A Special Class of Pure O-Sequences

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Abstract

The pure O-sequences of the form $(1, a, a, \ldots)$ are classified.
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1 Introduction

Let $x_1, \ldots, x_s$ represent distinct indeterminates with $\deg x_i = 1$, for $i = 1, \ldots, s$. A nonempty, finite set $\mathcal{A}$ of monomials in $x_1, \ldots, x_s$ is called an *order ideal of monomials* if for any $u \in \mathcal{A}$ and any monomial v that divides u, we have $v \in \mathcal{A}$. In particular, $1 \in \mathcal{A}$ for any order ideal of monomials $\mathcal{A}$. We say that $\mathcal{A}$ is *pure* if the maximal elements of $\mathcal{A}$, with respect to divisibility, all have the same degree. The *h-vector* of $\mathcal{A}$ is defined as $h(\mathcal{A}) = (h_0, h_1, \ldots, h_n)$, where

$$n = \max\{\deg u : u \in \mathcal{A}\} \text{ and } h_i = \left|\{u \in \mathcal{A} : \deg u = i\}\right|, \text{ for } 0 \leqslant i \leqslant n.$$

Clearly, $h_0 = 1$.

A finite sequence of positive integers $h = (h_0, h_1, \ldots, h_n)$ is called an *O-sequence* if there exists an order ideal of monomials $\mathcal{A}$ with $h = h(\mathcal{A})$. Finally, following Stanley [21], an O-sequence h is *pure* if there exists a pure order ideal of monomials $\mathcal{A}$ with $h = h(\mathcal{A})$. Equivalently, in the language of commutative algebra, pure O-sequences coincide with the Hilbert functions of (standard graded) artinian monomial level algebras. We refer to [1, 18] for an introduction to the theory of pure O-sequences both combinatorially and algebraically, and for some recent developments.

A classification of the possible O-sequences is essentially due to Macaulay (see [15] and [22, Theorem 2.2]). On the other hand, an explicit characterization of pure O-sequences

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seems entirely out of reach, despite much effort by many researchers. A notably long-standing problem in this field is a conjecture of Stanley's, stating that the h-vector of any matroid complex is a pure O-sequence [21, 23]. Partial results in this direction have been obtained in, for instance, [2, 3, 4, 5, 8, 9, 13, 14, 16, 20].

The purpose of the present paper is to prove the following:

Theorem A. Let $n \geqslant 4$. Then a sequence $h = (1, a, a, \ldots, a, b) \in \mathbb{Z}_{>0}^{n+1}$ is a pure O-sequence if and only if $b \leqslant a \leqslant 2b$.

The proof of Theorem A is given in Section 1. In addition, in Section 2, as a supplement to Theorem A, the pure O-sequences $(1, a, b)$ and $(1, a, a, b)$ are classified (see Proposition 7).

Part of our motivation for considering pure O-sequences of the form $(1, a, a, \ldots, a, b)$ arises from the theory of δ-vectors of Castelnuovo polytopes [10, 12, 19], where it was shown that a sequence $(1, a, a, \ldots, a, b) \in \mathbb{Z}_{>0}^{n+1}$, with $n \geqslant 2$, is the h-vector of a Cohen–Macaulay graded domain if $b \leqslant a \leqslant (b+1)(n+1)$. In particular, Theorem A together with Proposition 7 (ii) guarantees that, when $n \geqslant 3$, any pure O-sequence $(1, a, a, \ldots, a, b)$ is the h-vector of a Cohen–Macaulay graded domain.

Finally, a problem of current interest in commutative algebra is to determine classes of artinian algebras that enjoy the so-called *Weak* (or *Strong*) *Lefschetz Properties* [1, 6]. The results of this paper imply that, over a field of characteristic zero, *any* artinian monomial level algebra with Hilbert function given by a pure O-sequence of the form $(1, a, a, \ldots, a, b)$ has the Strong Lefschetz Property.

2 Proof of Theorem A

Our proof of Theorem A is divided into several lemmata.

Lemma 1. *Let $h = (1, a, a, \ldots, a, b) \in \mathbb{Z}_{>0}^{n+1}$ be a pure O-sequence, where $n \geqslant 3$. Then $b \leqslant a$.*

Proof. This result can easily be shown using [1, Proposition 3.6], but we present a self-contained proof since it appears of independent interest. Consider an artinian monomial level k-algebra $A = \bigoplus_{i=0}^{n} A_i$ with Hilbert function h, where $k = A_0$ is a field of characteristic zero. By Hibi-Hausel's g-theorem on the differentiability of a pure O-sequence through its first half (see [9, Theorem 1.1] and [7, Theorem 6.2]), we deduce that multiplication by a Zariski-general linear form L between consecutive graded pieces A_i and A_{i+1} is injective, for all indices $i \leqslant \lfloor n/2 \rfloor$.

Now note that, because A_i and A_{i+1} have the same k-vector space dimension in those degrees (namely, a), multiplication by L is in fact bijective. Finally, since the grading of A is standard, it is easy to see ([17, Proposition 2.1]) that if multiplication by L is surjective from some degree i to $i + 1$, then it is surjective from degree j to $j + 1$, for all $j \geqslant i$. The case $j = n - 1$ yields $b \leqslant a$. $\square$

Lemma 2. *Let $n \geqslant 2$. Then $(1, a, a, \ldots, a, b) \in \mathbb{Z}_{>0}^{n+1}$ is a pure O-sequence for any $b = \lceil a/2 \rceil, \ldots, a$.*

Proof. Partition a into exactly b parts of size $\leqslant 2$, say $a = 2s_1 + s_2$ with $s_1 + s_2 = b$, and consider the two sets of monomials $x_1x_2^{n-1}, \dots, x_{2s_1-1}x_{2s_1}^{n-1}$ and $y_1^n, \dots, y_{s_2}^n$. Then the pure O-sequence they generate is

$$\begin{aligned} & (1,0,0,\dots,0) + s_1 \cdot (0,2,2,\dots,2,1) + s_2 \cdot (0,1,1,\dots,1,1) \\ = \ & (1, 2s_1+s_2, 2s_1+s_2, \dots, 2s_1+s_2, s_1+s_2) \\ = \ & (1, a, a, \dots, a, b), \end{aligned}$$

as desired. $\square$

Lemma 3. *Let $n \geqslant 4$. Then $(1, a, a, \dots, a, b) \in \mathbb{Z}_{>0}^{n+1}$ cannot be a pure O-sequence if $2b < a$.*

Proof. Let $x_1, \dots, x_a$ be variables. Suppose that $h = (1, a, a, h_3, \dots, h_{n-1}, b) \in \mathbb{Z}_{>0}^{n+1}$ is a pure O-sequence, where $2b < a$ and $n \geqslant 4$. Let $\{u_1, \dots, u_b\}$ denote a set of monomials in $x_1, \dots, x_a$ of degree n that generates h. Let p_j be the number of variables x_i for which x_i divides u_j, but does not divide any of $u_1, \dots, u_{j-1}$. Since $2b < a$, we can assume that $p_1 \geqslant 3$. Let q_j be the number of quadratic monomials $x_ix_{i'}$ for which $x_ix_{i'}$ divides u_j, but does not divide any of $u_1, \dots, u_{j-1}$. Then

$$\sum_{j=1}^{b} p_j = \sum_{j=1}^{b} q_j = a. \tag{1}$$

Note that $q_j \geqslant p_j$, for each j. Furthermore, since $n \geqslant 4$, it follows that $q_1 > p_1$. This contradicts (1), completing the proof. $\square$

Combining Lemmata 1, 2, and 3 proves Theorem A.

Remark 4. Let $n \geqslant 4$, and $h = (1, h_1, \dots, h_n) \in \mathbb{Z}_{>0}^{n+1}$ be a pure O-sequence such that $h_1 = h_i$ for some $2 \leqslant i \leqslant n-2$. It follows from [9, Theorem 1.1] that $h_1 = h_2$.

Let $h_1 = h_2 = a$ and $h_n = b$. Arguing as in the proof of Lemma 3, we have $a \leqslant 2b$ and each $q_j \in \{1, 2\}$. Let $q_j = 2$ for $1 \leqslant j \leqslant b'$, and $q_j = 1$ for $b'+1 \leqslant j \leqslant b$. If $1 \leqslant j \leqslant b'$, then $u_j = x_{j_1}x_{j_2}^{n-1}$ with $j_1 \neq j_2$. If $b'+1 \leqslant j \leqslant b$, then either $u_j = x_{j_1}^n$ or $u_j = x_{j_1}x_{j_2'}^{n-1}$, where $1 \leqslant j_2' \leqslant b'$. Thus, each $h_i = 2b' + (b - b') = a$. Hence $h = (1, a, a, \dots, a, b)$.

Remark 5. The proof of Lemma 3, together with Remark 4, shows that all pure O-sequences of the form $(1, a, a, \dots, a, b)$ can be constructed starting from the two pure O-sequences $(1, 2, 2, \dots, 2, 1)$ and $(1, 1, \dots, 1, 1)$.

Remark 6. Interestingly from an algebraic standpoint, it follows from the argument of Lemma 1 that, over a field of characteristic zero, any artinian monomial level algebra with Hilbert function given by the pure O-sequence $(1, a, a, \dots, a, b)$ enjoys the Strong Lefschetz Property.

3 The pure O-sequences $(1, a, b)$ and $(1, a, a, b)$

As a supplement to Theorem A, we give the following result, which completes the characterization of pure O-sequences of the form $(1, a, \ldots, a, b)$.

Proposition 7. *Let a and b be positive integers.*

(i) The sequence $(1, a, b)$ is a pure O-sequence if and only if $\lceil a/2 \rceil \leqslant b \leqslant \binom{a+1}{2}$.

(ii) The sequence $(1, a, a, b)$ is a pure O-sequence if and only if $\lceil a/3 \rceil \leqslant b \leqslant a$.

Proof. (i) The proof is a simple exercise. See [1, Corollary 4.7] and [9, Example 1.2].

(ii) We want to determine when b monomials of degree 3 in a variables have a total of a degree 2 divisors. First note that each of these b monomials can involve at most 3 variables. This implies $\lceil a/3 \rceil \leqslant b$. The upper bound $b \leqslant a$ was proven in Lemma 1. Thus, it remains to show that $(1, a, a, b)$ is a pure O-sequence for each integer $b = \lceil a/3 \rceil, \ldots, a$.

Any degree 3 monomial is of one of the following three kinds: xyz (which generates the pure O-sequence $(1, 3, 3, 1)$); xy^2 (generating $(1, 2, 2, 1)$); and x^3 (generating $(1, 1, 1, 1)$). Further, any integer a in the range under consideration can be partitioned into *exactly* b parts of size $\leqslant 3$ (since, clearly, any integer $a - b$ satisfying $0 \leqslant a - b \leqslant 2b$ can be partitioned into *at most* b parts of size $\leqslant 2$). Hence write $a = 3t_1 + 2t_2 + t_3$, where the multiplicities t_i are nonnegative and sum up to b.

Now consider t_1 squarefree monomials of degree 3 in disjoint sets of variables, say

$$x_1x_2x_3, \ \ldots, \ x_{3t_1-2}x_{3t_1-1}x_{3t_1};$$

t_2 monomials of the form

$$y_1y_2^2, \ \ldots, \ y_{2t_2-1}y_{2t_2}^2;$$

and t_3 monomials of the form

$$z_1^3, \ \ldots, \ z_{t_3}^3.$$

The pure O-sequence generated by the above $t_1 + t_2 + t_3$ monomials is given by:

$$\begin{aligned} & (1,0,0,0) + t_1 \cdot (0,3,3,1) + t_2 \cdot (0,2,2,1) + t_3 \cdot (0,1,1,1) \\ = \;& (1, 3t_1 + 2t_2 + t_3, 3t_1 + 2t_2 + t_3, t_1 + t_2 + t_3) \\ = \;& (1, a, a, b). \end{aligned}$$

This concludes the proof of (ii). □

Remark 8. We wrap up by noting that while any pure O-sequence is an artinian level Hilbert function [1], the converse is far from being true, even for $n = 2$.

It is easy to see that the sequence $(1, a, b)$ is level if and only if $1 \leqslant b \leqslant \binom{a+1}{2}$. Also, $(1, a, a, b)$ is level for any b in the range $1 \leqslant b \leqslant a$. More generally, using the techniques of [11], it can be shown that for any $n \geqslant 3$, $(1, a, a, \ldots, a, b) \in \mathbb{Z}_{>0}^{n+1}$ is level whenever $1 \leqslant b \leqslant a$, over a field of any characteristic.

However, larger values of b may also be attained when $n \geqslant 3$. For instance, $(1, 13, 13, 14)$ is a level Hilbert function ([1, Chapter 3]). In fact, we remark here without proof that, in stark contrast to the case of pure O-sequences, it is possible to construct level sequences $(1, a, a, b)$ where the difference $b - a$ gets arbitrarily large and is asymptotic to a itself. This strongly suggests that an explicit characterization of level Hilbert functions might hard to achieve, even in the special case $(1, a, a, b)$.

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