Shellability of componentwise discrete polymatroids

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Abstract

In the present paper, motivated by a conjecture of Jahan and Zheng, we prove that componentwise polymatroidal ideals have linear quotients. This solves positively a conjecture of Bandari and Herzog. We introduce componentwise discrete polymatroids, as the combinatorial counterpart of componentwise polymatroidal ideals, and show that they are shellable multicomplexes.

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1 Componentwise linear quotients

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring with coefficients over a field K, and let $I \subset S$ be a monomial ideal. Let $\mathcal{G}(I)$ be the unique minimal set of monomial generators of I. We say that I has *linear quotients* if there exists an order $u_1, \dots, u_m$ of $\mathcal{G}(I)$ such that $(u_1, \dots, u_{j-1}) : u_j$ is generated by variables for $j = 2, \dots, m$.

For $j \geqslant 0$, let $I_{\langle j \rangle}$ be the monomial ideal generated by the monomials of degree j belonging to I. We say that I has *componentwise linear quotients* if $I_{\langle j \rangle}$ has linear quotients for all j. It is known that ideals with linear quotients have componentwise linear quotients [12, Corollary 2.8]. The converse is an open question [12]:

Conjecture 1. (Jahan–Zheng) Let I be a monomial with componentwise linear quotients. Then I has linear quotients.

The above conjecture is widely open. See [11] for some partial results.

2 Componentwise Polymatroidal Ideals

A monomial ideal I is called *polymatroidal* if the set of the exponent vectors of the minimal monomial generators of I is the set of bases of a discrete polymatroid [9]. Polymatroidal ideals have linear quotients. A monomial ideal I is *componentwise polymatroidal* if the

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component $I_{\langle j\rangle}$ is polymatroidal for all j. Hence, componentwise polymatroidal ideals are ideals with componentwise linear quotients. Therefore, a particular case of Conjecture 1 is:

Conjecture 2. (Bandari–Herzog) Let I be a componentwise polymatroidal ideal. Then I has linear quotients.

This conjecture was firstly considered in [1] and proved for ideals of componentwise Veronese type. Recently, Bandari and Qureshi [2] proved it in the two variables case and for componentwise polymatroidal ideals with strong exchange property.

We are going to prove Conjecture 2 in full generality.

For this aim, we recall some results from [2]. For a monomial $u = x_1^{a_1}\cdots x_n^{a_n} \in S$, we denote its *degree* by $\deg(u) = a_1 + \cdots + a_n$. Whereas, the x_i*-degree* of u is the integer $\deg_{x_i}(u) = a_i = \max\{j \geqslant 0 : x_i^j \text{ divides } u\}$.

Theorem 3. [2, Proposition 1.2] *Let $I \subset S$ be a monomial ideal. Then, the following conditions are equivalent.*

(i) *I is a componentwise polymatroidal ideal.*

(ii) *For all $u, v \in I$ with $\deg(u) \leqslant \deg(v)$ and with u not dividing v, and all i such that $\deg_{x_i}(v) > \deg_{x_i}(u)$ there exists an integer j with $\deg_{x_j}(v) < \deg_{x_j}(u)$ and such that $x_j(v/x_i) \in I$.*

Proposition 4. [2, Proposition 1.5] *Let $I \subset S$ be a componentwise polymatroidal ideal. Then the following property, called the dual exchange property, holds: For all $u, v \in I$ with $\deg(u) \leqslant \deg(v)$, and all i such that $\deg_{x_i}(v) < \deg_{x_i}(u)$ there exists an integer j with $\deg_{x_j}(v) > \deg_{x_j}(u)$ and such that $x_i(v/x_j) \in I$.*

We close this section with some examples.

Example 5. (a) Componentwise polymatroidal ideals in two variables were classified in [2]. Let $I \subset K[x, y]$ be a monomial ideal. We may assume that the minimal monomial generators of I do not have any common factor. In fact, if $I = uJ$ for a monomial $u \in S$ and a monomial ideal J, then I is componentwise polymatroidal if and only if J is such. It is proved in [2, Corollary 2.7] that $I \subset K[x, y]$ is a componentwise polymatroidal ideal if and only if I is a yx*-tight* ideal in the sense of [2, Definition 2.1].

(b) Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geqslant 0}$ and $d \geqslant 1$. The ideal of *Veronese type* $(\mathbf{a}, d)$ is

$$I_{\mathbf{a},d} \ = \ (x_1^{b_1}\cdots x_n^{b_n} \ : \ b_1 + \cdots + b_n = d, \ b_i \leqslant a_i, \ \text{for all } i).$$

Monomial ideals whose all components are of Veronese type are componentwise polymatroidal ideals, see also [1, Section 3].

(c) A monomial ideal I generated in a single degree has the strong exchange property if for all $u, v \in \mathcal{G}(I)$ all i such that $\deg_{x_i}(u) > \deg_{x_i}(v)$ and all j such that $\deg_{x_j}(u) < \deg_{x_j}(v)$, then $x_j(u/x_i)$ belongs to $\mathcal{G}(I)$. It is known that any such ideal

I is a polymatroidal ideal of the form $I = uI_{\mathbf{a},d}$ for some suitable monomial $u \in S$, $\mathbf{a} \in \mathbb{Z}^n_{\geqslant 0}$ and $d \geqslant 1$. Hence, ideals whose all components satisfy the strong exchange property are componentwise polymatroidal.

(d) Denote by $\mathfrak{m}$ the maximal ideal $(x_1, \dots, x_n)$. It is known that the product of polymatroidal ideals is polymatroidal. Let $1 \leqslant d_1 < \cdots < d_t$ be positive integers, $J_1, \dots, J_t$ be polymatroidal ideals generated in degrees $d_1, \dots, d_t$, respectively, such that $\mathfrak{m}^{d_{i+1}-d_i} J_i \subseteq J_{i+1}$ for $i = 1, \dots, t-1$. Let $I = J_1 + \cdots + J_t$. Then I is componentwise polymatroidal. Indeed,

$$I_{\langle j\rangle} = \begin{cases} J_i & \text{if } j = d_i, \text{ for some } i, \\ \mathfrak{m}^{j-d_i} J_i & \text{if } d_i < j < d_{i+1}, \text{ for some } i, \\ \mathfrak{m}^{j-d_t} J_t & \text{if } j \geqslant d_t, \end{cases}$$

is polymatroidal for all j.

(e) Let $u = x_{i_1} \cdots x_{i_d}$ and $v = x_{j_1} \cdots x_{j_d}$ be two monomials of the same degree d, with $1 \leqslant i_1 \leqslant \ldots \leqslant i_d \leqslant n$ and $1 \leqslant j_1 \leqslant \ldots \leqslant j_d \leqslant n$. We write $v \preceq_{\text{Borel}} u$ if $j_k \leqslant i_k$ for all k. The *principal Borel ideal generated by* u, denoted by $B(u)$, is the monomial ideal generated in degree d whose minimal generating set is

$$\mathcal{G}(B(u)) \ = \ \{v \in S \ : \ \deg(v) = \deg(u), \ v \preceq_{\text{Borel}} u\}.$$

It is known that $B(u)$ is polymatroidal. Let $u, v \in S$ be monomials of the same degree. It follows from the definition of $\succeq_{\text{Borel}}$ that $B(v) \subseteq B(u)$ if and only if $v \preceq_{\text{Borel}} u$. Notice that $\mathfrak{m}^\ell B(u) = B(ux_n^\ell)$ for any ℓ. We say that a monomial ideal I is *componentwise principal Borel* if all $I_{\langle j\rangle}$ are principal Borel ideals. From (d) and these considerations, it follows that I is componentwise principal Borel if and only if there exists monomials $u_1, \dots, u_t$ of degrees $d_1 < \cdots < d_t$, respectively, such that

$$u_i x_n^{d_{i+1}-d_i} \ \preceq_{\text{Borel}} \ u_{i+1},$$

for $i = 1, \dots, t-1$, and $I = B(u_1) + \cdots + B(u_t)$.

(f) Actually, componentwise polymatroidal ideals appeared implicitly for the first time in the work of Francisco and Van Tuyl [7], in connection to *ideals of fat points*. For $n \geqslant 1$, set $[n] = \{1, \dots, n\}$. Given a non-empty subset A of $[n]$, denote by P_A the polymatroidal ideal $(x_i \, : \, i \in A)$. Suppose that $A_1, \dots, A_t$ are non-empty subsets of $[n]$ such that $A_i \cup A_j = [n]$ for all $i \neq j$. It is shown in [7, Theorem 3.1] that

$$I = P_{A_1}^{k_1} \cap \cdots \cap P_{A_t}^{k_t}$$

is componentwise polymatroidal for all positive integers $k_1, \dots, k_t \geqslant 1$.

(g) Let I be a polymatroidal ideal generated in degree d. The *socle* of I is the monomial ideal $\operatorname{soc}(I) = (I : \mathfrak{m})_{\langle d-1\rangle}$. It is conjectured in [1, page 760], and proved in some special cases in [4], that $\operatorname{soc}(I)$ is again polymatroidal. It is noted in [4] that $(I : \mathfrak{m})$ is generated in at most two degrees $d-1$ and d, and that $(I : \mathfrak{m})_{\langle d\rangle} = I$. Thus

$$(I : \mathfrak{m}) = \operatorname{soc}(I) + I.$$

Furthermore, it follows by the very definition of colon ideal that $\mathfrak{m}(I : \mathfrak{m}) \subseteq I$. In particular, $\mathfrak{m} \cdot \mathrm{soc}(I) \subseteq I$. Hence, if $\mathrm{soc}(I)$ is polymatroidal, it would follow by the construction in (d) that $(I : \mathfrak{m})$ is componentwise polymatroidal.

(h) More generally, let I be a componentwise polymatroidal ideal. If the above conjecture about the socle of polymatroidal ideals is true, then $(I : \mathfrak{m})$ would be componentwise polymatroidal as well. Indeed,

$$\begin{aligned}(I : \mathfrak{m})_{\langle j\rangle} &= \{u \in S \ :\ \deg(u) = j,\ \text{and } ux_i \in I,\ \text{for all } i\}\\ &= \{u \in S \ :\ \deg(u) = j,\ \text{and } ux_i \in I_{\langle j+1\rangle},\ \text{for all } i\}\\ &= (I_{\langle j+1\rangle} : \mathfrak{m})_{\langle j\rangle}\\ &= \mathrm{soc}(I_{\langle j+1\rangle})\end{aligned}$$

would be a polymatroidal ideal, for all j.

3 Componentwise polymatroidal ideals have linear quotients

We are now ready to prove the main result in the paper.

Theorem 6. *Componentwise polymatroidal ideals have linear quotients.*

Proof. Let $I \subset S = K[x_1, \dots, x_n]$ be a componentwise polymatroidal ideal. We prove the theorem by induction on $|\mathcal{G}(I)|$, the number of minimal monomial generators of I. If $|\mathcal{G}(I)| = 1$, then I is a principal ideal and it has linear quotients.

Suppose $|\mathcal{G}(I)| > 1$. By induction, all componentwise polymatroidal ideals in S with less than $|\mathcal{G}(I)|$ generators have linear quotients. Furthermore, we may suppose that all monomials $u \in \mathcal{G}(I)$ have no common factor $w \neq 1$. Otherwise, we may consider the ideal I' with $\mathcal{G}(I') = \{u/w : u \in \mathcal{G}(I)\}$. Then I' is componentwise polymatroidal too, and I has linear quotients if and only if I' has linear quotients. Let $d = \alpha(I)$ be the *initial degree* of I. That is, $I_{\langle j\rangle} = 0$ for $0 \leqslant j < d$ and $I_{\langle d\rangle} \neq 0$. Let j be any integer such that x_j divides some monomial generator of $I_{\langle d\rangle}$. After a suitable relabeling, we may assume $j = 1$. Therefore, we can write

$$I = x_1 I_1 + I_2$$

for unique monomial ideals $I_1, I_2 \subset S$ such that

$$\begin{aligned}\mathcal{G}(x_1 I_1) &= \{u \in \mathcal{G}(I) \ :\ x_1 \text{ divides } u\},\\ \mathcal{G}(I_2) &= \{u \in \mathcal{G}(I) \ :\ x_1 \text{ does not divide } u\}.\end{aligned}$$

We are going to prove the following three facts:

(a) $I_2 \subseteq I_1$ as monomial ideals of S.

(b) $x_1 I_1$ is a componentwise polymatroidal ideal of S.

(c) I_2 is a componentwise polymatroidal ideal of $K[x_2, \dots, x_n]$.

Once we get these claims, the proof ends as follows. Since the monomials in $\mathcal{G}(I)$ have no common factor $\neq 1$, $|\mathcal{G}(x_1I_1)|$ and $|\mathcal{G}(I_2)|$ are strictly less than $|\mathcal{G}(I)|$. Items (b) and (c) together with our induction hypothesis imply that x_1I_1 and I_2 have linear quotients, with linear quotients orders, say $u_1,\dots,u_r$ of $\mathcal{G}(x_1I_1)$, and $v_1,\dots,v_s$ of $\mathcal{G}(I_2)$. We claim $u_1,\dots,u_r,v_1,\dots,v_s$ is a linear quotients order of I. Indeed, if $\ell\in[r]$, then $(u_1,\dots,u_{\ell-1}):u_\ell$ is generated by variables by our inductive hypothesis on x_1I_1. Whereas, if $\ell\in[s]$, using the inductive hypothesis on I_2, we obtain that the ideal

$$\begin{aligned}(u_1,\dots,u_r,v_1,\dots,v_{\ell-1}):v_\ell &= (u_1,\dots,u_r):v_\ell+(v_1,\dots,v_{\ell-1}):v_\ell\\ &= (x_1I_1:v_\ell)+(v_1,\dots,v_{\ell-1}):v_\ell\\ &= (x_1)+(v_1,\dots,v_{\ell-1}):v_\ell\end{aligned}$$

is generated by variables, because it is a sum of ideals generated by variables. Here, we have used the fact that $v_\ell\in\mathcal{G}(I_2)\subset I_1$ and x_1 does not divide v_ℓ to get the equality $(x_1I_1:v_\ell)=x_1(I_1:v_\ell)=x_1S=(x_1)$.

It remains to prove items (a), (b) and (c).

Proof of (a): It is enough to show that any monomial of $\mathcal{G}(I_2)$ is divided by some monomial of I_1. Let $v\in\mathcal{G}(I_2)$ and let $u\in x_1I_1$ with $\deg(u)=\alpha(I)$. Then $\deg(u)=\alpha(x_1I_1)=\alpha(I)$. Therefore $\deg(u)\leqslant\deg(v)$. Moreover $\deg_{x_1}(v)=0<\deg_{x_1}(u)$. By the dual exchange property (Proposition 4) we can find j with $\deg_{x_j}(v)>\deg_{x_j}(u)$ such that $x_1(v/x_j)\in I$. Then there is $w\in\mathcal{G}(I)$ that divides $x_1(v/x_j)$. If $w\in\mathcal{G}(I_2)$, then x_1 does not divide w and so w divides v/x_j, against the fact that v is a minimal generator of I. Hence $w\in\mathcal{G}(x_1I_1)$ and $w=x_1w'$ divides $x_1(v/x_j)$. Consequently $w'\in I_1$ divides v/x_j. Hence $w'\in I_1$ divides $v\in\mathcal{G}(I_2)$, as desired.

Proof of (b): Let $u,v\in x_1I_1$ with $\deg(u)\leqslant\deg(v)$, u not dividing v, and let i such that $\deg_{x_i}(v)>\deg_{x_i}(u)$. By Theorem 3(ii) it is enough to determine j with $\deg_{x_j}(v)<\deg_{x_j}(u)$ such that $x_j(v/x_i)\in x_1I_1$. Since $u,v\in I$, by Theorem 3 we can find j with $\deg_{x_j}(v)<\deg_{x_j}(u)$ such that $x_j(v/x_i)\in I$. We show now that $x_j(v/x_i)\in x_1I_1$. Note that x_1 divides $v\in x_1I_1$. If $i\neq 1$, then x_1 divides $x_j(v/x_i)$. Otherwise, if $i=1$, since x_1 divides $u\in x_1I_1$ and $\deg_{x_1}(v)>\deg_{x_1}(u)\geqslant 1$, we obtain $\deg_{x_1}(x_j(v/x_1))\geqslant 1$. Hence, in both cases x_1 divides $x_j(v/x_i)$. Now, if some $w\in\mathcal{G}(I_2)$ divides $x_j(v/x_i)$ then x_1w also divides $x_j(v/x_i)$. By item (a), $x_1w\in x_1I_2\subset x_1I_1$ and so $x_j(v/x_i)\in x_1I_1$. Otherwise, some $w\in\mathcal{G}(x_1I_1)$ divides $x_j(v/x_i)$ and again $x_j(v/x_i)\in x_1I_1$, as wanted.

Proof of (c): Let $u,v\in I_2$ with $\deg(u)\leqslant\deg(v)$, u not dividing v and let i such that $\deg_{x_i}(v)>\deg_{x_i}(u)$. Recall that we are regarding I_2 as an ideal of $K[x_2,\dots,x_n]$, hence $\deg_{x_1}(v)=\deg_{x_1}(u)=0$. By Theorem 3(ii) valid in I, there exists j with $\deg_{x_j}(v)<\deg_{x_j}(u)$ and such that $x_j(v/x_i)\in I$. Since $j\neq 1$, x_1 does not divide $x_j(v/x_i)$. Hence $x_j(v/x_i)\in I_2$, as desired. ☐

Example 7. By Examples 5(f), $I=P^2_{\{1,2,3\}}\cap P^2_{\{1,3,4\}}$ is componentwise polymatroidal. Notice that $\mathcal{G}(I)=\{x_1^2,x_1x_3,x_3^2,x_1x_2x_4,x_2x_3x_4,x_2^2x_4^2\}$ and $\alpha(I)=2$. A variable dividing a generator of least degree is for instance x_1. Using the notation in the proof of Theorem 6 and the *Macaulay2* [8] package [5], we checked that $I_1 = (x_1, x_3, x_2x_4)$, $I_2 = (x_3^2, x_2x_3x_4, x_2^2x_4^2)$ are componentwise polymatroidal ideals and $I_2 \subseteq I_1$. The ideal I_1 has linear quotients order x_1, x_3, x_2x_4. Whereas a linear quotients order of I_2 is $x_3^2, x_2x_3x_4, x_2^2x_4^2$. Hence, according to the proof of the theorem, a linear quotients order of $I = x_1I_1 + I_2$ is indeed $x_1^2, x_1x_3, x_1x_2x_4, x_3^2, x_2x_3x_4, x_2^2x_4^2$.

Unfortunately the product of componentwise polymatroidal ideals is not a componentwise polymatroidal ideal anymore [1]. However, we expect that

Conjecture 8. Each power of a componentwise polymatroidal ideal has linear quotients.

For a monomial ideal I, denote by $\text{HS}_j(I)$ the jth *homological shift ideal* of I [4]. That is, the monomial ideal generated by the monomials whose exponent vectors are the jth multigraded shifts appearing in the minimal multigraded free resolution of I. It is expected that $\text{HS}_j(I)$ is polymatroidal for all j, if I is polymatroidal. For some partial results on this conjecture see [3, 4, 6].

Question 9. Let I be a componentwise polymatroidal ideal. Is $\text{HS}_j(I)$ componentwise polymatroidal as well, for all j?

4 Componentwise Discrete Polymatroids

In this final section, we introduce the combinatorial counterpart of componentwise polymatroidal ideals, which we call *componentwise discrete polymatroids*.

For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_{\geqslant 0}^n$, denote by $\mathbf{a}[i] = a_i$ the ith component of $\mathbf{a}$. We set $|\mathbf{a}| = a_1 + \cdots + a_n$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geqslant 0}^n$. We write $\mathbf{a} \leqslant \mathbf{b}$ if $\mathbf{a}[i] \leqslant \mathbf{b}[i]$ for all i. We write $\mathbf{a} < \mathbf{b}$ if $\mathbf{a} \leqslant \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the canonical basis of $\mathbb{Z}_{\geqslant 0}^n$, that is $\mathbf{e}_i[j] = 0$ for all $j \neq i$ and $\mathbf{e}_i[i] = 1$. A *simplicial multicomplex* $\mathcal{M}$ on the vertex set $[n]$ is a finite subset of $\mathbb{Z}_{\geqslant 0}^n$ satisfying the following properties:

(a) If $\mathbf{a} \in \mathcal{M}$ and $\mathbf{b} \leqslant \mathbf{a}$, then $\mathbf{b} \in \mathcal{M}$.

(b) $\mathbf{e}_i \in \mathcal{M}$ for all i.

Any $\mathbf{a} \in \mathcal{M}$ is called a *face* of $\mathcal{M}$. A *facet* $\mathbf{a} \in \mathcal{M}$ is a face of $\mathcal{M}$ for which there is no $\mathbf{b} \in \mathcal{M}$ such that $\mathbf{a} < \mathbf{b}$. The set of facets of $\mathcal{M}$ is denoted by $\mathcal{F}(\mathcal{M})$. We set $\alpha(\mathcal{M}) = \min\{|\mathbf{a}| : \mathbf{a} \in \mathcal{F}(\mathcal{M})\}$ and $\omega(\mathcal{M}) = \max\{|\mathbf{a}| : \mathbf{a} \in \mathcal{F}(\mathcal{M})\}$. The dimension of $\mathcal{M}$ is $\dim(\mathcal{M}) = \max\{|\mathbf{a}| - 1 : \mathbf{a} \in \mathcal{M}\}$. Notice that $\dim(\mathcal{M}) = \omega(\mathcal{M}) - 1$.

For any $\mathbf{b}_1, \dots, \mathbf{b}_\ell \in \mathbb{Z}_{\geqslant 0}^n$, we denote by $\langle \mathbf{b}_1, \dots, \mathbf{b}_\ell \rangle$ the unique, smallest with respect to the inclusion, simplicial multicomplex containing $\mathbf{b}_1, \dots, \mathbf{b}_\ell$.

For $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}$, we set $\mathbf{x}^{\mathbf{a}} = \prod_i x_i^{\mathbf{a}[i]}$. The *facet ideal* of $\mathcal{M}$ is defined as

$$I(\mathcal{M}) \;=\; (\mathbf{x}^{\mathbf{a}} \;:\; \mathbf{a} \in \mathcal{F}(\mathcal{M})).$$

There is a natural bijection between monomial ideals of S and simplicial multicomplexes on vertex set $[n]$, defined by assigning to each monomial ideal $I \subset S$ the simplicial multicomplex $\mathcal{M}_I = \langle \mathbf{a} \in \mathbb{Z}_{\geqslant 0}^n : \mathbf{x}^{\mathbf{a}} \in \mathcal{G}(I) \rangle$.

Now, we introduce a special class of simplicial multicomplexes. A simplicial multicomplex $\mathcal{P}$ is called a *componentwise discrete polymatroid* if $I(\mathcal{P})$ is a componentwise polymatroidal ideal. To adhere to the classical terminology used for discrete polymatroids, we call the facets of $\mathcal{P}$ the *bases* of $\mathcal{P}$. Notice that a componentwise discrete polymatroid $\mathcal{P}$ is a discrete polymatroid if and only if $\alpha(\mathcal{P})=\omega(\mathcal{P})$.

We denote by $[n]^{\langle d\rangle}$ the discrete polymatroid $\{\mathbf{a}\in\mathbb{Z}_{\geqslant 0}^n:|\mathbf{a}|\leqslant d\}$. In particular $[n]^{\langle 1\rangle}=\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$. Whereas, given a non-empty finite set $A\subset\mathbb{Z}_{\geqslant 0}^n$ and an integer $j\geqslant 0$, we set $A_{\langle j\rangle}=\{\mathbf{a}\in A:|\mathbf{a}|\leqslant j\}$. Furthermore, if $A_1,A_2\subset\mathbb{Z}_{\geqslant 0}^n$ are non-empty finite sets, we define the sum as $A_1+A_2=\{\mathbf{a}_1+\mathbf{a}_2\ :\ \mathbf{a}_1\in A_1,\mathbf{a}_2\in A_2\}$.

Now, we can characterize componentwise discrete polymatroids.

Theorem 10. *The following conditions are equivalent:*

(i) *$\mathcal{P}$ is a componentwise discrete polymatroid.*

(ii) *For all $\alpha(\mathcal{P})\leqslant j\leqslant\omega(\mathcal{P})$, the simplicial multicomplex*

$$\bigcup_{k=\alpha(\mathcal{P})}^{j}(\mathcal{P}_{\langle k\rangle}+[n]^{\langle j-k\rangle})$$

is a discrete polymatroid.

(iii) *For all $\mathbf{a},\mathbf{b}\in\bigcup_{\ell=\alpha(\mathcal{P})}^{\omega(\mathcal{P})}\bigcup_{k=\alpha(\mathcal{P})}^{\ell}(\mathcal{P}_{\langle k\rangle}+[n]^{\langle \ell-k\rangle})$ with $\alpha(\mathcal{P})\leqslant|\mathbf{a}|\leqslant|\mathbf{b}|$ and $\mathbf{a}\not\leqslant\mathbf{b}$, and all i such that $\mathbf{b}[i]>\mathbf{a}[i]$, there is an integer j with $\mathbf{b}[j]<\mathbf{a}[j]$ such that $\mathbf{b}-\mathbf{e}_i+\mathbf{e}_j\in\bigcup_{\ell=\alpha(\mathcal{P})}^{\omega(\mathcal{P})}\bigcup_{k=\alpha(\mathcal{P})}^{\ell}(\mathcal{P}_{\langle k\rangle}+[n]^{\langle \ell-k\rangle})$.*

Proof. We first notice the following fact. Let $I\subset S$ be a monomial ideal, and let $\omega(I)=\max\{\deg(u):u\in\mathcal{G}(I)\}$. Then I is componentwise polymatroidal if and only if $I_{\langle j\rangle}$ is polymatroidal for $\alpha(I)\leqslant j\leqslant\omega(I)$. Only sufficiency needs a proof. Suppose that $I_{\langle j\rangle}$ is polymatroidal for $\alpha(I)\leqslant j\leqslant\omega(I)$. If $j>\omega(I)$, then $I_{\langle j\rangle}=\mathfrak{m}^{j-\omega(I)}I_{\langle\omega(I)\rangle}$ is polymatroidal for it is the product of two polymatroidal ideals.

It is easily seen that $I(\mathcal{P})_{\langle j\rangle}=I(\bigcup_{k=\alpha(\mathcal{P})}^{j}(\mathcal{P}_{\langle k\rangle}+[n]^{\langle j-k\rangle}))$ for all $\alpha(\mathcal{P})\leqslant j\leqslant\omega(\mathcal{P})$. Since, by definition, $I(\mathcal{P})$ is componentwise polymatroidal if and only if $I(\mathcal{P})_{\langle j\rangle}$ is polymatroidal for all $\alpha(\mathcal{P})\leqslant j\leqslant\omega(\mathcal{P})$, the equivalence (i)$\Leftrightarrow$(ii) follows at once.

The implication (i)$\Rightarrow$(iii) follows from Theorem 3. Conversely, assume that (iii) holds. Then, [9, Theorem 2.3] implies that $I(\mathcal{P})_{\langle j\rangle}$ is polymatroidal for all $\alpha(\mathcal{P})\leqslant j\leqslant\omega(\mathcal{P})$. This shows that (iii)$\Rightarrow$(ii) and concludes the proof. ∎

A simplicial multicomplex $\mathcal{M}$ is called *pure* if $|\mathbf{a}|=|\mathbf{b}|$ for all $\mathbf{a},\mathbf{b}\in\mathcal{F}(\mathcal{M})$. Whereas, $\mathcal{M}$ is called *shellable* if there exists an order $\mathbf{a}_1,\dots,\mathbf{a}_m$ of $\mathcal{F}(\mathcal{M})$ such that the simplicial multicomplex

$$\langle\mathbf{a}_1,\dots,\mathbf{a}_{j-1}\rangle\cap\langle\mathbf{a}_j\rangle$$

is pure of dimension $|\mathbf{a}_j|-1$ for all $j=2,\dots,m$. In this case, $\mathbf{a}_1,\dots,\mathbf{a}_m$ is called a *shelling order* of $\mathcal{M}$. It is well-known and easily seen that $\mathbf{a}_1,\dots,\mathbf{a}_m$ is a shelling order of $\mathcal{M}$ if and only if $\mathbf{x}^{\mathbf{a}_1},\dots,\mathbf{x}^{\mathbf{a}_m}$ is a linear quotients order of $I(\mathcal{M})$. Thus, Theorem 6 implies immediately

Corollary 11. *Componentwise discrete polymatroids are shellable.*

We end the paper with some natural questions.

Let $\mathcal{P}$ be a componentwise discrete polymatroid. Attached to $\mathcal{P}$ there are the following three monomial subalgebras of $S[t]$:

$$\begin{aligned} K[\mathcal{P}] &= K[\mathbf{x}^{\mathbf{a}}t \ : \ \mathbf{a} \in \mathcal{P}], \\ K[\mathcal{F}(\mathcal{P})] &= K[\mathbf{x}^{\mathbf{a}}t \ : \ \mathbf{a} \in \mathcal{F}(\mathcal{P})], \\ \mathcal{R}(I(\mathcal{P})) &= \bigoplus_{k\geqslant 0} I(\mathcal{P})^k t^k \ = \ K[x_1, \dots, x_n, \mathbf{x}^{\mathbf{a}}t \ : \ \mathbf{a} \in \mathcal{F}(\mathcal{P})]. \end{aligned}$$

We call $K[\mathcal{F}(\mathcal{P})]$ the *base ring* of $\mathcal{P}$. Whereas, $\mathcal{R}(I(\mathcal{P}))$ is the Rees algebra of $I(\mathcal{P})$. These three algebras are toric rings. It follows from a famous theorem of Hochster that if a toric ring is normal, then it is Cohen–Macaulay [10].

Question 12. Let $\mathcal{P}$ be a componentwise discrete polymatroid. Are the rings $K[\mathcal{P}]$, $K[\mathcal{F}(\mathcal{P})]$, $\mathcal{R}(I(\mathcal{P}))$ normal? Cohen–Macaulay?

The above question has a positive answer when $\mathcal{P}$ is actually a discrete polymatroid, see [9, Theorem 6.1], [9, Corollary 6.2] and [13, Proposition 3.11].

On the other hand, the following question is open even for discrete polymatroids.

Question 13. Let $\mathcal{P}$ be a componentwise discrete polymatroid. Are the rings $K[\mathcal{P}]$, $K[\mathcal{F}(\mathcal{P})]$, $\mathcal{R}(I(\mathcal{P}))$ Koszul?

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