Ascent sequences avoiding a triple of length-3 patterns

David Callan[a] Toufik Mansour[b]

Submitted: Jan 18, 2024; Accepted: Jan 22, 2025; Published: Mar 14, 2025

Abstract

An *ascent sequence* is a sequence $a_1a_2\cdots a_n$ consisting of non-negative integers satisfying $a_1 = 0$ and for $1 < i \leqslant n$, $a_i \leqslant \text{asc}(a_1a_2\cdots a_{i-1})+1$, where $\text{asc}(a_1a_2\cdots a_k)$ is the number of ascents in the sequence $a_1a_2\cdots a_k$. We say that two sets of patterns B and C are A-Wilf-equivalent if the number of ascent sequences of length n that avoid B equals the number of ascent sequences of length n that avoid C, for all $n \geqslant 0$. In this paper, we show that the number of A-Wilf-equivalences among triples of length-3 patterns is 62. The main tool is generating trees; bijective methods are also sometimes used. One case is of particular interest: ascent sequences avoiding the 3 patterns 100, 201 and 210 are easy to characterize, but it seems remarkably involved to show that, like 021-avoiding ascent sequences, they are counted by the Catalan numbers.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction

An *ascent*, short for *ascent index*, in an integer sequence $s_1s_2\cdots s_m$ is an index $1 \leqslant j \leqslant m-1$ such that $s_j < s_{j+1}$. An *ascent sequence* $a_1a_2\cdots a_n$ is one consisting of non-negative integers satisfying $a_1 = 0$ and for all i with $1 < i \leqslant n$, $a_i \leqslant \text{asc}(a_1a_2\cdots a_{i-1})+1$, where $\text{asc}(a_1a_2\cdots a_k)$ is the number of ascents in the sequence $a_1a_2\cdots a_k$. For example, the sequence 0102321401 is an ascent sequence, whereas 0104 is not. Bousquet-Mèlou, Claesson, Dukes, and Kitaev [2] connected ascent sequences to $(2+2)$-free posets. Since then, ascent sequences have been considered in a series of papers where connections to many other combinatorial structures have been found (see, for example, [4–7,9–11] as well as [8, Section 3.2.2]).

Let $a = a_1a_2\cdots a_n$ be any sequence and $\tau = \tau_1\cdots\tau_m$ be any *pattern*, that is, a word in $\{0,\ldots,\ell\}^m$ which contains each letter $0,1,\ldots,\ell$ for some $m \geqslant 1$, $\ell \geqslant 0$. We say the

[a]Department of Statistics, University of Wisconsin, Madison, USA (`callan@stat.wisc.edu`).
[b]Department of Mathematics, University of Haifa, 3103301 Haifa, Israel (`tmansour@univ.haifa.ac.il`).

sequence a contains τ if a has a subsequence that is order isomorphic to τ, that is, there is a subsequence $a_{f(1)}, a_{f(2)}, \ldots, a_{f(m)}$, where $1 \leqslant f(1) < f(2) < \cdots < f(m) \leqslant n$, such that $a_{f(i)} X a_{f(j)}$ if and only if $\tau_i X \tau_j$, for all $X \in \{<, >, =\}$ and $1 \leqslant i, j \leqslant m$. Otherwise, a is said to avoid τ. For instance, the ascent sequence 01013043351 has two occurrences of the pattern 110, namely, the subsequences 110 and 331, but avoids the pattern 3120. We denote the set of all ascent sequences that avoid a list of patterns $\tau^{(1)}, \ldots, \tau^{(s)}$ by $A_n(\tau^{(1)}, \ldots, \tau^{(s)})$ or $A_n(\{\tau^{(1)}, \ldots, \tau^{(s)}\})$. We say that two sets of patterns P and Q are A-Wilf-equivalent, denoted $P \overset{a}{\sim} Q$, if $|A_n(P)| = |A_n(Q)|$ for every n.

There are 13 patterns of length 3: 000, 001, 010, 100, 011, 101, 110, 012, 021, 102, 120, 201, and 210. The number of A-Wilf-equivalence classes among single patterns of length 3 is 9 [7]. The number of A-Wilf-equivalence classes among pairs of patterns of length 3 is 35 [1]. The main result of this paper can be formulated as follows.

Theorem 1. *The number of A-Wilf-equivalence classes among triples of length-3 patterns is* 62.

Section 2 describes generating trees, their succession rules, and the candidate Wilf classes. Table 1 in Section 2 lists all the candidate Wilf classes and either gives the corresponding succession rules for the generating trees or a reference to Section 3, which mostly uses bijective methods. The most difficult case, Class 61, is treated by both generating tree and bijective methods.

2 Generating trees

Let P be any set of patterns such that the length of each pattern is at least two. Define $\mathcal{A}(P) = \cup_{n=0}^{\infty} A_n(P)$. We will construct a pattern-avoidance generating tree $\mathcal{T}(P)$ (see [12]) for the class of pattern-avoiding ascent sequences $\mathcal{A}(P)$. Starting with the root 0 which stays at level 1, we construct in a recursive manner the non-root nodes of the tree $\mathcal{T}(P)$ such that the nth level of the tree consists of exactly the elements of $A_n(P)$ arranged so that the parent of an ascent sequence $a_1 \cdots a_n \in A_n(p)$ is the unique ascent sequence $a_1 \cdots a_{n-1} \in A_{n-1}(P)$. The children of $a_1 \cdots a_{n-1} \in A_{n-1}(P)$ are obtained from the set $\{a_1 \cdots a_{n-1} a_n \mid a_n = 0, 1, \ldots, \mathrm{asc}(a_1 \cdots a_{n-1}) + 1\}$ by applying the pattern-avoiding restrictions of the patterns in P. We arrange the nodes from the left to the right so that if $a = a_1 \cdots a_{n-1} i$ and $a' = a_1 \cdots a_{n-1} i'$ are children of the same parent $a_1 \cdots a_{n-1}$, then a appears on the left of a' if $i < i'$. Figure 1 presents the first few levels of $\mathcal{T}(\{011\})$. Clearly, the cardinality of $A_n(P)$ equals the number of nodes in the nth level of $\mathcal{T}(P)$.

For a given set of patterns P. Let $\mathcal{T}(P; a)$ denote the subtree consisting of the ascent sequence a as the root and its descendants in $\mathcal{T}(P)$. For any $a, a' \in \mathcal{T}(P)$, we say that the subtrees $\mathcal{T}(P; a)$ and $\mathcal{T}(P; a')$ are isomorphic, and write $\mathcal{T}(P; a) \cong \mathcal{T}(P; a')$, if these subtrees are isomorphic in the sense of plane (ordered) tree. We define an equivalence relation on the set of nodes of $\mathcal{T}(P)$ as follows. Let a and a' be two nodes in $\mathcal{T}(P)$, we say that a is *equivalent* to a', denoted by $a \sim a'$, if and only if $\mathcal{T}(P; a) \cong \mathcal{T}(P; a')$. Define $V[P]$ to be the set of all equivalence classes in the quotient set $\mathcal{T}(P)/\sim$. We will represent each equivalence class $[v]$ by the label of the unique node v which appears on

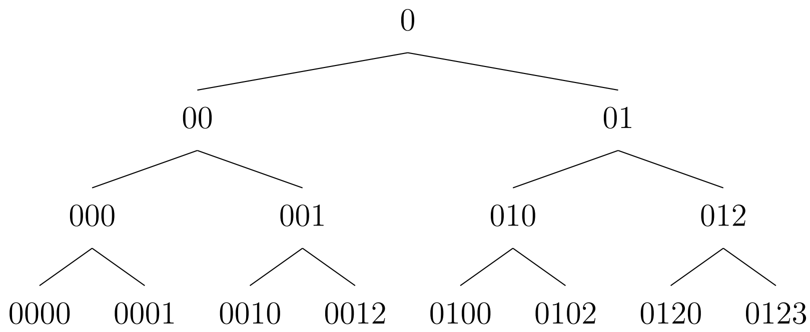

Figure 1: First four levels of $\mathcal{T}(\{011\})$

the tree $\mathcal{T}(P)$ as the left-most node at the lowest level among all other nodes in the same equivalence class. Let $\mathcal{T}[P]$ be the same tree $\mathcal{T}(P)$ where we replace each node a by its equivalence class label.

Example 2. Let $P = \{000, 001, 021\}$. The generating tree $\mathcal{T}[P]$ has a root a_0 and satisfies the following rules

$$a_m \rightsquigarrow b_m, c_m, a_{m+1}, \quad c_m \rightsquigarrow b_m,$$

where $a_m = 01 \cdots m$, $b_m = a_m 0$, and $c_m = a_m m$. To show that, we need to verify the succession rules of the generating tree. The children of a_m are $a_m j$, where $j = 0, 1, \ldots, m+1$, so a_m has only three children b_m, c_m, a_{m+1} in $\mathcal{T}[P]$. Note that any child of b_m contains a pattern in P, so there are no children for b_m in $\mathcal{T}[P]$. Also, any child of c_m that avoids P is $c_m 0$. But it is not hard to see $a = c_m 0 v \in A_n(P)$ if and only if $a' = a_m 0 v \in A_{n-1}(P)$ by removing the second occurrence of the letter m in a. Thus, $a_m \rightsquigarrow b_m, c_m, a_{m+1}$ and $c_m \rightsquigarrow b_m$.

The basic outline of the generating tree method is the following.

(1) Let P be any set of patterns and let D be any positive number (here we use $D = 8$).

(2) We find the first D levels of the generating tree $\mathcal{T}(P)$.

(3) By (2), we guess all the succession rules of $\mathcal{T}(P)$.

(4) Based on (3), we try to prove these succession rules (for instance, see Example 2). If we fail, then we increase D by 1 and go back to Step (2). Otherwise, the succession rules of the generating tree $\mathcal{T}[P]$ are found.

Let L be the set of all triples of patterns of length-3. A *candidate class* is a maximal subset C of L such that for any $P, P' \in C$, $|A_n(P)| = |A_n(P')|$, for all $n = 1, 2, \ldots, 11$. Table 2 in the Appendix shows all the 62 candidate classes of L. A candidate class is called *trivial* if it contains exactly one triple, otherwise, it is called *nontrivial*. Clearly, any

A-Wilf equivalence class is contained in a candidate class. There are 35 trivial candidate classes and 27 nontrivial candidate classes of triples of length-3 patterns.

To prove Theorem 1, we show that the 27 nontrivial candidate classes of triples of length-3 patterns are indeed 27 A-Wilf equivalences. To establish this, we use the generating tree method as described above. Table 1 describes all these 27 nontrivial candidate classes including the corresponding succession rules of the generating trees and explicit formulas (if they exist) for the generating function.

Table 1: Rules of generating trees for ascent sequences avoiding a triple of length-3 patterns.

Beginning of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
2	{000,001,012}	$0 \rightsquigarrow 00, 01;\ 01 \rightsquigarrow 010, 011;\ 011 \rightsquigarrow 010$	
	{000,010,012}	$0 \rightsquigarrow 00, 01;\ 00 \rightsquigarrow 01;\ 01 \rightsquigarrow 011$	$x + 2x^2 + 2x^3 + x^4$
3	{000,001,010}	$a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m m$	
	{000,001,011}	$a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m 0$	
	{000,010,011}	$a_0 \rightsquigarrow b_0, a_1;\ a_m \rightsquigarrow a_{m+1};\ b_m \rightsquigarrow b_{m+1}$, where $a_m = 01\cdots m$, $b_m = 0a_m$	
	{001,010,011}	$a_0 \rightsquigarrow 00, a_1;\ a_m \rightsquigarrow a_{m+1};\ 00 \rightsquigarrow 00$, where $a_m = 01\cdots m$	
	{001,010,012} {001,011,012}	$0 \rightsquigarrow 00, 01;\ 00 \rightsquigarrow 00;\ 01 \rightsquigarrow 01$	
	{010,011,012}	$a_m \rightsquigarrow a_{m+1}, 01$, where $a_m = 0^m$	$\frac{x(1+x)}{1-x}$
4	{000,012,101}	$0 \rightsquigarrow 00, 01;\ 00 \rightsquigarrow 001;\ 01 \rightsquigarrow 010, 001;$ $001 \rightsquigarrow 010$	
	{000,012,110}	$0 \rightsquigarrow 00, 01;\ 00 \rightsquigarrow 001;\ 01 \rightsquigarrow 001, 011;$ $001 \rightsquigarrow 011$	$x + 2x^2 + 3x^3 + 2x^4$
5	{000,012,021} {000,012,100} {000,012,102} {000,012,120} {000,012,201} {000,012,210}	$0 \rightsquigarrow 00, 01;\ 00 \rightsquigarrow 001;\ 01 \rightsquigarrow 001, 001;$ $001 \rightsquigarrow 0011$	$x + 2x^2 + 3x^3 + 3x^4$
6	{000,011,102}	$a_m \rightsquigarrow b_m, a_{m+1};\ c_m \rightsquigarrow c_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m 0$, $c_m = 0a_m$ $(b_0 = c_0)$	
	{000,011,120}	$a_0 \rightsquigarrow b_0, a_1;\ a_1 \rightsquigarrow b_1, a_2;\ a_m \rightsquigarrow a_{m+1};$ $b_m \rightsquigarrow b_{m+1}$, where $a_m = 01\cdots m$, $b_m = 0a_m$	
	{001,011,100}	$a_m \rightsquigarrow b_m, a_{m+1};\ b_0 \rightsquigarrow b_0$, where $a_m = 01\cdots m$, $b_m = 0a_m$	
	{001,011,120}	$a_m \rightsquigarrow b_m, a_{m+1};\ b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = 0a_m$	
	{001,012,100}	$0 \rightsquigarrow 00, 01;\ 00 \rightsquigarrow 00;\ 01 \rightsquigarrow 010, 01$	
	{001,012,110}	$0 \rightsquigarrow 00, 01;\ 00 \rightsquigarrow 00;\ 01 \rightsquigarrow 010, 010;$ $010 \rightsquigarrow 010$	
	{011,012,100}	$a_m \rightsquigarrow a_{m+1}, 01;\ 01 \rightsquigarrow 010$, where $a_m = 0^m$	$x + 2x^2 + \frac{3x^3}{1-x}$
7	{000,001,021}	$a_0 \rightsquigarrow c_0, a_1;\ a_m \rightsquigarrow c_m, b_m, a_{m+1};$ $b_m \rightsquigarrow c_m$, where $a_m = 01\cdots m$, $b_m = a_m m$, $c_m = a_m 0$	
	{000,001,120}	$a_0 \rightsquigarrow c_0, a_1;\ a_m \rightsquigarrow b_m, c_m, a_{m+1};$ $b_m \rightsquigarrow c_m$, where $a_m = 01\cdots m$, $b_m = a_m m$, $c_m = a_m(m-1)$	$x + 2x^2 + 3x^3 + \frac{4x^4}{1-x}$
8	{000,001,110}	$a_m \rightsquigarrow (b_m)^{m+1}, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m 0$	
	{000,011,021} {000,011,100} {000,011,101} {000,011,110} {000,011,201} {000,011,210}	$a_m \rightsquigarrow b_m, a_{m+1};\ b_m \rightsquigarrow b_{m+1}$, where $a_m = 01\cdots m$, $b_m = 0a_m$	
	{001,010,021} {001,010,100} {001,010,101} {001,010,102} {001,010,110} {001,010,120} {001,010,201} {001,010,210}	$a_m \rightsquigarrow b_m, a_{m+1};\ b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m m$	
	{001,011,021} {001,011,101} {001,011,102} {001,011,110} {001,011,201} {001,011,210}	$a_m \rightsquigarrow b_m, a_{m+1};\ b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m 0$	
	{001,012,021} {001,012,101} {001,012,102}		

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{001,012,120} {001,012,201} {001,012,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 01$; $010 \rightsquigarrow 010$	
	{010,011,021} {010,011,100} {010,011,101} {010,011,102} {010,011,110} {010,011,120} {010,011,201} {010,011,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow b_{m,j+1}$, where $a_m = 0^m$ and $b_{m,j} = 0^m 12 \cdots j$	
	{010,012,021} {010,012,100} {010,012,101} {010,012,102} {010,012,110} {010,012,120} {010,012,201} {010,012,210} {011,012,021} {011,012,101} {011,012,102} {011,012,110} {011,012,120} {011,012,201} {011,012,210}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 01$, where $a_m = 0^m$	$\frac{x}{(1-x)^2}$
10	{000,001,100} {000,001,101} {000,001,102} {000,001,201}	$a_m \rightsquigarrow b_{m,0}, \ldots, b_{m,m}, a_{m+1}$; $b_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,k-1}$, where $a_m = 01 \cdots m$, $b_{m,j} = a_m j$	
	{000,010,021} {000,010,100} {000,010,101} {000,010,102} {000,010,110} {000,010,120} {000,010,201} {000,010,210}		See [3]
14	{001,021,100}	$a_0 \rightsquigarrow c_0, a_1$; $c_0 \rightsquigarrow c_0$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $c_m \rightsquigarrow b_m, c_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$, $c_m = a_m m$	
	{001,021,110}	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow (b_m)^2, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$	
	{001,021,120}	$a_0 \rightsquigarrow b_0, a_1$; $b_0 \rightsquigarrow b_0$; $a_1 \rightsquigarrow 010, b_1, a_2$; $b_1 \rightsquigarrow 010, b_1$; $a_m \rightsquigarrow b_m, a_{m+1}$; $b_m r u b_m$, where $a_m = 01 \cdots m$, $b_m = a_m m$	
	{001,100,110}	$a_m \rightsquigarrow (b_m)^m, c_m, a_{m+1}$; $c_m \rightsquigarrow c_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$, $c_m = a_m m$	
	{001,100,120}	$a_0 \rightsquigarrow b_0, a_1$; $b_0 \rightsquigarrow b_0$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m, b_m$, where $a_m = 01 \cdots m$, $b_m = a_m(m-1)$, $c_m = a_m m$	
	{001,110,120}	$a_0 \rightsquigarrow 00, a_1$; $00 \rightsquigarrow 00$; $a_m \rightsquigarrow (b_m)^2, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01 \cdots m$, $b_m = a_m(m-1)$	
	{011,100,102}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow c_m, b_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = a_m 1 \cdots j$, $c_m = 01 \cdots m0$	
	{011,100,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,1} \rightsquigarrow c_m, b_{m,2}$; $c_m \rightsquigarrow b_{m+1,2}$; $b_{m,j} \rightsquigarrow b_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = a_m 1 \cdots j$, $c_m = a_m 10$	
	{011,102,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,1} \rightsquigarrow 010, b_{m,2}$; $010 \rightsquigarrow 010$; $b_{m,j} \rightsquigarrow b_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = a_m 1 \cdots j$	
	{012,100,101}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010, 01$, where $a_m = 0^m$	
	{012,100,110}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010, 010$; $010 \rightsquigarrow 0101$; $0101 \rightsquigarrow 0101$, where $a_m = 0^m$	
	{012,101,110}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010, 010$; $010 \rightsquigarrow 010$, where $a_m = 0^m$	$\frac{x(1+x^2)}{(1-x)^2}$
15	{001,021,101} {001,021,102} {001,021,201} {001,021,210}	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $b_m \rightsquigarrow b_m$; $c_m \rightsquigarrow b_m, c_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$, $c_m = a_m m$	
	{001,100,210}	$a_m \rightsquigarrow (b_m)^m, c_m, a_{m+1}$; $c_m \rightsquigarrow (b_m)^m, c_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$, $c_m = a_m m$	
	{001,101,110} {001,102,110} {001,110,201} {001,110,210}	$a_m \rightsquigarrow (b_m)^{m+1}, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$	
	{001,101,120} {001,102,120} {001,120,201}		

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{001,120,210}	$a_0 \rightsquigarrow b_0, a_1$; $b_0 \rightsquigarrow b_0$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $c_m \rightsquigarrow c_m$; $b_m \rightsquigarrow c_m, b_m$, where $a_m = 01 \cdots m$, $b_m = a_m(m-1)$, $c_m = a_m m$	
	{011,021,100} {011,100,101} {011,100,110} {011,100,201} {011,100,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow c_{m,j}, b_{m,j+1}$; $c_{m,j} \rightsquigarrow c_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$, $c_{m,j} = 0^m 102 \cdots j$	
	{011,021,102} {011,101,102} {011,102,110} {011,102,201} {011,102,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow c_j, b_{m,j+1}$; $c_m \rightsquigarrow c_m$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$, $c_m = 01 \cdots m0$	
	{011,021,120} {011,101,120} {011,110,120} {011,120,201} {011,120,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,1} \rightsquigarrow c_{m+1,1}, b_{m,2}$; $b_{m,j} \rightsquigarrow b_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$	
	{012,021,100} {012,100,102} {012,100,120} {012,100,201} {012,100,210}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010, 01$; $010 \rightsquigarrow 0101$; $0101 \rightsquigarrow 0101$, where $a_m = 0^m$	
	{012,021,101} {012,101,102} {012,101,120} {012,101,201} {012,101,210}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 010, 01$; $010 \rightsquigarrow 010$, where $a_m = 0^m$	
	{012,021,110} {012,102,110} {012,110,120} {012,110,201} {012,110,210}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 01, 011$; $011 \rightsquigarrow 011$, where $a_m = 0^m$	$\frac{x(1-x+x^2)}{(1-x)^3}$
16	{001,100,101} {001,100,102} {001,100,201}		See [3]
21	{001,101,210} {001,102,210} {001,201,210}	$a_m \rightsquigarrow (b_m)^m, c_m, a_{m+1}$; $b_m \rightsquigarrow b_m$; $c_m \rightsquigarrow (b_m)^m, c_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$, $c_m = a_m m$	$\frac{x(1-2x+2x^2)}{(1-x)^4}$
22	{000,021,101} {000,021,110}		See [3]
24	{000,101,102} {000,101,110}		See Subsection 3.1 See Subsection 3.1
	{001,101,102} {001,101,201} {001,102,201}	$a_m \rightsquigarrow b_{m,0}, \ldots, b_{m,m}, a_{m+1}$; $b_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,j}$, where $a_m = 01 \cdots m$, $b_{m,j} = a_m j$	
	{010,021,100} {010,021,101} {010,021,102} {010,021,110} {010,021,120} {010,021,201} {010,021,210} {010,100,101} {010,100,102} {010,100,110} {010,100,120} {010,100,201} {010,100,210} {010,101,102} {010,101,110} {010,101,120} {010,101,201} {010,101,210} {010,102,110} {010,102,120} {010,102,201} {010,102,210} {010,110,120} {010,110,201} {010,110,210} {010,120,201} {010,120,210} {010,201,210} {011,021,101} {011,021,110} {011,021,201} {011,021,210} {011,101,110}		

Continuation of Table 1			
Class	B triple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{011,101,201} {011,101,210} {011,110,201} {011,110,210} {011,201,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow b_{m,j+1}, b_{m,j+1}$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$	
	{012,021,102} {012,021,120} {012,021,201} {012,021,210} {012,102,120} {012,102,201} {012,102,210} {012,120,201} {012,120,210} {012,201,210}	$a_m \rightsquigarrow a_{m+1}, 01$; $01 \rightsquigarrow 01, 01$, where $a_m = 0^m$	$\frac{x}{1-2x}$
28	{000,021,100} {000,021,201} {000,021,210}		See Section 3.2
34	{000,100,101} {000,101,201}		See Section 3.3
41	{021,101,102}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow c_j, b_{m+1,j}, b_{m,j+1}$; $c_m \rightsquigarrow c_m$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$, $c_m = 01 \cdots m0$	
	{021,101,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,1} \rightsquigarrow c_m, b_{m+1,1}, b_{m,2}$; $b_{m,j} \rightsquigarrow b_{m+1,j}, b_{m,j+1}$; $c_m \rightsquigarrow c_{m+1}, b_{m,2}$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$, $c_m = a_m 10$	
	{021,102,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,1} \rightsquigarrow 010, b_{m+1,1}, b_{m,2}$; $b_{m,j} \rightsquigarrow b_{m+1,j}, b_{m,j+1}$; $010 \rightsquigarrow 010, 0101$; $0101 \rightsquigarrow 0101, 0101$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$	
	{100,102,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}$; $b_{m,j} \rightsquigarrow c_j, b_{m+1,j}, b_{m,j+1}$; $c_m \rightsquigarrow d_m$; $d_m \rightsquigarrow d_m$, where $a_m = 0^m$, $b_{m,j} = 0^m 12 \cdots j$, $c_m = 01 \cdots m(m-1)$, $d_m = c_m m$	
	{101,102,110}	$a_m \rightsquigarrow 010, a_0, a_{m+1}$; $010 \rightsquigarrow 010$, where $a_m = 01 \cdots m$	
	{101,102,120}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, (01)^2$; $010 \rightsquigarrow 010$	
	{102,110,120}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, 0, 01$; $010 \rightsquigarrow 010, 0101$; $0101 \rightsquigarrow 0101$	$\frac{x(1-2x+2x^2)}{(1-2x)(1-x)^2}$
42	{021,100,101} {021,100,110} {021,100,120} {021,101,110} {021,110,120}		See Section 3.4
43	{100,101,110}	$a_m \rightsquigarrow \epsilon^m, a_0, a_{m+1}$; $\epsilon \rightsquigarrow a_0$, where $a_m = 01 \cdots m$	
	{100,101,120}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow \epsilon, (01)^2$; $\epsilon \rightsquigarrow 0$	
	{101,110,120}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, 0, 01$; $010 \rightsquigarrow 010, 0$	$\frac{x(1-x+x^2)}{1-3x+2x^2-x^3}$
47	{021,102,201} {021,102,210}	={021,102} ={021,102}	See [1]
	{102,110,210}	$a_m \rightsquigarrow (010)^m, a_0, a_{m+1}$; $010 \rightsquigarrow 010, 0101$; $0101 \rightsquigarrow 0101$, where $a_m = 01 \cdots m$	$\frac{x(1-3x+4x^2-x^3)}{(1-2x)(1-x)^3}$
49	{101,102,210} {102,120,201} {102,120,210}		See Section 3.5
51	{101,120,201} {101,120,210}		See [1]
52	{021,100,201} {021,100,210} {021,110,201} {021,110,210}	={021,100} ={021,100} ={021,110} ={021,110}	See [1]
53	{021,101,201} {021,101,210} {021,120,201} {021,120,210} {100,101,210} {101,102,201} {101,110,201} {101,110,210}	={021,101} ={021,101} ={021,120} ={021,120}	See [1,3]
55	{100,120,201} {110,120,201}		See Theorem 4
56	{100,120,210} {110,120,210}		See Theorem 3
61	{021,201,210} {100,201,210} {110,201,210}	={021}	See [7] See Section 3.7
End of Table 1			

3 Classes not covered by Table 1

3.1 Class 24

From the results listed in Table 1, it remains to show first that

$$|A_n(000, 101, 102)| = |A_n(000, 101, 110)| = 2^{n-1}.$$

For any word $w = w_1w_2 \cdots w_n$ and integer k, we define $k + w$ as $(w_1 + k)(w_2 + k) \cdots (w_n + k)$. Let $a_n = |A_n(000, 101, 102)|$. Clearly, $a_1 = 1$ and $a_2 = 2$. So, from now on, we assume that $n \geqslant 3$. Note that any ascent sequence π in $A_n(000, 101, 102)$ can be decomposed as either $\pi = 0(1 + \pi')$, $\pi = 0(1 + \pi'')0$, or $\pi = 00(1 + \pi'')$ such that $\pi' \in A_{n-1}(000, 101, 102)$ and $\pi'' \in A_{n-2}(000, 101, 102)$. Hence, $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 1$ and $a_1 = 2$. By induction on n, we have $a_n = 2^{n-1}$.

For the case $A_n(000, 101, 110)$, based on a small modification of Proposition 15 in [1], the ascent sequences of $A_n(000, 101, 110)$ can be characterized as a generating tree with a root (2) and the rules

$$(k) \rightsquigarrow (1)^{k-1}, (k + 1); \quad (1) \rightsquigarrow (2).$$

Let $A_k(x)$ be the generating function for the number of nodes at level n in the subtree with a root (k), where the root stays at level 1. Hence, $A_1(x) = x + xA_2(x)$ and $A_k(x) = x+(k-1)xA_1(x)+xA_{k+1}(x)$. Define $A(x; v) = \sum_{k\geqslant 2} A_k(x)v^{k-2}$. Then $A_1(x) = x+xA_2(x)$ and

$$A(x; v) = \frac{x}{1 - v} + \frac{x}{(1 - v)^2}A_1(x) + \frac{x}{v}(A(x; v) - A_2(x)).$$

By taking $v = x$, we have

$$A_2(x) = \frac{x}{1 - x} + \frac{x}{(1 - x)^2}A_1(x).$$

Thus, from $A_1(x) = x + xA_2(x)$, we obtain that $A_2(x) = \frac{x}{1-2x}$, as required.

3.2 Class 28

Since an ascent sequence begins with a 0, if it contains either 201 or 210, then it also contains 021. Similarly, if an ascent sequence contains 100, then it also contains either 000 or 021. Hence,

$$\{000, 021, 100\} \overset{a}{\sim} \{000, 021, 201\} \overset{a}{\sim} \{000, 021, 210\}.$$

3.3 Class 34

Let $\tau = \{000, 101\}$. Then ascent sequences avoiding τ can be characterized as the sequences of nonnegative integers (letters) that satisfy:

- letters appear at most twice, to avoid 000,

- the first appearances of letters are consecutive nonnegative integers $0, 1, 2, \ldots$ in that order, to meet the ascent condition,
- for each repeated letter u, all entries between the two appearances of u are $> u$, to avoid 101 and 000.

Here is a bijection from $A_n(\tau)$ to $\mathcal{M}_n$, the Motzkin sequences of length n. Given a τ-avoider, turn the first appearance of each repeated letter into a U, the second appearance of each repeated letter into a D, and each nonrepeated letter into an F. For example, $w = 0102234453 \in A_{10}(\tau)$ becomes $UFDUDUUDFD$; see Figure 2. The result is the

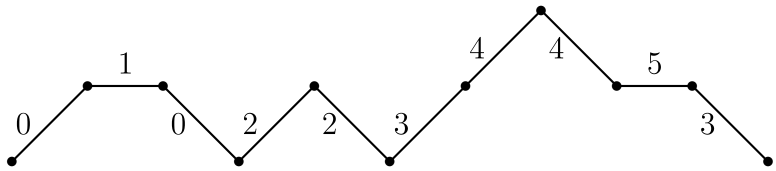

Figure 2: The Motzkin path $UFDUDUUDFD$

desired Motzkin path, with U representing an upstep $(1, 1)$, F a flatstep $(1, 0)$ and D a downstep $(1, -1)$. We leave the reader to verify that the itemized conditions above ensure the result is a Motzkin path of length n. To reverse the map, given a Motzkin path, label its weak-rise steps (U and F) with $0, 1, 2, \ldots$ left to right. Then give each downstep the label already on its matching upstep, as in Figure 2. Read all the labels, left to right, to get the corresponding τ-avoider.

From the preceding characterization, after a descent in a τ-avoider, the next entry (if present) is another descent bottom or a new left-to-right maximum. So τ-avoiders also avoid 100 and 201; hence $A_n(000, 100, 101) = A_n(000, 101, 201) = A_n(\tau)$.

3.4 Class 42

In this section, we show that

$$|\mathcal{A}_n(021, 100, 101)| = |\mathcal{A}_n(021, 100, 110)| = |\mathcal{A}_n(021, 100, 120)| = |\mathcal{A}_n(021, 101, 110)| \\ = |\mathcal{A}_n(021, 110, 120) = (n + 2)2^{n-3},$$

for all $n \geqslant 2$.

3.4.1 Class 42.1: {021,100,101}

Since an ascent sequence begins with 0, avoiding 021 implies all descents are to 0; avoiding 100 then implies there is at most one descent, and the ascent condition and avoidance of 101 further imply that all entries after a descent $v0$ are $> v$. Consequently, a (021,100,101)-avoider of length n is either a weakly increasing ascent sequence or, for some $a \geqslant 1,\ k \geqslant 2$, has the form $u_1 \ldots u_k\, 0\, v_1 \ldots v_{n-k-1}$ with $(u_i)_{i=1}^{k}$ a weakly increasing ascent sequence that ends at $u_k = a$ and $(v_i - a - 1)_{i=1}^{n-1-k}$ another weakly increasing ascent sequence, possibly empty. It follows that, for $n \geqslant 1$,

$$|A_n(021, 100, 101)| = 2^{n-1} + \sum_{k=2}^{n-1} \sum_{a=1}^{k-1} \binom{k-1}{a} \max(1, 2^{n-k-2}) = \max(1, (n+2)2^{n-3})\,.$$

3.4.2 Class 42.2: {021,100,110}

Avoiding 021 and 100 implies there is at most one descent, $v0$, and all later entries are $\geqslant v$; avoiding 110 implies no descent can occur after two adjacent equal nonzero entries. Consequently, a $(021, 100, 110)$-avoider of length $n \geqslant 1$ either has no descents or has the form $0^a 12 \dots (b-1) b\, 0\, y$ where $a \geqslant 1$, $b \geqslant 1$, y has length $c = n - a - b - 1 \geqslant 0$ and $(y_1 - b, y_2 - b, \dots, y_c - b) \in \mathcal{J}_c$ where $\mathcal{J}_n$ is the set of weakly increasing sequences of nonnegative integers $(u_i)_{i=1}^n$ such that all increases from one entry to the next in the sequence $(-1, u_1, u_2, \dots, u_n)$ are 0 or 1 except there may be a single jump of 2 (and thus u_1 may be 0 or 1). For example $\mathcal{J}_2 = \{00, 01, 02, 11, 12\}$. Due to the jump of 2, an element of $\mathcal{J}_n$ can be viewed as a pair of weakly increasing ascent sequences whose lengths sum to n, and so the counting sequence $(|\mathcal{J}_n|)_{n\geqslant 0}$ is the convolution $(1, 1, 2, 4, 8, \dots) * (1, 1, 2, 4, 8, \dots) = (1, 2, 5, 12, 28, \dots) = (X_n)_{n\geqslant 0}$ with $X_n := \max(1, (n+3)2^{n-2}) = \lceil (n+3)2^{n-2} \rceil$. Hence,

$$\begin{aligned} |A_n(021, 100, 110)| &= 2^{n-1} + \sum_{a=1}^{n-2} \sum_{b=1}^{n-1-a} X_{n-a-b-1} \\ &= 2^{n-1} + \sum_{k=2}^{n-1} (k-1) X_{n-k-1} \\ &= \max(1, (n+2)2^{n-3})\,. \end{aligned}$$

3.4.3 Class 42.3: {021,100,120}

As in Section 3.4.2, there is at most one descent, $v0$, and all later entries are $\geqslant v$, and then avoiding 120 implies $v = 1$. Consequently, a $(021, 100, 120)$-avoider of length $n \geqslant 1$ either has no descents or has the form $0^a 1^b\, 0\, y$ where $a \geqslant 1$, $b \geqslant 1$, y has length $c = n - a - b - 1$ and $(y_1 - 1, y_2 - 1, \dots, y_c - 1) \in \mathcal{J}_c$, leading to the same count as in Section 3.4.2.

3.4.4 Class 42.4: {021,101,110}

Again, all descents are to 0 and no descent occurs after a nonzero letter has appeared twice. Let $k \geqslant 3$ denote the position of the last 0 in a (021,101,110)-avoider w that is not weakly increasing. Then, for some $a \in [1, k-2]$, w has the form $0\, u\, 0\, v$ where u, of length $k-2$, consists of one copy of each of $1, 2, \dots, a$, in that order, interspersed with $k - 2 - a$ 0's $\big(\binom{k-2}{a}$ choices$\big)$, and $v - (a + 1)$ is a weakly increasing ascent sequence of length $n-k \geqslant 0$. Hence

$$|A_n(021,101,110)| = 2^{n-1} + \sum_{k=3}^{n}\sum_{a=1}^{k-2}\binom{k-2}{a}\max(1,2^{n-k-1}) = \max(1,(n+2)2^{n-3})\,.$$

3.4.5 Class 42.5: {021,110,120}

A $(021,101,110)$-avoider of length $n \geqslant 1$ either has no descents or has the form $0^a\,1\,0^b\,y$ where $a \geqslant 1,\ b \geqslant 1,\ y$ has length $c = n-a-b-1$ and $(y_1-1, y_2-1, \ldots, y_c-1) \in \mathcal{J}_c$, leading to the same count as in Section 3.4.2.

3.5 Class 49

In this subsection, we show that

$$\{101,102,210\} \overset{a}{\sim} \{102,120,201\} \overset{a}{\sim} \{102,120,210\}.$$

3.5.1 Class 49.1: {101,102,210}

The restrictions here force an avoiding ascent sequence $(w_i)_{i=1}^n$ to be either weakly increasing or to have precisely one descent, say from $w_r = a$ to $b \in [0, a-1]$ and thereafter to remain constant. Hence,

$$|A_n(101,102,210)| = 2^{n-1} + \sum_{a=1}^{n-2}\sum_{r=a+1}^{n-1} a\binom{r-1}{a} = (n-1)2^{n-2} + 1\,.$$

3.5.2 Classes 49:2-3: {102,120,201} and {102,120,210}

Ascent sequences avoiding 102 and 120 are treated using generating functions in [1, Proposition 13]. Here is a direct count. These restrictions force an avoiding ascent sequence $(w_i)_{i=1}^n$ to be either weakly increasing or to have first descent of size 1 unit, say from $w_r = a$ to $a-1$ and thereafter to be either a or $a-1$. Given $r \in [2, n-1]$ and $a \in [1, r-1]$, there are $\binom{r-1}{a}$ choices for $(w_i)_{i=1}^r$ and 2^{n-r-1} choices to fill out the ascent sequence. Hence,

$$|A_n(102,120)| = 2^{n-1} + \sum_{r=2}^{n-1}\sum_{a=1}^{r-1}\binom{r-1}{a}2^{n-r-1} = (n-1)2^{n-2} + 1\,.$$

The above characterization shows that (102,120)-avoiding ascent sequences also avoid 201 and 210 and so $A_n(102,120,201) = A_n(102,120,210) = A_n(102,120)$.

3.6 Classes 55 and 56

A (*weak*) *left to right maximum*, *LRmax* for short, in an ascent sequence $a = a_1a_2\cdots a_n$ is an entry a_i such that $a_i \geqslant a_j$ for all $j < i$. If $a_i = m$ we say m is the value of the LRmax. Thus for $a = 011022111$, the LRmax are a_1, a_2, a_3, a_5, a_6 with values $0,1,1,2,2$, respectively. So any ascent sequence $a \in A_n$ can be decomposed uniquely as $m_1\pi^{(1)}\cdots m_k\pi^{(k)}$, where $m_1, \ldots, m_k$ are all of the LRmax entries in a; thus $0 = m_1 \leqslant m_2 \leqslant \cdots \leqslant m_k \leqslant n-1$ and $m_i > \pi^{(i)}$ (entrywise). We call this the WLRmax decomposition of a (W for weakly). There is another decomposition: say $\mathbf{m}_1 < \mathbf{m}_2 < \cdots < \mathbf{m}_k$ are the first occurrences of the (distinct) LRmax values of an ascent sequence a. Then a can be decomposed as $\mathbf{m}_1\pi^{(1)}\mathbf{m}_2\pi^{(2)}\cdots\mathbf{m}_k\pi^{(k)}$ for some $k \geqslant 1$ with $\mathbf{m}_i \geqslant \pi^{(i)}$ (entrywise) for $1 \leqslant i \leqslant k$. This is the LRmax decomposition of a. For example, **0**00**1**011**3**02**5**110433 is an LRmax decomposition with the first occurrence of each left to right maximum value bolded.

Theorem 3. *We have* $\{100, 120, 210\} \overset{a}{\sim} \{110, 120, 210\}$.

Proof. Suppose $a \in A_n$ and $\mathbf{m}_1\pi^{(1)}\mathbf{m}_2\pi^{(2)}\cdots\mathbf{m}_k\pi^{(k)}$ is the LRmax decomposition of a. Then $a \in A_n(100, 120, 210)$ if and only if the following hold

- $\mathbf{m}_1 = 0 < \mathbf{m}_2 = 1 < \mathbf{m}_3 < \cdots < \mathbf{m}_k \leqslant n-1$ and $\pi^{(1)} = 00\cdots 0$;
- $\mathbf{m}_i\pi^{(i)} \geqslant \mathbf{m}_{i-1}$ and the number of occurrence of the letter $\mathbf{m}_{i-1}$ in $\pi^{(i)}$ is at most one, for all $i = 2, \ldots, k$;
- $\pi^{(i)}$ can be written as $\pi^{(i,1)}\mathbf{m}_i \cdots \pi^{(i,s_i)}\mathbf{m}_i\pi^{(i,s_i+1)}$ where $\pi^{(i,1)}\cdots\pi^{(i,s_i+1)}$ forms an increasing sequence.

Also, $a \in A_n(110, 120, 210)$ if and only if the following hold

- $\mathbf{m}_1 = 0 < \mathbf{m}_2 = 1 < \mathbf{m}_3 < \cdots < \mathbf{m}_k \leqslant n-1$ and $\pi^{(1)} = 00\cdots 0$;
- $\mathbf{m}_i\pi^{(i)} \geqslant \mathbf{m}_{i-1}$, for all $i = 2, \ldots, k$;
- $\pi^{(i)}$ can be written as $\theta^{(i)}\mathbf{m}_i \cdots \mathbf{m}_i$, where $\theta^{(i)}$ forms a nondecreasing sequence.

To show that there is a bijection f from $a \in A_n(100, 120, 210)$ to $b \in A_n(110, 120, 210)$, it suffices to construct a bijection f_i that maps $\mathbf{m}_i\pi^{(i)}$ of a to $\mathbf{m}_i\pi^{(i)}$ of b, for any $i = 2, 3, \ldots, k$. Note that $\pi^{(i)} = \pi^{(i,1)}\mathbf{m}_i \cdots \pi^{(i,s_i)}\mathbf{m}_i\pi^{(i,s_i+1)}$ of a, where $\pi^{(i,1)}\cdots\pi^{(i,s_i+1)}$ forms an increasing sequence. So, we map $\mathbf{m}_i\pi^{(i)}$ of a to $\mathbf{m}_i\beta^{(i)}$ of b as follows. Let $\beta^{(i)}$ be empty sequence, we read $\pi^{(i,j)}$ from $j = 1, \ldots, s_i + 1$,

- if $\pi^{(i,j)}$ is the empty sequence, then we append the maximal letter of $\pi^{(i,1)}\cdots\pi^{(i,j-1)}$ (if it exists) to $\beta^{(i)}$. Otherwise, if $\pi^{(i,1)}\cdots\pi^{(i,j)}$ is an empty sequence, then we append the letter $\mathbf{m}_{i-1}$ to $\beta^{(i)}$;
- if $\pi^{(i,j)}$ is not the empty sequence, then we append the sequence $\pi^{(i,j)}$ to $\beta^{(i)}$.

At the end we append d times the letter $\mathbf{m}_i$ to $\beta^{(i)}$, where $d = s_i - |\{j \mid \pi^{(i,j)} = \emptyset\}|$. Clearly, f_i is a bijection. For instance, let $\mathbf{m}_{i-1} = 1$ and $\mathbf{m}_i = 5$, then as an example $f_i(551235545) = 511233445$.

Hence, f forms a bijection from $A_n(100, 120, 210)$ to $A_n(110, 120, 210)$, as required. ☐

Similar to the arguments in the proof of Theorem 3, we have

Theorem 4. *We have* $\{100, 120, 201\} \overset{a}{\sim} \{110, 120, 201\}$.

Note that by our procedure as described in Section 2, one can find the succession rules of the generating trees $\mathcal{T}[\{100, 120, 201]$, $\mathcal{T}[\{100, 120, 210]$, $\mathcal{T}[\{110, 120, 201]$, and $\mathcal{T}[\{110, 120, 210]$. However, with the combinatorial proofs of Theorems 3 and 4 and tedious to find the succession rules of the generating trees, we did not try to find explicit formulas for the corresponding generating functions.

3.7 Class 61

In this section, we count $A_n(100, 201, 210)$ and show that

$$A_n(100, 201, 210) \text{ and } A_n(110, 201, 210)$$

are equinumerous, using both generating-tree and bijective approaches.

3.7.1 A bijection from $A_n(100, 201, 210)$ to $A_n(110, 201, 210)$

First, we give analogous characterizations of ascent sequences that avoid each of these triples. Obviously, weakly increasing ascent sequences avoid both these triples of patterns. Now suppose $r \geqslant 1$ *distinct* values, say $(m_i)_{i=1}^r$, occur as descent bottoms in an ascent sequence a. Split a at the *last* occurrence of each m_i to decompose a as $a = \pi_1 m_1 \pi_2 m_2 \ldots \pi_r m_r \pi_{r+1}$. Then we have the following obvious characterizations.

Lemma 5. *(i) An ascent sequence a avoids the patterns* 100, 201 *and* 210 *if and only if $\pi_1 m_1 \leqslant \pi_2 m_2 \leqslant \ldots \leqslant \pi_r m_r \leqslant \pi_{r+1}$ (entrywise) and each of $\pi_1, \ldots, \pi_{r+1}$ is weakly increasing. (ii) An ascent sequence a avoids the patterns* 110, 201 *and* 210 *if and only if $\pi_1 m_1 \leqslant \pi_2 m_2 \leqslant \ldots \leqslant \pi_r m_r \leqslant \pi_{r+1}$ (entrywise), π_{r+1} is weakly increasing and for each $i = 1, \ldots, r$, the entries in π_i greater than m_i are strictly increasing, and the entries in π_i that are no greater than m_i are weakly increasing.*

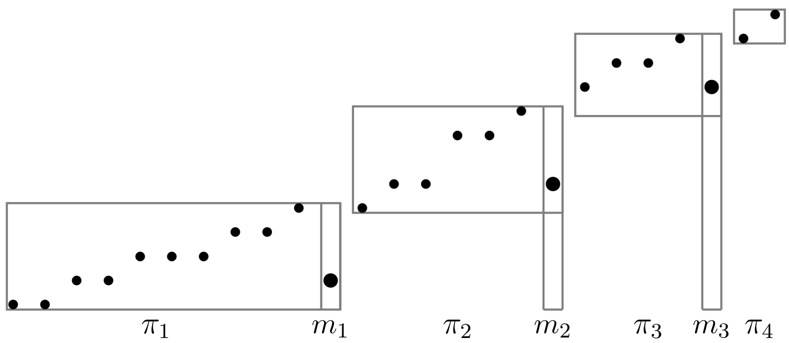

Figure 3: A (100,201,210)-avoiding ascent sequence: each π_i is weakly increasing

Sequences in $A_n(100, 201, 210)$ and $A_n(110, 201, 210)$ are depicted in Figures 3 and 4 respectively with a heavy dot denoting the last occurrence of each descent bottom value and a box enclosing each $\pi_i m_i$. A bijection between them is rather obvious (the two depicted avoiders correspond under this bijection).

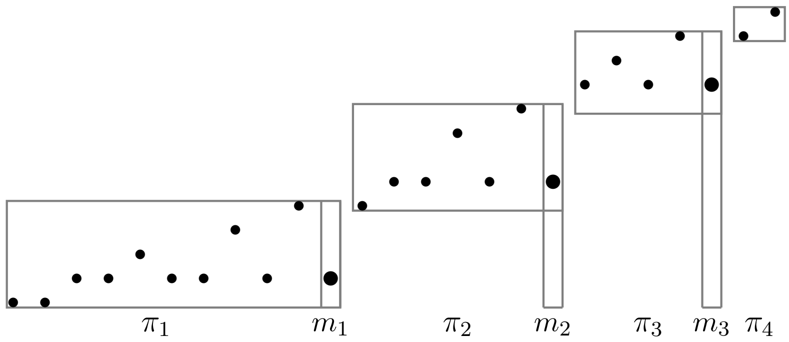

Figure 4: A (110,201,210)-avoiding ascent sequence: each subsequence $\{a_j \in \pi_i : a_j > m_i\}$ is strictly increasing

3.7.2 A generating tree for $A_n(100, 201, 210)$

Based on our algorithm, we find that the generating tree $\mathcal{T}[100, 201, 210]$ has a root $\alpha_{0,0}$ and satisfies the following rules:

$$\begin{aligned} \alpha_{a,m} &\rightsquigarrow \beta_{a+1}^{m-a}, \alpha_{a,m}, \alpha_{a,m+1}, \ldots, \alpha_{0,m+1}, \\ \beta_a &\rightsquigarrow \alpha_{a,a}, \ldots, \alpha_{0,a}, \end{aligned}$$

where $\alpha_{a,m} = 0101212 \cdots a(a-1)a\,(a+1)(a+2)\cdots m$ and $\beta_a = 0101212\cdots(a-1)(a-2)(a-1)a(a-1)$. To see the rules, we note that the children of $\alpha_{a,m}$ (respectively, β_a) are exactly $\alpha_{a,m}j$ with $j = a, a+1, \ldots, a+m+1$ (respectively, $\beta_a j$ with $j = a, a+1, \ldots, 2a$). Now, we show the following equivalences:

- $\mathcal{T}(\{100, 201, 210\}; \alpha_{a,m}j) \cong \mathcal{T}(\{100, 201, 210\}; \beta_{a+1})$, for all $j = a, a+1, \ldots, m-1$: Let $\pi = \alpha_{a,m}j\pi'$ be any ascent sequence that avoids $\{100, 201, 210\}$. So π' does not contain any letter from the set $\{0, 1, \ldots, m-1$. Thus, π avoids $\{100, 201, 210\}$ if and only if $\beta_{a+1}(1+a-m+\pi')$ avoids $\{100, 201, 210\}$, which proves the equivalence.

- $\mathcal{T}(\{100, 201, 210\}; \alpha_{a,m}m) \cong \mathcal{T}(\alpha_{a,m})$: Note that the ascent sequence $\pi = \alpha_{a,m}m\pi'$ avoids $\{100, 201, 210\}$ if and only if the ascent sequence $\alpha_{a,m}\pi'$ avoids $\{100, 201, 210\}$ (just remove the letter m).

- $\mathcal{T}(\{100, 201, 210\}; \alpha_{a,m}j) \cong \mathcal{T}(\{100, 201, 210\}; \alpha_{a+m+1-j,m+1})$, for all $j = m+1, m+2, \ldots, a+m+1$: Let $j = m+1+j'$ and $\pi = \alpha_{a,m}j\pi'$ be any ascent sequence that avoids $\{100, 201, 210\}$. Note that π' does not contain any letter from the set $\{0, 1, \ldots, j'\}$. So by removing the letters $0, 1, \ldots, j'$ from π' we obtain that π avoids $\{100, 201, 210\}$ if and only if the ascent sequence $\alpha_{a+m+1-j,m+1}(-j'+\pi') = \alpha_{a-j',m+1}(-j'+\pi')$ avoids $\{100, 201, 210\}$, which proved the equivalence.

Hence, the first rule is holding. Similarly, the second rule is holding.

Define $A_{a,m}(x)$ (respectively, $B_a(x)$) to be the generating function for the number of nodes at level n in the tree $\mathcal{T}(\{100,201,210\};\alpha_{a,a})$ (respectively, $\mathcal{T}(\{100,201,210\};\beta_a)$), where the root stay at level 1. Hence, the above rules can be translated to

$$A_{a,m}(x) = x + (m-a)xB_{a+1}(x) + xA_{a,m}(x) + x\sum_{j=0}^{a} A_{j,m+1}(x), \quad m \geqslant a \geqslant 0$$
$$B_a(x) = x + x\sum_{j=0}^{a} A_{j,a}(x), \quad a \geqslant 1.$$

Now, we define $A(x;v,u) = \sum_{a\geqslant 0}\sum_{m\geqslant a} A_{a,m}(x)v^a u^m$ and $B(x;v) = \sum_{a\geqslant 1} B_a(x)v^{a-1}$. Then the last two recurrences can be written as

$$\begin{aligned} A(x;v,u) &= \frac{x}{(1-u)(1-vu)} + xA(x;v,u) + \frac{x}{u(1-v)}(A(x;v,u) - A(x;1,vu)) \\ &\quad + \frac{xu}{(1-u)^2}B(x;vu), \qquad (1) \\ B(x;v) &= \frac{x}{1-v} + \frac{x}{v}(A(x;1,v) - A(x;0,0)). \qquad (2) \end{aligned}$$

By (2), (1) can be written as

$$\begin{aligned} A(x;v/u,u) &= \frac{x}{(1-u)(1-v)} + xA(x;v/u,u) + \frac{x}{u-v}(A(x;v/u,u) - A(x;1,v)) \\ &\quad + \frac{xu}{(1-u)^2}\left(\frac{x}{1-v} + \frac{x}{v}(A(x;1,v) - A(x;0,0))\right). \qquad (3) \end{aligned}$$

In order to solve this equation, we assume that $A(x;0,0) = C(x) - 1 = \frac{1-\sqrt{1-4x}}{2x} - 1$. By substituting $u = \frac{vx-v-x}{x-1}$ into (3), we obtain that

$$A(x;1,v) = \frac{x(xC(x)(vx-v-x) + v - xv + x^2)}{(v-1)(v^2(x-1)^2 - v(3x-1)(x-1) + x^3)}.$$

Hence, by substituting expression of $A(x;1,v)$ into (3) and replacing v with vu, we obtain

$$\begin{aligned} A(x;v,u) &= \frac{x((x-1)u^3v^2 + (2-3x)u^2v + (3x-1)u - x)\sqrt{1-4x}}{2((x-1)^2u^2v^2 - (3x-1)(x-1)uv + x^3)(1-uv)(1-u)^2} \\ &\quad + \frac{x((x-1)u^3v^2 + 2(1-x)u^2v^2 - x(2x-1)u^2v)}{2((x-1)^2u^2v^2 - (3x-1)(x-1)uv + x^3)(1-uv)(1-u)^2} \\ &\quad + \frac{x(2(2x-1)uv + (4x-1)(x-1)u - 2x^2 + x)}{2((x-1)^2u^2v^2 - (3x-1)(x-1)uv + x^3)(1-uv)(1-u)^2}. \end{aligned}$$

Note that the expression of $A(x;v,u)$ satisfies (3) and $A(x;0,0) = C(x) - 1$. Hence, $A(x;v,u)$ is the solution of (3), which, by Lemma 5, leads to the following result.

Theorem 6. *The number of ascent sequences in $A_n(100,201,210)$ is given by $\frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number.*

3.7.3 Decorated paths

To count $A_n(100, 201, 210)$ bijectively, we first give a bijection, presented in the next section, from $A_n(100, 201, 210)$ to a set $\mathcal{DP}_n$ that turns $w \in A_n(100, 201, 210)$ into a weakly increasing ascent sequence intertwined with a path of steps, $U = (1, 1)$, $F = (1, 0)$, $D = (1, -1)$, that we will call a *decorated path*, and $\mathcal{DP}_n$ denotes the set of decorated paths of size n. A member of $\mathcal{DP}_n$ is a string of symbols (or letters or entries), each symbol a nonnegative integer (number, for short) $0, 1, 2, \ldots$ or a step, $U = (1, 1)$, $F = (1, 0)$, $D = (1, -1)$, such as $0\,0\,F\,1\,U\,1\,2\,2\,D\,F\,3\,U\,D\,3$, with $n = \#U$'s $+ \#$ numbers, that satisfies certain conditions as follows. Ignoring the steps, the numbers must form a weakly increasing ascent sequence. Ignoring the numbers, the steps must form what we will call a *ternary* path, because they're counted by the ternary numbers $A001764 = (1, 3, 12, 55, 273, 1428, \ldots)$. We assume familiarity with basic notions about Dyck paths. A ternary path of size n consists of n each U, F, D steps such that ignoring the F's leaves a Dyck path, while if the D's are ignored, the F's and U's alternate, starting with an F. For example, $FUDFUD$, $FUFDUD$ and $FUFUDD$ are the 3 ternary paths of size 2, and $FUUDFD$ does not qualify because the F's and U's don't alternate.

The empty string is the sole member of $\mathcal{DP}_0$. For a nonempty string to qualify as a decorated path, the following further conditions must be satisfied by its entries regarding how the numbers and steps intertwine:

1. The first entry is 0.

2. If a step is immediately preceded by a number a, then the next number (if there is one) is determined: for aU the next number is also a, for aF and aD, the next number must be $a + 1$. So $0F0UD$ does not qualify, nor does

   $$0\,F\,1\,U\,1\,F\,D\,2\,2\,U\,D\,F\,3\,U\,D$$

   (since $2\,U\,D\,F\,3$ offends).

3. Each U is immediately preceded by a number; this condition is violated by $0FUD$ (U is preceded by F).

4. There are no consecutive UF's in the decorated path. So $0\,F\,1\,\underline{U\,F}\,1\,U\,D\,D$ does not qualify.

Recall that a weakly increasing ascent sequence has first entry 0 and each later entry is the same as, or one more than, its predecessor. Here are the first few sets of decorated paths:

$$\mathcal{DP}_0 = \{\epsilon\},\ \mathcal{DP}_1 = \{0\},\ \mathcal{DP}_2 = \{00, 01\},\ \mathcal{DP}_3 = \{000, 001, 011, 012, 0F1UD\},$$

and $\mathcal{DP}_4$ consists of 8 weakly increasing ascent sequences and

$$00F1UD,\ 0F1U1D,\ 0F1UD1,\ 0F11UD,\ 01F2UD,\ 0F12UD.$$

A *weak-rise* in a (100,201,210)-avoiding ascent sequence is an entry w_i such that $w_i \geqslant w_{i-1}$ or w_i starts the sequence. Thus every entry is a weak-rise or a descent bottom.

In a weakly increasing ascent sequence, every entry is a weak-rise and the jump from one entry to the next is at most 1. But in a (100,201,210)-avoiding ascent sequence, the jump from one weak-rise to the next weak-rise can be bigger than 1 due to the possibility that entries bracketing a descent bottom entry may be equal, thereby contributing an ascent without increasing the maximum entry. For a weak-rise w_i, define the *descent-induced excess* of w_i, denoted $e(w_i)$, as follows. If w_{i-1} is a descent bottom entry, then $e(w_i) = w_i - w_{i-2} \geqslant 0$, otherwise $e(w_i) = 0$ if $w_i = w_{i-1}$, and $e(w_i) = w_i - w_{i-1} - 1 \geqslant 0$ if $w_i > w_{i-1}$, where here we define $e(w_1) = 0$. Thus, in $w = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 1 & 2 & 1 & 2 & 3 & 2 & 5 & 6 & 6 & 5 & 6 & 8 & 9 \end{smallmatrix}\right)$ with descent bottoms at positions 3, 6, 10, we see that $e(w_7) = 5 - 3 = 2$ and $e(w_{12}) = 8 - 6 - 1 = 1$ while all other weak-rises have descent-induced excess of 0.

Lemma 7. *For each weak-rise w_k in $w \in A_n(100, 201, 210)$,*

$$\sum \{e(w_i) : \ i \leqslant k \ \text{and } w_i \ \text{a weak-rise}\} \leqslant \# \ \text{descents in } (w_i)_{i=1}^{k}.$$

Proof. This is a consequence of Lemma 5 and the defining condition for ascent sequences because each descent contributes at most one unit to later descent-induced excesses. ☐

3.7.4 A bijection from $A_n(100, 201, 210)$ to $\mathcal{DP}_n$

Every weakly increasing ascent sequence avoids 100, 201 and 210. For an avoider $w \in A_n(100, 201, 210)$ with no descents the decorated path has no steps and consists simply of w. Otherwise, let $r \geqslant 1$ denote the number of descents in w and for each descent $a_i > b_i$, record the size $d_i := a_i - b_i \geqslant 1$ of the descent, $1 \leqslant i \leqslant r$. Then (i) change each descent bottom entry to a U step, (ii) for each weak-rise entry w_k, insert $e(w_k)$ D steps just before w_k, and add $(\#\,\text{descents in } w) - \sum\{e(w_i) : \ w_i \text{ is weak-rise}\}$ D's at the end, and (iii) subtract from each numerical entry the total number of D's occurring to its left. This ensures that when two numbers are separated by one or more D's, the second number is one more than the first, and the predecessor of a U step, necessarily a number, is the same as the first number appearing after the U if there is one. For example, with $r = 5$ descents and $(d_i)_{i=1}^{r} = (1, 1, 1, 2, 3)$,

$$\begin{array}{llcccccccccccccccccc}
\text{(row 1)} & & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 3 & 6 & 4 & 8 & 8 & 9 & 6 & 9 \\
 & \xrightarrow{(i)} & 0 & 1 & U & 1 & 2 & U & 2 & 3 & U & 3 & 3 & 6 & U & 8 & 8 & 9 & U & 9 \\
 & \xrightarrow{(ii)} & 0 & 1 & U & 1 & 2 & U & 2 & 3 & U & 3 & 3 & D\,D\,6 & U & D\,D\,8 & 8 & 9 & U & 9\,D \\
\text{(row 4)} & \xrightarrow{(iii)} & 0 & 1 & U & 1 & 2 & U & 2 & 3 & U & 3 & 3 & D\,D\,4 & U & D\,D\,4 & 4 & 5 & U & 5\,D
\end{array}$$

To insert the Fs into this sequence (row 4) of numbers and U, D steps, first insert a dot after each D step and after each number whose successor is a D step or a larger number (but no dots after the last U):

$$0 \cdot 1\; U\; 1 \cdot 2\; U\; 2 \cdot 3\; U\; 3\; 3 \cdot D \cdot D \cdot 4\; U\; D \cdot D \cdot 4\; 4 \cdot 5\; U\; 5\; D$$

Lemma 8. *The number of dots between two successive Us is equal to the difference between the two corresponding descent tops in the original avoider (to the first descent top in the case of the first U).*

Proof. Say w_j and w_k are two successive descent tops in the original sequence, row 1 in the example above. Then

$$\begin{aligned} w_k - w_j &= (w_{j+2} - w_j) + \sum_{i=j+3}^{k} (w_i - w_{i-1}) \\ &= e(w_{j+2}) + \sum_{i=j+3}^{k} e(w_i) + \#\{i \in [j+3, k] : w_i > w_{i-1}\} \\ &= (\#\, D\text{'s between } w_{j+1} \text{ and } w_{j+2}) + (\#\, D\text{'s between } w_{j+2} \text{ and } w_k) + \\ &\quad\ (\#\,\text{numbers followed by a } D \text{ or a larger number}) \text{ [all in row 4]} \\ &= \#\text{ dots between the corresponding } U\text{'s}, \end{aligned}$$

where the last summands in the second and third equalities are equal because if two numbers are separated by one or more D's in row 4, then the first number is necessarily smaller than the second. ☐

By Lemma 5(i) and 8, the number of dots between the $(i-1)$-th U and the i-th U is equal to the number of possible values for d_i, $1 \leqslant i \leqslant r$. Now insert an F at the d_i-th dot, counted left to right, between the $(i-1)$-th U and the i-th U, $1 \leqslant i \leqslant r$, (and erase the dots) to obtain the desired decorated path in $\mathcal{DP}_n$. The example yields

$$0\ F\ 1\ U\ 1\ F\ 2\ U\ 2\ F\ 3\ U\ 3\ 3\ D\ F\ D\ 4\ U\ D\ D\ 4\ 4\ F\ 5\ U\ 5\ D\,.$$

We leave to the reader the straightforward verification that the resulting decorated path satisfies the conditions to be in $\mathcal{DP}_n$ and that the map is invertible.

3.7.5 A sum to count $\mathcal{DP}_n$

We now give an explicit multi-index sum to count $\mathcal{DP}_n$. There are 2^{n-1} elements of $\mathcal{DP}_n$ with no U, F, D steps. So suppose the underlying ternary path T of $w \in \mathcal{DP}_n$ has size $r \geqslant 1$ so that w contains $n-r$ numbers. A number must precede each U in w and for each adjacent UF in T, there must be at least one number between U and F in w. Say there are k UF's in T. There are $3r+1$ gaps between the steps in T (including at both ends). As we have just seen, $r+k+1$ gaps are necessarily occupied by numbers, namely, the first gap (contains 0), each gap before a U and each gap between a UF. Each of the remaining $2r-k$ gaps may or may not be occupied. Say $i \in [0, 2r-k]$ of them are occupied—$\binom{2r-k}{i}$ choices—giving $r+k+i+1$ occupied gaps.

Apart from the initial gap whose first number is 0, the first number in each of the other $r+k+i$ occupied gaps is determined by the last number in the preceding gap, by condition (2) in the definition of $\mathcal{DP}_n$. Also, a choice of positions among the $n-r-1$ nonfirst numbers (the first number is necessarily 0) for these $r+k+i$ determined numbers specifies how many numbers go into each occupied gap. All that's left in order to specify w is $n-r-(r+k+i+1)$ binary choices (each number is the same as or one unit more than its predecessor). There are $\frac{1}{k+1}\binom{r-1}{k}\binom{2r}{k}$ ternary paths of size r with k UF's. These observations yield the following sum to count $\mathcal{DP}_n$ (note the $r=k=i=0$ summand is 2^{n-1} as it should be):

$$|\mathcal{DP}_n| = \sum_{r=0}^{\frac{n-1}{2}} \sum_{k=0}^{r} \sum_{i=0}^{2r-k} \frac{1}{k+1}\binom{r-1}{k}\binom{2r}{k}\binom{2r-k}{i}\binom{n-r-1}{r+k+i} 2^{n-i-k-2r-1}. \quad (4)$$

By Theorem 6 and (4), we have the following combinatorial identity.

Corollary 9. *For all $n \geqslant 1$,*

$$\sum_{r=0}^{\frac{n-1}{2}} \sum_{k=0}^{r} \sum_{i=0}^{2r-k} \frac{1}{k+1}\binom{r-1}{k}\binom{2r}{k}\binom{2r-k}{i}\binom{n-r-1}{r+k+i} 2^{n-i-k-2r-1} = \frac{1}{n+1}\binom{2n}{n}.$$

3.7.6 A bijection ϕ from $\mathcal{DP}_n$ to Dyck n-paths

We first define a statistic X on $\mathcal{DP}_n$ that motivates our bijection ϕ. This statistic turns out to have the same distribution as the "first return" on Dyck paths. For $w \in \mathcal{DP}_n$, if w consists only of numbers, then $X(w) = 1 + \mathrm{asc}(w)$. Otherwise, place a dot in each gap between the letters of w starting with the last U and a dot at the end, as illustrated in Figure 5 below, and say a gap is "good" unless it is a gap between U and D or between two equal numbers. There are always at least two gaps and the "gap" at the very end is always good. Then X counts the good gaps, indicated by the circled dots in Figure 5. Say $w \in \mathcal{DP}_n$ is *primitive* if w is nonempty and $X(w) = 1$. Thus, for $n \geqslant 1$, the set of

Ex.1: $- - - -\ U \odot 3 \cdot 3 \odot 4 \odot D \odot 5 \cdot 5 \cdot 5 \odot D \odot D \odot 6 \cdot 6 \odot$

Ex.2: $- - - -\ U \cdot D \odot$

Figure 5: Two examples of terminal segments of decorated paths and their "good" gaps.

primitive decorated paths in $\mathcal{DP}_n$, denoted $\mathcal{PP}_n$, consists of those that end UD together with the all-numbers sequence 0^n.

In the next two sections, we will give two preliminary bijections ψ and ρ on which ϕ depends. The first, ψ, sends $\mathcal{PP}_n$ to $\mathcal{DP}_{n-1}$ for $n \geqslant 1$ and corresponds to de-elevating (removing the first and last step from) a primitive (precisely one return) Dyck path to obtain a 1-size-smaller Dyck path. The second, ρ, sends $\mathcal{DP}_n$, $n \geqslant 1$, to pairs of decorated paths (y, z) whose sizes sum to n where y is a primitive decorated path and z is an arbitrary decorated path. This corresponds to the first-return decomposition that splits a nonempty Dyck path into a pair of Dyck paths, the first with precisely one return, the second arbitrary.

With these two bijections in hand, it is clear we can recursively define a size-preserving bijection ϕ from decorated paths to Dyck paths. The base case is that ϕ sends the empty decorated path to the empty Dyck path. Then, for a nonempty decorated path w, with $\rho(w) = (y, z)$ where y is primitive,

$$\phi(w) = U \cdot \phi(\psi(y)) \cdot D \cdot \phi(z)\,,$$

where the dot denotes concatenation.

3.7.7 The bijection ψ from $\mathcal{PP}_n$ to $\mathcal{DP}_{n-1}$.

Suppose $w \in \mathcal{PP}_n$. If $w = 0^n$, then $\psi(w) = 0^{n-1}$. Otherwise, let w' denote the segment of w strictly between the last F and terminating UD. Note that steps in w' (if any) must all be Ds because Fs and Us alternate. Let I denote the list of numbers in w'. Then I is weakly increasing, and nonempty by condition (3) in Section 3.7.3 with last (and largest) entry a, say. Let x denote the letter immediately preceding the last F. Then x is a number or $x = D$ by condition (4). If I is a constant list, so $I = a^s$, then w' has the form $D^{r\geqslant 0}a^s$ since all D's must occur at the start by condition (2). We now consider 3 cases to facilitate showing that ψ is invertible.

1. I is a constant list a^s and $x = D$. Then w has the form on the left side below and ψ deletes the last occurrence of each of F, U and D, and moves the $D^{r\geqslant 0}$ factor to the end.
$$\cdots DFD^{r\geqslant 0}a^{s\geqslant 1}UD \;\rightarrow\; \cdots Da^{s\geqslant 1}D^{r\geqslant 0}\,.$$

2. I is a constant list a^s and x is a number. By condition (2) $x = a - 1$. Then w has the form on the left side below and ψ performs the same actions as in case 1.
$$\cdots (a-1)FD^{r\geqslant 0}a^{s\geqslant 1}UD \;\rightarrow\; \cdots (a-1)a^{s\geqslant 1}D^{r\geqslant 0}\,.$$

3. I is not a constant list. So $a-1$ must occur in I since I is part of a weakly increasing ascent sequence, and w has the form on the left side below. Here ψ interchanges the last U step and the $D^{r\geqslant 0}$ factor, deletes one a, and subtracts 1 from each remaining a.
$$\cdots F \cdots (a-1)D^{r\geqslant 0}a^{s\geqslant 1}UD \;\rightarrow\; \cdots F \cdots (a-1)U(a-1)^{s-1\geqslant 0}D^{r+1\geqslant 1}\,.$$

Next, we show ψ is invertible. A constant decorated path consists entirely of 0s: $0^{r\geqslant 0}$. A nonconstant decorated path ends with a number, cases (i) and (ii) in Figure 6 below, or a D step, cases (iii) - (vi), and these 6 cases are mutually exclusive and exhaustive. The cases 1,2,3 can thus be distinguished: case 1 when $\psi(w)$ is in case (i) or (iii), case 2 when $\psi(w)$ is in case (ii) or (iv), and case 3 when $\psi(w)$ is in case (v) or (vi). It is now clear how to reverse ψ.

Case	Terminal segment
(i)	$Da^{s\geqslant 1}$
(ii)	$(a-1)a^{s\geqslant 1}$
(iii)	$Da^{s\geqslant 1}D^{r\geqslant 1}$
(iv)	$(a-1)a^{s\geqslant 1}D^{r\geqslant 1}$
(v)	$Ua^{s\geqslant 1}D^{r\geqslant 1}$
(vi)	$UD^{r\geqslant 1}$

Figure 6: Cases by terminal segment.

3.7.8 The bijection $\boldsymbol{\rho}$ from $\boldsymbol{\mathcal{DP}_n}$ to $\bigcup_{i=1}^{n} \boldsymbol{\mathcal{PP}_i \times \mathcal{DP}_{n-i}}$, $\boldsymbol{n \geqslant 1}$

To define ρ, we need a notion of standardization. To *standardize* a string of numbers and steps with respect to a number a, denoted $\mathrm{St}_a(w)$, means to add $a-b$ to each number in the string where b is the first number in the string so that the first number becomes a. For example, $\mathrm{St}_2(3U44D) = 2U33D$ and $\mathrm{St}_a(\epsilon) = \epsilon$.

Now suppose $w \in \mathcal{DP}_n$, $n \geqslant 1$, and consider four cases to define $\rho(w)$ as a pair of decorated paths.

- If w consists entirely of numbers, then w can be written in the form $w = 0^j w_2$ where w_2 is empty or begins with the number 1, and $\rho(w) = (0^j, \mathrm{St}_0(w_2))$ of the form (all-numbers, all-numbers). For example, $\rho(001223) = (00, 0112)$.

- If the last U of w is (immediately) followed by a number, say a, so that w has the form $w_1 a^j w_2$ where w_1 ends with the last U, $j \geqslant 1$, and the first letter of w_2 is not a. Then $\rho(w) = (0^j, w_1 \cdot \mathrm{St}_a(w_2))$ of the form (all-numbers, not all-numbers). For example, $\rho(0F1U112D3) = (00, 0F1U1D2)$.

- If the last U of w has successor a D and this D is the last D in w, then w has the form $w_1 w_2$ where w_1 ends with UD and w_2 (possible empty) consists entirely of numbers. Here $\rho(w) = (w_1, \mathrm{St}_0(w_2))$ of the form (not all-numbers, all-numbers). For example, $\rho(0F12UD233) = (0F12UD, 011)$.

- If the last U of w has successor a D, denoted D_1, and D_1 is not the last D in w, let D_2 denote the second D after the last U and let U_2 denote its matching upstep in the underlying UFD path. By condition (3), U_2 is preceded by a number, say a. See Figure 7 below for a schematic example, where v denotes the string of 0 or more numbers between D_1 and D_2, and the arrows indicate matching steps.

  Split w after U_2 and after D_1 to write $w = w_1 w_2 w_3$, as shown. Then $\rho(w) = (\mathrm{St}_0(w_2), w_1 \cdot \mathrm{St}_a(w_3))$ of the form (not all-numbers, not all-numbers). For example, $\rho(0F1U1F222UD233D4) = (0F111UD, 0F1U122D3)$.

The four bulleted cases can be distinguished according as each of the two resulting decorated paths consists entirely of numbers or not, as indicated in each of the bulleted cases.

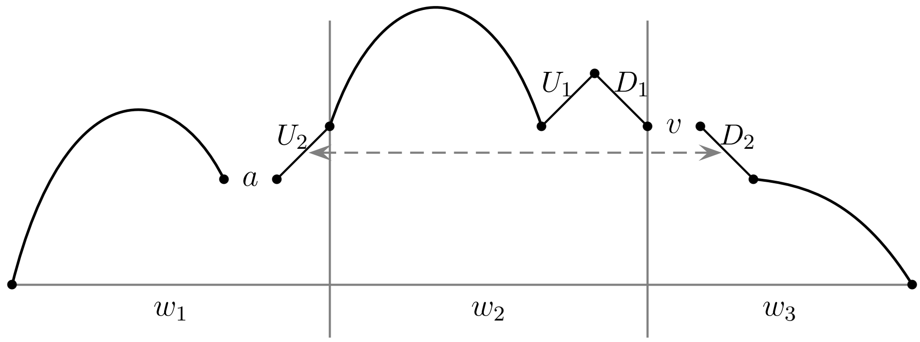

Figure 7: A decomposition for the bijection ρ.

It is clear the map ρ can then be reversed by looking at the position of the last U in the second member of $\rho(w)$ if it contains a U, otherwise by a suitable concatenation.

4 Further results

By using our algorithm as in the previous sections, one can show that the number aw_k of A-Wilf-equivalence classes of k length-3 patterns is given by

$$aw_4 = 74, \quad aw_5 = 61, \quad aw_6 = 47, \quad aw_7 = 35, \quad aw_8 = 25,$$
$$aw_9 = 18, \quad aw_{10} = 12, \quad aw_{11} = 7, \quad aw_{12} = 3, \quad aw_{13} = 1.$$

Acknowledgment

We are very grateful to two anonymous referees for a thorough reading of the paper that caught several shortcomings in the presentation. We appreciate their time and effort.

References

[1] A. M. Baxter and L. K. Pudwell. Ascent sequences avoiding pairs of patterns. *Electron. J. Combin.*, 22(1):#P1.58, 2015.

[2] M. Bousquet-Mèlou, A. Claesson, M. Dukes, and S. Kitaev. (2+2)-free posets, ascent sequences and pattern avoiding permutations. *J. Combin. Theory Ser. A*, 117(7):884–909, 2010.

[3] D. Callan and T. Mansour. Ascent sequences avoiding a triple of 3-letter patterns and Fibonacci numbers. *Art of Discret. Appl. Math.*, 7:#P2.10, 2024.

[4] D. Callan, T. Mansour, and M. Shattuck. Restricted ascent sequences and Catalan numbers. *Appl. Anal. Discrete Math.*, 8:288–303, 2014.

[5] M. Dukes and R. Parviainen. Ascent sequences and upper triangular matrices containing non-negative integers. *Electron. J. Combin.*, 17(1):#R53, 2010.

[6] M. Dukes, J. Remmel, S. Kitaev, and E. Steingrímsson. Enumerating (2 + 2)-free posets by indistinguishable elements. *J. Comb.*, 2(1):139–163, 2011.

[7] P. Duncan and E. Steingrímsson. Pattern avoidance in ascent sequences. *Electron. J. Combin.*, 18(1):#P226, 2011.

[8] S. Kitaev. Patterns in Permutations and Words. In: Monographs in Theoretical Computer Science (with a forward by J. Remmel). Springer-Verlag, 2011.

[9] S. Kitaev and J. Remmel. Enumerating (2+2)-free posets by the number of minimal elements and other statistics. *Discrete Appl. Math.*, 159:2098–2108, 2011.

[10] T. Mansour and M. Shattuck. Some enumerative results related to ascent sequences. *Discrete Math.*, 315–316:29–41, 2014.

[11] T. Mansour and M. Shattuck. Ascent sequences and Fibonacci numbers. *FILOMAT*, 29(4):703–712, 2015.

[12] J. West. Generating trees and forbidden subsequences. *Discrete Math.*, 157:363–374, 1996.

Appendix

Table 2: Ascent sequences avoiding a triples of length-3 patterns.

Beginning of Table 2		
Class	B triple	$\{\lvert\mathcal{A}_n(B)\rvert\}_{n=1}^{11}$
1	{000,011,012}	1,2,2,0,0,0,0,0,0,0,0
2	{000,001,012},{000,010,012}	1,2,2,1,0,0,0,0,0,0,0
3	{000,001,010},{000,001,011},{000,010,011} {001,010,011},{001,010,012},{001,011,012} {010,011,012}	1,2,2,2,2,2,2,2,2,2,2
4	{000,012,101},{000,012,110}	1,2,3,2,0,0,0,0,0,0,0
5	{000,012,021},{000,012,100},{000,012,102} {000,012,120},{000,012,201},{000,012,210}	1,2,3,3,0,0,0,0,0,0,0
6	{000,011,102},{000,011,120},{001,011,100} {001,011,120},{001,012,100},{001,012,110} {011,012,100}	1,2,3,3,3,3,3,3,3,3,3
7	{000,001,021},{000,001,120}	1,2,3,4,4,4,4,4,4,4,4
8	{000,001,110},{000,011,021},{000,011,100} {000,011,101},{000,011,110},{000,011,201} {000,011,210},{001,010,021},{001,010,100} {001,010,101},{001,010,102},{001,010,110} {001,010,120},{001,010,201},{001,010,210} {001,011,021},{001,011,101},{001,011,102} {001,011,110},{001,011,201},{001,011,210} {001,012,021},{001,012,101},{001,012,102} {001,012,120},{001,012,201},{001,012,210} {010,011,021},{010,011,100},{010,011,101} {010,011,102},{010,011,110},{010,011,120} {010,011,201},{010,011,210},{010,012,021} {010,012,100},{010,012,101},{010,012,102} {010,012,110},{010,012,120},{010,012,201} {010,012,210},{011,012,021},{011,012,101} {011,012,102},{011,012,110},{011,012,120} {011,012,201},{011,012,210}	1,2,3,4,5,6,7,8,9,10,11
9	{000,001,210}	1,2,3,5,7,9,11,13,15,17,19
10	{000,001,100},{000,001,101},{000,001,102} {000,001,201},{000,010,021},{000,010,100} {000,010,101},{000,010,102},{000,010,110} {000,010,120},{000,010,201},{000,010,210}	1,2,3,5,8,13,21,34,55,89,144
11	{000,201,210}	1,2,4,10,25,66,177,488,1368,3900,11258
12	{000,100,201}	1,2,4,10,26,74,218,672,2130,6945,23145
13	{000,100,210}	1,2,4,10,26,75,228,738,2501,8857,32543
14	{001,021,100},{001,021,110},{001,021,120} {001,100,110},{001,100,120},{001,110,120} {011,100,102},{011,100,120},{011,102,120} {012,100,101},{012,100,110},{012,101,110}	1,2,4,6,8,10,12,14,16,18,20
15	{001,021,101},{001,021,102},{001,021,201} {001,021,210},{001,100,210},{001,101,110} {001,101,120},{001,102,110},{001,102,120} {001,110,201},{001,110,210},{001,120,201} {001,120,210},{011,021,100},{011,021,102} {011,021,120},{011,100,101},{011,100,110} {011,100,201},{011,100,210},{011,101,102} {011,101,120},{011,102,110},{011,102,201} {011,102,210},{011,110,120},{011,120,201} {011,120,210},{012,021,100},{012,021,101} {012,021,110},{012,100,102},{012,100,120} {012,100,201},{012,100,210},{012,101,102} {012,101,120},{012,101,201},{012,101,210} {012,102,110},{012,110,120},{012,110,201} {012,110,210}	1,2,4,7,11,16,22,29,37,46,56
16	{001,100,101},{001,100,102},{001,100,201}	1,2,4,7,12,20,33,54,88,143,232
17	{000,021,102}	1,2,4,8,11,18,29,47,76,123,199
18	{000,102,120}	1,2,4,8,12,20,32,52,84,136,220
19	{000,021,120}	1,2,4,8,13,23,39,67,114,194,329
20	{000,102,110}	1,2,4,8,14,24,40,66,108,176,286
21	{001,101,210},{001,102,210},{001,201,210}	1,2,4,8,15,26,42,64,93,130,176
22	{000,021,101},{000,021,110}	1,2,4,8,15,28,51,92,164,290,509
23	{000,101,120}	1,2,4,8,15,29,56,108,208,401,773
24	{000,101,102},{000,101,110},{001,101,102} {001,101,201},{001,102,201},{010,021,100} {010,021,101},{010,021,102},{010,021,110} {010,021,120},{010,021,201},{010,021,210} {010,100,101},{010,100,102},{010,100,110}	

Continuation of Table 2		
Class	B triple	$\{\lvert\mathcal{A}_n(B)\rvert\}_{n=1}^{11}$
	{010,100,120},{010,100,201},{010,100,210} {010,101,102},{010,101,110},{010,101,120} {010,101,201},{010,101,210},{010,102,110} {010,102,120},{010,102,201},{010,102,210} {010,110,120},{010,110,201},{010,110,210} {010,120,201},{010,120,210},{010,201,210} {011,021,101},{011,021,110},{011,021,201} {011,021,210},{011,101,110},{011,101,201} {011,101,210},{011,110,201},{011,110,210} {011,201,210},{012,021,102},{012,021,120} {012,021,201},{012,021,210},{012,102,120} {012,102,201},{012,102,210},{012,120,201} {012,120,210},{012,201,210}	1,2,4,8,16,32,64,128,256,512,1024
25	{000,110,120}	1,2,4,8,17,37,84,195,465,1131,2809
26	{000,102,210}	1,2,4,9,17,33,61,112,202,361,639
27	{000,102,201}	1,2,4,9,17,35,69,139,277,555,1109
28	{000,021,100},{000,021,201},{000,021,210}	1,2,4,9,18,37,73,143,275,523,983
29	{000,100,102}	1,2,4,9,18,38,78,163,337,701,1453
30	{000,100,120}	1,2,4,9,19,43,98,231,552,1345,3329
31	{000,120,201}	1,2,4,9,19,43,99,236,568,1394,3462
32	{000,120,210}	1,2,4,9,19,43,99,236,570,1410,3547
33	{000,101,210}	1,2,4,9,20,45,101,227,510,1146,2575
34	{000,100,101},{000,101,201}	1,2,4,9,21,51,127,323,835,2188,5798
35	{000,110,201}	1,2,4,9,21,51,128,331,876,2360,6446
36	{000,100,110}	1,2,4,9,22,58,163,486,1526,5019,17208
37	{000,110,210}	1,2,4,9,22,58,164,492,1555,5143,17706
38	{021,102,110}	1,2,5,12,25,48,89,164,305,576,1105
39	{021,100,102}	1,2,5,12,26,53,105,206,404,795,1571
40	{100,102,110}	1,2,5,12,27,57,117,237,477,957,1917
41	{021,101,102},{021,101,120},{021,102,120} {100,102,120},{101,102,110},{101,102,120} {102,110,120}	1,2,5,12,27,58,121,248,503,1014,2037
42	{021,100,101},{021,100,110},{021,100,120} {021,101,110},{021,110,120}	1,2,5,12,28,64,144,320,704,1536,3328
43	{100,101,110},{100,101,120},{101,110,120}	1,2,5,12,28,65,151,351,816,1897,4410
44	{100,101,102}	1,2,5,12,29,70,169,408,985,2378,5741
45	{100,110,120}	1,2,5,12,30,78,210,581,1648,4778,14120
46	{102,110,201}	1,2,5,13,31,69,147,305,623,1261,2539
47	{021,102,201},{021,102,210},{102,110,210}	1,2,5,13,32,74,163,347,722,1480,3005
48	{100,102,210}	1,2,5,13,33,80,187,426,953,2104,4599
49	{101,102,210},{102,120,201},{102,120,210}	1,2,5,13,33,81,193,449,1025,2305,5121
50	{100,102,201}	1,2,5,13,33,82,201,489,1185,2866,6925
51	{101,120,201},{101,120,210}	1,2,5,13,33,82,202,497,1224,3017,7439
52	{021,100,201},{021,100,210},{021,110,201} {021,110,210}	1,2,5,13,34,88,224,560,1376,3328,7936
53	{021,101,201},{021,101,210},{021,120,201} {021,120,210},{100,101,210},{101,102,201} {101,110,201},{101,110,210}	1,2,5,13,34,89,233,610,1597,4181,10946
54	{100,101,201}	1,2,5,13,35,97,275,794,2327,6905,20705
55	{100,120,201},{110,120,201}	1,2,5,13,35,98,284,845,2567,7932,24857
56	{100,120,210},{110,120,210}	1,2,5,13,35,98,284,847,2589,8085,25725
57	{100,110,201}	1,2,5,13,35,98,285,856,2638,8297,26529
58	{100,110,210}	1,2,5,13,36,106,330,1079,3682,13040,47702
59	{102,201,210}	1,2,5,14,39,104,265,650,1547,3596,8205
60	{101,201,210}	1,2,5,14,41,122,365,1094,3281,9842,29525
61	{021,201,210},{100,201,210},{110,201,210}	1,2,5,14,42,132,429,1430,4862,16796,58786
62	{120,201,210}	1,2,5,14,42,133,440,1507,5304,19074,69787
End of Table 2		