A note on inverting the dijoin of oriented graphs

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Abstract

For an oriented graph D and a set $X \subseteq V(D)$, the inversion of X in D is the graph obtained from D by reversing the orientation of each edge that has both endpoints in X. Define the inversion number of D, denoted $\mathrm{inv}(D)$, to be the minimum number of inversions required to obtain an acyclic oriented graph from D. The dijoin, denoted $D_1 \to D_2$, of two oriented graphs D_1 and D_2 is constructed by taking vertex-disjoint copies of D_1 and D_2 and adding all edges from D_1 to D_2. We show that $\mathrm{inv}(D_1 \to D_2) > \mathrm{inv}(D_1)$, for any oriented graphs D_1 and D_2 such that $\mathrm{inv}(D_1) = \mathrm{inv}(D_2) \geqslant 1$. This resolves a question of Aubian, Havet, Hörsch, Klingelhoefer, Nisse, Rambaud and Vermande. Our proof proceeds via a natural connection between the graph inversion number and the subgraph complementation number.

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1 Introduction

Given an oriented graph D and a set $X \subseteq V(D)$, the *inversion* of X in D is the oriented graph obtained from D by reversing the orientation of each edge that has both endpoints in X. In this case, we say that we *invert* X in D. Given a family of sets $X_1, \dots, X_k \subseteq V(D)$, the *inversion* of $X_1, \dots, X_k$ in D is the oriented graph obtained by inverting each set in turn: inverting X_1 in D, then X_2 in the resulting oriented graph, and so on. Note that the order in which we perform these inversions does not impact the final oriented graph.

If inverting $X_1, \dots, X_k$ in D produces an acyclic oriented graph, then these sets form a *decycling family* of D. The inversion number was introduced by Belkechine [4] and early results on the topic were obtained by Belkechine, Bouaziz, Boudabbous and Pouzet [5].

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Given oriented graphs D_1 and D_2, the *dijoin* from D_1 to D_2, denoted by $D_1 \to D_2$ is the oriented graph constructed from vertex-disjoint copies of D_1 and D_2 by adding all edges uv where $u \in V(D_1)$ and $v \in V(D_2)$. Bang-Jensen, Costa Ferreira da Silva and Havet [3] observed that if D_1 and D_2 are oriented graphs then $\mathrm{inv}(D_1 \to D_2) \leqslant \mathrm{inv}(D_1) + \mathrm{inv}(D_2)$, and conjectured that equality holds for all D_1, D_2. They proved that the conjecture holds if $\mathrm{inv}(D_1) = \mathrm{inv}(D_2) = 1$.

However, this conjecture was shown to be false by two simultaneous papers [1, 2]. The authors of [2] provide a whole family of counterexamples, showing that for every odd $k \geqslant 3$ there is a tournament D_1 with $\mathrm{inv}(D_1) = k$ such that for any oriented graph D_2 with $\mathrm{inv}(D_2) \geqslant 1$, we have $\mathrm{inv}(D_1 \to D_2) \leqslant \mathrm{inv}(D_1) + \mathrm{inv}(D_2) - 1$. Thus the trivial upper bound on the inversion number of a dijoin is not always tight.

For a trivial lower bound, it is easy to see that $\mathrm{inv}(D_1 \to D_2) \geqslant \max\{\mathrm{inv}(D_1), \mathrm{inv}(D_2)\}$. As a first step towards investigating the tightness of this lower bound, Aubian, Havet, Hörsch, Klingelhoefer, Nisse, Rambaud and Vermande asked the following question.

Question 1 ([2, Problem 5.5])**.** Does there exist a non-acyclic oriented graph D such that

$$\mathrm{inv}(D \to D) = \mathrm{inv}(D)?$$

We answer this question in the negative. In fact, we prove the following slightly more general result.

Theorem 2. *Let D_1 and D_2 be oriented graphs such that* $\mathrm{inv}(D_1) = \mathrm{inv}(D_2) \geqslant 1$. *Then* $\mathrm{inv}(D_1 \to D_2) > \mathrm{inv}(D_1)$.

In order to prove this theorem, we use a natural connection between the subgraph complementation number, as studied by Buchanan, Purcell and Rombach in [6], and the inversion number, which we believe may be useful in future research on this topic. This allows us to deduce that the inversion number of a digraph D is either $\mathrm{tmr}(D)$ or $\mathrm{tmr}(D)+1$, where $\mathrm{tmr}(D)$ is the minimum rank across a family of matrices (see Section 3). The same connection is made in [7], however (by using the results in [6]) we are able to classify when $\mathrm{inv}(D) = \mathrm{tmr}(D) + 1$, which is a vital ingredient in our proof. We discuss this further in Section 3, after first noting some easy observations in Section 2. We prove Theorem 2 in Section 4. Some open problems and conjectures are given in Section 5.

2 Preliminaries

In this section, we recall some definitions and notation pertaining to oriented graphs, and present some basic results on the inversion number of oriented graphs which will be useful in later sections.

Recall that an *oriented* graph is a pair $D = (V, E)$, where V is a collection of vertices and $E \subseteq V^{(2)}$ is a collection of ordered pairs of distinct vertices such that, for any $u, v \in V$ at most one of uv and vu is in E. For $u, v \in V$, we write uv to denote the *edge oriented (or directed) from u to v*. An oriented graph can be viewed as the result of assigning a direction to, or orienting, each edge of a suitable simple graph. A *tournament* is an oriented graph where exactly one of uv and vu is present for all $u \neq v \in V$, i.e. an orientation of a complete graph. An oriented graph D_1 is a *subgraph* of D_2, denoted $D_1 \subseteq D_2$, if $V(D_1) \subseteq V(D_2)$ and $E(D_1) \subseteq E(D_2)$.

The *out-neighbourhood* of a vertex v, denoted $N^+(v)$, is the set of all vertices $u \in V$ such that $vu \in E$. The *in-neighbourhood* of v, denoted $N^-(v)$, is the set of all vertices $u \in V$ such that $uv \in E$. A vertex v is a *source* if $N^-(v)$ is empty, and v is a *sink* if $N^+(v)$ is empty.

We first make the simple observation that removing vertices and edges from an oriented graph cannot increase the inversion number. Indeed, after removing the same vertices and edges from a decycling family of the initial oriented graph, it is a decycling family of the subgraph.

Observation 3. *Let D_1 and D_2 be tournaments and suppose that $D_1 \subseteq D_2$. Then* $\mathrm{inv}(D_1) \leqslant \mathrm{inv}(D_2)$.

The following result can help to reduce a problem about oriented graphs to a problem about tournaments only.

Proposition 4. *For every oriented graph D, there is a tournament D^* on the same vertex set with $D \subseteq D^*$ and* $\mathrm{inv}(D^*) = \mathrm{inv}(D)$.

Proof. Let $k := \mathrm{inv}(D)$. Let U be the acyclic oriented graph that is reached from D after applying a decycling family $X_1, X_2, \ldots, X_k$. There exists a transitive tournament U^* on the same vertex set with $U \subseteq U^*$. Inverting the sets $X_1, X_2, \ldots, X_k$ in U^* gives a tournament D^* with $D \subseteq D^*$. Clearly $\mathrm{inv}(D^*) \leqslant k$ and so by Observation 3, $\mathrm{inv}(D^*) = k$. □

We will also use the following simple results that consider the effect that the removal of a single vertex has on the inversion number of an oriented graph.

Proposition 5. *Let D be an oriented graph on $n \geqslant 2$ vertices. If $v \in V(D)$ is a sink or a source, then* $\mathrm{inv}(D - v) = \mathrm{inv}(D)$.

Proof. By Observation 3, $\mathrm{inv}(D - v) \leqslant \mathrm{inv}(D)$. Now, since v is a sink or a source, a decycling family of $D - v$ is also a decycling family of D. Hence, $\mathrm{inv}(D - v) \geqslant \mathrm{inv}(D)$. □

Given an oriented graph D, we say that $u, v \in V(D)$ are *twin* vertices if $N^+(u) \setminus \{v\} = N^+(v) \setminus \{u\}$ and $N^-(u) \setminus \{v\} = N^-(v) \setminus \{u\}$.

Proposition 6. *Let D be an oriented graph on $n \geqslant 2$ vertices. If $u, v \in V(D)$ are twin vertices, then* $\mathrm{inv}(D - v) = \mathrm{inv}(D)$.

Proof. By Observation 3, $\mathrm{inv}(D - v) \leqslant \mathrm{inv}(D)$. Now, suppose $X_1, \ldots, X_k$ is a decycling family of $D - v$. For $1 \leqslant i \leqslant k$, let $Y_i = X_i$ if $u \notin X_i$, and $Y_i = X_i \cup \{v\}$ if $u \in X_i$. Then $Y_1, \ldots, Y_k$ is a decycling family of D, and $\mathrm{inv}(D - v) \geqslant \mathrm{inv}(D)$. □

Finally, we have the following trivial upper bound on the inversion number of an oriented graph.

Proposition 7. *For an oriented graph D of order $n \geqslant 1$, we have* $\operatorname{inv}(D) \leqslant n-1$.

Proof. Note that the inversion number of an oriented graph on one vertex is clearly zero, and assume the statement of the proposition is true for oriented graphs of order $n-1$. Let v be a vertex in D, and let $X = N^+(v) \cup \{v\}$. Inverting X in D, gives an oriented graph D' in which v is a sink. Hence, using Proposition 5, $\operatorname{inv}(D') = \operatorname{inv}(D'-v) \leqslant n-2$, and $\operatorname{inv}(D) \leqslant \operatorname{inv}(D') + 1 \leqslant n-1$. □

3 Subgraph complementation and tournament minimum rank

In this section, we discuss a natural connection between the subgraph complementation number, as studied in [6], and the inversion number. Importantly, we will show that the inversion number is closely related to the lowest rank of a matrix from a particular set of matrices, a key step in our proof of Theorem 2.

3.1 Background on subgraph complementation

In order to state the results of Buchanan, Purcell and Rombach [6], we first require the following definitions.

Given an (undirected) graph G of order $n \geqslant 1$ and a set $X \subseteq V(G)$, the *subgraph complementation* of X in G is the graph obtained from G by complementing the edges in $G[X]$. In this case, we say that we *complement* X in G. Given a family of sets $X_1, \dots, X_k \subseteq V(G)$, the *subgraph complementation* of $X_1, \dots, X_k$ in G is the graph obtained by complementing each set in turn: complementing X_1 in G, then X_2 in the resulting graph, and so on. Note that the order in which we perform these subgraph complementations does not impact the final graph.

If complementing $X_1, \dots, X_k$ in G results in the empty graph $\overline{K_n}$, then these sets form a *subgraph complementing system* of G. The *subgraph complementation number* of G, denoted by $c_2(G)$, is the minimum number of sets in a subgraph complementing system of G.

Note that $\mathcal{F}$ is a subgraph complementing system of G if and only if each pair of adjacent vertices appears together in an odd number of sets in $\mathcal{F}$, while each pair of non-adjacent vertices appears together in an even number of sets in $\mathcal{F}$.

Let $\mathcal{M}(G)$ be the collection of all $n \times n$ matrices with entries in $\{0,1\}$ that can be obtained from the adjacency matrix[1] of G by altering diagonal entries. In this paper the rank of a matrix is always taken over $\mathbb{F}_2$ and we will refer to the rank of a matrix taken over $\mathbb{F}_2$ as simply the rank. Define the *minimum rank* of a graph G, denoted by $\operatorname{mr}(G)$, to be the minimum rank of a matrix in $\mathcal{M}(G)$.

[1] Recall that the *adjacency matrix* of G, denoted $A(G)$, is the $n \times n$ matrix such that $A_{i,j} = 1$ whenever $ij \in E(G)$ and 0 otherwise (including on the diagonal).

Buchanan, Purcell, and Rombach [6] showed that the quantities $\mathrm{mr}(G)$ and $c_2(G)$ cannot differ by more than 1. In addition, they characterised the graphs for which they differ.

Lemma 8 ([6] Corollary 4.7 and Theorem 4.12). *Let G be a graph. Then either*

1. $c_2(G) = \mathrm{mr}(G)$, *or*

2. $c_2(G) = \mathrm{mr}(G) + 1$, *in which case* $\mathrm{mr}(G)$ *is even.*

Moreover, if G has at least one edge, then $c_2(G) = \mathrm{mr}(G) + 1$ if and only if there is a unique matrix $M \in \mathcal{M}(G)$ of minimum rank and all of the diagonal entries of this matrix are equal to zero.

Although it will not be directly relevant for our application of the result, the interested reader might like to know that the proof of Lemma 8 uses an equivalent form of the problem. A d-dimensional *faithful orthogonal representation* of a graph G over the field $\mathbb{F}_2$ is a function $\phi : V(G) \to \mathbb{F}_2^d$ where non-adjacent vertices are assigned orthogonal vectors; that is, $\phi(u)\cdot\phi(v) = 0$ for all $uv \notin E(G)$, and adjacent vertices are assigned non-orthogonal vectors; that is, $\phi(u)\cdot\phi(v) = 1$. The d-dimensional faithful orthogonal representations of G over $\mathbb{F}_2$ are in bijective correspondence with the subgraph complementation systems of G, where a representation ϕ corresponds to the system $\{X_1, \dots, X_d\}$ with $v \in V(G)$ included in X_i if and only if the ith entry of $\phi(v)$ is 1. This approach bears a strong similarity to the analysis used in [2] to prove that for every odd $k \geqslant 3$ there is a tournament D_1 with $\mathrm{inv}(D_1) = k$ such that for any oriented graph D_2 with $\mathrm{inv}(D_2) \geqslant 1$, we have $\mathrm{inv}(D_1 \to D_2) \leqslant \mathrm{inv}(D_1) + \mathrm{inv}(D_2) - 1$.

3.2 Tournament minimum rank

Let D be a tournament on n vertices and T a transitive tournament on the same vertex set. Define $G_{D,T}$ to be the (undirected) graph on the same vertex set as D with the edge ij present if and only if the edge between vertices i and j has opposite orientations in D and T. Clearly, a series of inversions that takes D to T corresponds exactly to a subgraph complementing system of $G_{D,T}$.

Let $\mathcal{T}$ be the collection of all (labelled) n-vertex transitive tournaments and define

$$\mathcal{M}^*(D) = \bigcup_{T \in \mathcal{T}} \mathcal{M}(G_{D,T}).$$

Define the *tournament minimum rank* of a tournament D, denoted $\mathrm{tmr}(D)$, as:

$$\mathrm{tmr}(D) := \min\{\mathrm{rank}(M) : M \in \mathcal{M}^*(D)\}.$$

Equivalently, $\mathrm{tmr}(D) = \min_{T \in \mathcal{T}} \mathrm{mr}(G_{D,T})$.

The following result is a direct consequence of Lemma 8 and provides a useful relationship between $\mathrm{inv}(D)$ and $\mathrm{tmr}(D)$.

Corollary 9 (Corollary to Lemma 8)**.** *Let D be a tournament. Then either*

1. $\mathrm{inv}(D) = \mathrm{tmr}(D)$*, or*
2. $\mathrm{inv}(D) = \mathrm{tmr}(D) + 1$*, in which case* $\mathrm{tmr}(D)$ *is even.*

Moreover, if D is not transitive, then $\mathrm{inv}(D) = \mathrm{tmr}(D) + 1$ *if and only if every matrix $M \in \mathcal{M}^*(D)$ with minimum rank has every diagonal entry equal to zero.*

Proof. Let T be an n-vertex transitive tournament. If transforming D into T requires ℓ inversions, then $c_2(G_{D,T}) = \ell$. So, by the definition of the inversion number,

$$\mathrm{inv}(D) = \min_{T \in \mathcal{T}} c_2(G_{D,T}).$$

By Lemma 8, we have that $\mathrm{mr}(G_{D,T}) \leqslant c_2(G_{D,T}) \leqslant \mathrm{mr}(G_{D,T}) + 1$, and so

$$\mathrm{tmr}(D) = \min_{T \in \mathcal{T}} \mathrm{mr}(G_{D,T}) \leqslant \min_{T \in \mathcal{T}} c_2(G_{D,T}) \leqslant \min_{T \in \mathcal{T}} \mathrm{mr}(G_{D,T}) + 1 = \mathrm{tmr}(D) + 1.$$

Furthermore, the equality $\mathrm{inv}(D) = \mathrm{tmr}(D)+1$ holds if and only if $c_2(G_{D,T}) = \mathrm{mr}(G_{D,T})+1$ for every $T \in \mathcal{T}$ with $\mathrm{mr}(G_{D,T}) = \mathrm{tmr}(D)$. This immediately implies that, if $\mathrm{inv}(D) = \mathrm{tmr}(D) + 1$, then $\mathrm{tmr}(D)$ must be even.

Moreover, if D is not transitive, then $G_{D,T}$ contains at least one edge for any $T \in \mathcal{T}$ and Lemma 8 tells us that, if $c_2(G_{D,T}) = \mathrm{mr}(G_{D,T}) + 1$, then there is a unique matrix of minimum rank in $\mathcal{M}(G_{D,T})$ and it has all zeroes on the diagonal. Hence, if $\mathrm{inv}(D) = \mathrm{tmr}(D) + 1$, this is true of every T with $\mathrm{mr}(G_{D,T}) = \mathrm{tmr}(D)$ and every matrix in $\mathcal{M}^*(D)$ of minimum rank has zeros on the diagonal. ☐

It is interesting to note that all of the examples in [1, 2] of pairs of graphs D_1, D_2 with $\mathrm{inv}(D_1 \to D_2) < \mathrm{inv}(D_1) + \mathrm{inv}(D_2)$ have that D_i is a tournament with $\mathrm{inv}(D_i) = \mathrm{tmr}(D_i) + 1$ for at least one i. In fact, the following result holds.

Theorem 10. *Let D_1 be a tournament with* $\mathrm{inv}(D_1) = \mathrm{tmr}(D_1) + 1$*, and let D_2 be any oriented graph with* $\mathrm{inv}(D_2) \geqslant 1$*. Then* $\mathrm{inv}(D_1 \to D_2) \leqslant \mathrm{inv}(D_1) + \mathrm{inv}(D_2) - 1$.

This can be proved using a similar argument to that used in [2]. Alternatively, we can directly apply Corollary 9, as follows.

Proof. Let D_1 be a tournament satisfying $\mathrm{inv}(D_1) = \mathrm{tmr}(D_1)+1$. Applying Proposition 4, let D_2^* be a tournament containing D_2 with $\mathrm{inv}(D_2^*) = \mathrm{inv}(D_2)$. We see that

$$\begin{aligned}\mathrm{inv}(D_1 \to D_2) \leqslant \mathrm{inv}(D_1 \to D_2^*) &\leqslant \mathrm{tmr}(D_1 \to D_2^*) + 1 \\ &\leqslant \mathrm{tmr}(D_1) + \mathrm{tmr}(D_2^*) + 1 \\ &\leqslant \mathrm{inv}(D_1) + \mathrm{inv}(D_2^*) = \mathrm{inv}(D_1) + \mathrm{inv}(D_2).\end{aligned}$$

Suppose for a contradiction that we have equality. Then

$$\mathrm{inv}(D_1 \to D_2^*) = \mathrm{tmr}(D_1 \to D_2^*) + 1, \tag{1}$$

$$\mathrm{tmr}(D_1 \to D_2^*) = \mathrm{tmr}(D_1) + \mathrm{tmr}(D_2^*), \quad \text{and} \tag{2}$$

$$\mathrm{inv}(D_2^*) = \mathrm{tmr}(D_2^*). \tag{3}$$

Using (3) and $\mathrm{inv}(D_2^*) \geqslant 1$ (so D_2^* is not transitive), Corollary 9 tells us that there is some minimum rank matrix $M_2 \in \mathcal{M}^*(D_2^*)$ with a non-zero diagonal entry. Then, letting M_1 be a minimum rank matrix in $\mathcal{M}^*(D_1)$, the matrix

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

is a matrix in $\mathcal{M}^*(D_1 \to D_2^*)$ with a non-zero entry on the diagonal and rank $\mathrm{tmr}(D_1) + \mathrm{tmr}(D_2^*)$, which is equal to $\mathrm{tmr}(D_1 \to D_2^*)$ by (2). Applying Corollary 9 again, this implies that $\mathrm{inv}(D_1 \to D_2^*) = \mathrm{tmr}(D_1 \to D_2^*)$, contradicting (1). □

4 Proof of Theorem 2

In Problem 5.5 of [2], Aubian, Havet, Hörsch, Klingelhoefer, Nisse, Rambaud, and Vermande ask whether there exists a non-acyclic oriented graph D such that $\mathrm{inv}(D \to D) = \mathrm{inv}(D)$. The goal of this section is to prove Theorem 2, and answer this question in the negative. We restate it below for convenience.

Theorem 2. *Let D_1 and D_2 be oriented graphs such that* $\mathrm{inv}(D_1) = \mathrm{inv}(D_2) \geqslant 1$. *Then* $\mathrm{inv}(D_1 \to D_2) > \mathrm{inv}(D_1)$.

In fact, we may focus our attention exclusively on the tournament case.

Lemma 11. *Let D_1 and D_2 be tournaments such that* $\mathrm{inv}(D_1) = \mathrm{inv}(D_2) \geqslant 1$. *Then* $\mathrm{inv}(D_1 \to D_2) > \mathrm{inv}(D_1)$.

Before we prove Lemma 11, we demonstrate why this suffices to prove Theorem 2.

Proof of Theorem 2. Suppose for a contradiction that there exist oriented graphs D_1, D_2 with $\mathrm{inv}(D_1) = \mathrm{inv}(D_2) = \mathrm{inv}(D_1 \to D_2)$. Let $k := \mathrm{inv}(D_1)$.

Apply Proposition 4 to obtain a tournament $(D_1 \to D_2)^*$ with $(D_1 \to D_2) \subseteq (D_1 \to D_2)^*$ and

$$\mathrm{inv}((D_1 \to D_2)^*) = \mathrm{inv}(D_1 \to D_2) = k.$$

Since $(D_1 \to D_2)^*$ contains every edge of $D_1 \to D_2$, it is the dijoin of two tournaments E_1 and E_2, where $E_1 \supseteq D_1$ and $E_2 \supseteq D_2$. By Observation 3, both E_1 and E_2 have inversion number at least k. Hence

$$k = \mathrm{inv}((D_1 \to D_2)^*) = \mathrm{inv}(E_1 \to E_2) \geqslant \mathrm{inv}(E_1) \geqslant k$$

and

$$k = \mathrm{inv}((D_1 \to D_2)^*) = \mathrm{inv}(E_1 \to E_2) \geqslant \mathrm{inv}(E_2) \geqslant k.$$

Therefore, we have two tournaments E_1, E_2 with $\mathrm{inv}(E_1) = k = \mathrm{inv}(E_2) = \mathrm{inv}(E_1 \to E_2)$, contradicting Lemma 11. □

In order to prove Lemma 11, we will require the following lemma about the structure of certain symmetric matrices. Call an $n \times m$ matrix with entries in $\{0, 1\}$ a *staircase matrix* if its entries increase down each column and decrease along each row (so that the 1s form the shape of a staircase in the bottom left).

Lemma 12. *Let M be a symmetric $(n+m) \times (n+m)$ matrix with entries in $\{0, 1\}$ of the form*

$$\begin{bmatrix} A & C \\ C^T & B \end{bmatrix}$$

where A is a symmetric $n \times n$ matrix, B is a symmetric $m \times m$ matrix and C is an $n \times m$ staircase matrix. If $m \geqslant \operatorname{rank}(A) + 1$, then one of the following holds:

1. $\operatorname{rank}(M) \geqslant \operatorname{rank}(A) + 1$, *or*
2. *there are two adjacent columns of B which are identical, or*
3. *the final column of B contains only zeroes.*

Proof. Suppose that M is a matrix of the given form. Clearly $\operatorname{rank}(M) \geqslant \operatorname{rank}(A)$, so suppose that $\operatorname{rank}(M) = \operatorname{rank}(A) = k$. It must therefore be possible to write each row of B as a linear combination of rows of C over $\mathbb{F}_2$.

Since $\operatorname{rank}(M) = k$, it follows that the staircase C must contain at most k distinct non-zero columns ('steps'). Since $m \geqslant k + 1$, this means that either C contains two consecutive columns with the same entries or C contains a zero column. We split into two cases.

Case 1. Suppose that C contains two adjacent columns with the same entries, say column i and $i + 1$. Since each row of B can be written as a linear combination of rows of C over $\mathbb{F}_2$, this implies that columns i and $i + 1$ of B must also contain the same entries.

Case 2. Suppose that C contains a zero column. Since C is a staircase matrix, the final column of C must be a zero column. Since each row of B can be written as a linear combination of rows of C, this implies that the final column of B also contains only zeroes.

□

We are now armed with all the tools necessary to prove Lemma 11.

Proof of Lemma 11. Suppose for a contradiction that there exist non-transitive tournaments D_1 and D_2 with $\operatorname{inv}(D_1) = \operatorname{inv}(D_2) = \operatorname{inv}(D_1 \to D_2)$. Take D_1, D_2 to be tournaments with this property such that $|V(D_1)| + |V(D_2)|$ is minimal, and let $n_1 := |V(D_1)|$ and $n_2 := |V(D_2)|$. Let $n := n_1 + n_2$ and $k := \operatorname{inv}(D_1)$. Note that Proposition 7 tells us that $k + 1 \leqslant n_1, n_2$.

Our goal is to show that every minimum rank matrix in $\mathcal{M}^*(D_1 \to D_2)$ has rank k and only zero entries on the diagonal. Then, by Corollary 9, we are able to conclude that $\mathrm{inv}(D_1 \to D_2) = k+1$ to obtain a contradiction.

Suppose that M is a matrix of minimum rank in $\mathcal{M}^*(D_1 \to D_2)$, where $M \in \mathcal{M}(G_{D_1 \to D_2, T})$ for some transitive tournament T. Note that T naturally induces an order $\prec$ on its vertices, with the source as the first vertex and the sink as the last vertex. We fix a different ordering $\phi : V(T) \to [n]$ of the vertices of T (and thus of $D_1 \to D_2$), which is obtained by first taking all vertices in the copy of D_1 in the order induced by the natural ordering on T, and then taking all vertices in the copy of D_2 in the order induced by T. That is, $\phi(u) < \phi(v)$ if and only if either $u \in V(D_1)$ and $v \in V(D_2)$, or $u, v \in V(D_1)$ and $u \prec v$, or $u, v \in V(D_2)$ and $u \prec v$.

Note that permuting both the rows and the columns of M by a given permutation does not change the rank of M or the diagonal entries. Therefore, we may, and will, assume that our matrix M has rows and columns ordered according to the vertex ordering ϕ.

Now, since M has minimum rank, by Corollary 9, $\mathrm{rank}(M) \in \{k-1, k\}$. Moreover, by our choice of vertex order, M has the form

$$\begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \tag{4}$$

where A is an $n_1 \times n_1$ symmetric matrix, B is an $n_2 \times n_2$ symmetric matrix and C is an $n_1 \times n_2$ staircase matrix. To see that C is indeed a staircase matrix, consider a 1 in C, and suppose it corresponds to the edge uv where $u \in V(D_1)$ and $v \in V(D_2)$. Since this entry, which we denote $C_{u,v}$, is a 1, we have $v \prec u$. An entry $C_{u,v'}$ to the left of $C_{u,v}$ corresponds to an edge between u and some vertex $v' \in V(D_2)$ with $v' \prec v$. Hence, $v' \prec v \prec u$ and the entry $C_{u,v'}$ must also be a 1. Similarly an entry below $C_{u,v}$ corresponds to the edge between v and some vertex u' with $u \prec u'$, it must also be a 1 as we have $v \prec u \prec u'$.

Clearly $\mathrm{rank}(A) \leqslant \mathrm{rank}(M) \leqslant k$. Since $A \in \mathcal{M}^*(D_1)$, by Corollary 9,

$$\mathrm{rank}(A) \geqslant \mathrm{tmr}(D_1) \geqslant \mathrm{inv}(D_1) - 1 = k - 1.$$

The corresponding inequalities also hold for B, and thus $\mathrm{rank}(A), \mathrm{rank}(B) \in \{k-1, k\}$.

Claim 13. $\mathrm{rank}(A) = \mathrm{rank}(B) = k - 1$ *and* $\mathrm{rank}(M) = k$.

Proof. First suppose, in order to obtain a contradiction, that $\mathrm{rank}(A) = \mathrm{rank}(M)$. Since $n_2 \geqslant k+1 \geqslant \mathrm{rank}(A)+1$, by Lemma 12 we can immediately deduce that either there are two adjacent columns of B that have the same entries, or the final column of B contains only zeroes.

Suppose there are two adjacent columns of B with the same entries, and let these correspond to the vertices u and v. Let $i = \phi(u)$ and note that $\phi(v) = i+1$. By definition of ϕ, a vertex $w \in D_2$ is in $N^+(u)$ if and only if either $\phi(w) < i$ and $B_{\phi(w),i} = 1$, or $\phi(w) > i$ and $B_{\phi(w),i} = 0$. Similarly, a vertex $w \in D_2$ is in $N^+(v)$ if and only if either $\phi(w) < i+1$ and $B_{\phi(w),i+1} = 1$, or $\phi(w) > i+1$ and $B_{\phi(w),i+1} = 0$. Since $B_{j,i} = B_{j,i+1}$ for all j, we see that $N^+(u) \setminus \{v\} = N^+(v) \setminus \{u\}$. In particular, u and v are twin vertices in D_2. Let $D_2' = D_2 - u$. By Proposition 6, $\mathrm{inv}(D_2') = \mathrm{inv}(D_2)$.

Otherwise, suppose that the final column of B is all zeros. This means that the vertex u with $\phi(u) = n$ is a sink in D_2. Let $D_2' = D_2 - u$. By Proposition 5, $\text{inv}(D_2') = \text{inv}(D_2)$.

In either case, by Observation 3,

$$k = \text{inv}(D_2) = \text{inv}(D_2') \leqslant \text{inv}(D_1 \to D_2') \leqslant \text{inv}(D_1 \to D_2) = k.$$

In particular, $\text{inv}(D_2') = \text{inv}(D_1 \to D_2') = k$ and D_2' has one fewer vertex than D_2, contradicting the minimality of $|V(D_1)| + |V(D_2)|$. Hence, $\text{rank}(A) < \text{rank}(M)$.

Now suppose $\text{rank}(B) = \text{rank}(M)$. This follows along the same lines as the previous case, the only difference being that we must apply Lemma 12 to M with rows and columns in reverse. Either there are two adjacent columns of A that have the same entries or the first column of A contains only zeroes, corresponding to D_1 containing twin vertices or a source vertex, respectively. The proof then proceeds as before, and thus $\text{rank}(B) < \text{rank}(M)$. Therefore, $\text{rank}(A) = \text{rank}(B) = k - 1$ and $\text{rank}(M) = k$. □

By Claim 13, $\text{inv}(D_1) = \text{rank}(A)+1$ and $\text{inv}(D_2) = \text{rank}(B)+1$. Hence, by Corollary 9, A and B (and thus M) must have zero entries on the diagonal. Therefore, every matrix of $\mathcal{M}^*(D_1 \to D_2)$ of minimum rank has rank k and every diagonal entry equal to zero. By Corollary 9, we conclude that $\text{inv}(D_1 \to D_2) = k + 1$, which is a contradiction. □

5 Open problems

In light of Theorem 10, and the fact that all of the examples in [1, 2] of pairs of oriented graphs D_1, D_2 with $\text{inv}(D_1 \to D_2) < \text{inv}(D_1)+\text{inv}(D_2)$ can be obtained by an application of this theorem, we ask whether these are all such examples.

Question 14. Do there exist tournaments D_1, D_2 with $\text{inv}(D_i) = \text{tmr}(D_i)$ for $i = 1, 2$ and $\text{inv}(D_1 \to D_2) < \text{inv}(D_1) + \text{inv}(D_2)$?

Note that by Corollary 9, for any tournament D, if $\text{inv}(D)$ is even, then $\text{inv}(D) = \text{tmr}(D)$. Hence, a negative answer to this question would disprove the following pair of similar conjectures (the latter of which is strictly stronger than the former).

Conjecture 15 ([1, Conjecture 8.9])**.** For all $\ell, r \in \mathbb{N}$ with $\ell \geqslant 3$ or $r \geqslant 3$ there exist oriented graphs D_1 and D_2 with $\text{inv}(D_1) = \ell$ and $\text{inv}(D_2) = r$, but $\text{inv}(D_1 \to D_2) < \text{inv}(D_1) + \text{inv}(D_2)$.

Conjecture 16 ([2, Conjecture 5.3])**.** For all $\ell \geqslant 3$ there exists an oriented graph D_1 with $\text{inv}(D_1) = \ell$ such that for all D_2 with $\text{inv}(D_2) \geqslant 1$, we have $\text{inv}(D_1 \to D_2) < \text{inv}(D_1) + \text{inv}(D_2)$.

One approach to answering Question 14 would be to bound the tournament minimum rank of the dijoin of two tournaments.

Question 17. Do there exist tournaments D_1, D_2 with $\text{tmr}(D_1 \to D_2) < \text{tmr}(D_1) + \text{tmr}(D_2)$?

A negative answer to this question would be a very strong result that would also answer Question 14 in the negative, and therefore resolve the two conjectures above. Moreover, we could immediately conclude that

$$\mathrm{inv}(D_1) + \mathrm{inv}(D_2) - 2 \leqslant \mathrm{inv}(D_1 \to D_2) \leqslant \mathrm{inv}(D_1) + \mathrm{inv}(D_2).$$

for all oriented graphs D_1, D_2, by a simple application of Proposition 4 and Corollary 9.

One place to start would be to answer Question 17 in the special case when one of the tournaments is $\overrightarrow{C_3}$, the directed cycle on three vertices.

Conjecture 18. For all tournaments D, we have $\mathrm{tmr}(D \to \overrightarrow{C_3}) = \mathrm{tmr}(D) + 1 = \mathrm{tmr}(\overrightarrow{C_3} \to D)$.

We can generalise the idea of dijoins to sequences of graphs. Given a finite sequence $D_1, \ldots, D_k$ of oriented graphs, the *k-join* of $D_1, \ldots, D_k$, denoted by $[D_1, \ldots, D_k]$, is the oriented graph constructed from vertex-disjoint copies of $D_1, \ldots, D_k$ by adding all edges uv where $u \in V(D_i)$ and $v \in V(D_j)$ for $i < j$. For ease of notation, we write $[D]_k$ for the k-join of k copies of the same oriented graph D.

Pouzet, Kaddour and Thatte [7] proved that $\mathrm{inv}\left(\left[\overrightarrow{C_3}\right]_k\right) = k$ for all k, where $\overrightarrow{C_3}$ is the directed cycle on three vertices. Further to this, Alon, Powierski, Savery, Scott and Wilmer [1] proved that if $D_1, D_2, \ldots, D_k$ are oriented graphs with $\mathrm{inv}(D_i) \leqslant 2$ for all i and $\mathrm{inv}(D_i) = 2$ for at most one i, then

$$\mathrm{inv}([D_1, D_2, \ldots, D_k]) = \sum_{i=1}^{k} \mathrm{inv}(D_i).$$

They conjecture that the condition that $\mathrm{inv}(D_i) = 2$ for at most one i is unnecessary.

Conjecture 19 ([1, Conjecture 8.8])**.** Let $k \in \mathbb{N}$, and let $D_1, \ldots, D_k$ be oriented graphs satisfying $\mathrm{inv}(D_i) \leqslant 2$ for all i. Then $\mathrm{inv}([D_1, \ldots, D_k]) = \sum_{i=1}^{k} \mathrm{inv}(D_i)$.

We remark that $\mathrm{inv}(D) = \mathrm{tmr}(D)$ for every tournament with $\mathrm{inv}(D) \leqslant 2$, and so a negative answer to Question 17 would immediately lead to a proof of this conjecture. In addition, combined with Corollary 9, it would give an affirmative answer to the following question, yielding a more general result.

Question 20. Let $k \in \mathbb{N}$, and let $D_1, \ldots, D_k$ be oriented graphs such that, for every i, either $\mathrm{inv}(D_i) = 1$ or $\mathrm{inv}(D_i)$ is even. Is $\mathrm{inv}([D_1, \ldots, D_k]) = \sum_{i=1}^{k} \mathrm{inv}(D_i)$?

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