The image of a tropical linear space

Joshua Mundinger

Submitted: Jun 8, 2024; Accepted: Feb 25, 2025; Published: Mar 14, 2025
© The author. Released under the CC BY-ND license (International 4.0).

Abstract

Given a tropical linear space $L \subseteq \mathbb{T}^n$ and a matrix $A \in \mathbb{T}^{m \times n}$, the image AL of L under A is typically not a tropical linear space. We introduce a tropical linear space $\mathrm{tropim}_A(L)$, the tropical image, containing AL. We show under mild hypotheses that $\mathrm{tropim}_A(L)$ is realizable if L is and apply the tropical image to construct the stable sum of two tropical linear spaces without a disjoint pair of bases.

Mathematics Subject Classifications: 14T15

1 Introduction

In tropical geometry, a single polynomial defines a tropical hypersurface, but not every intersection of hypersurfaces is a tropical variety. Morphisms of tropical varieties carry similar difficulties. For instance, the image of a tropicalized morphism (a tuple of tropicalized polynomials) typically is not a tropical variety, let alone equal to the tropicalization of the image. Can the image of a tropical morphism be extended to a tropical variety?

In this paper, we give an algebraic treatment of this question in the case of a linear map on a tropical linear space, drawing on the algebraic treatment of tropical linear spaces of [6] and the construction of Stiefel tropical linear spaces in [5]. Given a matrix A and a tropical linear space L over a semifield S, we introduce a tropical linear space $\mathrm{tropim}_A(L)$, called the *tropical image*. The tropical Plücker coordinates of the tropical image are determined by the minors of A and the coordinates of L, analogously to the classical situation.

Theorem. The tropical image $\mathrm{tropim}_A(L)$ enjoys the following properties:

- it contains the set-theoretic image AL;
- its rank is at most the rank of L;
- the underlying matroid of $\mathrm{tropim}_A(L)$ is induced from the underlying matroid of L via the bipartite graph underlying A;

University of Wisconsin-Madison (jmundinger@wisc.edu).

- if A has a non-zero maximal minor, then the image $\mathrm{tropim}_A\left(S^E\right)$ of the free S-module S^E is the Stiefel troipcal linear space associated to A^T;
- if $L = \mathrm{trop}\,(\Lambda)$ is realizable over a sufficiently large field, then $\mathrm{tropim}_A\,(L)$ is realizable and is equal to the tropicalization of an image of Λ;
- the stable sum $L +_{st} L'$ of tropical linear spaces is the tropical image of $L \oplus L'$ under the matrix $\begin{bmatrix} I & I \end{bmatrix}$ representing addition.

The tropical image allows us to unify and generalize constructions appearing in the literature, for instance by extending Stiefel tropical linear spaces to all matrices, or the stable sum to all pairs of linear spaces. The tropical image is constructed by first constructing a linear space containing the set-theoretic graph of a linear function, and then projecting. The tropical graph coincides with an iterated tropical modification. By giving an algebraic treatment and connecting to the matroid literature the linear case, we are able to give examples of unexpected phenomena in tropical geometry, for instance an example of a tropical modification $L' \to L$ where L and L' are tropical linear spaces but the function corresponding to the modification is *non*-linear.

2 Preliminaries

2.1 Modules over idempotent semifields

A *semiring* is a set with two binary operations that satisfy the axioms of a (commutative) ring except for the existence of additive inverses. The additive identity will be denoted 0, and the multiplicative identity 1. A *semifield* is a semiring where every nonzero element has a multiplicative inverse. A semiring is *idempotent* if and only if $1+1 = 1$. Idempotent semirings have a canonical ordering given by $a \leqslant b$ if and only if $a+b = b$; if this ordering is total, the semiring is said to be *totally ordered.* In tropical geometry, the main example of a totally ordered semifield is the tropical semifield $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with the operations of maximum and the usual real addition.

A *module* N over a semifield S is an abelian monoid with a homomorphism $S \to End(N)$. The dual of an S-module N is $N^\vee := \mathrm{Hom}_S(N, S)$. If S is an idempotent semifield and $E = \{e_1, e_2, \ldots, e_n\}$ is a finite set, the free S-module with basis E will be denoted S^E. Unlike free modules over rings, S^E has a unique basis up to permutation and scaling [7, Proposition 2.2.2]. The dual $(S^E)^\vee$ of a free module is also free; the dual basis to E will usually be denoted $\{x_1, x_2, \ldots, x_n\}$ and satisfies

$$x_i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Remark 1. The category of totally ordered idempotent semifields is equivalent to the category of totally ordered abelian groups via $S \mapsto S^\times$.

Definition 2 ([7], Definition 2.3.1)**.** Let S be a totally ordered idempotent semifield, and let $f \in (S^E)^\vee$ be a linear form. The *tropical hyperplane* defined by f is the submodule

$$\left\{ v = \sum_{i=1}^{n} v_i e_i \in S^E : f(v) = f\left(\sum_{i \neq j} v_i e_i\right) \text{ for all } j \right\}.$$

If $f = \sum_{i=1}^{n} f_i x_i$, then the condition that $v = \sum_{i=1}^{n} v_i e_i$ is in the tropical hyperplane defined by f if $f(v)$ tropically vanishes, that is, the maximum of $\{f_i v_i\}_{i=1}^{n}$ is achieved twice.

2.2 Tropical linear spaces

Tropical linear spaces, introduced by Speyer [16], are defined by tropical Plücker coordinates, which satisfy the tropicalization of the classical quadratic Plücker relations. Over the tropical semifield, they are equivalent to valuated matroids, going back to work of Murota. In that context, the results in this section appear in [11]; our main reference below is [7], which works over a general idempotent semifield. Here and below, $\binom{E}{d}$ will denote the set of subsets of E of size d.

Definition 3 ([7], Definition 4.1.1)**.** Let S be a totally ordered idempotent semifield. A *tropical Plücker vector* of rank d on ground set E is a nonzero vector $w \in S^{\binom{E}{d}}$ satisfying the *tropical Plücker relations*: for any $J \in \binom{E}{d+1}$ and $K \in \binom{E}{d-1}$,

$$\sum_{i \in J-K} w_{J-i} w_{K+i} = \sum_{i \in J-K, i \neq j} w_{J-i} w_{K+i}$$

for all $j \in J - K$.

Definition 4 ([7], Theorem 4.2.1)**.** Let S be a totally ordered idempotent semifield, and let $w \in S^{\binom{E}{d}}$ be a tropical Plücker vector. Then the *tropical linear space* associated to w is the intersection of the tropical hyperplanes defined by

$$\sum_{i \in J} w_{J-i} x_i$$

over all $J \in \binom{E}{d+1}$. The tropical linear space associated to w is denoted L_w.

Tropical linear spaces determine their tropical Plücker coordinates up to a scalar in $S^\times$ (first proven over $\mathbb{T}$ in [16] and [10], and for any S in [7, §6.2]). Tropical linear spaces are submodules of the ambient free module, since tropical hyperplanes are. The following lemma gives a generating set for a tropical linear space, known as the *valuated cocircuits*. The hypothesis that the idempotent semifield S is totally ordered is essential.

Lemma 5. *[6, Proposition 4.1.9] Let S be a totally ordered idempotent semifield, and let $w \in S^{\binom{E}{d}}$ be a tropical Plücker vector. Then L_w is generated as an S-module by*

$$\left\{ \sum_{i \in E-K} w_{K+i} e_i : K \in \binom{E}{d-1} \right\}.$$

2.3 Subspaces and the tropical incidence relations

For two tropical Plücker vectors w and z on ground set E, when is $L_w \subseteq L_z$? This occurs if and only if w and z satisfy the *tropical incidence relations*, the tropical analogue of incidence relations for Plücker vectors. These relations first appeared in [8] over $\mathbb{T}$ but hold over any idempotent semifield.

Lemma 6. *Let S be a totally ordered idempotent semifield, and let $w \in S^{\binom{E}{d}}$ and $z \in S^{\binom{E}{e}}$ be tropical Plücker vectors. Then $L_w \subseteq L_z$ if and only if*

$$\sum_{i \in A-B} z_{A-i} w_{B+i} = \sum_{i \in A-B, i \neq j} z_{A-i} w_{B+i} \tag{1}$$

for all $A \in \binom{E}{e+1}$, $B \in \binom{E}{d-1}$, and $j \in A - B$.

Proof. The relations (1) exactly require that for any $B \in \binom{E}{d-1}$ and $A \in \binom{E}{e+1}$, the valuated cocircuit $\sum_{i \in E-B} w_{B+i} e_i$ of w is contained in the hyperplane defined by $\sum_{i \in A} z_{A-i} x_i$, one of the hyperplanes defining L_z. Thus, the relations (1) hold if and only if all the valuated cocircuits of w are contained in L_z. By Lemma 5, L_w is generated as an S-module by its valuated cocircuits, so since L_z is an S-module, $L_w \subseteq L_z$. □

2.4 Exterior algebra and operations on Plücker vectors

If S is an idempotent semifield, the *tropical Grassmann algebra* or *exterior algebra* $\bigwedge^* S^E$ on S^E is the S-algebra quotient of the symmetric algebra on S^E by the relations $e_i^2 \sim 0$ [7, §3.1]. If $I = \{i_1, \dots, i_d\} \subset E$, let $e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d}$. The dth graded piece $\bigwedge^d S^E$ is free with basis $\{e_I \mid I \subseteq E, |I| = d\}$. We will consider a tropical Plücker vector of rank d as an element of $\bigwedge^d S^E$.

There are three operations of interest on the exterior algebra that preserve the tropical Plücker relations. The first is the multiplication $\wedge$. The second is the Hodge star $\star : \bigwedge^d S^E \to \bigwedge^{|E|-d} (S^E)^\vee$, which maps e_I to x_{E-I}. Combining these gives the third operation $\cdot : \bigwedge^d S^E \times \bigwedge^{d'} S^n \to \bigwedge^{d+d'-n} S^E$ defined by $w \cdot w' = \star(\star w \wedge \star w')$.

Lemma 7. *Let S be a totally ordered idempotent semifield.*

(a) If $w \in \bigwedge^d S^E$ and $w' \in \bigwedge^{d'} S^E$ are tropical Plücker vectors such that $w \wedge w' \neq 0$, then $w \wedge w'$ is a tropical Plücker vector.

(b) If $w \in \bigwedge^d S^E$ is a tropical Plücker vector, then so is $\star w$.

(c) If $w \in \bigwedge^d S^E$ and $w' \in \bigwedge^{d'} S^E$ are tropical Plücker vectors such that $w \cdot w' \neq 0$, then $w \cdot w'$ is a tropical Plücker vector.

Proof. (a) is [7, Proposition 5.1.2]. (b) can be checked from the definition, and (c) follows from (a) and (b). □

Remark 8. For $\varphi \in (S^E)^\vee$, the tropical linear space $L_{\star\varphi}$ is the tropical hyperplane defined by φ, for

$$\sum_{i\in E} (\star\varphi)_{E-i} x_i = \sum_{i \in E} \varphi(e_i) x_i = \varphi,$$

and hence $\sum_{i=1}^n c_i e_i$ is in the hyperplane defined by φ if and only if it is in the tropical linear space with tropical Plücker coordinates $\star\varphi$.

All three operations, when they result in a nonzero vector, have geometric interpretations: $L_{w\wedge w'}$ is the stable sum of L_w and $L_{w'}$ [5], $L_{\star w}$ is the tropical orthogonal dual of L_w, and $L_{w\cdot w'}$ is the stable intersection of L_w and $L_{w'}$ [16]. Speyer introduced the stable intersection over $\mathbb{T}$ when all tropical Plücker coordinates are non-zero [16] and gave a geometric interpretation in terms of polyhedral complexes.

The orthogonal dual reverses inclusions, while stable sum and stable intersection preserve inclusions. Over $\mathbb{T}$, this follows from polyhedral geometry; we provide an algebraic proof over any totally ordered idempotent semifield.

Lemma 9. *Let S be a totally ordered idempotent semifield.*

- *(a) If $w \in \bigwedge^d S^E$ and $z \in \bigwedge^e S^E$ are tropical Plücker vectors and $L_w \subseteq L_z$, then $L_{\star z} \subseteq L_{\star w}$.*
- *(b) If $w \in \bigwedge^d S^E, w' \in \bigwedge^{d'} S^E, z \in \bigwedge^e S^E$, and $z' \in \bigwedge^{e'} S^E$ are tropical Plücker vectors such that $z \wedge z' \neq 0$, $L_w \subseteq L_z$, and $L_{w'} \subseteq L_{z'}$, then $w \wedge w' \neq 0$, and $L_{w\wedge w'} \subseteq L_{z \wedge z'}$.*
- *(c) If $w \in \bigwedge^d S^E, w' \in \bigwedge^{d'} S^E, z \in \bigwedge^e S^E$, and $z' \in \bigwedge^{e'} S^E$ are tropical Plücker vectors such that $w \cdot w' \neq 0$, $L_w \subseteq L_z$, and $L_{w'} \subseteq L_{z'}$, then $z \cdot z' \neq 0$, and $L_{w\cdot w'} \subseteq L_{z\cdot z'}$.*

Proof. We use the tropical incidence relations of Lemma 6. For (a), the Plücker vectors w and z satisfy the tropical incidence relations if and only if $\star z$ and $\star w$ do, for

$$\sum_{i\in A-B} z_{A-i} w_{B+i} = \sum_{i \in B^c - A^c} (\star w)_{B^c - i} (\star z)_{A^c + i}$$

and dropping a term from the right-hand side is exactly dropping a term from the left-hand side. Alternatively, (a) follows from the characterization of tropical orthogonal duality of [7, Corollary 4.4.4].

The proof of (b) is a generalization of [7, Proposition 5.1.2]. If $z \wedge z' \neq 0$, [3, Theorem 6.5] implies $w \wedge w' \neq 0$. If $A \in \binom{E}{e+e'+1}$ and $B \in \binom{E}{d+d'-1}$, then

$$\sum_{i\in A-B} (z\wedge z')_{A-i} (w \wedge w')_{B+i} = \sum_{i \in A-B} \sum_{\substack{C \sqcup C' = A-i \\ D \sqcup D' = B+i}} z_C z'_{C'} w_D w'_{D'}. \tag{2}$$

Collecting terms with $i \in D$ and $i \in D'$ separately gives the equal expression

$$\sum_{\substack{J \sqcup C' = A \\ K \sqcup D' = B}} (z'_{C'} w'_{D'}) \left(\sum_{i \in J-K} z_{J-i} w_{K+i} \right) + \sum_{\substack{C \sqcup J' = A \\ D \sqcup K' = B}} (z_C w_D) \left(\sum_{i\in J'-K'} z'_{J'-i} w'_{K'+i} \right)$$

Further, the terms with fixed $j \in A - B$ in (2) are exactly the terms with $j \in J - K$ or $j \in J' - K'$ in the above sums. Since $L_w \subseteq L_z$ and $L_{w'} \subseteq L_{z'}$, the sums above are equal when terms for j are dropped. Thus, the tropical incidence relations for $w \wedge w'$ and $z \wedge z'$ hold, so $L_{w\wedge w'} \subseteq L_{z\wedge z'}$. Finally, (c) follows from (a) and (b). □

2.5 Matroids

A *matroid* M on a finite ground set E is a collection of *bases*, subsets of E, satisfying the strong exchange axiom: if I and J are bases of M and $i \in I - J$, then there exists $j \in J - I$ such that $I - i + j$ and $J - j + i$ are bases of M. Matroids may be defined in a number of cryptomorphic ways, and standard matroid terminology will be used in various remarks and examples in this paper; in particular, we will use the deletion $M\backslash F$ and contraction M/F for $F \subseteq E$. A standard reference for matroids is [12].

Matroids are equivalent to tropical linear spaces over the two-element idempotent semifield $\mathbb{B} = \{0, 1\}$, with additive identity 0 and multiplicative identity 1 [3, Theorem 1.5]: a vector $w \in \bigwedge \mathbb{B}^E$ is a tropical Plücker vector if and only if it is the indicator vector of the bases of a matroid. Further, a tropical Plücker vector w over an idempotent semifield S defines a matroid, by taking the bases to be those coordinates of w that are non-zero. As the tropical linear space L_w recovers w up to a scalar [7, §6.2], this matroid only depends on L_w and is called the *underlying matroid* of L_w. The underlying matroid of L_w corresponds to the pushforward of L_w under the canonical semiring homomorphism $S \to \mathbb{B}$.

2.6 Minors of tropical linear spaces

Frenk defined the minors of a tropical linear space in [6], following the definition of the minors of a valuated matroid given by Dress and Wenzel [4]. The following is essentially [1, Lemma 6.4], translated to the language of totally ordered idempotent semifields. It shows that coordinate subspaces and projections of tropical linear spaces are again tropical linear spaces.

Recall that if M is a matroid on E and $F \subseteq E$, then $M \setminus F$ is the deletion of M with respect to F, and $M/F = (M^* \setminus F)^*$ is the contraction of M to $E - F$ [12, 3.1].

Lemma 10. *Let S be a totally ordered idempotent semifield, and let $w \in \bigwedge^d S^E$ be a tropical Plücker vector. Let M be the underlying matroid of w. Let $F \subseteq E$ be fixed, and let $J \subseteq E - F$. Define $z \in S^{\binom{F}{d-|J|}}$ by*

$$z_I = w_{J\cup I}.$$

(a) If $|J| < \operatorname{rk} M/F$ or $|J| > \operatorname{rk} M\backslash F$, then z is zero.

(b) if $|J| = \operatorname{rk} M/F$, then z is either zero or the Plücker vector of $\pi_F(L_w)$, where $\pi_F : S^E \to S^F$ is the coordinate projection map;

(c) if $|J| = \operatorname{rk} M\backslash F$, then z is either zero or the Plücker vector of $S^F \cap L_w$.

Proof. (a) If $|J| < \operatorname{rk} M/F$ or $|J| > \operatorname{rk} M\backslash F$, then there is no $I \subseteq F$ such that $J \cup I$ is a basis for M.

(b) If $|J| = \operatorname{rk} M/F$ and there is $I \subseteq F$ such that $w_{I\cup J} \neq 0$, then $|I| = \operatorname{rk} M\backslash(E-F)$, so J is a basis for M/F. Then apply [1, Lemma 6.4].

(c) This follows from the dual argument to (b) and the dual part of [1, Lemma 6.4]. □

Remark 11. Over $\mathbb{B}$, Lemma 10 shows that if M is the matroid associated to $L_w \subseteq \mathbb{B}^E$, then $\pi_F(L_w)$ is the tropical linear space corresponding to the matroid $M\backslash(E-F)$ and $\mathbb{B}^F \cap L_w$ is the tropical linear space corresponding to the matroid $M/(E-F)$. Thus, when $|J| = \operatorname{rk} M\backslash F$, z is nonzero exactly when J is a basis of $M\backslash F$, and dually for M/F. See [3, Theorem 4.1] for more details.

3 Linear extensions

3.1 Linear extensions of tropical linear spaces

In tropical geometry, graphs of regular functions on a balanced polyhedral complex are generally not balanced; in particular, the graph of a linear function $\varphi \in (S^E)^\vee$ is typically not a tropical linear space. However, there is a natural balanced polyhedral complex containing the set-theoretic graph. In his doctoral thesis, Frenk studied extensions of tropical linear spaces over $\mathbb{T}$, translating between polyhedral and algebraic definitions of this balanced polyhedral complex [6, §4.2.2]. We generalize Frenk's construction, connecting it to matroidal notions and to the important operation of tropical modification. These methods allow us to construct an explicit example of a tropical linear space that is the tropical modification of a tropical linear space along a *non-linear* rational function.

Definition 12. An *extension* of a tropical linear space $L_w \subseteq S^E$ is a tropical linear space $L_z \subseteq S^{E\cup P}$ for P disjoint from E such that $\pi_E(L_z) = L_w$. An *elementary* extension of $L_w \subseteq S^E$ is an extension $L_z \subseteq S^{E\cup P}$ such that $|P| = 1$.

Because projection corresponds to the restricted matroid, the underlying matroid of an extension is an extension of the underlying matroid. Elementary extensions of matroids were studied by Crapo [12, Chapter 7.2].

For $\varphi \in (S^E)^\vee$, the set-theoretic graph of φ is contained in the tropical hyperplane defined by $\varphi + x_p \in (S^{E+p})^\vee$, which has tropical Plücker coordinates $\star_{E+p}(\varphi + x_p)$. This suggests defining the "graph" of φ on a tropical linear space $L_w \subseteq S^E$ as the stable intersection of this tropical hyperplane with $L_w \oplus S$.

Definition 13. Let S be a totally ordered idempotent semifield. A *linear extension* of a tropical linear space $L_w \subseteq S^E$ is an elementary extension $L_z \subseteq S^{E+p}$ of L_w such that for some $\varphi \in (S^E)^\vee$, z is of the form

$$z = (w \wedge e_p) \cdot \star_{E+p}(\varphi + x_p) = \star_{E+p}(\star_E w \wedge (\varphi + x_p)).$$

Such an L_z will be denoted $L_w +_\varphi p$.

By Lemma 7, the vector $(w \wedge e_p) \cdot \star_{E+p}(\varphi + x_p)$ is always a tropical Plücker vector, and for $I \subseteq E$ of size $\operatorname{rk} w$, $((w \wedge e_p) \cdot \star_{E+p}(\varphi + x_p))_I = w_I$. Thus, Lemma 10 implies that a linear extension of L_w projects onto L_w. The following lemma shows that $L_w +_\varphi p$ contains the set-theoretic graph of φ on L_w.

Lemma 14. *[6, Proposition 4.2.12] Let S be a totally ordered idempotent semifield, and $L_w \subseteq S^E$ be a tropical linear space. Let $\varphi \in (S^E)^\vee$. If $w \cdot \star\varphi \neq 0$, then*

$$L_w +_\varphi p = \{v + \varphi(v)e_p \mid v \in L_w\} \cup \{v + ae_p \mid v \in L_{w\cdot\star\varphi}, a \leqslant \varphi(v)\};$$

if $w \cdot \star\varphi = 0$, then

$$L_w +_\varphi p = \{v + \varphi(v)e_p \mid v \in L_w\}.$$

Proof. Let $z = (w \wedge e_p) \cdot \star_{E+p}(\varphi + x_p)$. First suppose that $w \cdot \star\varphi \neq 0$. Let d be the rank of L_w. If $A \subseteq E + p$ is of size $d - 1$ and contains p, then

$$\begin{aligned}\sum_{i \in (E+p)-A} z_{A+i} e_i &= \sum_{i \in (E+p)-A} (\star_E w \wedge (\varphi + x_p))_{(E+p)-A-i} e_i \\ &= \sum_{i \in E-(A-p)} \star_E(\star_E w \wedge \varphi)_{A-p+i} e_i,\end{aligned}$$

which is the cocircuit of $L_{w\cdot\star\varphi}$ associated to $A - p$. If $A \subseteq E$ is of size $d - 1$,

$$\begin{aligned}\sum_{i \in (E+p)-A} z_{A+i} e_i &= \sum_{i \in E-A} (\star_E w \wedge (\varphi + x_p))_{(E+p)-A-i} e_i \\ &\quad + (\star_E w \wedge (\varphi + x_p))_{E-A} e_p \\ &= \sum_{i \in E-A} w_{A+i} e_i + \varphi\left(\sum_{i \in E-A} w_{A+i} e_i\right) e_p,\end{aligned}$$

which is exactly $v + \varphi(v)e_p$ for v the cocircuit of L_w associated to A. If $w \cdot \star\varphi \neq 0$, the same arguments show that the cociruits of z are exactly $v + \varphi(v)e_p$ for v a cocircuit of L_w, and hence are included in the right-hand side.

Now we prove the reverse inclusion. If $v \in L_w$, then v is a linear combination of cocircuits of L_w. Since φ is linear, $v + \varphi(v)e_p$ is thus a linear combination of terms $v' + \varphi(v')e_p$ where v' is a cocircuit of L_w. Hence, $v + \varphi(v)e_p \in L_w +_\varphi p$. Now suppose that $w \cdot \star\varphi \neq 0$, $v \in L_{w\cdot\star\varphi}$, and $a \leqslant \varphi(v)$. If $\varphi(v) = 0$, there is nothing to prove, so assume $\varphi(v) \neq 0$. Then

$$v + ae_p = a\varphi(v)^{-1}(v + \varphi(v)e_p) + v$$

since addition is idempotent, showing that $v + ae_p \in L_w +_\varphi p$. □

A basic property of linear extensions is monotonicity:

Lemma 15. *Let S be a totally ordered idempotent semifield, and $L_w \subseteq L_z \subseteq S^E$ be tropical linear spaces. Let $\varphi \in (S^E)^\vee$. Then*

$$L_w +_\varphi p \subseteq L_z +_\varphi p.$$

Proof. Let $\star$ denote the Hodge star on S^{E+p}. By Lemma 9, the wedge and its dual operation are both monotonic, so $L_{w\wedge e_p} \subseteq L_{z\wedge e_p}$ and

$$L_{(w\wedge e_p)\cdot\star(\varphi+x_p)} \subseteq L_{(z\wedge e_p)\cdot\star(\varphi+x_p)},$$

as desired. □

3.2 Tropical modification and linearity

Lemma 14 implies that linear extensions over $\mathbb{T}$ are *tropical modifications* in the sense of [14]. This section is self-contained and will not be required for following sections.

Theorem 16. *Let $L \subseteq \mathbb{T}^E$ be a tropical linear space, and let $\varphi \in (\mathbb{T}^E)^\vee$ be a dual vector. Then the intersection of the linear extension $L +_\varphi p$ with the tropical torus $(\mathbb{T}^{E+p})^\times$ is equal as a set to the tropical modification of $L \cap (\mathbb{T}^E)^\times$ along φ, which is equal to $\operatorname{div}_L(\varphi)$.*

Proof. If w, w' are tropical Plücker vectors with all nonzero coordinates, then Speyer showed $L_{w\cdot w'} \cap (\mathbb{T}^\times)^E$ is the geometric stable intersection

$$\lim_{\tau\to(1,\ldots,1)} (L_w \cap \tau L_{w'}) \cap (\mathbb{T}^\times)^E,$$

where the limit is over an open subset of the torus $(\mathbb{T}^\times)^E = \mathbb{R}^E$ [16, Theorem 4.11]. Speyer's proof also goes through when w, w' are arbitrary tropical Plücker vectors such that $w \cdot w' \neq 0$. The tropical modification of L along φ in the torus is exactly the set-theoretic graph of φ on L along with the undergraph on $\operatorname{div}_L(\varphi)$. By [14, Proposition 2.12], $\operatorname{div}_L(\varphi)$ is exactly the geometric stable intersection of the hyperplane defined by φ and L in the torus. This shows $(L +_\varphi p) \cap (\mathbb{T}^\times)^E$ is the tropical modification of $L \cap (\mathbb{T}^\times)^E$ by φ. □

In [14, Proposition 2.25], Shaw proves that every rank-preserving nontrivial extension of a matroid corresponds to a tropical modification of the corresponding tropical linear spaces. However, tropical modifications by linear functions are special amoing matroid extensions. We recall the relevant matroid theory: given a matroid M on ground set E, and given $X \subseteq E$, the *principal extension* of M with respect to X, denoted $M +_F p$, can be described as the matroid on $E + p$ with independent sets

$$\{I \mid I \in \mathcal{I}(M)\} \cup \{I \cup p \mid I \in \mathcal{I}(M), \operatorname{cl}_M(I) \not\supseteq \operatorname{cl}_M(F)\},$$

where $\mathcal{I}(M)$ is the set of independent sets of M [12, Proposition 7.2.5].

Lemma 17. *Let S be a totally ordered idempotent semifield, $L_w \subseteq S^E$ a tropical linear space with underlying matroid M, and $\varphi \in (S^E)^\vee$. If*

$$F = \{i \in E \mid \varphi(e_i) \neq 0\},$$

then the underlying matroid of the linear extension $L_w +_\varphi p$ is the principal extension $M +_F p$.

Proof. The bases of $(w \wedge e_p) \cdot \star_{E+p}(\varphi + x_p)$ are exactly I when $I \subseteq E$ is a basis of M and $K \cup p$ when $\varphi(\sum_{i \in E-K} w_{K+i} e_i) \neq 0$. Hence, $K \cup p$ is a basis of $L_w +_\varphi p$ if and only if the cocircuit of M supported in $E - K$, i.e. $E - \mathrm{cl}_M(K)$, intersects F. Hence, the underlying matroid of $L_w +_\varphi p$ is exactly the prinipcal extension $M +_F p$. $\square$

It is known that not every elementary extension of matroids is principal. Hence, there are some elementary tropical modifications $L' \to L$ where L and L' are both tropical linear spaces, but the rational function associated to the modification $L' \to L$ is *not* linear.

Example 18. Let $w = e_{123} + e_{124} + e_{134} + e_{234} \in \bigwedge^3 \mathbb{B}^4$ be the tropical Plücker vector corresponding to the uniform matroid $U_{3,4}$. Let $z = (e_1 + e_2) \wedge (e_3 + e_4)$; then $L_z = \langle e_1 + e_2 \rangle \oplus \langle e_3 + e_4 \rangle \subseteq L_w$. I claim that

$$\mathrm{div}_{L_w}\left(\frac{(x_1 + x_2)(x_3 + x_4)}{x_1 + x_2 + x_3 + x_4}\right) = L_z.$$

At the level of algebraic cycles, we have

$$\begin{aligned}\mathrm{div}_{L_w}\left(\frac{(x_1 + x_2)(x_3 + x_4)}{x_1 + x_2 + x_3 + x_4}\right) &= \mathrm{div}_{L_w}(x_1 + x_2) + \mathrm{div}_{L_w}(x_3 + x_4) \\ &\quad - \mathrm{div}_{L_w}(x_1 + x_2 + x_3 + x_4).\end{aligned}$$

All of these divisors, of linear spaces by linear functions, may be computed according to Theorem 16. Their tropical Plücker vectors are

$$\begin{aligned} w_1 = w \cdot \star(x_1 + x_2) &= e_{13} + e_{14} + e_{23} + e_{24} + e_{34}, \\ w_2 = w \cdot \star(x_3 + x_4) &= e_{12} + e_{13} + e_{14} + e_{23} + e_{24}, \\ w_3 = w \cdot \star(x_1 + x_2 + x_3 + x_4) &= e_{12} + e_{13} + e_{14} + e_{23} + e_{24} + e_{34}, \end{aligned}$$

respectively. By computing the valuated cocircuits via Lemma 5, and then taking the tropical span, it may be checked that the faces of L_{w_i} (as a polyhedral complex) are given by the following:

$$\begin{aligned} L_{w_1} &= \{(\alpha, \beta, \alpha, \alpha) \mid \alpha \geqslant \beta\} \cup \{(\beta, \alpha, \alpha, \alpha) \mid \alpha \geqslant \beta\} \\ &\quad \cup \{(\alpha, \alpha, \beta, \beta) \mid \alpha \geqslant \beta\}; \\ L_{w_2} &= \{(\alpha, \alpha, \beta, \alpha) \mid \alpha \geqslant \beta\} \cup \{(\alpha, \alpha, \alpha, \beta) \mid \alpha \geqslant \beta\} \\ &\quad \cup \{(\beta, \beta, \alpha, \alpha) \mid \alpha \geqslant \beta\}; \\ L_{w_3} &= \{(\beta, \alpha, \alpha, \alpha) \mid \alpha \geqslant \beta\} \cup \{(\alpha, \beta, \alpha, \alpha) \mid \alpha \geqslant \beta\} \\ &\quad \cup \{(\alpha, \alpha, \beta, \alpha) \mid \alpha \geqslant \beta\} \cup \{(\alpha, \alpha, \alpha, \beta) \mid \alpha \geqslant \beta\}. \end{aligned}$$

Hence at the level of tropical cycles, we have

$$L_{w_1} + L_{w_2} - L_{w_3} = \{(\alpha, \alpha, \beta, \beta) \mid \alpha, \beta \in \mathbb{T}\} = \langle e_1 + e_2 \rangle \oplus \langle e_3 + e_4 \rangle.$$

Thus, L_z is a linear divisor of L_w, giving rise to a tropical modification $L_{w + z \wedge e_p} \to L_w$, associated to a non-linear rational function. In terms of the underlying matroids, the extension $L_{w+z\wedge e_p}$ of L_w corresponds to an elementary extension of matroids with the non-principal modular cut $\mathcal{M} = \{\{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$.

Remark 19. A similar phenomenon was noted by Shaw [14, Example 2.29], where the moduli space $\mathcal{M}_{0,5}^{trop}$ of tropical rational curves with five marked points cannot be realized as a sequence of modifications along regular functions.

4 Tropical images

4.1 Tropical graphs and tropical images

Extending our work in §3.1, we come to the main definitions of the paper: how to define the image of a tropical linear space.

Definition 20. Let S be a totally ordered idempotent semifield, and $w \in \bigwedge^d S^E$ be a tropical Plücker vector. Let $A = \{a_{ij}\} \in S^{F \times E}$ be a matrix with columns indexed by E and rows indexed by $F = \{f_1, \ldots, f_m\}$. Let $\{x_1, \ldots, x_n\}$ denote the dual basis to S^E and $\{y_1, \ldots, y_m\}$ denote the dual basis to S^F. Let $\rho_j = \sum_{i \in E} a_{ji} x_i \in (S^E)^\vee$ be the form associated with the jth row of A. The *tropical graph* of A on L_w is the tropical linear space in $S^{E \sqcup F}$ with tropical Plücker vector

$$g(w, A) = (w \wedge f_F) \cdot \star(\rho_1 + y_1) \cdot \star(\rho_2 + y_2) \cdot \ldots \cdot \star(\rho_m + y_m) \in \bigwedge^d S^{E \sqcup F},$$

where $f_F = f_1 \wedge f_2 \wedge \cdots \wedge f_m$.

By definition, the tropical graph of A on L_w is the iterated linear extension

$$(\cdots((L_w +_{\rho_1} f_1) +_{\rho_2} f_2) \cdots +_{\rho_m} f_m).$$

This tropical linear space does not depend on the order of the extensions, as the wedge product and its dual operation are commutative.

Definition 21. Let S be a totally ordered idempotent semifield, $A \in S^{F \times E}$ be a matrix, and $w \in \bigwedge^d S^E$ be a tropical Plücker vector. The *tropical image* of L_w under A, denoted $\operatorname{tropim}_A(L_w)$, is the projection

$$\pi_F\left(L_{g(w,a)}\right)$$

of the tropical graph of A on L_w onto the codomain of A.

By Lemma 10, the tropical image is a tropical linear space.

Lemma 22. *Let S be a totally ordered idempotent semifield, $L_w \subseteq S^E$ a tropical linear space, and $A \in S^{F \times E}$ a matrix. Then*

(a) $AL_w \subseteq \operatorname{tropim}_A(L_w)$*;*

(b) $\operatorname{rk} \operatorname{tropim}_A(L_w) \leqslant \operatorname{rk} L_w$*;*

(c) if $L_w \subseteq L_z \subseteq S^E$ is another tropical linear space, then

$$\operatorname{tropim}_A(L_w) \subseteq \operatorname{tropim}_A(L_z).$$

Proof. (a) By Lemma 14, the tropical graph of A on L_w, equal to the linear extension $(\cdots((L_w +_{\rho_1} f_1) +_{\rho_2} f_2) \cdots +_{\rho_m} f_m)$, contains $v + \sum_{i=1}^m \rho_i(v) f_i$ for all $v \in L_w$. Hence, the tropical image contains $\sum_{i=1}^m \rho_i(v) f_i = Av$ for all $v \in L_w$.

(b) The tropical graph of A on L_w has the same rank as L_w, and projecting onto a coordinate subspace does not increase rank.

(c) By Lemma 15, linear extensions are monotonic, and the coordinate projection π_F is monotonic. $\square$

The following lemma gives a more explicit formulation of the Plücker coordinates of the tropical image.

Lemma 23. *Let S be a totally ordered idempotent semifield, $w \in \bigwedge^d S^E$ be a tropical Plücker vector, and $A \in S^{F \times E}$ be a matrix. The tropical Plücker coordinates z of* $\operatorname{tropim}_A(L_w)$ *are*

$$z_J = \sum_{I \subseteq E-K} \operatorname{tdet}(A_{JI})\, w_{I \cup K},$$

where K is any basis for the contraction of the tropical graph of A on L_w to E, and $\operatorname{tdet}(A_{JI})$ *is the tropical $J \times I$ minor of A.*

Proof. Let $g(w, A)$ denote the tropical Plücker vector of the tropical graph of A on L_w. By Lemma 10, if K is a basis for the contraction of $g(w, A)$ to E, then $J \mapsto g(w, A)_{K \cup J}$ for $J \subseteq F$ are tropical Plücker coordinates for $\operatorname{tropim}_A(L_w)$. By definition, if $\star$ is the Hodge star on $S^{E \sqcup F}$ and ρ_j denotes the form associated to the jth row of A,

$$\begin{aligned} g(w, A)_{K \cup J} &= [(w \wedge f_F) \cdot \star(\rho_1 + y_1) \cdot \ldots \cdot \star(\rho_m + y_m)]_{K \cup J} \\ &= \left[(\star_E w) \wedge \bigwedge_{j=1}^{m} (\rho_j + y_j) \right]_{(E-K) \cup (F-J)} \end{aligned}$$

by duality. Since $\star_E w$ is supported in E and $\rho_j + y_j$ is supported in $E \cup j$ for all $j \in F$, the only terms that contribute to y_{F-J} in the above wedge product are $\rho_j + y_j$ for $j \in F - j$. Hence,

$$\begin{aligned} g(w, A)_{K \cup J} &= \left[(\star_E w) \wedge \bigwedge_{j \in J} (\rho_j + y_j) \right]_{E-K} \\ &= \sum_{I \subseteq E-K} (\star_E w)_{E-K-I} (\bigwedge_{j \in J} \rho_j)_I \\ &= \sum_{I \subseteq E-K} w_{I \cup K} \operatorname{tdet}(A_{JI}), \end{aligned}$$

as desired. $\square$

Lemma 23 shows that if $A \in S^{F \times E}$ has a nonzero maximal minor, then $\operatorname{tropim}_A(S^E)$ is equal to the Stiefel tropical linear space associated to A^T (see [5]). Even if A has no nonzero maximal minor, we have

Corollary 24. *For arbitrary $A \in S^{F\times E}$, if B is a submatrix of A formed by the columns of a maximal non-zero minor of A, then*

$$\operatorname{tropim}_A\left(S^E\right) = \operatorname{tropim}_B\left(S^E\right),$$

and hence is a Stiefel tropical linear space. In particular, $\operatorname{tropim}_A\left(S^E\right)$ *has rank equal to the size of the largest nonzero minor of A.*

4.2 The underlying matroid of the tropical image

In this section, we compute the underlying matroid of the tropical image. Given a matroid M on E and a bipartite graph Γ on $E \sqcup F$, the *induced matroid* $\Gamma(M)$ is the matroid on F whose independent sets are exactly the $J \subseteq F$ that have a perfect matching in Γ to an independent set of M (see [12, Chapter 12.2]).

Theorem 25. *Suppose M is a matroid on E corresponding to the tropical linear space $L_w \subseteq \mathbb{B}^E$. Let Γ be a bipartite graph on $E \sqcup F$, with incidence matrix $A \in \mathbb{B}^{F\times E}$. Then the induced matroid $\Gamma(M)$ corresponds to the tropical linear space* $\operatorname{tropim}_A(L_w) \subseteq \mathbb{B}^F$.

Proof. Non-zero terms in the expansion of $\operatorname{tdet}(A_{JI})$ correspond to bijections $f : J \to I$ such that $A_{j,f(j)} = 1$ for all $j \in J$, i.e. j and $f(j)$ are adjacent in Γ for all $j \in J$. Hence, $\operatorname{tdet}(A_{JI}) = 1$ if and only if there is a matching from J to I in Γ. By Lemma 23, the tropical Plücker coordinates of $\operatorname{tropim}_A(L_w)$ are

$$z_J = \sum_{I\subseteq E-K} \operatorname{tdet}(A_{JI})\, w_{K\cup I}, \tag{3}$$

where K is a basis for the contraction of the tropical graph to E. By Lemma 10, K is a minimal subset of E such that (3) does not vanish for all J, and hence the bases of $\operatorname{tropim}_A(L_w)$ are the bases of $\Gamma(M)$. ☐

Corollary 26. *The tropical linear spaces associated to transversal matroids are exactly the tropical images of free $\mathbb{B}$-modules.*

Proof. Transversal matroids are the induced matroids of free matroids. ☐

Corollary 26 also follows from the description of Stiefel tropical linear spaces as the tropical image of free modules.

Remark 27. On [6, p. 108], Frenk investigates the *valuated linking system* associated to a bipartite graph with edges weighted by elements of a semifield. If A is the weighted incidence matrix of the weighted bipartite graph, Frenk's definition assigns to the pair (I, J) the weight $\operatorname{tdet}(A_{JI})$. Thus, the tropical image under A is, in Frenk's language, the image under the valuated linking system associated to the weighted graph of A.

Surprisingly, an iterated tropical image may not be a tropical image: it may be that $\operatorname{tropim}_B(\operatorname{tropim}_A(L_w))$ is not equal to $\operatorname{tropim}_C(L_w)$ for any matrix C!

Example 28. Consider the $\mathbb{B}$-matrices

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad\qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The tropical image $\mathrm{tropim}_B(\mathrm{tropim}_A(\mathbb{B}^3))$ is exactly the rank 2 truncation of the transversal matroid $\mathrm{tropim}_B(\mathbb{B}^3)$. This truncation has three cyclic flats of rank 1, namely $\{1,2\}$, $\{3,4\}$, and $\{5,6\}$. By a result of Brylawski, a rank r transversal matroid has at most $\binom{r}{k}$ rank-k cyclic flats [2]. Thus, $\mathrm{tropim}_B(\mathrm{tropim}_A(\mathbb{B}^3))$ is not transversal, and hence is not of the form $\mathrm{tropim}_C(\mathbb{B}^3)$ for any matrix C.

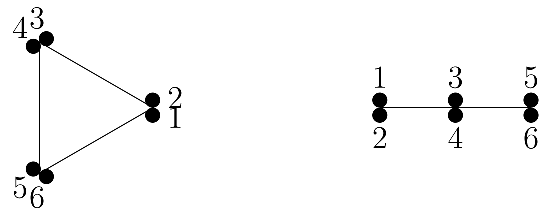

Figure 1: Geometric representations of the matroids $\mathrm{tropim}_B(\mathbb{B}^3)$ and $\mathrm{tropim}_B(\mathrm{tropim}_A(\mathbb{B}^3))$ of Example 28.

4.3 Realizability

Recall that if k is a field, S a totally ordered idempotent semifield, and $\mathrm{val} : k \to S$ is a surjective valuation, then we may *tropicalize* a linear subspace $\Lambda \subseteq k^n$ by taking the valuation of every element of Λ coordinate-wise. The result $\mathrm{trop}(\Lambda)$ is the *tropicalization* of Λ. The tropical Plücker coordinates of $\mathrm{trop}(\Lambda)$ are the image of the Plücker coordinates of Λ under val [15]. A tropical linear space is *realizable* with respect to a particular valuation if it is a tropicalization of some linear space under that valuation. For matroids, a realizable tropical linear space via $k \to \mathbb{B}$ is exactly a representable matroid over k.

Let M be a matroid on E and Γ a bipartite graph on $E \sqcup F$. Piff and Welsh proved in 1970 that if M is representable over a sufficiently large field, then $\Gamma(M)$ is representable over that field as well [12, Proposition 12.2.16]. Fink and Rincón also observed that a Stiefel tropical linear space in $\mathbb{T}^n$ is the tropicalization of the image of a general lift of its matrix. Theorem 31 below is a common generalization of these results and gives a criterion for realizability of the tropical image.

Given a field κ and a totally ordered semifield S, let $\kappa\{t^S\}$ denote the field of formal series $\sum_{s \in T} \alpha_s t^s$, where $T \subseteq S^\times$ is well-ordered and $\alpha_s \in \kappa$ for all $s \in T$. There is a surjective valuation $\mathrm{val} : \kappa\{t^S\} \to S$ that sends a series $\sum_{s \in T} \alpha_s t^s$ to $(\min\{s \in T \mid \alpha_s \neq 0\})^{-1}$ (here the inverse is to accord with our convention of maximum for addition).

Poonen showed that every valuation that is trivial on its initial field factors through such a valuation [13]. The main technical tool of our analysis of realizability is Lemma 30, which describes when solutions to equations on valuations in S may be lifted to $\kappa\{t^S\}$. Proving the existence of solutions requires the following lemma on the non-vanishing of polynomials.

Lemma 29. *[9, Lemma 2] Let κ be a field, and let $f \in \kappa[x_1, \ldots, x_n]$ be a polynomial. Suppose the degree of f in each x_i alone is less than $|\kappa|$. Then there exists $\alpha \in \kappa^n$ such that $f(\alpha) \neq 0$.*

Lemma 30. *Let S be a totally ordered idempotent semifield, κ a field, and let $\mathrm{val} : \kappa\{t^S\} \to S$ be the standard valuation. Let $f_1, f_2, \ldots, f_m \in \kappa\{t^S\}[x_1, \ldots, x_n]$ and let $a \in S^n$. If d_i is the maximum degree of f_i in each x_j alone, and $|\kappa| > \sum_{i=1}^m d_i$, then there exists a non-empty Zariski open subset of κ^n such that if the leading coefficients of $\alpha \in \mathrm{val}^{-1}(a)$ lie in that subset, then*

$$\mathrm{val}(f_i)(a) = \mathrm{val}(f_i(\alpha))$$

for all i.

Proof. Suppose $f_i = \sum_{\mathbf{u}} c^i_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ for $c^i_{\mathbf{u}} \in \kappa\{t^S\}$. Let $g_i \in \kappa[x_1, \ldots, x_n]$ be the sum of the leading terms of those $c^i_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ such that $\mathrm{val}(f_i)(a) = \mathrm{val}(c^i_{\mathbf{u}}) a_1^{u_1} \cdots a_n^{u_n}$. Then $\alpha \in \mathrm{val}^{-1}(a)$ satisfies $\mathrm{val}(f_i)(a) = \mathrm{val}(f_i(\alpha))$ if and only if g_i does not vanish on the leading coefficients of α. The degree of g_i in a single indeterminate is at most d_i, so by Lemma 29, there is a point α in κ^n such that $g_1 g_2 \cdots g_m$ does not vanish, i.e. where no g_i vanishes. ☐

Theorem 31. *Let S be a totally ordered idempotent semifield, and $\mathrm{val} : \kappa\{t^S\} \to S$ be the standard valuation. Let $\Lambda \subseteq \kappa\{t^S\}^E$ be a rank d linear subspace, and $L = \mathrm{trop}(\Lambda)$. Let $A \in S^{F \times E}$ be a matrix. If $|\kappa| > \binom{|E|+|F|}{d}$, then $\mathrm{tropim}_A(L)$ is realizable over $\kappa\{t^S\}$, and for $\Delta \in \mathrm{val}^{-1}(A)$ with generic leading coefficients we have*

$$\mathrm{tropim}_A(L) = \mathrm{trop}(\Delta\Lambda).$$

Proof. Because coordinate projection commutes with tropicalization, it suffices to show for $\Delta \in \mathrm{val}^{-1}(A)$ with generic leading coefficients that the tropicalization of the graph of Δ on Λ is the tropical graph of A on L.

Let $e_1, \ldots, e_n, f_1, \ldots, f_m$ be the standard basis of $\kappa\{t^S\}^{E \sqcup F}$ and let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be the dual basis. Let p be a Plücker vector for Λ. The (classical) graph of a matrix $X = [X_{ji}]_{i \in E, j \in F}$ on $\Lambda \subseteq \kappa\{t^S\}$ has Plücker vector

$$\star_{E \sqcup F} \left(\star_E p \wedge \bigwedge_{j=1}^{m} \left(y_j - \sum_{i=1}^{n} X_{ji} x_i \right) \right), \tag{4}$$

where $\star_{E \sqcup F}$ and $\star_E$ denote the classical Hodge stars on $\kappa\{t^S\}^{E \sqcup F}$ and $\kappa\{t^S\}^E$. For each $I \in \binom{E \sqcup F}{d}$, let $f_I \in \kappa\{t^S\}[X_{ji}]$ be the coefficient of e_I in (4). If $J = I \cap E$ and $K = I \cap F$, then

$$f_I = \pm \left(\star_E p \wedge \bigwedge_{j=1}^{m} \left(y_j - \sum_{i=1}^{n} X_{ji} x_i \right) \right)_{E-J \cup F-K}$$
$$= \pm \left(\star_E p \wedge \bigwedge_{j \in K} \left(- \sum_{i=1}^{n} X_{ji} x_i \right) \right)_{E-J},$$

so the terms of f_I are of the form $\pm p_{J\cup\sigma(K)} \prod_{j\in K} X_{j,\sigma(j)}$ for an injective function $\sigma : K \to (E - J)$. Hence, every coefficient of f_I is of the form $\pm p_{J'}$ for some $J' \subseteq E$. Comparing with the definition of tropical image shows $I \mapsto \mathrm{val}(f_I)(A)$ is a tropical Plücker vector for the tropical graph of A on L. The degree of f_I in each indeterminate is at most 1, and by hypothesis $|\kappa| > |\{f_I : I \subseteq E \sqcup F, |I| = d\}|$. By Lemma 30, there is a non-empty Zariski open subset of $\kappa^{F\times E}$ such that if the leading coefficients of $\Delta \in \mathrm{val}^{-1}(A)$ lie in that open subset, then the tropical graph of A on L is the tropicalization of the graph of Δ on Λ. □

5 Stable sum

In this section, we show the stable sum of tropical linear spaces is a tropical image under addition. This also provides a generalization of the stable sum to tropical linear spaces L_w and L_z when $w \wedge z = 0$, generalizing matroid unions.

The tropical addition map $+ : S^{E\sqcup E} \to S^E$, coinciding with the tropicalization of classical addition, has matrix

$$A_+ = \begin{bmatrix} I_E & I_E \end{bmatrix},$$

where I_E is the identity matrix on S^E. The underlying bipartite graph of this matrix is the same graph used to define the matroid union [12, Theorem 12.3.1], suggesting investigating the tropical image under A_+ over an arbitrary semifield.

Theorem 32. *Let S be a totally ordered idempotent semifield. Let L_w and L_z be tropical linear spaces in S^E. If $w \wedge z \neq 0$, then*

$$\mathrm{tropim}_{A_+} (L_w \oplus L_z) = L_{w\wedge z},$$

where A_+ is the matrix of the addition map $+ : S^{E\sqcup E} \to S^E$.

Proof. Write the domain of the sum map as $S^{E'\sqcup E''}$ for $E' = \{e'_1, \ldots, e'_n\}$ and $E'' = \{e''_1, \ldots, e''_n\}$ two copies of E. The underlying bipartite graph of addition is the graph on $(E' \sqcup E'') \sqcup E$ where a vertex in E is exactly adjacent to its two copies in $E' \sqcup E''$. By Theorem 25 and [12, Theorem 12.3.1], the underlying matroid of $\mathrm{tropim}_{A_+} (L_w \oplus L_z)$ is the matroid union of the underlying matroids of L_w and L_z. Because $w \wedge z \neq 0$, these matroids have disjoint bases, so $\mathrm{rk}\,\mathrm{tropim}_+ (L_w \oplus L_z) = \mathrm{rk}\, L_w + \mathrm{rk}\, L_z = \mathrm{rk}\, L_w \oplus L_z$. As the ranks of $L_w \oplus L_z$ and its tropical image are the same, Lemma 23 implies that the tropical Plücker coordinates of $\mathrm{tropim}_+ (L_w \oplus L_z)$ are

$$J \mapsto \sum_{I \subseteq E' \sqcup E''} \mathrm{tdet}\left((A_+)_{JI}\right) w_{I \cap E'} z_{I \cap E''} = \sum_{J = J' \sqcup J''} w_{J'} z_{J''}$$

since $\mathrm{tdet}\left((A_+)_{JI}\right)$ is 1 if and only if the copies of $I \cap E'$ and $I \cap E''$ in E are a partition of J. This shows that the tropical Plucker vector of the tropical image is exactly $w \wedge z$, as desired. □

The stable sum (and dually, stable intersection) of tropical linear spaces $L_w \subseteq S^E$ and $L_z \subseteq S^E$ has so far only been defined when $w \wedge z \neq 0$ [5]. Theorem 32 suggests the following definition for arbitrary L_w and L_z:

Definition 33. Let $L_w \subseteq S^E$ and $L_z \subseteq S^E$ be tropical linear spaces. Then the *stable sum* of L_w and L_z is

$$L_w +_{st} L_z = \mathrm{tropim}_{A_+} (L_w \oplus L_z),$$

By Theorem 25, the underlying matroid of $L_w +_{st} L_z$ is the union of the underlying matroids of L_w and L_z, even when $w \wedge z = 0$. If $w \wedge z = 0$, then the stable sum is equal to a rank-additive stable sum of subspaces:

Corollary 34. *Let L_w and L_z be tropical linear spaces in S^E. Then if $L_{w'}$ and $L_{z'}$ are subspaces of L_w and L_z such that $w' \wedge z'$ is non-zero and has rank equal to $L_w +_{st} L_z$, then*

$$L_{w' \wedge z'} = L_w +_{st} L_z.$$

Proof. Since $L_{w'} \oplus L_{z'} \subseteq L_w \oplus L_z$, by Lemma 22,

$$L_{w'} +_{st} L_{z'} \subseteq L_w +_{st} L_z. \tag{5}$$

But by Theorem 32, $L_{w'} +_{st} L_{z'} = L_{w' \wedge z'}$, and so the tropical linear spaces in (5) have the same rank. Hence, they are equal. □

The stable sum of realizable tropical linear spaces has the following "stable" interpretation: combining Theorem 31 and 32 shows that if Λ_1 and Λ_2 are transverse subspaces of $\kappa\{t^S\}^E$,

$$\mathrm{trop}(\Lambda_1) +_{st} \mathrm{trop}(\Lambda_2) = \mathrm{trop}(\gamma_1 \Lambda_1 + \gamma_2 \Lambda_2)$$

for generic $(\gamma_1, \gamma_2) \in (\kappa^\times)^{E \sqcup E}$. Dualizing and observing that $\mathrm{trop}(\gamma\Lambda) = \mathrm{trop}(\Lambda)$ for any $\gamma \in (\kappa^\times)^E$ shows that if Λ_1 and Λ_2 intersect transversely, then

$$\mathrm{trop}(\Lambda_1) \cap_{st} \mathrm{trop}(\Lambda_2) = \mathrm{trop}(\Lambda_1 \cap \gamma\Lambda_2)$$

for generic $\gamma \in (\kappa^\times)^E$. This result is known for tropical varieties of all degrees [10, Theorem 3.6.1], but the proof is entirely in the language of balanced polyhedral complexes. The methods of this paper provide a new and algebraic proof in the linear case.

Acknowledgements

The author thanks Noah Giansiracusa for invaluable discussion and guidance. Thanks to the referee for useful comments.

References

[1] Matthew Baker and Nathan Bowler. Matroids over hyperfields, arXiv:1601.01204v5, 2017.

[2] Thomas H. Brylawski. An affine representation for transversal geometries. *Studies in Appl. Math.*, 54(2):143–160, 1975. doi:10.1002/sapm1975542143

[3] Colin Crowley, Noah Giansiracusa, and Joshua Mundinger. A module-theoretic approach to matroids. *J. Pure Appl. Algebra*, 224(2):894–916, 2020. doi:10.1016/j.jpaa.2019.06.016

[4] Andreas W. M. Dress and Walter Wenzel. Valuated matroids. *Adv. Math.*, 93(2):214–250, 1992. doi:10.1016/0001-8708(92)90028-J

[5] Alex Fink and Felipe Rincón. Stiefel tropical linear spaces. *J. Combin. Theory Ser. A*, 135:291–331, 2015. doi:10.1016/j.jalgebra.2020.02.042

[6] Bart Frenk. *Tropical varieties, maps and gossip.* PhD thesis, Eindhoven University of Technology, 2013.

[7] Jeffrey Giansiracusa and Noah Giansiracusa. A Grassmann algebra for matroids. *Manuscripta Math.*, 156(1-2):187–213, 2018. doi:10.1007/s00229-017-0958-z

[8] Mohammad Moinul Haque. Tropical incidence relations, polytopes, and concordant matroids, arXiv:1211.2841v2, 2012.

[9] Bernt Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, 5:85–90, 1973. doi:10.1112/blms/5.1.85

[10] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, 2015. doi:10.1090/gsm/161

[11] Kazuo Murota and Akihisa Tamura. On circuit valuation of matroids. *Adv. in Appl. Math.*, 26(3):192–225, 2001. doi:10.1006/aama.2000.0716

[12] James Oxley. *Matroid theory* (2nd edition), volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, 2011. doi:10.1093/acprof:oso/9780198566946.001.0001

[13] Bjorn Poonen. Maximally complete fields. *Enseign. Math. (2)*, 39(1-2):87–106, 1993.

[14] Kristin M. Shaw. A tropical intersection product in matroidal fans. *SIAM J. Discrete Math.*, 27(1):459–491, 2013. doi:10.1137/110850141

[15] David Speyer and Bernd Sturmfels. The tropical Grassmannian. *Adv. Geom.*, 4(3):389–411, 2004. doi:10.1515/advg.2004.023

[16] David E. Speyer. Tropical linear spaces. *SIAM J. Discrete Math.*, 22(4):1527–1558, 2008. doi:10.1137/080716219