

For example, let $\alpha = (5, 2, 1, 4, 3, 3, 2, 6, 2, 3)$ and let $S = \{2, 3, 5, 8\}$. Then the coarsening of α with respect to S is

$$\Theta(\alpha, S) = (5, 2 + 1 + 4, 3 + 3, 2, 6 + 2, 3) = (5, 7, 6, 2, 8, 3).$$

Therefore $(5, 7, 6, 2, 8, 3) \succeq (5, 2, 1, 4, 3, 3, 2, 6, 2, 3)$. Note that as long as α is a composition (rather than a weak composition), there is a unique subset producing each distinct coarsening.

When we expand an immaculate function indexed by a composition into the complete homogeneous basis for NSym, some of the terms are indexed by weak compositions. Since $H_0 = 1$, we usually simply delete the zeros and consider the indices to be shorter compositions. However, the proofs in this section are aided by keeping track of the positions of the zeros in the indexing compositions. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a weak composition whose subsequence of nonzero parts is $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_s}$ (with $j_1 < j_2 < \ldots < j_s$), let $flat(\alpha)$ be the composition $(\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_s})$. It will be useful to think of $flat(\alpha)$ as a coarsening; to do so, we standardize notation for the coarsening set.

In this section, all immaculate functions will be indexed by compositions (rather than weak compositions or integer sequences). To indicate this shift, we use α rather than μ as our index for immaculate functions going forward. This means that all indices appearing in the first row of the matrix $\mathbb{M}_\alpha$ used to construct the immaculate function $\mathfrak{S}_\alpha$ (see Theorem 8) are positive integers. Thus the Δ values indexing the H functions will always begin with a positive integer. Therefore all of the weak compositions appearing in the H expansions in this section will begin with a nonzero part.

Definition 53 (allowable flat coarsening subset). Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a weak composition with $\alpha_1 > 0$ and let the set $\{i_1, \ldots, i_j\}$ be the collection of indices such that $\alpha_{i_1} = 0, \alpha_{i_2} = 0, \ldots \alpha_{i_j} = 0$. Let $Z = \{i_1 - 1, \ldots, i_j - 1\} \subseteq [k-1]$. Then an *allowable flat coarsening subset* for α is any subset of $[k-1]$ containing Z.

For example, let $\alpha = (5, 0, 3, 0, 1, 5, 0, 4)$ and consider the composition $(5, 3, 1, 9)$ obtained by flattening and coarsening α. Since $\alpha_2 = 0, \alpha_4 = 0$, and $\alpha_7 = 0$, any allowable flat coarsening subset for α must contain the subset $\{1, 3, 6\}$. The only allowable flat coarsening subset producing $(5, 3, 1, 9)$ is $S = \{1, 3, 6, 7\}$. Although the coarsening of α with respect to the set $Q = \{2, 3, 6, 7\}$ also produces the composition $(5, 0+3+0, 1, 5+0+4) = (5, 3, 1, 9)$, Q is not an allowable flat coarsening subset since $\alpha_2 = 0$ but $2 - 1 \notin Q$.

Notice that for any weak composition α and any allowable flat coarsening subset S, the coarsening $\Theta(\alpha, S)$ will be a composition. The main reason for introducing the allowable flat coarsening subset terminology is to provide a unique way to represent the coarsening of a weak composition.

Lemma 54. *Given a weak composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ such that $\alpha_1 \neq 0$, and a composition α' such that α' is a coarsening of $flat(\alpha)$, there is a unique allowable flat coarsening subset S such that $\Theta(\alpha, S) = \alpha'$.*

Proof. Suppose S and $\hat{S}$ are two distinct allowable flat coarsening subsets such that $\Theta(\alpha, S) = \alpha' = \Theta(\alpha, \hat{S})$. Let s be the smallest integer that is in exactly one of S and $\hat{S}$. Without loss of generality, assume that $s \in S$ and $s \notin \hat{S}$. Furthermore, we have $\alpha_{s+1} > 0$, since if $\alpha_{s+1} = 0$ then s must be a member of any allowable flat coarsening subset for α, including $\hat{S}$. Now, let s' be the smallest of the set of consecutive positive integers in S that ends with s. Then

$$\alpha_{s'} + \alpha_{s'+1} + \cdots + \alpha_s < \alpha_{s'} + \alpha_{s'+1} + \cdots + \alpha_s + \alpha_{s+1},$$

since $\alpha_{s+1} > 0$. The positive integer $\alpha_{s'} + \alpha_{s'+1} + \cdots + \alpha_s$ is one of the parts of $\Theta(\alpha, \hat{S})$, say the i^{th} part. Then $\alpha_{s'} + \alpha_{s'+1} + \cdots + \alpha_s + \alpha_{s+1}$ is the i^{th} part of $\Theta(\alpha, S)$. But this means that $\Theta(\alpha, \hat{S}) \neq \Theta(\alpha, S)$, since their i^{th} parts do not agree. Therefore there must be a unique allowable flat coarsening subset S such that $\Theta(\alpha, S) = \alpha'$. $\square$

6.1 Ribbon expansions of immaculate functions

The ribbon basis expands positively into the immaculate basis via standard immaculate tableaux [5], but the expansion of the immaculate basis into the ribbon basis is only known for certain special cases. In particular, Campbell provides the following formulas; one for the ribbon expansion of immaculate functions indexed by rectangles and one for immaculate functions indexed by products of two rectangles satisfying certain size conditions.

Theorem 55. *[8] The ribbon expansion of an immaculate function indexed by a rectangle (m^k) is given by*

$$\mathfrak{S}_{(m^k)} = \sum_{\sigma \in S_k} \epsilon(\sigma) R_{(m-1+\sigma_1, m-2+\sigma_2, \ldots, m-k+\sigma_k)},$$

with the convention that R_α vanishes if α contains any nonpositive parts.

Theorem 56. *[8] The ribbon expansion of an immaculate function indexed by the product $\alpha = (a^b, c^d)$ of rectangles satisfying $b \leqslant c$ and $b \leqslant a$ is given by*

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_k} \epsilon(\sigma) R_{(\alpha_1 - 1+\sigma_1, \alpha_2 - 2+\sigma_2, \ldots, \alpha_k - k+\sigma_k)},$$

with the convention that R_α vanishes if α contains any nonpositive parts.

It is not true in general that

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_k} \epsilon(\sigma) R_{(\alpha_1 - 1+\sigma_1, \alpha_2 - 2+\sigma_2, \ldots, \alpha_k - k+\sigma_k)}. \tag{18}$$

For example,

$$\begin{aligned} \mathfrak{S}_{(1,3,1)} &= H_{(1,3,1)} - H_{(2,2,1)} + H_{(3,2)} - H_{(1,4)} \\ &= R_{(1,3,1)} - R_{(2,2,1)} + R_{(3,2)} - R_{(2,3)}. \end{aligned}$$

One open problem is to classify the compositions for which Equation (18) is true. In this section, we provide a partial solution to this question by developing a large class of compositions for which this expansion applies.

Let α be a composition. We describe a function f_i from a tunnel hook covering γ of α to a tunnel hook covering $f_i(\gamma)$ of α. Recall that s_i is the transposition $(i, i+1)$ so that if $\sigma = \sigma_1 \cdots \sigma_n$ then

$$\sigma s_i = \sigma_1 \sigma_2 \cdots \sigma_{i-1} \sigma_{i+1} \sigma_i \sigma_{i+2} \cdots \sigma_n.$$

Definition 57 (function $\mathbf{f_i(\gamma)}$). Let α be a composition and $\gamma \in THC_\alpha$. Let σ be the permutation associated to γ (see Procedure 28 and Lemma 31). Set $f_i(\gamma)$ to be the tunnel hook covering of μ whose associated permutation is σs_i. If Δ is the sequence associated to the tunnel hook covering γ (see Procedure 23), then let $f_i(\Delta)$ denote the sequence associated to the tunnel hook covering $f_i(\gamma)$.

It is immediate that f_i is an involution, since $\sigma s_i s_i = \sigma$. We now describe how applying f_i to a tunnel hook covering impacts the tunnel hooks. Note that Lemma 29 implies that if (p_i, q_i) and (p_{i+1}, q_{i+1}) are distinct terminal cells of a tunnel hook covering, then $p_i - q_i \neq p_{i+1} - q_{i+1}$. Therefore we only need to consider the cases $p_i - q_i < p_{i+1} - q_{i+1}$ and $p_i - q_i > p_{i+1} - q_{i+1}$.

Proposition 58. *Let α be a composition and $\gamma \in THC_\alpha$. Let $\mathfrak{h}(1, \tau_1), \ldots, \mathfrak{h}(k, \tau_k)$ be the tunnel hooks in γ with $\tau_j = (p_j, q_j)$. Then the terminal cells of $f_i(\gamma)$ are*

$$\begin{cases} (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, (p_i + 1, q_i + 1), \tau_{i+2}, \ldots, \tau_k) & \text{if } p_i - q_i < p_{i+1} - q_{i+1}, \\ (\tau_1, \ldots, \tau_{i-1}, (p_{i+1} - 1, q_{i+1} - 1), \tau_i, \tau_{i+2}, \ldots, \tau_k) & \text{if } p_i - q_i > p_{i+1} - q_{i+1}. \end{cases}$$

Proof. Assume $(\tau_1, \ldots, \tau_{i-1}, (p_i, q_i), (p_{i+1}, q_{i+1}), \tau_{i+2}, \ldots, \tau_k)$ are the terminal cells for γ with associated permutation σ. Recall Proposition 31 implies that $\sigma_i = p_i - q_i + 1$. First assume $p_i - q_i < p_{i+1} - q_{i+1}$ (which also means $\sigma_i < \sigma_{i+1}$). Then applying s_i to the associated permutation σ corresponds to selecting terminal cell (p_{i+1}, q_{i+1}) for the i^{th} tunnel hook. In this case the i^{th} tunnel hook now covers cell (p_i, q_i) and therefore by Lemma 26(C), $(p_i + 1, q_i + 1)$ becomes the terminal cell for the tunnel hook corresponding to $(f_i(\sigma))_{i+1} = p_i - q_i + 1$ (writing $f_i(\sigma)$ to denote the permutation associated to $f_i(\gamma)$).

If $p_i - q_i > p_{i+1} - q_{i+1}$, then $f_i(\gamma)$ has permutation $f_i(\sigma)$ with $(f_i(\sigma))_i < (f_i(\sigma))_{i+1}$. Therefore applying the involution f_i to γ effectively "undoes" the operation in the first situation. So the terminal cell for the i^{th} tunnel hook is $(p_{i+1} - 1, q_{i+1} - 1)$ and the terminal cell for the $(i+1)^{th}$ tunnel hook is (p_i, q_i), as desired. □

Lemma 59.

$$-\prod_{\mathfrak{h}(r,\tau_r) \in \gamma} \epsilon(\mathfrak{h}(r, \tau_r)) = \prod_{\mathfrak{h}(r,\tau_r) \in f_i(\gamma)} \epsilon(\mathfrak{h}(r, \tau_r))$$

Proof. The non-commutative determinant of $\mathbb{M}_\alpha$ can be computed according to the formula

$$\mathfrak{det}(\mathbb{M}_\alpha) = \sum_{\sigma \in S_k} \epsilon(\sigma) (\mathbb{M}_\alpha)_{1,\sigma_1} (\mathbb{M}_\alpha)_{2,\sigma_2} \cdots (\mathbb{M}_\alpha)_{k,\sigma_k}. \tag{19}$$

Each tunnel hook covering corresponds to one of the permutations in the summation of Equation (19). The map f_i replaces the tunnel hook corresponding to σ with the tunnel hook corresponding to σs_i, which multiplies the sign by -1. □

We now use tunnel hook coverings to prove that a larger class of compositions than those described in Campbell [8] satisfies Equation (18).

Theorem 60. *Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition such that $\alpha_i \geqslant i$ for $1 \leqslant i \leqslant k$. Then*

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_k} \epsilon(\sigma) R_{(\alpha_1 - 1 + \sigma_1, \alpha_2 - 2 + \sigma_2, \ldots, \alpha_k - k + \sigma_k)}.$$

Proof. Recall the complete homogeneous noncommutative symmetric functions expand into the ribbon basis according to the following formula. Let α be a composition. Then

$$H_\alpha = \sum_{\beta \succeq \alpha} R_\beta, \tag{20}$$

where $\succeq$ is the refinement ordering on compositions.

As in Procedure 23, we set $\Delta_r = \Delta(\mathfrak{h}(r, \tau_r))$ and $\Delta(\gamma) = (\Delta_1, \Delta_2, \cdots, \Delta_k)$, where $\Delta(\gamma)$ is the sequence corresponding to the tunnel hook covering γ. Therefore the homogeneous expansion of the immaculate functions becomes

$$\begin{aligned}
\mathfrak{S}_\alpha &= \sum_{\gamma \in THC_\alpha} \left(\prod_{r=1}^{k} \epsilon(\mathfrak{h}(r, \tau_r)) \ H_{\Delta_r} \right) \\
&= \sum_{\gamma \in THC_\alpha} \left(\prod_{r=1}^{k} \epsilon(\mathfrak{h}(r, \tau_r)) \right) H_{\Delta(\gamma)} \\
&= \sum_{\gamma \in THC_\alpha} \left(\prod_{r=1}^{k} \epsilon(\mathfrak{h}(r, \tau_r)) \right) \left(\sum_{\beta \succeq flat(\Delta(\gamma))} R_\beta \right), \qquad \text{by Equation (20).}
\end{aligned}$$

Note that since $\alpha_r \geqslant r$ for all $1 \leqslant r \leqslant k$, it is not possible for all the cells in row r to be covered by tunnel hooks originating in rows 1 through $r-1$, so every tunnel hook starts within the GBPR diagram $D_\alpha^{(0)}$. Therefore $\Delta(\mathfrak{h}(r, \tau_r)) > 0$ for all $1 \leqslant r \leqslant k$ for every tunnel hook covering γ. Every indexing composition β in the ribbon expansion can be thought of as a pair (Δ, S) where $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_k)$ is the sequence corresponding to a tunnel hook covering γ and S is the unique allowable flat coarsening subset such that $\beta = \Theta(\Delta, S)$.

We describe a sign-reversing involution on the pairs (Δ, S) with $S \neq \emptyset$, proving that every term with less than k parts cancels out. For an arbitrary pair (Δ, S) corresponding to a term $R_{\Theta(\Delta,S)}$, let i be the smallest element in S and set $g(\Delta, S) = (f_i(\Delta), S)$. We will show that $R_{\Theta(f_i(\Delta),S)}$ cancels out $R_{\Theta(\Delta,S)}$. The map f_i is a sign-reversing involution (by Definition 57 and Lemma 59), so that $R_{\Theta(f_i(\Delta),S)}$ and $R_{\Theta(\Delta,S)}$ appear with opposite signs. Any coarsening set S is an allowable flat coarsening subset for any indexing composition of a tunnel hook covering of α (since all parts of α are nonzero).

Since $(f_i(\Delta))_j = \Delta_j$ for all $j \neq i, i+1$ and $(f_i(\Delta))_{i+1} + (f_i(\Delta))_i = \Delta_i + \Delta_{i+1}$, we have $\Theta(f_i(\Delta), S) = \Theta(\Delta, S)$. Thus the ribbon function indexed by $\Theta(f_i(\Delta), S)$ cancels out the ribbon function indexed by $\Theta(\Delta, S)$ since they occur with opposite signs. Therefore the only terms appearing in the ribbon expansion after cancellation are exactly the R_β such that the length of the composition β equals k, completing the proof. $\square$

We next use tunnel hooks to provide a combinatorial proof of Campbell's ribbon expansion of immaculate functions [8]. First, we provide a lemma that will be useful in our proof.

Lemma 61. *Let γ be a tunnel hook covering of the diagram of a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$. Let j be the smallest value such that $\alpha_i = \alpha_j$ for all i such that $j \leqslant i \leqslant k$. If $\mathfrak{h}(r, \tau_r)$ is a tunnel hook of γ with $j \leqslant r \leqslant k$ and such that $\mathfrak{h}(r, \tau_r)$ includes at least one cell that is purple in the GBPR diagram D_α, then either $\mathfrak{h}(r, \tau_r)$ consists of a single purple cell or there exists a tunnel hook $\mathfrak{h}(s, \tau_s)$ in γ with $r \leqslant s \leqslant k$ such that $\Delta(\mathfrak{h}(s, \tau_s)) < 0$.*

Proof. Assume that $\mathfrak{h}(r, \tau_r)$ is a tunnel hook with $j \leqslant r \leqslant k$ containing at least one cell that is purple in the GBPR diagram D_α. Note that since α is a composition, all cells in D_α are either blue or purple. If $\mathfrak{h}(r, \tau_r)$ consists of just one cell then this cell is either red (meaning $\mathfrak{h}(r, \tau_r) < 0$) or purple, so we are done. Therefore we may assume that $\mathfrak{h}(r, \tau_r)$ contains more than one cell. We will show that there exists a tunnel hook $\mathfrak{h}(s, \tau_s)$ in γ with $r \leqslant s \leqslant k$ such that $\Delta(\mathfrak{h}(s, \tau_s)) < 0$. First, note that for any two cells in a tunnel hook, if one is strictly north of the other, then it must also be weakly west of the other. Since $\alpha_i = \alpha_j$ for all $j \leqslant i \leqslant k$, if a cell in row i of the tunnel hook $\mathfrak{h}(r, \tau_r)$ is blue in the diagram D_α, then all cells of $\mathfrak{h}(r, \tau_r)$ in higher (more northernly) rows must also be blue in the diagram D_α. This means that in order to contain a cell that is purple in D_α, the southernmost row of the tunnel hook $\mathfrak{h}(r, \tau_r)$ must contain a cell that is purple in D_α. Let $D^{(t)}_{\alpha/\nu^{(t)}} \setminus D_\alpha$ be the set of all red cells in $D^{(t)}_{\alpha/\nu^{(t)}}$. Note that these are precisely the cells converted from purple to red during the construction of the first t tunnel hooks. If any cells in row r are red and $\mathfrak{h}(r, \tau_r)$ terminates in row r, then $\Delta(\mathfrak{h}(r, \tau_r)) < 0$ and we are done. Therefore assume that either no cells in row r are red or the tunnel hook $\mathfrak{h}(r, \tau_r)$ terminates in a row higher than row r. If no cells in row r are red, $\mathfrak{h}(r, \tau_r)$ must contain exactly one purple cell in row r, immediately to the right of a cell that is blue in D_α. This means that the terminal cell of $\mathfrak{h}(r, \tau_r)$ must be in a higher row than row r since otherwise $\mathfrak{h}(r, \tau_r)$ would consist of only one cell.

We may now assume $\mathfrak{h}(r, \tau_r)$ terminates in a row higher than row r. Then $\mathfrak{h}(r, \tau_r)$ includes at least one purple or red cell in row $r+1$ (the cell immediately above the initial cell in $\mathfrak{h}(r, \tau_r)$), which creates at least one red cell in row $r+1$ during Step 2(c) of Procedure 23.

We have now shown that any tunnel hook $\mathfrak{h}(r, \tau_r)$ such that $j \leqslant r \leqslant k$ including a cell that is purple in D_α and also consisting of more than one cell must result in at least one red cell in row $r+1$. Now the tunnel hook $\mathfrak{h}(r+1, \tau_{r+1})$ includes the red cell in row $r+1$ and therefore can only be nonnegative if it introduces a red cell in row $r+2$. Iterating this argument shows that under the conditions on α in this lemma, once a red cell is introduced, there must be an additional red cell in a higher row. Therefore there exists a positive integer $r \leqslant s \leqslant k$ such that $\Delta(\mathfrak{h}(s, \tau_s)) < 0$. ☐

We are now ready to use tunnel hook coverings to prove Campbell's Rectangle Theorem.

Theorem 62. *[8] Let $\alpha = (m^k)$ be a rectangle with k rows each of length m. Then*

$$\mathfrak{S}_{(m^k)} = \sum_{\sigma \in S_n} \epsilon(\sigma) R_{(m-1+\sigma_1, m-2+\sigma_2, \ldots, m-k+\sigma_k)},$$

with the convention that R_β vanishes if β contains any nonpositive parts.

Proof. Let $\alpha = (m^k)$ be a rectangle with k rows each of length m. Letting $j = 1$ in Lemma 61 implies that if γ is a tunnel hook covering of (m^k) such that $\Delta(\mathfrak{h}(r, \tau_r)) \geqslant 0$ for all $1 \leqslant r \leqslant k$, then every tunnel hook $\mathfrak{h}(r, \tau_r)$ in γ either consists of a single cell that is purple in D_α (so that $\Delta(\mathfrak{h}(r, \tau_r)) = 0$) or is contained entirely within the rectangle (m^k) (so that $\Delta(\mathfrak{h}(r, \tau_r)) > 0$).

We claim that for the rectangle (m^k), we cannot have two different tunnel hook coverings $\gamma \neq \gamma'$ such that $flat(\Delta(\gamma)) = flat(\Delta(\gamma'))$ and both $(\Delta(\gamma))_i \geqslant 0$ and $(\Delta(\gamma'))_i \geqslant 0$ for all i such that $1 \leqslant i \leqslant k$. To see this, assume $flat(\Delta(\gamma)) = flat(\Delta(\gamma'))$ for some $\gamma \neq \gamma'$ such that $(\Delta(\gamma))_i \geqslant 0$ and $(\Delta(\gamma'))_i \geqslant 0$ for all $1 \leqslant i \leqslant k$. Let j be the smallest positive integer such that $(\Delta(\gamma))_j \neq (\Delta(\gamma'))_j$. If both $(\Delta(\gamma))_j > 0$ and $(\Delta(\gamma'))_j > 0$, then their flattenings must differ at the corresponding position. One of $(\Delta(\gamma))_j$ or $(\Delta(\gamma'))_j$ must therefore be zero. Assume without loss of generality that $(\Delta(\gamma))_j = 0$. Then the tunnel hook of γ with lowest row j must include a purple cell in row j of D_α. But then the tunnel hook of γ' with lowest row j must also contain a purple cell in row j of D_α, and therefore Lemma 61 implies that $(\Delta(\gamma'))_j = 0$, contradicting the assumption that $(\Delta(\gamma))_j \neq (\Delta(\gamma'))_j$. Therefore there is only one tunnel hook covering γ such that $(\Delta(\gamma))_i \geqslant 0$ for all $1 \leqslant i \leqslant k$ with flattening $flat(\Delta(\gamma))$.

We may therefore write the complete homogeneous expansion of $\mathfrak{S}_{(m^k)}$ using the flattenings $flat(\Delta)$ as the subscripts, keeping track of the flattenings by associating the term $flat(\Delta)$ with the pair (Δ, S), where S is the unique allowable flat coarsening subset (see Lemma 54) such that $flat(\Delta) = \Theta(\Delta, S)$. Since only one tunnel hook covering produces a given flattened composition, no entries cancel out in the complete homogeneous expansion.

To prove that the ribbon expansion satisfies Equation (18), we first ignore any tunnel hook covering γ such that $\Delta(\gamma)$ includes negative parts since $H_z = 0$ when $z < 0$. Next, we construct a sign-reversing involution on pairs (Δ, S) where Δ is a weak composition obtained from a tunnel hook covering of (m^k) and $S \neq \emptyset$ is a nontrivial allowable flat coarsening subset for Δ.

Consider an arbitrary tunnel hook covering γ of α described by tunnel cells $(\tau_1, \tau_2, \ldots, \tau_i, \tau_{i+1}, \ldots, \tau_k)$ and corresponding indexing weak composition $\Delta(\gamma) = (\Delta_1, \Delta_2, \ldots, \Delta_k)$ such that $\Delta_i \geqslant 0$ for all i. Let R_β be a term in the ribbon expansion of $\mathfrak{S}_\alpha$ with $\beta = \Theta(\Delta, S)$ such that $S = \{j_1, \ldots, j_s\}$ is a nontrivial allowable flat coarsening subset for Δ. Let i be the index of the smallest element of S. If τ_i is not contained within the rectangle (m^k), then $\Delta_i = 0$ by Lemma 61. But if so, then $i - 1 \in S$ (by the definition of allowable flat coarsening subset) which contradicts the assumption that i is the smallest element of S. Therefore τ_i must be contained within the rectangle (m^k) and we must have $\Delta_i > 0$.

Next, set $g(\Delta, S) = (f_i(\Delta), S)$, where $f_i(\Delta)$ is the sequence obtained by applying f_i to the tunnel hook covering with associated sequence Δ. This map is sign-reversing by Lemma 59. Since f_i is an involution and the set S is unchanged, we must prove that $f_i(\Delta)$ is a weak composition and also that S is an allowable flat coarsening subset for $f_i(\Delta)$. If so, we claim that the ribbon term indexed by $\Theta(\Delta, S)$ is cancelled out by the ribbon term indexed by $\Theta(f_i(\Delta, S)$ and therefore every R_β whose indexing composition β has length less than k will be cancelled out in the ribbon expansion.

To prove our claim, first note that Lemmas 31 and 37 imply that $(f_i(\Delta))_j = \Delta_j$ for $j \neq i, i+1$. Therefore to see that $f_i(\Delta)$ is a weak composition, we must prove that $(f_i(\Delta))_i \geqslant 0$ and $(f_i(\Delta))_{i+1} \geqslant 0$.

Recall that $\Delta_i > 0$ since i is the smallest element in the allowable flat coarsening subset S. This means that the i^{th} tunnel hook in γ is contained within the diagram (m^k). Since the map f_i changes the terminal cell but not the initial cell, Lemma 61 implies that the i^{th} tunnel hook in $f_i(\gamma)$ will also be contained entirely within (m^k). Therefore $(f_i(\Delta))_i > 0$ and we now only need to prove that $(f_i(\Delta))_{i+1} \geqslant 0$. The only way for $(f_i(\Delta))_{i+1} < 0$ is if the initial cell of the $(i+1)^{th}$ tunnel hook of $f_i(\gamma)$ is red. But red cells can only be created when earlier tunnel hooks cover purple or red cells. Since all earlier tunnel hooks are either contained entirely within the (m^k) or consist of a single purple cell by Lemma 61, no red cells appear in row $i+1$ and therefore the smallest possible value for $(f_i(\Delta))_{i+1}$ is 0.

Next, to prove that S is an allowable flat coarsening subset for $f_i(\Delta)$, we must show that if $(f_i(\Delta))_r = 0$, then $r-1 \in S$. First consider the case that $r \neq i, i+1$. Recall that $(f_i(\Delta))_r = \Delta_r$ for $r \neq i, i+1$. If $(f_i(\Delta))_r = 0$, then $\Delta_r = 0$, and therefore $r-1 \in S$ since S is an allowable flat coarsening subset for Δ. Note that we have already shown $(f_i(\Delta))_i > 0$, so the last case we need to consider is if $(f_i(\Delta))_{i+1} = 0$. But $i \in S$, so S is an allowable flat coarsening subset for $f_i(\Delta)$ and the proof is complete. $\square$

We now combine the classes of compositions introduced in Theorems 60 and 62 to produce a larger class of compositions for which Equation (18) applies, therefore generalizing Campbell's results.

Theorem 63. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition for which there exists J with $1 \leqslant J \leqslant k$ such that $\alpha_\ell \geqslant \ell$ for $1 \leqslant \ell \leqslant J$ and $\alpha_\ell = J$ for $J+1 \leqslant \ell \leqslant k$. Then*

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_k} \epsilon(\sigma) R_{(\alpha_1 - 1 + \sigma_1, \alpha_2 - 2 + \sigma_2, \ldots, \alpha_k - k + \sigma_k)},$$

with the convention that R_β vanishes if β contains any nonpositive parts.

Proof. As in the proof of Theorem 62, we must prove that if R_β (where β is a composition) is a term appearing in the ribbon expansion of $\mathfrak{S}_\alpha$ (after cancellation) then the length of β is k. To see this, we again describe a sign-reversing involution on pairs (Δ, S) where Δ is a weak composition obtained from a tunnel hook covering γ of α and $S \neq \emptyset$ is a nontrivial allowable flat coarsening subset for Δ.

Let $g(\Delta, S) = (f_i(\Delta), S)$ where i is the smallest element in S. As in the proof of Theorem 62, it is enough to show that $f_i(\Delta)$ is a weak composition and S is an allowable flat coarsening subset for $f_i(\Delta)$. To see that $f_i(\Delta)$ is a weak composition, we only need to show $(f_i(\Delta))_i \geqslant 0$ and $(f_i(\Delta))_{i+1} \geqslant 0$, since $(f_i(\Delta))_j = \Delta_j$ for $j \neq i, i+1$ by Lemmas 31 and 37.

First assume $i < J$. Then $(f_i(\Delta))_i > 0$ and $(f_i(\Delta))_{i+1} > 0$ since $\alpha_i \geqslant i$ and $\alpha_{i+1} \geqslant i+1$. Next, consider the case $i = J$. Then $\alpha_i \geqslant i$, so $(f_i(\Delta))_i > 0$ and we must show $(f_i(\Delta))_{i+1} \geqslant 0$. The only way $(f_i(\Delta))_{i+1} < 0$ is if there are red cells in row $i+1$ after the construction of the first i tunnel hooks of $f_i(\gamma)$. But since $i = J$ and all rows of α higher than row $i(= J)$ have length J, all the tunnel hooks in rows 1 through i remain completely inside the original diagram $\mathcal{D}_\alpha$. Therefore there are no red cells anywhere in the diagram after the construction of the first i tunnel hooks of $f_i(\gamma)$. So $(f_i(\Delta))_{i+1} \geqslant 0$.

Finally, assume $i > J$. Then $\alpha_i = J = \alpha_{i+1}$ and in fact $\alpha_r = J$ for all $r \geqslant i$. Therefore Lemma 61 applies to all tunnel hooks of $f_i(\gamma)$ constructed in rows i and above. Since we assumed that $\Delta_r \geqslant 0$ for all $1 \leqslant r \leqslant k$, any tunnel hook $\mathfrak{h}_r$ (where $\mathfrak{h}_r = \mathfrak{h}(r, \tau_r)$) in γ (with $r \geqslant i$) such that $\Delta_r > 0$ must lie entirely inside the diagram $\mathcal{D}_\alpha$. (Otherwise by Lemma 61, there would be a value ℓ such that $\Delta_\ell < 0$, a contradiction.) Since i is the smallest element of S, we must have $\Delta_i > 0$ because if $\Delta_i = 0$ then $i - 1 \in S$ by the allowable flat coarsening subset definition. Therefore since $\Delta_i > 0$, the tunnel hook $\mathfrak{h}_i$ must lie entirely within the diagram $\mathcal{D}_\alpha$. But this means that there are blue cells in row i when the tunnel hook h'_i of $f_i(\gamma)$ originating in row i is constructed. Therefore $(f_i(\Delta))_i > 0$. Every cell of $\mathfrak{h}'_i$ lying in a row higher than row i must be weakly west of the initial cell of $\mathfrak{h}'_i$ and every row above row i has the same length as row i. Therefore $\mathfrak{h}'_i$ is contained entirely inside the diagram $\mathcal{D}_\alpha$. Thus no red cells are created during the construction of $\mathfrak{h}'_i$. Since no red cells are created during the constructions of the tunnel hooks in rows 1 to $i-1$ either, there are no red cells in row $i+1$ of $f_i(\gamma)$ when the tunnel hook $\mathfrak{h}'_{i+1}$ is constructed. Therefore $(f_i(\Delta))_{i+1} \geqslant 0$.

Next, to show S is an allowable flat coarsening subset for $f_i(\Delta)$, we must prove that if $(f_i(\Delta))_r = 0$, then $r - 1 \in S$. Consider first the case that $r \neq i, i+1$. Since S is an allowable flat coarsening subset for Δ and $(f_i(\Delta))_r = \Delta_r$ for $r \neq i, i+1$, if $(f_i(\Delta))_r = 0$, then $\Delta_r = 0$, and therefore $r - 1 \in S$. Note that we have already shown $(f_i(\Delta))_i > 0$, so the last case we need to consider is if $(f_i(\Delta))_{i+1} = 0$. But since $i \in S$, our proof is complete. Therefore S is an allowable flat coarsening subset for $f_i(\Delta)$. $\square$

To see that Campbell's rectangles (a^b, c^d) with $b \leqslant c$ and $b \leqslant a$ are contained in the set of compositions for which Theorem 63 applies, let $\alpha_1 = \alpha_2 = \ldots = \alpha_b = a$ and $\alpha_{b+1} = \alpha_{b+2} = \ldots = \alpha_{b+d} = c$. Set $J = c$ in Theorem 63. For $1 \leqslant \ell \leqslant b$, we have $\alpha_\ell = a \geqslant b \geqslant \ell$. For $b \leqslant \ell \leqslant c$, we have $\alpha_\ell = c \geqslant \ell$. For $c+1 \leqslant \ell \leqslant k$, we have $\alpha_\ell = c$. Therefore Campbell's rectangles satisfy the hypotheses of Theorem 63 with $J = c$, but are certainly not the only compositions satisfying these hypotheses.

The compositions appearing in Theorem 63 are not the full set for which Equation (18) is true. For example,

$$\mathfrak{S}_{1123} = R_{1123} - R_{1132} - R_{1213} + R_{1231} + R_{1312} - R_{1321}$$

but $(1, 1, 2, 3)$ does not satisfy the conditions in Theorem 63 since $\alpha_2 < 2$ but $\alpha_3 \neq \alpha_2$. We, like Campbell, leave the full classification as an open problem.

7 Future Directions

In addition to classifying the compositions for which Equation (18) is true, it would be valuable to identify a combinatorial formula for the expansion of an arbitrary immaculate function into the ribbon basis. Duality could then be used to transform this formula into a formula in QSym for the expansion of an arbitrary fundamental quasisymmetric function into the dual immaculate quasisymmetric basis. This would open up a new approach to Schur positivity since there is a combinatorial formula for the expansion of dual immaculate quasisymmetric functions into the Young quasisymmetric Schur functions, and any symmetric function that expands positively into the Young quasisymmetric Schur functions is Schur positive.

For example, one of the major open problems in algebraic combinatorics is to find a combinatorial formula for the Schur expansion of Macdonald polynomials. Macdonald polynomials can be expanded combinatorially into fundamental quasisymmetric functions using statistics on tableaux [14, 15]. A formula for the expansion of the fundamental quasisymmetric functions into the dual immaculate quasisymemtric functions could be used to construct a combinatorial formula for the expansion of Macdonald polynomials into Young quasisymmetric Schur functions. There would still be signs and cancellation with this approach, but since both steps involve statistics on tableaux, the goal would be to construct a sign-reversing involution on these diagrams to prove Schur positivity.

There are cancellations inherent in Theorem 1 since there are also cancellations in the Jacobi-Trudi formula and in the special rim hook formula [10]. All of the compositions exhibiting cancellation for $n \leqslant 5$ are of the form $(\alpha_1, \alpha_2, \ldots, \alpha_j, \alpha_j + 1, \alpha_{j+2}, \ldots, \alpha_k)$ for some j with $\alpha_j < j$. One can then produce a cancellation-free formula for these situations by ruling out the corresponding tunnel hook coverings. The problem of finding a cancellation-free combinatorial formula for the homogeneous basis expansion of the immaculates then reduces to identifying the cancellations that fall outside of this pattern and determining which tunnel hook coverings should be ruled out to avoid them.

Another interesting problem that could potentially be attacked using the combinatorics introduced in this article is the problem of dual immaculate multiplication. One of the things that makes this problem significantly harder than the classical Littlewood-Richardson situation is the fact that boxes can be removed as well as added. The GBPR diagrams introduced in this paper provide a method for representing this "growth and decay" within the tableau format. In fact, GBPR diagrams could be extended to any situation in which a weak composition is skewed by a partition.

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