# Low Rank Groups of Lie Type Acting Point- and Line-Primitively on Finite Generalised Quadrangles

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**Abstract**

Suppose we have a finite thick generalised quadrangle whose automorphism group $G$ acts primitively on both the set of points and the set of lines. Then $G$ must be almost simple. In this paper, we show that $\mathrm{soc}(G)$ cannot be isomorphic to ${}^2\mathrm{B}_2(2^{2m+1})$ or ${}^2\mathrm{G}_2(3^{2m+1})$ where $m$ is a positive integer.

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## 1 Introduction

A generalised polygon is a type of incidence structure introduced by Jacques Tits (1959) [18] to realise groups of Lie type as symmetries (more precisely, automorphism groups) of geometric objects. Let $n$ be a positive integer. A generalised $n$-gon is an incidence structure whose incidence graph is a bipartite graph with diameter $n$ and girth $2n$. We say that a generalised $n$-gon has order $(s, t)$ if every line has $s + 1$ points incident with it and every point is incident with $t + 1$ lines. Furthermore, a generalised $n$-gon is said to be thick if it has order $(s, t)$ where $s, t > 1$. Feit and Higman showed in [6] that thick generalised $n$-gons exist if and only if $n \in \{3, 4, 6, 8\}$. Our focus from now will only be on thick generalised polygons. Therefore, we refer to a thick generalised polygon as simply a generalised polygon. The examples of generalised polygons that arise from groups of Lie type via the construction of Tits are called *classical*. Since then, many non-classical examples of projective planes (generalised 3-gons) and generalised quadrangles (4-gons) have been found [19, Section 3.7]. In the case of generalised hexagons (6-gons) and generalised octagons (8-gons), the only known examples are the classical ones.

Many attempts have been made to construct new examples. These attempts involve introducing various symmetry conditions on the automorphism groups and analysing the possible groups acting on generalised polygons with these symmetry conditions. Buekenhout and Van Maldeghem (1994) [5] showed that if a group acts distance-transitively on a

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generalised $n$-gon ($n \geqslant 4$), then in fact, it must act point-primitively on that generalised $n$-gon. We summarise the results on generalised hexagons and octagons in Table 1 and the results on generalised quadrangles in Table 2. Here, we let $\Gamma$ be a generalised $n$-gon (generalised hexagon or octagon in Table 1 and generalised quadrangle in Table 2) and $G \leqslant \mathsf{Aut}(\Gamma)$. For a point $\alpha$ in $\Gamma$, we write $G_\alpha$ to denote the point-stabiliser of $\alpha$ in $G$. Also, $q$ is assumed to be a prime power. Finally, the column of assumptions refers to the action of $G$ on $\Gamma$.

Table 1: Summary of results on generalised hexagons and octagons.

| **Assumptions** | **Conclusion** | **Reference** |
|---|---|---|
| Point-primitive, line-primitive and flag-transitive | $G$ is almost simple of Lie type | [15] |
| Point-primitive | $G$ is almost simple of Lie type | [3] |
| Point-primitive and $\mathrm{soc}(G) \cong \mathsf{PSL}_n(q)$ for $n \geqslant 2$ | $G_\alpha$ acts irreducibly on $V = \mathbb{F}_q^n$ | [8] |
| Point-primitive and $\mathrm{soc}(G) \cong {}^2\mathrm{B}_2(2^{2m+1})$, ${}^2\mathrm{G}_2(3^{2m+1})$ or ${}^2\mathrm{F}_4(2^{2m+1})$ | $\mathrm{soc}(G) \cong {}^2\mathrm{F}_4(2^{2m+1})$ and $\Gamma$ is the classical generalised octagon or its dual | [13] |

Table 2: Summary of results on generalised quadrangles.

| **Assumptions** | **Conclusion** | **Reference** |
|---|---|---|
| Point-primitive and line-primitive | $G$ is almost simple | [2] |
| Point-primitive line-primitive and flag-transitive | $G$ is almost simple of Lie type | [2] |
| Point-primitive, flag-transitive and $\mathrm{soc}(G) \cong A_n$ with $n \geqslant 5$ | $G \leqslant S_6$ and $\Gamma$ is the unique generalised quadrangle of order $(2, 2)$ | [2] |
| Point-primitive | $\mathrm{soc}(G)$ is not isomorphic to a sporadic group | [1] |
| Point-primitive and $\mathrm{soc}(G) \cong \mathsf{PSL}_2(q)$ for $q \geqslant 4$ | $q = 9$ and $\Gamma$ is the symplectic quadrangle $W(2)$ | [7] |
| Point-primitive and line-primitive | $\mathrm{soc}(G)$ is not isomorphic to $\mathsf{PSU}_3(q)$ for $q \geqslant 3$ | [12] |

The main theorem in this paper (Theorem 1) is motivated by the work of Morgan and Popiel [13] on generalised hexagons and octagons, where they showed that if $G$ acts point-primitively on a generalised hexagon or an octagon $\Gamma$ with socle $\mathrm{soc}(G) \cong {}^2\mathrm{B}_2(2^{2m+1})$, ${}^2\mathrm{G}_2(3^{2m+1})$ or ${}^2\mathrm{F}_4(2^{2m+1})$, then $\mathrm{soc}(G) \cong {}^2\mathrm{F}_4(2^{2m+1})$ and $\Gamma$ is the classical generalised octagon or its dual.

**Theorem 1.** *Let $m$ be a positive integer. An almost simple group with socle isomorphic to ${}^2\mathrm{B}_2(2^{2m+1})$ or ${}^2\mathrm{G}_2(3^{2m+1})$ cannot act primitively on both the set of points and the set of lines of a generalised quadrangle.*

The case where $\mathrm{soc}(G) \cong {}^2\mathrm{F}_4(2^{2m+1})$ is not included in this paper as it still requires further analysis and the techniques needed to classify generalised quadrangles in this scenario may be different.

# 2 Preliminaries

We develop some preliminary definitions and results from incidence geometry and group theory.

## 2.1 Incidence Geometry

An incidence geometry is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ is called the set of points, $\mathcal{L}$ is the set of lines disjoint from $\mathcal{P}$ and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{I}$ is the incidence relation. We say that a point $\alpha \in \mathcal{P}$ is incident with a line $L \in \mathcal{L}$ if $(\alpha, L) \in \mathcal{I}$. This pair $(\alpha, L)$ is called a flag. We say two points $\alpha, \beta \in \mathcal{P}$ are collinear if there exists a line $L \in \mathcal{L}$ such that $(\alpha, L) \in \mathcal{I}$ and $(\beta, L) \in \mathcal{I}$. Finally, the dual of an incidence geometry $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the incidence geometry $(\mathcal{P}', \mathcal{L}', \mathcal{I}')$ where the set of points $\mathcal{P}' = \mathcal{L}$, the set of lines $\mathcal{L}' = \mathcal{P}$ and the incident pair $(L, \alpha) \in \mathcal{I}'$ precisely when $(\alpha, L) \in \mathcal{I}$.

We may also view an incidence structure as a graph. More precisely, given an incidence structure $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, we construct a graph called the *incidence graph* of $\Gamma$ in the following way: we take the vertex set to be $\mathcal{P} \cup \mathcal{L}$ and we join an edge between $\alpha \in \mathcal{P}$ and $L \in \mathcal{L}$ if $(\alpha, L) \in \mathcal{I}$. Note that the disjointness of $\mathcal{P}$ and $\mathcal{L}$ ensures that the incidence graph does not contain loops. Finally, an automorphism of $\Gamma$ is a bijective map $\theta : \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ that sends points to points and lines to lines as well as preserving the incidence relation. The group of automorphisms of $\Gamma$ is denoted by $\mathsf{Aut}(\Gamma)$. While we have defined generalised polygons via this incidence graph, we will often think of them as incidence geometries. For more information on incidence structures and generalised polygons, we refer to [10, 14, 19].

From here on, a generalised quadrangle is assumed to be finite, *i.e.*, it has a finite set of points and set of lines. Furthermore, a generalised quadrangle with order $(s, t)$ has $(s + 1)(st + 1)$ points and $(t + 1)(st + 1)$ lines [14, (1.2.1)].

**Lemma 2** ([7, Corollary 2.3])**.** *Let $G$ be a group acting on a generalised quadrangle $\Gamma$ and suppose $g \in G$. Let $\Gamma_g = (\mathcal{P}_g, \mathcal{L}_g, \mathcal{I}_g)$ be the fixed substructure of $g$. If $|\mathcal{P}_g| \geqslant 2$, $|\mathcal{L}_g| \geqslant 2$ and $\Gamma_g$ admits an automorphism group $H$ that is transitive on both points and lines, then $\Gamma_g$ is a generalised quadrangle of order $(s', t')$ for some positive integers $s'$ and $t'$.*

The next number-theoretic lemma concerns the solutions of $s$ and $t$ given a certain number of points and lines.

**Lemma 3.** *Let $a, b, s, t$ be positive integers and $p$ be a prime. If*

$$(s+1)(st+1) = p^a \text{ and } (t+1)(st+1) = p^b, \tag{1}$$

*then $p = 2$. Furthermore, we have: $(s = 1$ and $b = a/2 + 1)$ or $(t = 1$ and $a = b/2 + 1)$.*

*Proof.* Let $d := \gcd(s+1, t+1, st+1)$. Then $d$ divides $t(s+1)-(st+1) = t-1$. Therefore, $d$ divides $t+1-(t-1) = 2$. Hence, $d = 1$ or $d = 2$. Since $s+1, t+1, st+1 \geqslant 2$, it follows that $p$ divides $s+1, t+1$ and $st+1$ and so, $p$ divides $d$. From which, we deduce that $d = 2$ and $p = 2$. Now, suppose that $s+1 \neq 2$ and $t+1 \neq 2$. Then $s+1 \equiv 0 \pmod 4$ and $t+1 \equiv 0 \pmod 4$. Thus, $st+1 \equiv (-1)(-1)+1 \equiv 2 \pmod 4$ and so, $st+1 = 2$. This is a contradiction since $st+1 > s+1 > 2$. Therefore, $s+1 = 2$ or $t+1 = 2$. Without loss of generality, say $t = 1$. Substituting for $t$ and $p$ in (1), we obtain

$$(s+1)^2 = 2^a \text{ and } 2(s+1) = 2^b.$$

Hence, $s+1 = 2^{a/2} = 2^{b-1}$, from which, we obtain $b = a/2+1$. If we consider the case where $s = 1$, then by the same argument, we obtain $a = b/2 + 1$. □

## 2.2 Group Theory

### 2.2.1 Notation

Let $n$ be a positive integer and $q$ be a prime power. We denote the cyclic group of order $n$ by $C_n$, the dihedral group of order $2n$ by $D_n$, the elementary abelian group of order $q$ by $E_q$. Given two groups $H$ and $K$, we write $H.K$ to mean an extension of $H$ by $K$, *i.e.*, a group $G$ with a normal subgroup $M$ where $M \cong H$ and $G/M \cong K$. When the extension is a split extension, we write $H : K$ (or $H \rtimes K$).

### 2.2.2 Elementary Results

In our study of groups of Lie type, we will come across subgroups that are semidirect products of two groups with coprime order. A classical result that will be useful in studying these subgroups is the Schur-Zassenhaus Theorem, which can be found in [9, Section 3B].

**Theorem 4** (Schur-Zassenhaus Theorem)**.** *Let $G$ be a group with a normal subgroup $N$ such that $|N|$ and $|G/N|$ are coprime. Then there exists a subgroup $K \leqslant G$ such that $G = N \rtimes K$,* i.e.*, $G = NK$ and $N \cap K = 1$. We call $K$ a* complement *of $N$ in $G$. Moreover, all complements of $N$ in $G$ are conjugate to $K$.*

**Corollary 5.** *Let* $G = N \rtimes K$ *where* $\gcd(|N|, |K|) = 1$*. Suppose* $M \leqslant G$ *with* $|M| = |K|$*. Then* $M = K^x$ *for some* $x \in G$*.*

*Proof.* Observe that $N \cap M = 1$ since they have coprime orders. Thus, $|NM| = |N||M|/|N \cap M| = |N||K| = |G|$. Hence, $M$ is a complement of $N$ in $G$. By the Schur-Zassenhaus Theorem, all complements of $N$ are conjugate and so, $M = K^x$ for some $x \in G$. □

**Corollary 6.** *Let* $G = N \rtimes K$ *where* $\gcd(|N|, |K|) = 1$ *and suppose* $g \in G$ *such that* $|g|$ *divides* $|K|$*. Then* $g \in K^h$ *for some* $h \in G$*. In particular, the number of conjugacy classes of elements with order* $|g|$ *in* $G$ *is at most the number of conjugacy classes of elements with order* $|g|$ *in* $K$*.*

*Proof.* First, observe that $g$ acts on $N$ by conjugation. Since $|g|$ divides $|K|$, it follows that $g \notin N$. Hence, we have the subgroup $M := N \rtimes \langle g \rangle \leqslant G$. Note that $KM = K(N \rtimes \langle g \rangle) = KN\langle g \rangle = G$ and $|G| = |K||M|/|K \cap M|$. Thus,

$$|K \cap M| = \frac{|K||M|}{|G|} = \frac{|K||N||\langle g \rangle|}{|G|} = \frac{|K||N||g|}{|K||N|} = |g|.$$

By Corollary 5, we deduce that $K \cap M$ is a complement of $N$ in $M$. By the Schur-Zassenhaus Theorem, all complements of $N$ in $M$ are conjugate. Therefore, $K \cap M = \langle g \rangle^x$ for some $x \in M$. Thus, $\langle g \rangle \leqslant K^{x^{-1}}$, whence, $g \in K^h$, where $h = x^{-1}$. The number of conjugacy classes of elements with order $|g|$ is at most the number in $K$ because any element of $G$ with order $|g|$ is conjugate to an element of $K$. □

### 2.2.3 Permutation Groups

Our investigation involves groups of Lie type acting on generalised quadrangles. The following lemma provides a formula for calculating the number of fixed points of an element with respect to a group action.

**Lemma 7** (Formula for the Number of Fixed Points [11, Lemma 2.5])**.** *Let* $G$ *be a finite group acting transitively on a set* $\Omega$*. Let* $\alpha \in \Omega$ *and* $g \in G$*. Then the number of fixed points of* $g$*, denoted* $\pi(g)$*, is given by*

$$\pi(g) = \frac{|\Omega||g^G \cap G_\alpha|}{|g^G|}. \tag{2}$$

**Lemma 8** ([12, Lemma 2.4])**.** *Let* $G$ *be a group acting transitively on a set* $\Omega$ *and* $g$ *be a non-identity element in* $G$*. Consider a point* $\alpha \in \Omega_g$*, where* $\Omega_g$ *is the set of fixed points of* $g$*. Then* $\mathbf{C}_G(g)$ *acts transitively on* $\Omega_g$ *if and only if* $g^G \cap G_\alpha = g^{G_\alpha}$, i.e., *the conjugacy class of* $g$ *in* $G$ *does not split into multiple conjugacy classes in* $G_\alpha$*. In this case,* $|\Omega_g| = |\mathbf{C}_G(g) : \mathbf{C}_G(g) \cap G_\alpha|$*.*

## 3 Suzuki Groups

### 3.1 Structure of Suzuki Groups

In this subsection, we provide some background information on the family of Suzuki groups. Throughout this section, we denote a Suzuki group by ${}^2\mathrm{B}_2(q)$, where $q = 2^{2m+1}$ for some positive integer $m$. We start with the following result which can be found in [16, Theorem 9].

**Theorem 9** (Maximal Subgroups of a Suzuki Group)**.** *Let $G = {}^2\mathrm{B}_2(q)$ and $H$ be a maximal subgroup of $G$. Then $H$ is isomorphic to one of the following:*

- *(i) $M \cong E_q.E_q.C_{q-1}$;*
- *(ii) $B_0 = \mathbf{N}_G(A_0) \cong D_{q-1}$, where $A_0 \cong C_{q-1}$;*
- *(iii) $A_1 \cong C_{q+\sqrt{2q}+1}$, $A_2 \cong C_{q-\sqrt{2q}+1}$;*
- *(iv) $B_i = \mathbf{N}_G(A_i) \cong C_{q\pm\sqrt{2q}+1} : C_4$ for $i \in \{1, 2\}$; and*
- *(v) $N_0 \cong {}^2\mathrm{B}_2(q_0)$, where $q_0 = 2^{n_0} > 2$ and $q = q_0^{r_0}$ for some prime $r_0$.*

*Moreover, there is only one conjugacy class of each type of maximal subgroup.*

The next lemma shows that all the involutions in a Suzuki group are conjugate.

**Lemma 10** (Conjugacy Class of Involutions)**.** *Let $G = {}^2\mathrm{B}_2(q)$ and $H$ be a maximal subgroup of $G$. Then $G$ and $H$ both have exactly one conjugacy class of involutions.*

*Proof.* It was shown in [16, Proposition 7] that all the involutions are conjugate in $G$. Furthermore, if $H \cong E_q.E_q.C_{q-1}$, then a cyclic subgroup of $H$ of order $q-1$ permutes the set of involutions in $H$ transitively [16, Proposition 8]. Therefore, we only need to consider the cases:[1] $H \cong D_{q-1} = C_{q-1} : C_2$ and $H \cong C_{q\pm\sqrt{2q}+1} : C_4$. Since $q$ is a power of 2, we have that $\gcd(q-1, 2) = 1$ and $\gcd(q \pm \sqrt{2q} + 1, 4) = 1$. Hence, by Corollary 6, we conclude that all the involutions are conjugate in $G$. □

The following lemma provides information about the centraliser of an involution in $G$.

**Lemma 11** (Centraliser of an Involution)**.** *Let $G = {}^2\mathrm{B}_2(q)$ with maximal subgroups $H, K \leqslant G$ where $H \cong D_{q-1}$ and $K \cong C_{q\pm\sqrt{2q}+1} : C_4$. Let $g \in G$, $h \in H$ and $k \in K$ be involutions. Then*

$$|\mathbf{C}_G(g)| = q^2, \ |\mathbf{C}_H(h)| = 2, \ \text{and } |\mathbf{C}_K(k)| = 4.$$

[1] Note that the subfield case: $H \cong {}^2\mathrm{B}_2(q_0)$, we have that $H$ is a Suzuki group and so, it follows from [16, Proposition 7].

*Proof.* Consider $G = {}^2\mathrm{B}_2(q)$. Note that $\mathbf{C}_G(g) \leqslant Q$ where $Q$ is a Sylow 2-subgroup of order $q^2$ [16, Proposition 1]. Moreover, $g \in \mathbf{Z}(Q)$ [16, Proposition 7]. Therefore, $|\mathbf{C}_G(g)| = |Q| = q^2$.

Next, we find $\mathbf{C}_H(h) = \mathbf{C}_G(h) \cap H$. Observe that $|\mathbf{C}_H(h)| = |\mathbf{C}_G(h) \cap H| \leqslant 2$ since $|\mathbf{C}_G(h)| = q^2$ and $|H| = 2(q-1)$. Note that $h \in \mathbf{C}_H(h)$. Hence, $\mathbf{C}_H(h) \cong C_2$. Similarly, we have $|\mathbf{C}_K(k)| = |\mathbf{C}_G(k) \cap K| \leqslant 4$ since $|\mathbf{C}_G(k)| = q^2$ and $|K| = 4(q \pm \sqrt{2q}+1)$. Let us now write $K = \langle x \rangle : \langle y \rangle$, where $|x| = q \pm \sqrt{2q}+1$ and $|y| = 4$. Using Corollary 6, we find that $k = (y^2)^z$ for some $z \in K$, whence, $y^z \in \mathbf{C}_K(k)$. Since $|y| = 4$, we have that $\mathbf{C}_K(k) \cong C_4$. □

**Lemma 12** (Number of Fixed Points of an Involution)**.** *Let $G = {}^2\mathrm{B}_2(q)$ act primitively on a set $\Omega$ and take a point $\alpha \in \Omega$. Suppose $g \in G_\alpha$ is an involution. Then*

$$|\Omega_g| = \begin{cases} q^2/2 = 2^{4m+1} & \text{if } G_\alpha \cong D_{q-1}, \\ q^2/4 = 2^{4m} & \text{if } G_\alpha \cong C_{q\pm\sqrt{2q}+1} : C_4, \\ (q/q_0)^2 = 2^{2n_0(r_0-1)} & \text{if } G_\alpha \cong {}^2\mathrm{B}_2(q_0) \text{ where } q_0 = 2^{n_0} > 2 \\ & \quad \text{and } q = q_0^{r_0} \text{ for some prime } r_0. \end{cases}$$

*Furthermore, $\mathbf{C}_G(g)$ acts transitively on $\Omega_g$, the set of points fixed by $g$.*

*Proof.* Since $G_\alpha$ contains a single class of involutions (Lemma 10), we conclude that $g^G \cap G_\alpha = g^{G_\alpha}$. Thus, applying Lemma 8, we find that $\mathbf{C}_G(g)$ acts transitively on $\Omega_g$ and $|\Omega_g| = |\mathbf{C}_G(g) : \mathbf{C}_G(g) \cap G_\alpha|$. Using Lemma 11, we obtain the desired formulae for $|\Omega_g|$. □

### 3.2 Suzuki Groups Acting on Generalised Quadrangles

We now prove Theorem 1 in the case where the socle is isomorphic to ${}^2\mathrm{B}_2(q)$.

*Proof.* Suppose that an almost simple group with socle ${}^2\mathrm{B}_2(q)$ acts primitively on the point-set and the line-set of a generalised quadrangle $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ with order $(s,t)$ where $s,t \geqslant 2$. Then $|\mathcal{P}| = (s+1)(st+1)$. Note that $s+1 \geqslant 3$ and $st+1 \geqslant 3$. Therefore, their product cannot be a prime. The argument in [13, Section 3] shows that the socle of this almost simple group acts primitively on both $\mathcal{P}$ and $\mathcal{L}$. Therefore, it suffices to consider the Suzuki group ${}^2\mathrm{B}_2(q)$ acting primitively on both $\mathcal{P}$ and $\mathcal{L}$. Let $G = {}^2\mathrm{B}_2(q)$. For a point $\alpha \in \mathcal{P}$ and a line $L \in \mathcal{L}$, their respective stabilisers $G_\alpha$ and $G_L$ are maximal subgroups of $G$. By Theorem 9, a maximal subgroup of ${}^2\mathrm{B}_2(q)$ is isomorphic to one of the following:

(i) (Parabolic) $E_q.E_q.C_{q-1}$;

(ii) (Dihedral) $D_{q-1}$;

(iii) (Frobenius) $C_{q\pm\sqrt{2q}+1} : C_4$; or

(iv) (Subfield) ${}^2\mathrm{B}_2(q_0)$, where $q_0 = 2^{n_0} > 2$ and $q = q_0^{r_0}$ for some prime $r_0$.

Moreover, there is only one conjugacy class of each type of maximal subgroup, see also [4, Table 8.16].

Note that if $G_\alpha \cong E_q.E_q.C_{q-1}$, then by [16], the action of $G$ is 2-transitive. Thus, a pair of collinear points can be mapped to a pair of non-collinear points, which is a contradiction. Therefore, we only need to investigate the cases: $D_{q-1}$, $C_{q\pm\sqrt{2q}+1} : C_4$ and ${}^2\mathrm{B}_2(q_0)$ for the point and line-stabiliser and show that there are no generalised quadrangles in those scenarios. To this end, suppose $G_\alpha$ is isomorphic to $D_{q-1}$, $C_{q\pm\sqrt{2q}+1} : C_4$ or ${}^2\mathrm{B}_2(q_0)$ and $G_L$ is isomorphic to $D_{q-1}$, $C_{q\pm\sqrt{2q}+1} : C_4$ or ${}^2\mathrm{B}_2(q_1)$, where $q_i = 2^{n_i} > 2$ and $q_i^{r_i} = q$ for primes $r_i$ and $i = 1, 2$. Observe that both $G_\alpha$ and $G_L$ contain an involution, say $g \in G_\alpha$ and $h \in G_L$. Since all the involutions in $G$ are conjugate (Lemma 10), there exists a $k \in G$ such that $g = h^k$. Thus, $g \in G_L^k$. Since $G_L^k = G_{L'}$ where $L' = L^k \in \mathcal{L}$, we can take $L$ to be $L'$ and assume that we have an involution $g \in G_\alpha \cap G_L$. Let $\mathcal{P}_g$ and $\mathcal{L}_g$ be the set of points and the set of lines fixed by $g$, respectively. By Lemma 12, we find that the number of fixed points and lines are

$$|\mathcal{P}_g| = 2^a \text{ and } |\mathcal{L}_g| = 2^b,$$

where $a \in A := \{4m, 4m+1, 2n_0(r_0-1)\}$ and $b \in B := \{4m, 4m+1, 2n_1(r_1-1)\}$. By Lemma 2, the fixed substructure, $(\mathcal{P}_g, \mathcal{L}_g, \mathcal{I}_g)$ is a generalised quadrangle of order $(s', t')$ for some positive integers $s'$ and $t'$. Therefore,

$$|\mathcal{P}_g| = (s'+1)(s't'+1) = 2^a \text{ and } |\mathcal{L}_g| = (t'+1)(s't'+1) = 2^b.$$

By Lemma 3, we have that $b = a/2 + 1$ or dually, $a = b/2 + 1$. Let us suppose that $b = a/2+1$. Thus, $a$ is even and so, $a = 4m$ or $2n_0(r_0-1)$. If $a = 4m$, then $b-1 = a/2 = 2m$. Hence, $b = 2m+1$, which is odd. However, the only odd element in $B$ is $4m+1$ and so, we have a contradiction. Next, if $a = 2n_0(r_0-1)$, then $b = a/2+1 = n_0(r_0-1)+1$. However, since $r_0$ is an odd prime, it follows that $b$ is odd. Therefore, $b = 4m+1$. Consequently, we find $4m+1 = b = n_0r_0 - n_0 + 1 < n_0r_0 = 2m+1$, which is a contradiction. The case where $a = b/2+1$ is analogous. Therefore, the Suzuki group ${}^2\mathrm{B}_2(q)$ cannot act primitively on both the set of points and the set of lines of a generalised quadrangle. Consequently, an almost simple group with socle isomorphic to ${}^2\mathrm{B}_2(q)$ cannot act primitively on both the set of points and the set of lines of a generalised quadrangle. □

# 4 Ree Groups

We provide some background information on the family of small Ree groups. Throughout this section, we denote a small Ree group by ${}^2\mathrm{G}_2(q)$, where $q = 3^{2m+1}$ for some positive integer $m$.

### 4.1 Structure of Ree Groups

We have the following theorem regarding the maximal subgroups of ${}^2\mathrm{G}_2(q)$ (see [4, Table 8.43]).

**Theorem 13** (Maximal Subgroups of a Small Ree Group)**.** *Let $G = {}^2\mathrm{G}_2(q)$ and $H$ be a maximal subgroup of $G$. Then $H$ is isomorphic to one of the following:*

*(i)* $E_q.E_q.E_q.C_{q-1}$*;*

*(ii)* $C_2 \times \mathsf{PSL}_2(q)$*;*

*(iii)* $(E_4 \times D_{(q+1)/4}) : C_3$*;*

*(iv)* $C_{q\pm\sqrt{3q}+1} : C_6$*; and*

*(v)* ${}^2\mathrm{G}_2(q_0)$*, where* $q_0 = 3^{n_0} > 3$ *and* $q = q_0^{r_0}$ *for some prime* $r_0$*.*

*Moreover, there is only one conjugacy class of each type of maximal subgroup.*

Analogous to the Suzuki groups, elements of order 3 play a crucial role in studying the action of a small Ree group on a generalised quadrangle. First, we recall a definition from group theory. For a group $G$ and $g \in G$, we say that $g$ is *real* in $G$ if $g$ is conjugate to its inverse in $G$, *i.e.*, there exists an element $h \in G$ such that $g^{-1} = g^h$. We may omit the "in $G$" part and simply refer to $g$ as a real element.

The following lemma is useful for analysing the maximal subgroup isomorphic to $(E_4 \times D_{(q+1)/4}) : C_3$.

**Lemma 14.** *Let $H = KR \cong (E_4 \times D_{(q+1)/4}) : C_3$ where $K \cong E_4 \times D_{(q+1)/4}$ and $R = \langle y \rangle \cong C_3$. Then $y$ centralises an involution in $H$.*

*Proof.* Note that $\langle y \rangle$ acts by conjugation on the set of involutions in $H$. We count the number of involutions in $H$. Since $q + 1 = 3^{2m+1} + 1 = 3(3^2)^m + 1 \equiv 3 + 1 \pmod 8 \equiv 4 \pmod 8$, it follows that $(q+1)/4$ is odd and thus, $D_{(q+1)/4}$ has $(q+1)/4$ involutions. Since an involution in $K$ is the product of an element of $E_4$ and an involution in $D_{(q+1)/4}$, or is just an involution in $E_4$, the number of involutions in $K$ is $4(q+1)/4+3 = q+4$. Now, all the involutions in $H$ are in $K$ because $K \trianglelefteq H$ and $\gcd(|K|, |R|) = 1$. Therefore, $H$ has $q+4$ involutions. Focusing on the action of $\langle y \rangle$ on the set of involutions, suppose that $\langle y \rangle$ has no fixed points, *i.e.*, $y$ does not centralise any involution. Then the orbits of $\langle y \rangle$ must have length divisible by 3. This implies that $q+4$ is divisible by 3, which is a contradiction as $q = 3^{2m+1}$. Therefore, $y$ has a fixed point, *i.e.*, $y$ centralises an involution. □

**Lemma 15** (Conjugacy Classes of Elements of Order Three)**.** *Let $G = {}^2\mathrm{G}_2(q)$ and $H$ be a maximal subgroup of $G$. Then the following statements hold:*

*(i) In $G$, there is one conjugacy class of real elements of order 3 and two conjugacy classes of non-real elements of order 3.*

*(ii) If $H \cong C_2 \times \mathsf{PSL}_2(q)$, $H \cong (E_4 \times D_{(q+1)/4}) : C_3$, or $H \cong C_{q\pm\sqrt{3q}+1} : C_6$, then there are two $H$-conjugacy classes of non-real elements of order 3.*

*Proof.* First, we find that the centre of a Sylow 3-subgroup of $G$ contains one conjugacy class of elements with order 3 with representative labelled $X$ [20, Chapter III, Paragraph 4]. Also, in [20, Chapter III, Paragraph 7], we find that there are two conjugacy classes of elements of order 3 with representatives labelled $T$ and $T^{-1}$. This yields (i).

Now we focus on (ii). From [20, Chapter III, Paragraphs 1 and 3], we deduce that an order 3 element in $G$ is not real precisely when it centralises an involution. Therefore, it suffices to show the following two points:

(a) An element of order 3 in $H$ centralises an involution.

(b) There are at most two conjugacy classes of elements with order 3 in $H$.

Indeed, once we have established that there are at most two conjugacy classes, it follows that we have exactly two conjugacy classes since the elements are not real.

Let us suppose $H \cong C_2 \times \mathsf{PSL}_2(q)$. Note that all the order 3 elements of $H$ are in $\mathsf{PSL}_2(q)$. Therefore, the involution in $C_2$ is centralised by the order 3 elements of $H$. Also, there are exactly two conjugacy classes of elements with order 3 in $\mathsf{PSL}_2(q)$ (see [17, Chapter 3, (6.3) (iii)]) and hence, two conjugacy classes in $H$.

Next, we consider $H = KR \cong (E_4 \times D_{(q+1)/4}) : C_3$ where $K \cong E_4 \times D_{(q+1)/4}$ and $R = \langle y \rangle \cong C_3$. Observe that $\gcd(|E_4 \times D_{(q+1)/4}|, |C_3|) = \gcd(2(q+1), 3) = 1$. Hence, by Corollary 6, any element of order 3 must be conjugate in $H$ to $y$ or $y^{-1}$. So there are at most two conjugacy classes of elements with order 3. Furthermore, $y$ centralises an involution in $H$ by Lemma 14.

Finally, let us suppose that $H = KR \cong C_{q\pm\sqrt{3q}+1} : C_6$ where $K \cong C_{q\pm\sqrt{3q}+1}$ and $R = \langle y \rangle \cong C_6$. We find that $\gcd(|C_{q\pm\sqrt{3q}+1}|, |C_6|) = \gcd(q \pm \sqrt{3q} + 1, 6) = 1$ since $q \pm \sqrt{3q} + 1 \equiv 1 \pm 1 + 1 \pmod 2 \equiv 1 \pmod 2$ and $q \pm \sqrt{3q} + 1 \equiv 1 \pmod 3$. There are two conjugacy classes of elements with order 3 in $R$, namely, $(y^2)^R$ and $(y^4)^R$. Using Corollary 6 again, we find that the number of conjugacy classes of elements with order 3 in $H$ is at most 2. Moreover, we note that $y^2$ centralises $y^3$, which is an involution. □

**Lemma 16** (Centralisers of Elements of Order Three). *Let $G = {}^2\mathrm{G}_2(q)$, $x \in G$ be a real element of order 3 and $y \in G$ be a non-real element of order 3. Then*

$$|\mathbf{C}_G(x)| = q^3 \text{ and } |\mathbf{C}_G(y)| = 2q^2.$$

*Consider maximal subgroups $H_1$, $H_2$ and $H_3$ of $G$ where $H_1 \cong C_2 \times \mathsf{PSL}_2(q)$, $H_2 \cong (E_4 \times D_{(q+1)/4}) : C_3$ and $H_3 \cong C_{q\pm\sqrt{3q}+1} : C_6$. Suppose $h_i \in H_i$ are elements of order 3 for $i \in \{1, 2, 3\}$. Then*

$$|\mathbf{C}_{H_1}(h_1)| = 2q, \ |\mathbf{C}_{H_2}(h_2)| = 6 \text{ and } |\mathbf{C}_{H_3}(h_3)| = 6.$$

*Proof.* From [20, Chapter III, Paragraphs 2 and 3], we obtain $|\mathbf{C}_G(y)| = 2q^2$ and $|\mathbf{C}_G(x)| = q^3$, respectively.

From [17, Chapter 3, (6.4) (i)], we find that $|\mathbf{C}_{\mathsf{PGL}_2(q)}(x)| = q$ when $x$ is an element of order 3. Since $\mathsf{PSL}_2(q)$ has index 2 in $\mathsf{PGL}_2(q)$ and $C := \mathbf{C}_{\mathsf{PGL}_2(q)}(x)$ has odd order, it follows that $C \leqslant \mathsf{PSL}_2(q)$. Now, back to $H_1$. Since the $C_2$ in $H_1$ also centralises $h_1$, we obtain $|\mathbf{C}_{H_1}(h_1)| = 2q$.

Next, observe that $|\mathbf{C}_{H_2}(h_2)| = |\mathbf{C}_G(h_2) \cap H_2| \leqslant 6$ since $\gcd(|\mathbf{C}_G(h_2)|, |H_2|) = \gcd(2q^2, 4 \cdot 2(q+1)/4 \cdot 6) = \gcd(2q^2, 12(q+1)) = 6$. Furthermore, $h_2$ centralises an involution in $H_2$ (Lemma 14). Therefore, $|\mathbf{C}_{H_2}(h_2)| = 6$.

Finally, we consider $H_3$. By the argument as above, we see that $\gcd(|\mathbf{C}_G(h_3)|, |H_3|) = \gcd(2q^2, 6(q \pm \sqrt{3q}+1)) = 6$, whence, $|\mathbf{C}_{H_3}(h_3)| = |\mathbf{C}_G(h_3) \cap H_3| \leqslant 6$. Since $h_3$ lies in some subgroup isomorphic to $C_6$ in $H_3$ (Corollary 6), it follows that $h_3$ is centralised by 6 elements. Therefore, $|\mathbf{C}_{H_3}(h_3)| = 6$. □

**Lemma 17** (Number of Fixed Points of Order Three Elements)**.** *Let $G = {}^2\mathrm{G}_2(q)$ act primitively on a set $\Omega$ and take a point $\alpha \in \Omega$. Suppose we have a non-real element $g \in G_\alpha$ of order 3. Then*

$$|\Omega_g| = \begin{cases} q & \text{if } G_\alpha \cong C_2 \times \mathsf{PSL}_2(q), \\ q^2/3 & \text{if } G_\alpha \cong (E_4 \times D_{(q+1)/4}) : C_3 \text{ or } G_\alpha \cong C_{q \pm \sqrt{3q}+1} : C_6 \\ (q/q_0)^2 & \text{if } G_\alpha \cong {}^2\mathrm{G}_2(q_0) \text{ where } q_0 = 3^{n_0} > 3 \text{ and } q = q_0^{r_0} \text{ for some prime } r_0. \end{cases}$$

*Furthermore, $\mathbf{C}_G(g)$ acts transitively on $\Omega_g$, the set of points fixed by $g$.*

*Proof.* By Lemma 15, $G_\alpha$ has precisely two conjugacy classes of order 3 non-real elements. Since $G$ also has two conjugacy classes of these elements, the conjugacy classes do not split, *i.e.*, $g^G \cap G_\alpha = g^{G_\alpha}$. Thus, applying Lemma 8, we find that $\mathbf{C}_G(g)$ acts transitively on $\Omega_g$ and $|\Omega_g| = |\mathbf{C}_G(g) : \mathbf{C}_G(g) \cap G_\alpha|$. Using Lemma 16, we obtain the desired formulae for $|\Omega_g|$. □

### 4.2 Ree Groups Acting on Generalised Quadrangles

We now prove Theorem 1 in the case where the socle is isomorphic to ${}^2\mathrm{G}_2(q)$.

*Proof.* Suppose that an almost simple group with socle ${}^2\mathrm{G}_2(q)$ acts primitively on the point-set and the line-set of a generalised quadrangle $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ with order $(s,t)$ where $s, t \geqslant 2$. Then $|\mathcal{P}| = (s+1)(st+1)$. Note that $s+1 \geqslant 3$ and $st+1 \geqslant 3$. Therefore, their product cannot be a prime. The argument in [13, Section 4] shows that the socle of this almost simple group acts primitively on both $\mathcal{P}$ and $\mathcal{L}$. Therefore, it suffices to consider the Ree group ${}^2\mathrm{G}_2(q)$ acting primitively on both $\mathcal{P}$ and $\mathcal{L}$. Let $G = {}^2\mathrm{G}_2(q)$. For a point $\alpha \in \mathcal{P}$ and a line $L \in \mathcal{L}$, their respective stabilisers $G_\alpha$ and $G_L$ are maximal subgroups of $G$. We use the list of maximal subgroups from Theorem 13.

If $G_\alpha \cong E_q.E_q.E_q.C_{q-1}$, then the action of $G$ is 2-transitive [20, Theorem (v)]. Thus, a pair of collinear points can be mapped to a pair of non-collinear points, which is a contradiction. Therefore, we only need to investigate the cases: $C_2 \times \mathsf{PSL}_2(q)$, $(E_4 \times D_{(q+1)/4}) : C_3$, $C_{q\pm\sqrt{3q}+1} : C_6$ and ${}^2\mathrm{G}_2(q_0)$ for the point and line-stabiliser and show that there are no generalised quadrangles in these scenarios. To this end, suppose $G_\alpha$ is isomorphic to $C_2 \times \mathsf{PSL}_2(q)$, $(E_4 \times D_{(q+1)/4}) : C_3$, $C_{q\pm\sqrt{2q}+1} : C_4$ or ${}^2\mathrm{G}_2(q_0)$ and $G_L$ is isomorphic to $C_2 \times \mathsf{PSL}_2(q)$, $(E_4 \times D_{(q+1)/4}) : C_3$, $C_{q\pm\sqrt{2q}+1} : C_4$ or ${}^2\mathrm{G}_2(q_1)$, where $q_i = 3^{n_i} > 3$ and $q_i^{r_i} = q$ for primes $r_i$ and $i = 1, 2$. By Lemma 15, both $G_\alpha$ and $G_L$ contain a non-real element of order 3, say $g \in G_\alpha$ and $h \in G_L$. Moreover, there are precisely two classes of non-real elements of order 3 in $G$, and so $g$ is conjugate to $h$ or $h^{-1}$. Without loss of generality, suppose $g$ is conjugate to $h$, and so, we can write $g = h^k$ for some $k \in G$. Hence, $g \in G_L^k$. Since $G_L^k = G_{L'}$ where $L' = L^k \in \mathcal{L}$, we can take $L$ to be $L'$ and assume that we have an order 3 non-real element $g \in G_\alpha \cap G_L$. Let $\mathcal{P}_g$ and $\mathcal{L}_g$ be the set of points and the set of lines fixed by $g$, respectively. By Lemma 17, we find that the number of fixed points and lines are

$$|\mathcal{P}_g| = 3^a \text{ and } |\mathcal{L}_g| = 3^b,$$

where $a \in A := \{2m+1, 4m+1, 2n_0(r_0-1)\}$ and $b \in B := \{2m+1, 4m+1, 2n_1(r_1-1)\}$. By Lemma 2, the fixed substructure, $(\mathcal{P}_g, \mathcal{L}_g, \mathcal{I}_g)$ is a generalised quadrangle of order $(s', t')$ for some positive integers $s'$ and $t'$. Therefore,

$$|\mathcal{P}_g| = (s'+1)(s't'+1) = 3^a \text{ and } |\mathcal{L}_g| = (t'+1)(s't'+1) = 3^b.$$

However, there are no solutions for $s'$ and $t'$ by Lemma 3 and so, we have a contradiction. Therefore, the small Ree group ${}^2\mathrm{G}_2(q)$ cannot act primitively on both the set of points and the set of lines of a generalised quadrangle. Consequently, an almost simple group with socle isomorphic to ${}^2\mathrm{G}_2(q)$ cannot act primitively on both the set of points and the set of lines of a generalised quadrangle. □

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## References

[1] John Bamberg and James Evans. No sporadic almost simple group acts primitively on the points of a generalised quadrangle. *Discrete Mathematics*, 344(4):112291, 2021.

[2] John Bamberg, Michael Giudici, J. Morris, Gordon Royle, and Pablo Spiga. Generalised quadrangles with a group of automorphisms acting primitively on points and lines. *Journal of Combinatorial Theory Series A*, 119(7):1479–1499, 2012.

[3] John Bamberg, S.P. Glasby, Tomasz Popiel, Cheryl E. Praeger, and Csaba Schneider. Point-primitive generalised hexagons and octagons. *Journal of Combinatorial Theory, Series A*, 147:186–204, 2017.

[4] John N. Bray, Derek F. Holt, and Colva M. Roney-Dougal. *The Maximal Subgroups of the Low-Dimensional Finite Classical Groups.* London Mathematical Society Lecture Note Series. Cambridge University Press, 2013.

[5] Francis Buekenhout and Hendrik Van Maldeghem. Finite distance-transitive generalized polygons. *Geometriae Dedicata*, 52:41–51, 1994.

[6] Walter Feit and Graham Higman. The nonexistence of certain generalized polygons. *Journal of Algebra*, 1(2):114–131, 1964.

[7] Tao Feng and Jianbing Lu. On finite generalized quadrangles with psl(2,q) as an automorphism group. *Des. Codes Cryptography*, 91(6):2347–2364, 3 2023.

[8] Stephen Glasby, Emilio Pierro, and Cheryl Praeger. Point-primitive generalised hexagons and octagons and projective linear groups. *Ars Mathematica Contemporanea*, 21, 04 2021.

[9] I.M. Isaacs. *Finite Group Theory.* Graduate studies in mathematics. American Mathematical Society, 2008.

[10] G. Kiss and T. Szonyi. *Finite Geometries.* CRC Press, 2019.

[11] Martin W. Liebeck and Jan Saxl. Minimal Degrees of Primitive Permutation Groups, with an Application to Monodromy Groups of Covers of Riemann Surfaces. *Proceedings of the London Mathematical Society*, s3-63(2):266–314, 09 1991.

[12] Jianbing Lu, Yingnan Zhang, and Hanlin Zou. Nonexistence of generalized quadrangles admitting a point-primitive and line-primitive automorphism group with socle $\mathrm{psu}(3, q)$, $q \geqslant 3$. *Journal of Algebraic Combinatorics*, 60:871–898, 2024.

[13] Luke Morgan and Tomasz Popiel. Generalised polygons admitting a point-primitive almost simple group of suzuki or ree type. *Electronic Journal of Combinatorics*, 23, 2 2016. #P1.34.

[14] S.E. Payne, J.A. Thas, and European Mathematical Society. *Finite Generalized Quadrangles.* EMS series of lectures in mathematics. European Mathematical Society, 2009.

[15] Csaba Schneider and Hendrik Van Maldeghem. Primitive flag-transitive generalized hexagons and octagons. *Journal of Combinatorial Theory, Series A*, 115(8):1436–1455, 2008.

[16] Michio Suzuki. On a class of doubly transitive groups. *Annals of Mathematics*, 75(1):105–145, 1962.

[17] Michio Suzuki. *Group Theory I.* A series of comprehensive studies in mathematics. Springer, 1982.

[18] Jacques Tits. Sur la trialité et certains groupes qui s'en déduisent. *Publications Mathématiques de l'IHÉS*, 2:13–60, 1959.

[19] H. Van Maldeghem. *Generalized Polygons.* Modern Birkhäuser Classics. Springer Basel, 2012.

[20] Harold N. Ward. On ree's series of simple groups. *Transactions of the American Mathematical Society*, 121(1):62–89, 1966.