The Norton-Balanced Condition for Q-Polynomial Distance-Regular Graphs

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Abstract

Let Γ denote a Q-polynomial distance-regular graph, with vertex set X and diameter $D \geqslant 3$. The standard module V has a basis $\{\hat{x} | x \in X\}$, where $\hat{x}$ denotes column x of the identity matrix $I \in \mathrm{Mat}_X(\mathbb{C})$. Let E denote a Q-polynomial primitive idempotent of Γ. The eigenspace EV is spanned by the vectors $\{E\hat{x} | x \in X\}$. It was previously known that these vectors satisfy a condition called the balanced set condition. In this paper, we introduce a variation on the balanced set condition called the Norton-balanced condition. The Norton-balanced condition involves the Norton algebra product on EV. We define Γ to be Norton-balanced whenever Γ has a Q-polynomial primitive idempotent E such that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. We show that Γ is Norton-balanced in the following cases: (i) Γ is bipartite; (ii) Γ is almost bipartite; (iii) Γ is dual-bipartite; (iv) Γ is almost dual-bipartite; (v) Γ is tight; (vi) Γ is a Hamming graph; (vii) Γ is a Johnson graph; (viii) Γ is the Grassmann graph $J_q(2D, D)$; (ix) Γ is a halved bipartite dual-polar graph; (x) Γ is a halved Hemmeter graph; (xi) Γ is a halved hypercube; (xii) Γ is a folded-half hypercube; (xiii) Γ has q-Racah type and affords a spin model. Some theoretical results about the Norton-balanced condition are obtained, and some open problems are given.

Mathematics Subject Classifications: 05E30

1 Introduction

This paper is about a family of finite undirected graphs, said to be distance-regular [7]. We will investigate a type of distance-regular graph, called Q-polynomial [7, Chapter 8]. The Q-polynomial property was introduced in 1973 by Delsarte [18] in his work on coding theory and design theory. Since that beginning, the property has been linked to many topics, such as orthogonal polynomials [3, p. 260], [27, 42]; spin models [9, 13, 14, 33, 34];

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q-deformed enveloping algebras [22, 24, 47]; partially ordered sets [1, 4, 16, 17, 20, 32, 36]; tridiagonal pairs [10, 21, 23]; free fermions [5, 6, 11]; and the double affine Hecke algebra [28, 29, 30]. Comprehensive treatments of the Q-polynomial property can be found in [2, 3, 7, 15, 46].

For a Q-polynomial distance-regular graph, the adjacency matrix has a distinguished primitive idempotent called a Q-polynomial idempotent. There is a characterization of the Q-polynomial primitive idempotents, called the balanced set characterization [38, Theorem 1.1]. Over the next few paragraphs, we will describe this characterization in order to motivate our main topic.

Throughout this section, $\Gamma = (X, \mathcal{R})$ denotes a distance-regular graph with vertex set X, adjacency relation $\mathcal{R}$, and diameter $D \geqslant 3$ (formal definitions start in Section 2).

Let V denote the $\mathbb{R}$-vector space consisting of the column vectors with coordinates indexed by X and all entries in $\mathbb{R}$. The vector space V becomes a Euclidean space with bilinear form $\langle u, v\rangle = u^t v$ for $u, v \in V$. For $x \in X$ define a vector $\hat{x} \in V$ that has x-coordinate 1 and all other coordinates 0. The vectors $\{\hat{x} | x \in X\}$ form an orthonormal basis for V. The adjacency matrix A of Γ acts on V by left multiplication.

Let E denote a primitive idempotent of Γ. The matrix E is the orthogonal projection onto the eigenspace EV of A. By construction, the subspace EV is spanned by the vectors $\{E\hat{x} | x \in X\}$.

Let ∂ denote the path-length distance function for Γ. For $x \in X$ and $0 \leqslant i \leqslant D$, define the set $\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}$. According to the balanced set characterization [38, Theorem 1.1], E is Q-polynomial if and only if the following (i), (ii) hold:

(i) the vectors $\{E\hat{x} | x \in X\}$ are mutually distinct;

(ii) for $x, y \in X$ and $0 \leqslant i, j \leqslant D$,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} - \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} E\hat{z} \in \mathrm{Span}\{E\hat{x} - E\hat{y}\}.$$

For the rest of this section, assume that E is Q-polynomial. We mention a special case of the balanced set dependency. Pick $x, y \in X$ and write $i = \partial(x, y)$. Define

$$x_y^- = \sum_{z \in \Gamma(x) \cap \Gamma_{i-1}(y)} \hat{z}, \qquad x_y^+ = \sum_{z \in \Gamma(x) \cap \Gamma_{i+1}(y)} \hat{z},$$

where $\Gamma(x) = \Gamma_1(x)$ and $\Gamma_{-1}(x) = \emptyset = \Gamma_{D+1}(x)$. Then

$$Ex_y^- - Ey_x^- \in \mathrm{Span}\{E\hat{x} - E\hat{y}\}, \qquad Ex_y^+ - Ey_x^+ \in \mathrm{Span}\{E\hat{x} - E\hat{y}\}.$$

The vectors $\{E\hat{x} | x \in X\}$ satisfy another type of linear dependency, known as the symmetric balanced set dependency [41, Theorem 2.6]. Let $x, y \in X$ and $0 \leqslant i, j \leqslant D$. According to the symmetric balanced set dependency,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} + \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} E\hat{z} \in \mathrm{Span}\{Ex_y^- + Ey_x^-, Ex_y^+ + Ey_x^+, E\hat{x} + E\hat{y}\}.$$

Comparing the balanced set dependency with its symmetric version, we find that for $x, y \in X$ and $0 \leqslant i, j \leqslant D$,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} \in \operatorname{Span}\{Ex^-_y, Ex^+_y, E\hat{x}, E\hat{y}\}.$$

It could happen that for all $x, y \in X$ the vectors Ex^-_y, Ex^+_y, $E\hat{x}$, $E\hat{y}$ are linearly dependent. We now consider some situations where this occurs.

The set of vectors $\{E\hat{x} | x \in X\}$ is called strongly balanced [39, Section 2] whenever for all $x, y \in X$ and $0 \leqslant i, j \leqslant D$,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} \in \operatorname{Span}\{E\hat{x}, E\hat{y}\}.$$

According to [39, Theorems 1, 3] the following are equivalent:

(i) the set $\{E\hat{x} | x \in X\}$ is strongly balanced;

(ii) E is dual-bipartite or almost dual-bipartite (see Section 3 below).

We now recall the Norton algebra structure on EV [8, Proposition 5.2]. For $u \in V$ and $x \in X$ let u_x denote the x-coordinate of u. So $u = \sum_{x \in X} u_x \hat{x}$. For $u, v \in V$ define a vector $u \circ v = \sum_{x \in X} u_x v_x \hat{x}$. The Norton algebra consists of the $\mathbb{R}$-vector space EV, together with the product

$$u \star v = E(u \circ v) \qquad\qquad (u, v \in EV).$$

The Norton product $\star$ is commutative, and nonassociative in general.

We now introduce the Norton-balanced condition. The set of vectors $\{E\hat{x} | x \in X\}$ is called Norton-balanced whenever for all $x, y \in X$ and $0 \leqslant i, j \leqslant D$,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} \in \operatorname{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}.$$

Let us clarify the Norton-balanced condition. By our above comments, the following are equivalent:

(i) the set $\{E\hat{x} | x \in X\}$ is Norton-balanced;

(ii) for all $x, y \in X$ we have $Ex^-_y, Ex^+_y \in \operatorname{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}$.

We say that Γ is Norton-balanced whenever Γ has a Q-polynomial primitive idempotent E such that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced.

Next, we describe our results. We have two kinds of results; some are about examples, and some are more theoretical. We first describe the results about examples. This will be done over the next four paragraphs.

Assume that Γ is Q-polynomial. Using some elementary arguments, we show that Γ is Norton-balanced in the following cases: (i) Γ is bipartite; (ii) Γ is almost bipartite; (iii) Γ is dual-bipartite; (iv) Γ is almost dual-bipartite; (v) Γ is tight.

The combinatorial structure of Γ is described by some well-known parameters called the intersection numbers. We show that in general, Γ being Norton-balanced is not a condition on the intersection numbers alone. To do this, we consider the Hamming graph $H(D, 4)$ and a Doob graph with diameter D. These graphs have the same intersection numbers. We show that $H(D, 4)$ is Norton-balanced and the Doob graph is not.

The book [2, Chapter 6.4] gives a list of the known infinite families of Q-polynomial distance-regular graphs with unbounded diameter. For each listed graph, every Q-polynomial structure is described. We examine these Q-polynomial structures. For each listed graph $\Gamma = (X, \mathcal{R})$ and each Q-polynomial primitive idempotent E of Γ, we determine if the set $\{E\hat{x}|x \in X\}$ is Norton-balanced or not. In summary form, our conclusion is that Γ is Norton-balanced in the following cases: (vi) Γ is a Hamming graph; (vii) Γ is a Johnson graph; (viii) Γ is the Grassmann graph $J_q(2D, D)$; (ix) Γ is a halved bipartite dual-polar graph; (x) Γ is a halved Hemmeter graph; (xi) Γ is a halved hypercube; (xii) Γ is a folded-half hypercube.

The Norton-balanced condition was inspired by our recent work with Nomura on spin models [34]. We show that Γ is Norton-balanced in the following case: (xiii) Γ has q-Racah type and affords a spin model.

We will describe our theoretical results after a definition and some comments.

We define Γ to be reinforced whenever the following (i), (ii) hold for $2 \leqslant i \leqslant D$:

(i) for $x, y \in X$ at distance $\partial(x, y) = i$, the average valency of the induced subgraph $\Gamma(x) \cap \Gamma_{i-1}(y)$ is independent of x and y;

(ii) for $x, y \in X$ at distance $\partial(x, y) = i-1$, the average valency of the induced subgraph $\Gamma(x) \cap \Gamma_i(y)$ is independent of x and y.

If Γ is distance-transitive then Γ is reinforced. Assume for the moment that Γ is reinforced. For $2 \leqslant i \leqslant D$ let z_i denote the average valency mentioned in (i), and note that $a_1 - z_i$ is the average valency mentioned in (ii). In Lemma 41 we give a formula $z_i = z_2\alpha_i + a_1\beta_i$, where α_i, β_i are determined by the intersection numbers.

We now describe our theoretical results. This will be done over the next three paragraphs. Let E denote a Q-polynomial primitive idempotent of Γ.

Consider the following two conditions on E:

(i) the set $\{E\hat{x}|x \in X\}$ is Norton balanced;

(ii) for $x, y \in X$ the vectors Ex^-_y, Ex^+_y, $E\hat{x}$, $E\hat{y}$ are linearly dependent.

By our earlier comments, (i) implies (ii). We display an example for which (ii) holds but not (i). We show that (i) is implied by (ii) together with a certain restriction on the coefficients in the linear dependence.

Let λ denote an indeterminate. For $2 \leqslant i \leqslant D-1$ we define a quadratic polynomial $\Phi_i(\lambda)$ whose coefficients are determined by the intersection numbers of Γ. Pick $x, y \in X$

at distance $\partial(x, y) = i$. Assuming that Γ is reinforced, we compute the inner products between Ex^-_y, Ex^+_y, $E\hat{x}$, $E\hat{y}$ in terms of the intersection numbers and z_i, z_{i+1}. Using these inner products and a Cauchy-Schwarz inequality, we show that $\Phi_i(z_2) \geqslant 0$, with equality if and only if Ex^-_y, Ex^+_y, $E\hat{x}$, $E\hat{y}$ are linearly dependent. We show that if Γ is reinforced and the set $\{E\hat{x}|x \in X\}$ is Norton-balanced, then $\Phi_i(z_2) = 0$ for $2 \leqslant i \leqslant D - 1$.

We say that E is a dependency candidate (or DC) whenever there exists $\xi \in \mathbb{C}$ such that $\Phi_i(\xi) = 0$ for $2 \leqslant i \leqslant D-1$. Note that E being DC is a condition on the intersection numbers of Γ. If Γ is reinforced and the set $\{E\hat{x}|x \in X\}$ is Norton-balanced, then E is DC. In our main theoretical result Theorem 128, we display a necessary and sufficient condition on the intersection numbers of Γ, for E to be DC. Using Theorem 128 we show that for certain examples Γ is not Norton-balanced.

In the previous paragraphs, we often assumed that Γ is reinforced; this was done for clarity and simplicity. In the main body of the paper, we sometimes use a more general argument that involves weaker hypotheses.

This paper is organized as follows. Section 2 contains some preliminaries. Sections 3, 4 contain basic information about a distance-regular graph Γ and its Q-polynomial primitive idempotents E. In Section 5 we recall the Norton algebra. In Section 6 we introduce the Norton-balanced condition. In Section 7 we give some examples that satisfy the Norton-balanced condition. In Section 8 we give some linear algebraic consequences of the Norton-balanced condition. In Section 9 we recall some parameters related to the Q-polynomial property. In Section 10 we discuss a 4-vertex configuration called a kite, and we introduce the reinforced condition. In Sections 11–14 we consider a pair of vertices x, y of Γ, and investigate the potential linear dependence of Ex^-_y, Ex^+_y, $E\hat{x}$, $E\hat{y}$. In Section 15 we introduce the polynomials $\Phi_i(\lambda)$. In Section 16 we discuss the DC condition. Sections 17–29 are about examples. Section 30 is about the case in which Γ affords a spin model. Section 31 contains some directions for future research.

2 Preliminaries

We now begin our formal argument. The following concepts and notation will be used throughout the paper. Let $\mathbb{R}$ denote the field of real numbers. Let X denote a nonempty finite set. The elements of X are called *vertices*. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with rows and columns indexed by X and all entries in $\mathbb{R}$. Let $I \in \mathrm{Mat}_X(\mathbb{R})$ denote the identity matrix. Let $V = \mathbb{R}^X$ denote the $\mathbb{R}$-vector space consisting of the column vectors with coordinates indexed by X and all entries in $\mathbb{R}$. The algebra $\mathrm{Mat}_X(\mathbb{R})$ acts on V by left multiplication. We endow V with a bilinear form $\langle\, ,\, \rangle$ such that $\langle u, v\rangle = u^t v$ for all $u, v \in V$, where t denotes transpose. Note that $\langle u, v\rangle = \langle v, u\rangle$ for $u, v \in V$. For $u \in V$ we abbreviate $\|u\|^2 = \langle u, u\rangle$. We have $\|u\|^2 \geqslant 0$, with equality if and only if $u = 0$. The bilinear form turns V into a Euclidean space. For $B \in \mathrm{Mat}_X(\mathbb{R})$ we have $\langle Bu, v\rangle = \langle u, B^t v\rangle$ for all $u, v \in V$. For $x \in X$ define a vector $\hat{x} \in V$ that has x-coordinate 1 and all other coordinates 0. The vectors $\{\hat{x}|x \in X\}$ form an orthonormal basis for V. The vector $\mathbf{1} = \sum_{x\in X} \hat{x}$ has all coordinates 1. Let $J \in \mathrm{Mat}_X(\mathbb{R})$ have all entries 1. Note that $J\hat{x} = \mathbf{1}$ for all $x \in X$. For $B, C \in \mathrm{Mat}_X(\mathbb{R})$ their entrywise product

$B \circ C \in \mathrm{Mat}_X(\mathbb{R})$ has (x, y)-entry $B_{x,y}C_{x,y}$ for all $x, y \in X$. For a positive $q \in \mathbb{R}$ let $q^{\frac{1}{2}}$ denote the positive square root of q.

3 Distance-Regular Graphs

In this section, we review some definitions and basic concepts concerning distance-regular graphs. See [2, 3, 7, 15, 46] for more information. Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and adjacency relation $\mathcal{R}$. For an integer $n \geqslant 0$, a *path of length* n in Γ is a sequence of vertices $\{x_i\}_{i=0}^n$ such that x_{i-1}, x_i are adjacent for $1 \leqslant i \leqslant n$. This path is said to *connect* x_0, x_n. For $x, y \in X$ let $\partial(x, y)$ denote the length of a shortest path that connects x, y. We call $\partial(x, y)$ the *distance between* x *and* y. The integer $D = \max\{\partial(x, y) | x, y \in X\}$ is called the *diameter* of Γ. For an integer $i \geqslant 0$ and $x \in X$ define the set $\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For $x \in X$ we call $|\Gamma(x)|$ the *valency of* x. For an integer $k \geqslant 0$, we say that Γ is *regular with valency* k whenever each vertex in X has valency k. We say that Γ is *distance-regular* whenever for all integers h, i, j $(0 \leqslant h, i, j \leqslant D)$ and all vertices $x, y \in X$ at distance $\partial(x, y) = h$, the cardinality $p^h_{i,j} = |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of x and y. The integers $p^h_{i,j}$ are called the *intersection numbers* of Γ. For the rest of this paper, we assume that Γ is distance-regular with $D \geqslant 3$. Note that Γ is regular with valency $k = p^0_{1,1}$. By construction $p^h_{i,j} = p^h_{j,i}$ for $0 \leqslant h, i, j \leqslant D$. By the triangle inequality the following holds for $0 \leqslant h, i, j \leqslant D$:

(i) $p^h_{i,j} = 0$ if one of h, i, j is greater than the sum of the other two;

(ii) $p^h_{i,j} \neq 0$ if one of h, i, j is equal to the sum of the other two.

We abbreviate

$$c_i = p^i_{1,i-1} \quad (1 \leqslant i \leqslant D), \qquad a_i = p^i_{1,i} \quad (0 \leqslant i \leqslant D), \qquad b_i = p^i_{1,i+1} \quad (0 \leqslant i \leqslant D-1).$$

We have $b_0 = k$. Note that $a_0 = 0$ and $c_1 = 1$. By [7, Lemma 4.1.6] we have

$$c_{i-1} \leqslant c_i \quad (2 \leqslant i \leqslant D), \qquad b_{i-1} \geqslant b_i \quad (1 \leqslant i \leqslant D-1), \qquad b_i \geqslant c_{D-i} \quad (0 \leqslant i \leqslant D-1).$$

Observe that $k = c_i + a_i + b_i$ $(0 \leqslant i \leqslant D)$, where $c_0 = 0$ and $b_D = 0$. For $0 \leqslant i \leqslant D$ define $k_i = p^0_{i,i}$ and note that $k_i = |\Gamma_i(x)|$ for all $x \in X$. We have $k_0 = 1$ and $k_1 = k$. By [3, p. 195] we have

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad (0 \leqslant i \leqslant D).$$

The graph Γ is called *bipartite* whenever $a_i = 0$ for $0 \leqslant i \leqslant D$. The graph Γ is called *almost bipartite* whenever $a_i = 0$ for $0 \leqslant i \leqslant D-1$ and $a_D \neq 0$. The graph Γ is called an *antipodal 2-cover* whenever $k_D = 1$. This occurs if and only if $k_i = k_{D-i}$ $(0 \leqslant i \leqslant D)$ if and only if $b_i = c_{D-i}$ $(0 \leqslant i \leqslant D)$; see [7, Proposition 4.2.2].

We recall the Bose-Mesner algebra of Γ. For $0 \leqslant i \leqslant D$ define $A_i \in \mathrm{Mat}_X(\mathbb{R})$ that has (x, y)-entry

$$(A_i)_{x,y} = \begin{cases} 1, & \text{if } \partial(x,y) = i; \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

For $x \in X$ we have

$$A_i \hat{x} = \sum_{y \in \Gamma_i(x)} \hat{y}. \tag{1}$$

We call A_i the ith *distance matrix* of Γ. We abbreviate $A = A_1$ and call this the *adjacency matrix* of Γ. Observe that (i) $A_0 = I$; (ii) $J = \sum_{i=0}^{D} A_i$; (iii) $A_i^t = A_i$ $(0 \leqslant i \leqslant D)$; (iv) $A_i A_j = \sum_{h=0}^{D} p_{i,j}^h A_h$ $(0 \leqslant i, j \leqslant D)$. Therefore the matrices $\{A_i\}_{i=0}^D$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{R})$, called the *Bose-Mesner algebra* of Γ. The matrix A generates M [46, Corollary 3.4]. The matrices $\{A_i\}_{i=0}^D$ are symmetric and mutually commute, so they can be simultaneously diagonalized over $\mathbb{R}$. Consequently M has a second basis $\{E_i\}_{i=0}^D$ such that (i) $E_0 = |X|^{-1} J$; (ii) $I = \sum_{i=0}^{D} E_i$; (iii) $E_i^t = E_i$ $(0 \leqslant i \leqslant D)$; (iv) $E_i E_j = \delta_{i,j} E_i$ $(0 \leqslant i, j \leqslant D)$. We call $\{E_i\}_{i=0}^D$ the *primitive idempotents* of Γ. The primitive idempotent E_0 is called *trivial*.

For $0 \leqslant i \leqslant D$ let θ_i denote the eigenvalue of A for E_i. We have $AE_i = \theta_i E_i = E_i A$. We have $A = \sum_{i=0}^{D} \theta_i E_i$. The scalars $\{\theta_i\}_{i=0}^D$ are mutually distinct because A generates M. We have

$$V = \sum_{i=0}^{D} E_i V \qquad \text{(orthogonal direct sum)}.$$

For $0 \leqslant i \leqslant D$ the subspace $E_i V$ is the eigenspace of A for the eigenvalue θ_i. By [7, p. 128] we have $\theta_0 = k$.

We recall the Krein parameters of Γ. For $0 \leqslant i, j \leqslant D$ we have $A_i \circ A_j = \delta_{i,j} A_i$. Therefore M is closed under $\circ$. Consequently, there exist scalars $q_{i,j}^h \in \mathbb{R}$ $(0 \leqslant h, i, j \leqslant D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{i,j}^h E_h \qquad (0 \leqslant i, j \leqslant D).$$

The scalars $q_{i,j}^h$ are called the *Krein parameters* of Γ. By construction $q_{i,j}^h = q_{j,i}^h$ for $0 \leqslant h, i, j \leqslant D$. By [3, p. 69] we have $q_{i,j}^h \geqslant 0$ for $0 \leqslant h, i, j \leqslant D$. By [46, Lemma 5.15] we have $q_{i,i}^0 = \dim(E_i V)$ for $0 \leqslant i \leqslant D$.

Next, we describe a feature of the Krein parameters that will play a role in our main results. In this description, we will use the following notation. For $u \in V$ and $x \in X$ let u_x denote the x-coordinate of u. So $u = \sum_{x \in X} u_x \hat{x}$. For $u, v \in V$ define a vector $u \circ v \in V$ that has x-coordinate $u_x v_x$ for all $x \in X$. So $u \circ v = \sum_{x \in X} u_x v_x \hat{x}$. We have

$$\hat{x} \circ \hat{y} = \delta_{x,y} \hat{x} \qquad (x, y \in X). \tag{2}$$

For $v \in V$ we have $\mathbf{1} \circ v = v$.

Lemma 1. (See [8, Proposition 5.1].) *The following hold for* $0 \leqslant h, i, j \leqslant D$.

(i) *Assume that* $q^h_{i,j} \neq 0$. *Then* $E_h\big(E_iV \circ E_jV\big)$ *spans* E_hV.

(ii) *Assume that* $q^h_{i,j} = 0$. *Then* $E_h\big(E_iV \circ E_jV\big) = 0$.

We recall the Q-polynomial property. The ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents is called *Q-polynomial* whenever the following hold for $0 \leqslant h, i, j \leqslant D$:

(i) $q^h_{i,j} = 0$ if one of h, i, j is greater than the sum of the other two;

(ii) $q^h_{i,j} \neq 0$ if one of h, i, j is equal to the sum of the other two.

Assume that the ordering $\{E_i\}_{i=0}^D$ is Q-polynomial. We abbreviate

$$c^*_i = q^i_{1,i-1} \;\; (1 \leqslant i \leqslant D), \qquad a^*_i = q^i_{1,i} \;\; (0 \leqslant i \leqslant D), \qquad b^*_i = q^i_{1,i+1} \;\; (0 \leqslant i \leqslant D-1).$$

We emphasize that $c^*_i \neq 0$ $(1 \leqslant i \leqslant D)$ and $b^*_i \neq 0$ $(0 \leqslant i \leqslant D-1)$. By [46, Lemma 5.15] we have $a^*_0 = 0$ and $c^*_1 = 1$. The Q-polynomial ordering $\{E_i\}_{i=0}^D$ is called *dual-bipartite* (resp. *almost dual-bipartite*) whenever $a^*_i = 0$ for $0 \leqslant i \leqslant D$ (resp. $a^*_i = 0$ for $0 \leqslant i \leqslant D-1$ and $a^*_D \neq 0$).

A primitive idempotent E of Γ is called *Q-polynomial* whenever there exists a Q-polynomial ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of Γ such that $E = E_1$. For the rest of this paragraph, assume that E is Q-polynomial. By construction, E is nontrivial. We say that E is *dual-bipartite* (resp. *almost dual-bipartite*) whenever the corresponding Q-polynomial ordering $\{E_i\}_{i=0}^D$ is dual-bipartite (resp. almost dual-bipartite). By [7, p. 241] and [19, Theorems 1.1, 1.2], E is dual-bipartite if and only if Γ is an antipodal 2-cover.

We say that Γ is *Q-polynomial* whenever there exists a Q-polynomial ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of Γ. We say that Γ is *dual-bipartite* (resp. *almost dual-bipartite*) whenever there exists a Q-polynomial ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of Γ that is dual-bipartite (resp. almost dual-bipartite).

For the rest of this paper, we assume that Γ is Q-polynomial. To avoid trivialities, we always assume that the valency $k \geqslant 3$.

4 Some eigenspace geometry

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. By construction, the eigenspace EV is spanned by the vectors $\{E\hat{x} | x \in X\}$. In this section, we discuss the geometry of these vectors. We will describe the inner product $\langle E\hat{x}, E\hat{y}\rangle$ for $x, y \in X$. We will also display some linear dependencies among $\{E\hat{x} | x \in X\}$.

The matrix E is contained in the Bose-Mesner algebra M. Therefore, there exist $\theta^*_i \in \mathbb{R}$ $(0 \leqslant i \leqslant D)$ such that

$$E = |X|^{-1} \sum_{i=0}^{D} \theta^*_i A_i. \tag{3}$$

By [3, p. 260] the scalars $\{\theta^*_i\}_{i=0}^D$ are mutually distinct. By [46, Lemma 3.9] we have $\theta^*_0 = \dim(EV)$. The scalars $\{\theta^*_i\}_{i=0}^D$ are called the *dual eigenvalues* of Γ associated with E. For notational convenience, let θ^*_{-1} and θ^*_{D+1} denote indeterminates. By [7, p. 128],

$$c_i\theta^*_{i-1} + a_i\theta^*_i + b_i\theta^*_{i+1} = \theta_1\theta^*_i \qquad (0 \leqslant i \leqslant D). \tag{4}$$

The following result is well known; see for example [7, Proposition 4.4.1].

Lemma 2. (See [7, Proposition 4.4.1].) *Pick $x, y \in X$ and write $i = \partial(x, y)$. Then the following* (i)–(iii) *hold:*

(i) $\langle E\hat{x}, E\hat{y}\rangle = |X|^{-1}\theta^*_i$;

(ii) $\|E\hat{x}\|^2 = \|E\hat{y}\|^2 = |X|^{-1}\theta^*_0$;

(iii) θ^*_i/θ^*_0 *is the cosine of the angle between $E\hat{x}$ and $E\hat{y}$.*

Corollary 3. *For distinct $x, y \in X$ we have $E\hat{x} \neq E\hat{y}$.*

Proof. Write $i = \partial(x, y)$ and note that $i \neq 0$. The dual eigenvalues $\{\theta^*_j\}_{j=0}^D$ are mutually distinct, so $\theta^*_i \neq \theta^*_0$. The result follows in view of Lemma 2(iii). □

As we consider additional consequences of Lemma 2, we will treat separately the case in which Γ is an antipodal 2-cover.

Lemma 4. *Assume that Γ is not an antipodal 2-cover. Then the following hold:*

(i) $\theta^*_0 > \theta^*_i > -\theta^*_0 \quad (1 \leqslant i \leqslant D)$;

(ii) *for distinct $x, y \in X$ the vectors $E\hat{x}, E\hat{y}$ are linearly independent.*

Proof. (i) Pick $x, y \in X$ at distance $\partial(x, y) = i$. Using Lemma 2 and trigonometry we obtain $\theta^*_0 > \theta^*_i \geqslant -\theta^*_0$, with equality on the right if and only if $E\hat{x} + E\hat{y} = 0$. Suppose this equality occurs. The vertices x, y uniquely determine each other by $E\hat{x} + E\hat{y} = 0$ and Corollary 3, so $k_i = 1$. Now $k_D = 1$ in view of [7, Proposition 5.1.1(i)]. Consequently Γ is an antipodal 2-cover, for a contradiction. We have shown that $\theta^*_0 > \theta^*_i > -\theta^*_0$.
(ii) By (i) and Lemma 2. □

Lemma 5. *Assume that Γ is an antipodal 2-cover. Then the following hold.*

(i) $\theta^*_0 > \theta^*_i > -\theta^*_0 \quad (1 \leqslant i \leqslant D-1)$ *and* $\theta^*_D = -\theta^*_0$;

(ii) *for distinct $x, y \in X$ the vectors $E\hat{x}, E\hat{y}$ are linearly independent if $\partial(x, y) \neq D$, and $E\hat{x} + E\hat{y} = 0$ if $\partial(x, y) = D$.*

Proof. Similar to the proof of Lemma 4, except that $\theta^*_D = -\theta^*_0$ by [7, Theorem 8.2.4]. □

Next, we display some linear dependencies among the vectors $\{E\hat{x} | x \in X\}$.

Lemma 6. *Let $x \in X$. Then*

$$\sum_{y \in \Gamma(x)} E\hat{y} = \theta_1 E\hat{x}. \tag{5}$$

Moreover for $0 \leqslant i \leqslant D$,

$$\sum_{y \in \Gamma_i(x)} E\hat{y} \in \operatorname{Span}\{E\hat{x}\}. \tag{6}$$

Proof. The equation (5) holds, because each side is equal to $EA\hat{x}$. To verify (6), note that $A_i = f_i(A)$, where f_i is a polynomial with real coefficients and degree i. Using (1) we obtain

$$\sum_{y \in \Gamma_i(x)} E\hat{y} = EA_i\hat{x} = Ef_i(A)\hat{x} = f_i(\theta_1)E\hat{x} \in \operatorname{Span}\{E\hat{x}\}.$$

□

5 The Norton algebra

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. In this section, we recall the Norton algebra product $\star$ on the vector space EV. For $x, y \in X$ we compute $E\hat{x} \star E\hat{y}$.

Definition 7. (See [8, Proposition 5.2].) The *Norton algebra EV* consists of the $\mathbb{R}$-vector space EV, together with the product

$$u \star v = E(u \circ v) \qquad\qquad (u, v \in EV). \tag{7}$$

We call $\star$ the *Norton product.*

The Norton product $\star$ is commutative, and nonassociative in general.

As we investigate $\star$ it is natural to consider $E\hat{x} \star E\hat{y}$ for all $x, y \in X$. In the next two lemmas we discuss some extremal cases.

Lemma 8. (See [44, Lemma 3.2].) *For $x \in X$,*

$$E\hat{x} \star E\hat{x} = |X|^{-1} a_1^* E\hat{x}.$$

Lemma 9. *The following* (i)–(iv) *are equivalent:*

(i) $u \star v = 0$ *for all* $u, v \in EV$*;*

(ii) $E\hat{x} \star E\hat{y} = 0$ *for all* $x, y \in X$*;*

(iii) $E\hat{x} \star E\hat{x} = 0$ *for all* $x \in X$*;*

(iv) $a_1^* = 0$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear. The implication (iii) $\Rightarrow$ (iv) is from Lemma 8, and the implication (iv) $\Rightarrow$ (i) is from Lemma 1. □

Corollary 10. *Assume that E is dual-bipartite or almost dual-bipartite. Then the equivalent conditions* (i)–(iv) *in Lemma 9 all hold.*

Proof. We assume that $a_i^* = 0$ for $0 \leqslant i \leqslant D-1$, so $a_1^* = 0$. □

Let $x, y \in X$. Shortly, we will review a formula for $E\hat{x} \star E\hat{y}$ that appeared in [44, Theorem 3.7].

Lemma 11. *For $x, y \in X$ and $0 \leqslant i, j \leqslant D$,*

$$A_i\hat{x} \circ A_j\hat{y} = \sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} \hat{z}. \tag{8}$$

Proof. By (1) and (2). □

Lemma 12. *Pick $0 \leqslant h, i, j \leqslant D$ and $x, y \in X$ at distance $\partial(x, y) = h$. Then:*

(i) $\|A_i\hat{x} \circ A_j\hat{y}\|^2 = p^h_{i,j}$;

(ii) $A_i\hat{x} \circ A_j\hat{y} = 0$ *if and only if* $p^h_{i,j} = 0$.

Proof. (i) By (8) and since $|\Gamma_i(x) \cap \Gamma_j(y)| = p^h_{i,j}$.

(ii) By (i) above. □

Definition 13. (See [44, Definition 3.5].) Pick $x, y \in X$ and write $i = \partial(x, y)$. Define

$$x_y^- = A\hat{x} \circ A_{i-1}\hat{y} = \sum_{z \in \Gamma(x) \cap \Gamma_{i-1}(y)} \hat{z},$$
$$x_y^0 = A\hat{x} \circ A_i\hat{y} = \sum_{z \in \Gamma(x) \cap \Gamma_i(y)} \hat{z},$$
$$x_y^+ = A\hat{x} \circ A_{i+1}\hat{y} = \sum_{z \in \Gamma(x) \cap \Gamma_{i+1}(y)} \hat{z},$$

where we understand

$$A_{-1} = 0, \qquad \Gamma_{-1}(x) = \emptyset, \qquad A_{D+1} = 0, \qquad \Gamma_{D+1}(x) = \emptyset.$$

We clarify the meaning of Definition 13. Pick $x, y \in X$. If $\partial(x, y) = D$ then $x_y^+ = 0$. If $\partial(x, y) = 1$ then $x_y^- = \hat{y}$. If $x = y$ then $x_y^0 = 0$ and $x_y^- = 0$.

Lemma 14. *For $x, y \in X$ we have*

$$x_y^- + x_y^0 + x_y^+ = A\hat{x}, \tag{9}$$

$$Ex_y^- + Ex_y^0 + Ex_y^+ = \theta_1 E\hat{x}. \tag{10}$$

Proof. Assertion (9) follows from (1) (with $i = 1$). Assertion (10) follows from (5). $\square$

Proposition 15. (See [44, Theorem 3.7].) *For $x, y \in X$ we have*

$$E\hat{x} \star E\hat{y} = \frac{(\theta^*_{i-1} - \theta^*_i)Ex^-_y + (\theta^*_{i+1} - \theta^*_i)Ex^+_y + (\theta_1 - \theta_2)\theta^*_i E\hat{x} + (\theta_2 - \theta_0)E\hat{y}}{|X|(\theta_1 - \theta_2)} \qquad (11)$$

where $i = \partial(x, y)$. We recall that θ^*_{-1} and θ^*_{D+1} are indeterminates.

We mention some special cases of Proposition 15.

Corollary 16. (See [44, Corollary 3.8].) *The following* (i)–(iii) *hold.*

(i) *For $x \in X$,*

$$E\hat{x} \star E\hat{x} = \frac{\theta_1\theta^*_1 - \theta_2\theta^*_0 + \theta_2 - \theta_0}{|X|(\theta_1 - \theta_2)} E\hat{x}.$$

(ii) *For $x, y \in X$ at distance $\partial(x, y) = 1$,*

$$E\hat{x} \star E\hat{y} = \frac{(\theta^*_2 - \theta^*_1)Ex^+_y + (\theta_1 - \theta_2)\theta^*_1 E\hat{x} + (\theta_2 - \theta_0 + \theta^*_0 - \theta^*_1)E\hat{y}}{|X|(\theta_1 - \theta_2)}.$$

(iii) *For $x, y \in X$ at distance $\partial(x, y) = D$,*

$$E\hat{x} \star E\hat{y} = \frac{(\theta^*_{D-1} - \theta^*_D)Ex^-_y + (\theta_1 - \theta_2)\theta^*_D E\hat{x} + (\theta_2 - \theta_0)E\hat{y}}{|X|(\theta_1 - \theta_2)}.$$

Comparing Lemma 8 and Corollary 16(i), we obtain

$$a^*_1 = \frac{\theta_1\theta^*_1 - \theta_2\theta^*_0 + \theta_2 - \theta_0}{\theta_1 - \theta_2}.$$

6 The Norton-balanced condition

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. In Section 4, we considered some linear dependencies among the vectors $\{E\hat{x} | x \in X\}$. In the present section, we return to this topic. We will review the balanced set condition [38] and some variations [39, 41]. Then we will introduce the Norton-balanced condition.

Lemma 17. (Balanced set condition [38, Theorem 1.1].) *For $x, y \in X$ and $0 \leqslant i, j \leqslant D$,*

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} - \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} E\hat{z} \in \operatorname{Span}\{E\hat{x} - E\hat{y}\}.$$

We emphasize some special cases of Lemma 17. For $x, y \in X$,

$$Ex_y^+ - Ey_x^+ \in \operatorname{Span}\{E\hat{x} - E\hat{y}\}, \qquad Ex_y^- - Ey_x^- \in \operatorname{Span}\{E\hat{x} - E\hat{y}\}.$$

Next, we describe the symmetric balanced set condition.

Lemma 18. (Symmetric balanced set condition [41, Theorem 2.6].) *For $x, y \in X$ and $0 \leqslant i, j \leqslant D$,*

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} + \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} E\hat{z} \in \operatorname{Span}\{Ex_y^- + Ey_x^-, Ex_y^+ + Ey_x^+, E\hat{x} + E\hat{y}\}.$$

Combining Lemmas 17 and 18, we obtain the following result.

Lemma 19. *For $x, y \in X$ and $0 \leqslant i, j \leqslant D$,*

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} \in \operatorname{Span}\{Ex_y^-, Ex_y^+, E\hat{x}, E\hat{y}\}.$$

It could happen that for all $x, y \in X$ the vectors Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent. We now consider some situations where this occurs.

Definition 20. (See [39, Section 2].) The set of vectors $\{E\hat{x} | x \in X\}$ is called *strongly balanced* whenever for all $x, y \in X$ and $0 \leqslant i, j \leqslant D$,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} \in \operatorname{Span}\{E\hat{x}, E\hat{y}\}.$$

Lemma 21. (See [39, Theorems 1, 3].) *The following are equivalent:*

(i) *the set $\{E\hat{x} | x \in X\}$ is strongly balanced;*

(ii) *E is dual-bipartite or almost dual-bipartite.*

We now introduce the Norton-balanced condition.

Definition 22. The set of vectors $\{E\hat{x} | x \in X\}$ is called *Norton-balanced* whenever for all $x, y \in X$ and $0 \leqslant i, j \leqslant D$,

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} \in \operatorname{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}.$$

Let us clarify the above definition.

Lemma 23. *The following are equivalent:*

(i) *the set $\{E\hat{x} | x \in X\}$ is Norton-balanced;*

(ii) *for all $x, y \in X$ we have $Ex_y^-, Ex_y^+ \in \operatorname{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}$.*

Proof. By Lemma 19. □

Lemma 24. *Let $x, y \in X$ and write $i = \partial(x, y)$.*

(i) *Assume that $i \in \{0, 1, D\}$. Then $Ex_y^-, Ex_y^+ \in \mathrm{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}$.*

(ii) *Assume that $2 \leqslant i \leqslant D-1$. Then $Ex_y^- \in \mathrm{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}$ if and only if $Ex_y^+ \in \mathrm{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}$.*

Proof. (i) By (5) and Corollary 16.
(ii) By Proposition 15. □

Remark 25. The Norton-balanced condition is not a condition on the intersection numbers alone. We show this with an example. The example involves the Hamming graph $H(D, 4)$ [2, p. 355] and a Doob graph of diameter D [2, p. 387]. These graphs have the same intersection numbers, but are not isomorphic. They both have a Q-polynomial structure with eigenvalue sequence $\theta_i = 3D - 4i$ $(0 \leqslant i \leqslant D)$. For either graph, let $E = E_1$ denote the primitive idempotent associated with θ_1. As we will see, the set $\{E\hat{x} | x \in X\}$ is Norton-balanced for $H(D, 4)$ but not for the Doob graph.

7 The Norton-balanced condition; first examples

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. In this section, we show that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced in the following cases: Γ is bipartite; Γ is almost bipartite; E is dual-bipartite; E is almost dual-bipartite; Γ is tight.

Lemma 26. *Assume that Γ is bipartite or almost bipartite. Let $1 \leqslant i \leqslant D-1$ and $x, y \in X$ at distance $\partial(x, y) = i$. Then*

$$Ex_y^+ = \theta_1 E\hat{x} - Ex_y^-. \tag{12}$$

Moreover,

$$E\hat{x} \star E\hat{y} = \frac{(\theta^*_{i-1} - \theta^*_{i+1})Ex_y^- + (\theta_1\theta^*_{i+1} - \theta_2\theta^*_i)E\hat{x} + (\theta_2 - \theta_0)E\hat{y}}{|X|(\theta_1 - \theta_2)}. \tag{13}$$

Proof. By Lemma 12 and $a_i = 0$ we have $x_y^0 = 0$. This and Lemma 14 imply (12). To get (13), evaluate Proposition 15 using (12). □

Proposition 27. *Assume that Γ is bipartite or almost bipartite. Then the set $\{E\hat{x} | x \in X\}$ is Norton-balanced.*

Proof. We invoke Lemma 23. For $x, y \in X$ we show that

$$Ex_y^-, Ex_y^+ \in \mathrm{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}. \tag{14}$$

First assume that $\partial(x, y) \in \{0, 1, D\}$. Then (14) holds by Lemma 24(i). Next assume that $2 \leqslant \partial(x, y) \leqslant D-1$. Then (14) holds by Lemma 24(ii) and (13). The result follows. □

Proposition 28. *Assume that E is dual-bipartite or almost dual-bipartite. Then the set $\{E\hat{x} | x \in X\}$ is Norton-balanced.*

Proof. By Lemma 21 and Definitions 20, 22. $\square$

Next, we recall what it means for Γ to be tight. The tight concept was introduced in [26], and discussed further in [35]. Assume for the moment that Γ is not bipartite. By [35, Theorem 1.3], $a_D = 0$ if and only if $a^*_D = 0$.

Definition 29. (See [35, Theorem 1.3].) We say that Γ is *tight* whenever Γ is not bipartite and $a_D = a^*_D = 0$.

We bring in some notation. Write

$$E_D = |X|^{-1} \sum_{i=0}^{D} \varrho_i A_i, \qquad \varrho_i \in \mathbb{R}.$$

For notational convenience, let ϱ_{-1} and ϱ_{D+1} denote indeterminates.

Lemma 30. *Assume that Γ is tight. Pick distinct $x, y \in X$ and write $i = \partial(x, y)$. Then*

$$(\varrho_{i-1} - \varrho_i) E x^-_y + (\varrho_{i+1} - \varrho_i) E x^+_y = \big(\theta_{D-1} - \theta_1\big) \varrho_i E\hat{x}. \tag{15}$$

Proof. We first show that

$$E\Big(E_D \hat{y} \circ (A - \theta_{D-1} I)\hat{x}\Big) = 0. \tag{16}$$

We have $E_D\hat{y} \in E_D V$. Moreover,

$$A - \theta_{D-1} I = \sum_{j=0}^{D} (\theta_j - \theta_{D-1}) E_j = \sum_{\substack{0 \leqslant j \leqslant D \\ j \neq D-1}} (\theta_j - \theta_{D-1}) E_j.$$

Therefore,

$$(A - \theta_{D-1} I)\hat{x} \in \sum_{\substack{0 \leqslant j \leqslant D \\ j \neq D-1}} E_j V.$$

For $0 \leqslant j \leqslant D$ such that $j \neq D-1$, we have $q^1_{D,j} = 0$ and therefore $E\big(E_D V \circ E_j V\big) = 0$ in view of Lemma 1. By these comments we get (16). By (16) and the construction,

$$\begin{aligned}
0 &= |X| E\Big(E_D\hat{y} \circ (A - \theta_{D-1} I)\hat{x}\Big) \\
&= |X| E\Big(E_D\hat{y} \circ \big(x^-_y + x^0_y + x^+_y - \theta_{D-1}\hat{x}\big)\Big) \\
&= E\Big(\varrho_{i-1} x^-_y + \varrho_i x^0_y + \varrho_{i+1} x^+_y - \theta_{D-1}\varrho_i \hat{x}\Big) \\
&= \varrho_{i-1} E x^-_y + \varrho_i E x^0_y + \varrho_{i+1} E x^+_y - \theta_{D-1} \varrho_i E\hat{x}.
\end{aligned}$$

In the previous line, we eliminate Ex^0_y using (10), and routinely obtain (15). $\square$

Lemma 31. *Assume that* Γ *is tight. Then the set* $\{E\hat{x}|x \in X\}$ *is Norton-balanced.*

Proof. We invoke Lemma 23. For $x, y \in X$ we show that

$$Ex_y^-, Ex_y^+ \in \text{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}. \tag{17}$$

First assume that $\partial(x, y) \in \{0, 1, D\}$. Then (17) holds by Lemma 24(i). Next assume that $2 \leqslant \partial(x, y) \leqslant D - 1$. The equations (11), (15) give a linear system in the unknowns Ex_y^-, Ex_y^+. For this linear system the coefficient matrix is invertible; indeed

$$\det \begin{pmatrix} \theta^*_{i-1} - \theta^*_i & \theta^*_{i+1} - \theta^*_i \\ \varrho_{i-1} - \varrho_i & \varrho_{i+1} - \varrho_i \end{pmatrix} \neq 0$$

by [26, p. 183]. By linear algebra, the linear system has a unique solution for Ex_y^-, Ex_y^+. Examining the solution, we routinely obtain (17). □

8 The vectors Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. In the previous section, we displayed some examples for which the set $\{E\hat{x}|x \in X\}$ is Norton-balanced. Later in the paper we will discuss some more examples. In this section, we will develop some methods that will facilitate the discussion.

Lemma 32. *Assume that the set* $\{E\hat{x}|x \in X\}$ *is Norton-balanced. Then for* $x, y \in X$ *the vectors* Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ *are linearly dependent.*

Proof. By Lemma 23 and linear algebra. □

Consider the converse to Lemma 32. For the moment, assume that for all $x, y \in X$ the vectors Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent. It is not necessarily the case that the set $\{E\hat{x}|x \in X\}$ is Norton-balanced; the next result gives a counterexample.

Lemma 33. *Assume that* $a_1^* = 0$ *and* $a_2^* \neq 0$. *Then:*

(i) *the set* $\{E\hat{x}|x \in X\}$ *is not Norton-balanced;*

(ii) *for* $x, y \in X$ *we have*

$$0 = (\theta^*_{i-1} - \theta^*_i)Ex_y^- + (\theta^*_{i+1} - \theta^*_i)Ex_y^+ + (\theta_1 - \theta_2)\theta^*_i E\hat{x} + (\theta_2 - \theta_0)E\hat{y}$$

where $i = \partial(x, y)$.

Proof. (i) We assume that $\{E\hat{x}|x \in X\}$ is Norton-balanced, and get a contradiction. By Lemma 9 and Definitions 20, 22 the set $\{E\hat{x}|x \in X\}$ is strongly balanced. By this and Lemma 21, E is dual-bipartite or almost dual-bipartite. This contradicts $a_2^* \neq 0$.

(ii) By Lemma 9, the left-hand side of (11) is equal to zero. □

Pick distinct $x, y \in X$. Our next general goal is to investigate the potential linear dependence among the vectors Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$. We will consider the following situations:

(i) Ex_y^-, $E\hat{x}$, $E\hat{y}$ are linearly dependent;

(ii) Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent;

(iii) Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent, but not (i), (ii).

As we discuss these situations, we will need some parameters β, γ, γ^* associated with E. We will also need some facts about a certain 4-vertex configuration called a kite. We review these topics in the next two sections.

9 The parameters β, γ, γ^*

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. In this section, we discuss some parameters β, γ, γ^* associated with E.

By [2, p. 283] the scalars

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{18}$$

are equal and independent of i for $2 \leqslant i \leqslant D-1$. We denote this common value by $\beta+1$. By [2, p. 283] there exist real numbers γ, γ^* such that both

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}, \qquad \gamma^* = \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1} \tag{19}$$

for $1 \leqslant i \leqslant D-1$. The recurrences (19) can be solved in closed form. We will focus on the sequence $\{\theta^*_i\}_{i=0}^D$; the sequence $\{\theta_i\}_{i=0}^D$ is similar. Let $\mathbb{C}$ denote the field of complex numbers. There exists $0 \neq q \in \mathbb{C}$ such that $\beta = q + q^{-1}$. Note that $q = 1$ iff $\beta = 2$, and $q = -1$ iff $\beta = -2$. By [2, p. 286] we have

case	θ^*_i closed form	γ^*
$\beta \neq \pm 2$	$\theta^*_i = a + bq^i + cq^{-i}$	$(2-\beta)a$
$\beta = 2$	$\theta^*_i = a + bi + ci^2$	$2c$
$\beta = -2$	$\theta^*_i = a + b(-1)^i + ci(-1)^i$	$4a$

In the above table, the a, b, c are appropriate complex numbers. The case $\gamma^* = 0$ becomes important later in the paper. We now examine this case.

Lemma 34. *We refer to the above table.*

(i) *Assume that $\beta \neq 2$. Then $\gamma^* = 0$ if and only if $a = 0$.*

(ii) *Assume that $\beta = 2$. Then $\gamma^* = 0$ if and only if $c = 0$.*

Proof. Immediate from the above table. □

A theorem of Leonard [3, p. 260] gives detailed formulas for the intersection numbers and Krein parameters of Γ. In [45, Section 20] these formulas are derived using the theory of Leonard pairs. Later in the paper we will invoke the formulas, using the notation of [45, Section 20]. The details of the formulas depend on the case of β shown in the table above Lemma 34. For each case, there are some subcases as shown in the table below.

case	subcases
$\beta \neq \pm 2$	q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, affine q-Krawtchouk, dual q-Krawtchouk
$\beta = 2$	Racah, Hahn, dual Hahn, Krawtchouk
$\beta = -2$	Bannai/Ito

Remark 35. In the theory of Leonard pairs, for the case $\beta \neq \pm 2$ there is a subcase called type IA in [3, p. 260] and quantum q-Krawtchouk in [45, Example 20.4]. We did not include this subcase in the above table, because the subcase does not occur for Q-polynomial distance-regular graphs [15, Proposition 5.8].

10 Kites

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. In this section we discuss a certain 4-vertex configuration in Γ, called a kite. We also explain what it means for Γ to be reinforced.

Definition 36. (See [41, Section 1].) For $2 \leqslant i \leqslant D$, an *$i$-kite* in Γ is a 4-tuple of vertices (x, y, z, w) such that

$$\partial(x,y) = i, \qquad \partial(x,z) = 1, \qquad \partial(y,z) = i-1,$$
$$\partial(x,w) = 1, \qquad \partial(y,w) = i-1, \qquad \partial(z,w) = 1.$$

Definition 37. For $2 \leqslant i \leqslant D$ define

$$z_i = \frac{\text{number of } i\text{-kites in } \Gamma}{|X| k_i c_i}.$$

We call z_i the *ith kite number* of Γ.

Shortly we will give a combinatorial interpretation of z_i.

Definition 38. Pick $2 \leqslant i \leqslant D$ and $x, y, z \in X$ such that

$$\partial(x,y) = i, \qquad \partial(x,z) = 1, \qquad \partial(y,z) = i-1.$$

Define

$$\zeta_i(x,y,z) = |\Gamma(x) \cap \Gamma_{i-1}(y) \cap \Gamma(z)|.$$

Note that $\zeta_i(x,y,z)$ is the number of vertices $w \in X$ such that (x,y,z,w) is an i-kite. We call ζ_i the *ith kite function.*

Lemma 39. *Referring to Definition 38, the scalar* $\zeta_i(x,y,z)$ *is an integer and* $0 \leqslant \zeta_i(x,y,z) \leqslant a_1$.

Proof. By construction and since $a_1 = |\Gamma(x) \cap \Gamma(z)|$. □

Note 40. For $2 \leqslant i \leqslant D$ the scalar z_i has the following combinatorial interpretation. Let Ω_i denote the set of 3-tuples of vertices (x,y,z) such that

$$\partial(x,y) = i, \qquad \partial(x,z) = 1, \qquad \partial(y,z) = i-1.$$

Note that $|\Omega_i| = |X|k_i c_i$. We have

$$z_i = \frac{\sum_{(x,y,z)\in\Omega_i} \zeta_i(x,y,z)}{|\Omega_i|}.$$

In other words, z_i is the average value of $\zeta_i(x,y,z)$ over all $(x,y,z) \in \Omega_i$. We have $0 \leqslant z_i \leqslant a_1$ in view of Lemma 39.

Lemma 41. (See [41, Theorem 2.11].) *For* $2 \leqslant i \leqslant D$ *we have* $z_i = z_2\alpha_i + a_1\beta_i$, *where*

$$\alpha_i = \frac{(\theta^*_1 - \theta^*_2)(\theta^*_0 + \theta^*_1 - \theta^*_{i-1} - \theta^*_i)}{(\theta^*_0 - \theta^*_2)(\theta^*_{i-1} - \theta^*_i)}, \tag{20}$$

$$\beta_i = \frac{(\theta^*_0 - \theta^*_1)(\theta^*_2 - \theta^*_i) - (\theta^*_1 - \theta^*_2)(\theta^*_1 - \theta^*_{i-1})}{(\theta^*_0 - \theta^*_2)(\theta^*_{i-1} - \theta^*_i)}. \tag{21}$$

We mention some handy facts about the scalars $\{\alpha_i\}_{i=2}^D$, $\{\beta_i\}_{i=2}^D$.

Lemma 42. *We have*

$$\alpha_i + \beta_i = \frac{\theta^*_1 - \theta^*_i}{\theta^*_{i-1} - \theta^*_i} \qquad (2 \leqslant i \leqslant D).$$

Proof. Use (20), (21). □

Lemma 43. *For* $2 \leqslant i \leqslant D-1$ *we have*

$$\alpha_i\alpha_{i+1} = \frac{(\beta+2)(\theta^*_1 - \theta^*_2)^2(\theta^*_0 - \theta^*_i)(\theta^*_1 - \theta^*_i)}{(\theta^*_0 - \theta^*_2)^2(\theta^*_{i-1} - \theta^*_i)(\theta^*_i - \theta^*_{i+1})}.$$

Proof. Use (20) and the table above Lemma 34. □

In Note 40 we discussed some averages. Next, we refine these averages.

Definition 44. Let $2 \leqslant i \leqslant D$ and $x, y \in X$ at distance $\partial(x,y) = i$. Note that $c_i = |\Gamma(x) \cap \Gamma_{i-1}(y)|$. Define

$$\zeta_i(x,y,*) = c_i^{-1} \sum_{z \in \Gamma(x)\cap\Gamma_{i-1}(y)} \zeta_i(x,y,z).$$

In other words, $\zeta_i(x,y,*)$ is the average value of $\zeta_i(x,y,z)$ over all $z \in \Gamma(x) \cap \Gamma_{i-1}(y)$.

In the next result, we clarify the meaning of $\zeta_i(x, y, *)$.

Lemma 45. *Referring to Definition 44, the scalar $\zeta_i(x, y, *)$ is the average valency of the induced subgraph $\Gamma(x) \cap \Gamma_{i-1}(y)$.*

Proof. By the last sentence in Definition 44. □

Let us emphasize a few points.

Lemma 46. *Let $2 \leqslant i \leqslant D$ and $x, y \in X$ at distance $\partial(x, y) = i$. The following are equivalent:*

(i) *for $z \in \Gamma(x) \cap \Gamma_{i-1}(y)$ the integer $\zeta_i(x, y, z)$ is independent of z;*

(ii) *$\zeta_i(x, y, z) = \zeta_i(x, y, *)$ for all $z \in \Gamma(x) \cap \Gamma_{i-1}(y)$;*

(iii) *the induced subgraph $\Gamma(x) \cap \Gamma_{i-1}(y)$ is regular.*

Proof. By Definitions 38, 44 and Lemma 45. □

Lemma 47. *For $2 \leqslant i \leqslant D$,*

$$z_i = \frac{1}{|X| k_i} \sum_{\substack{x,y \in X \\ \partial(x,y)=i}} \zeta_i(x, y, *).$$

Proof. By Note 40 and Definition 44. □

Pick an integer i $(2 \leqslant i \leqslant D)$. It could happen that $\zeta_i(x, y, *)$ is independent of x, y $\big(x, y \in X, \partial(x, y) = i\big)$.

Lemma 48. *For $2 \leqslant i \leqslant D$ the following are equivalent:*

(i) *$\zeta_i(x, y, *)$ is independent of x, y $\big(x, y \in X, \partial(x, y) = i\big)$;*

(ii) *$\zeta_i(x, y, *) = z_i$ for all $x, y \in X$ at distance $\partial(x, y) = i$.*

Proof. By Lemma 47. □

Recall the notion of distance-transitivity from [7, p. 136].

Lemma 49. *Assume that Γ is distance-transitive. Then for $2 \leqslant i \leqslant D$ the equivalent conditions* (i), (ii) *hold in Lemma 48.*

Proof. Routine. □

We have a comment.

Lemma 50. Pick $2 \leqslant i \leqslant D$ and $x, y, z \in X$ such that

$$\partial(x,y) = i, \qquad \partial(x,z) = 1, \qquad \partial(y,z) = i-1.$$

Then

$$a_1 - \zeta_i(x,y,z) = |\Gamma(x) \cap \Gamma_i(y) \cap \Gamma(z)|.$$

Proof. Routine using Definition 38. □

Definition 51. Let $2 \leqslant i \leqslant D$ and $y, z \in X$ at distance $\partial(y,z) = i-1$. Note that $b_{i-1} = |\Gamma_i(y) \cap \Gamma(z)|$. Define

$$\zeta_i(*,y,z) = \frac{1}{b_{i-1}} \sum_{x \in \Gamma_i(y) \cap \Gamma(z)} \zeta_i(x,y,z).$$

In other words, $\zeta_i(*,y,z)$ is the average value of $\zeta_i(x,y,z)$ over all $x \in \Gamma_i(y) \cap \Gamma(z)$.

In the next result, we clarify the meaning of $\zeta_i(*,y,z)$.

Lemma 52. *Referring to Definition 51, the scalar* $a_1 - \zeta_i(*,y,z)$ *is equal to the average valency of the induced subgraph* $\Gamma_i(y) \cap \Gamma(z)$.

Proof. By Lemma 50 and the last sentence in Definition 51. □

Let us emphasize a few points.

Lemma 53. *Let* $2 \leqslant i \leqslant D$ *and* $y, z \in X$ *at distance* $\partial(y,z) = i-1$. *The following are equivalent:*

(i) *for* $x \in \Gamma_i(y) \cap \Gamma(z)$ *the integer* $\zeta_i(x,y,z)$ *is independent of* x*;*

(ii) $\zeta_i(x,y,z) = \zeta_i(*,y,z)$ *for all* $x \in \Gamma_i(y) \cap \Gamma(z)$*;*

(iii) *the induced subgraph* $\Gamma_i(y) \cap \Gamma(z)$ *is regular.*

Proof. By Definition 51 and Lemma 52. □

Lemma 54. *For* $2 \leqslant i \leqslant D$,

$$z_i = \frac{1}{|X| k_{i-1}} \sum_{\substack{y,z \in X \\ \partial(y,z) = i-1}} \zeta_i(*,y,z).$$

Proof. By Note 40 and Definition 51, along with the fact that $k_i c_i = k_{i-1} b_{i-1}$. □

Pick an integer i $(2 \leqslant i \leqslant D)$. It could happen that $\zeta_i(*,y,z)$ is independent of y, z $\big(y, z \in X, \partial(y,z) = i-1\big)$.

Lemma 55. *For* $2 \leqslant i \leqslant D$ *the following are equivalent:*

(i) $\zeta_i(*, y, z)$ *is independent of* y, z $\big(y, z \in X, \partial(y, z) = i - 1\big)$*;*

(ii) $\zeta_i(*, y, z) = z_i$ *for all* $y, z \in X$ *at distance* $\partial(y, z) = i - 1$.

Proof. By Lemma 54. □

Lemma 56. *Assume that* Γ *is distance-transitive. Then for* $2 \leqslant i \leqslant D$ *the equivalent conditions* (i), (ii) *hold in Lemma 55.*

Proof. Routine. □

Definition 57. The graph Γ is said to be *reinforced* whenever the following (i), (ii) holds for $2 \leqslant i \leqslant D$:

(i) $\zeta_i(x, y, *)$ is independent of x, y $\big(x, y \in X, \partial(x, y) = i\big)$;

(ii) $\zeta_i(*, y, z)$ is independent of y, z $\big(y, z \in X, \partial(y, z) = i - 1\big)$.

Lemma 58. *Assume that* Γ *is distance-transitive. Then* Γ *is reinforced.*

Proof. By Lemmas 49 and 56. □

Lemma 59. *Assume that the kite function* ζ_i *is constant for* $2 \leqslant i \leqslant D$*. Then* Γ *is reinforced.*

Proof. For $2 \leqslant i \leqslant D$ we have $\zeta_i(x, y, z) = z_i$ for all $x, y, z \in X$ such that

$$\partial(x, y) = i, \qquad \partial(x, z) = 1, \qquad \partial(y, z) = i - 1.$$ □

11 When are $Ex^-_y, E\hat{x}, E\hat{y}$ linearly dependent

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. Pick $2 \leqslant i \leqslant D$ and $x, y \in X$ at distance $\partial(x, y) = i$. In this section, our goal is to obtain a necessary and sufficient condition for the vectors Ex^-_y, $E\hat{x}$, $E\hat{y}$ to be linearly dependent. In view of Lemma 5, we assume that $i \neq D$ if Γ is an antipodal 2-cover. By Lemmas 4, 5 the vectors $E\hat{x}$, $E\hat{y}$ are linearly independent.

Lemma 60. *We have*

$$|X|\langle Ex^-_y, E\hat{x}\rangle = c_i\theta^*_1, \qquad |X|\langle Ex^-_y, E\hat{y}\rangle = c_i\theta^*_{i-1}.$$

Proof. By Lemma 2(i) and the construction. □

Lemma 61. *For* $z \in \Gamma(x) \cap \Gamma_{i-1}(y)$ *we have*

$$|X|\langle Ex^-_y, E\hat{z}\rangle = \theta^*_0 + \zeta_i(x, y, z)\theta^*_1 + \big(c_i - \zeta_i(x, y, z) - 1\big)\theta^*_2. \tag{22}$$

Proof. By construction, $|\Gamma(x) \cap \Gamma_{i-1}(y)| = c_i$. Also by construction, any two distinct vertices in $\Gamma(x) \cap \Gamma_{i-1}(y)$ are at distance 1 or 2 in Γ. By Definition 38, the vertex z is adjacent to exactly $\zeta_i(x, y, z)$ vertices in $\Gamma(x) \cap \Gamma_{i-1}(y)$. The result follows in view of Lemma 2(i). $\square$

Lemma 62. *We have*

$$|X| c_i^{-1} \|E x_y^-\|^2 = \theta_0^* + \zeta_i(x, y, *)\theta_1^* + \big(c_i - \zeta_i(x, y, *) - 1\big)\theta_2^*.$$

Proof. Compute the average of (22) over all $z \in \Gamma(x) \cap \Gamma_{i-1}(y)$. Evaluate the result using Definitions 13, 44. $\square$

For notational convenience, define

$$r_i = c_i \frac{\theta_0^*\theta_1^* - \theta_{i-1}^*\theta_i^*}{\theta_0^{*2} - \theta_i^{*2}}, \qquad s_i = c_i \frac{\theta_0^*\theta_{i-1}^* - \theta_1^*\theta_i^*}{\theta_0^{*2} - \theta_i^{*2}}. \tag{23}$$

Lemma 63. *For $r, s \in \mathbb{R}$ the following are equivalent:*

(i) $E x_y^- - rE\hat{x} - sE\hat{y}$ *is orthogonal to each of* $E\hat{x}$, $E\hat{y}$*;*

(ii) *both*

$$c_i\theta_1^* = r\theta_0^* + s\theta_i^*, \qquad c_i\theta_{i-1}^* = r\theta_i^* + s\theta_0^*.$$

(iii) $r = r_i$ *and* $s = s_i$.

Proof. (i) ⇔ (ii) Use Lemma 2(i),(ii) and Lemma 60.
(ii) ⇔ (iii) By linear algebra and $\theta_0^{*2} \neq \theta_i^{*2}$. $\square$

Definition 64. Define the real number

$$z_i^- = \frac{r_i\theta_1^* + s_i\theta_{i-1}^* - \theta_0^* - (c_i - 1)\theta_2^*}{\theta_1^* - \theta_2^*}.$$

Lemma 65. *We have*

$$\|E x_y^- - r_i E\hat{x} - s_i E\hat{y}\|^2 = |X|^{-1} c_i (\theta_1^* - \theta_2^*)\big(\zeta_i(x, y, *) - z_i^-\big). \tag{24}$$

Proof. By Lemmas 60, 62, 63 each side of (24) is equal to

$$\langle E x_y^- - r_i E\hat{x} - s_i E\hat{y}, E x_y^- \rangle.$$

$\square$

Lemma 66. *We have*

$$\frac{\zeta_i(x, y, *) - z_i^-}{\theta_1^* - \theta_2^*} \geqslant 0. \tag{25}$$

Proof. By Lemma 65. □

Lemma 67. *The following hold.*

(i) *Assume that* $\theta^*_1 > \theta^*_2$. *Then* $\zeta_i(x, y, *) \geqslant z^-_i$.

(ii) *Assume that* $\theta^*_1 < \theta^*_2$. *Then* $\zeta_i(x, y, *) \leqslant z^-_i$.

Proof. By Lemma 66. □

Proposition 68. *The following are equivalent:*

(i) *equality holds in* (25)*;*

(ii) $Ex^-_y = r_i E\hat{x} + s_i E\hat{y}$*;*

(iii) *the vectors* Ex^-_y, $E\hat{x}$, $E\hat{y}$ *are linearly dependent;*

(iv) $\zeta_i(x, y, z) = z^-_i$ *for all* $z \in \Gamma(x) \cap \Gamma_{i-1}(y)$*;*

(v) $\zeta_i(x, y, *) = z^-_i$.

Assume that (i)–(v) *hold. Then* z^-_i *is an integer and* $0 \leqslant z^-_i \leqslant a_1$.

Proof. (i) ⇔ (ii) By Lemma 65.

(ii) ⇒ (iii) Clear.

(iii) ⇒ (ii) The vectors $E\hat{x}$, $E\hat{y}$ are linearly independent, so there exist $r, s \in \mathbb{R}$ such that $Ex^-_y = rE\hat{x} + sE\hat{y}$. The vector $Ex^-_y - rE\hat{x} - sE\hat{y}$ is equal to zero, so it is orthogonal to $E\hat{x}$ and $E\hat{y}$. We have $r = r_i$ and $s = s_i$ by Lemma 63.

(ii) ⇒ (iv) For $z \in \Gamma(x) \cap \Gamma_{i-1}(y)$ we take the inner product of $E\hat{z}$ with each side of $Ex^-_y = r_i E\hat{x} + s_i E\hat{y}$; this yields

$$\theta^*_0 + \zeta_i(x, y, z)\theta^*_1 + \big(c_i - \zeta_i(x, y, z) - 1\big)\theta^*_2 = r_i\theta^*_1 + s_i\theta^*_{i-1}.$$

Solve this equation for $\zeta_i(x, y, z)$ to find $\zeta_i(x, y, z) = z^-_i$.

(iv) ⇒ (v) By Definition 44.

(v) ⇒ (i) Clear.

We have shown that (i)–(v) are equivalent. We now assume that (i)–(v) hold. By (iv) and Lemma 39, the scalar z^-_i is an integer and $0 \leqslant z^-_i \leqslant a_1$. □

Lemma 69. *Referring to Proposition 68, assume that the equivalent conditions* (i)–(v) *hold. Then for all integers* j $(i \leqslant j \leqslant D)$ *we have*

$$\theta^*_{j-1} = \theta^*_j \frac{\theta^*_0\theta^*_1 - \theta^*_{i-1}\theta^*_i}{\theta^{*2}_0 - \theta^{*2}_i} + \theta^*_{j-i} \frac{\theta^*_0\theta^*_{i-1} - \theta^*_1\theta^*_i}{\theta^{*2}_0 - \theta^{*2}_i}; \tag{26}$$

$$\theta^*_{j-i+1} = \theta^*_{j-i} \frac{\theta^*_0\theta^*_1 - \theta^*_{i-1}\theta^*_i}{\theta^{*2}_0 - \theta^{*2}_i} + \theta^*_j \frac{\theta^*_0\theta^*_{i-1} - \theta^*_1\theta^*_i}{\theta^{*2}_0 - \theta^{*2}_i}. \tag{27}$$

Proof. To obtain (26), pick $w \in X$ such that $\partial(x, w) = j$ and $\partial(y, w) = j - i$. Take the inner product of $E\hat{w}$ with each term in $Ex_y^- = r_i E\hat{x} + s_i E\hat{y}$. Evaluate the resulting equation using Lemma 2(i) and (23). To obtain (27), repeat the calculation using $\partial(x, w) = j - i$ and $\partial(y, w) = j$. □

Lemma 70. *Assume that* $2 \leqslant i \leqslant D - 1$. *Then the following are equivalent:*

(i) *the equations* (26), (27) *hold for all integers* j $(i \leqslant j \leqslant D)$;

(ii) $\gamma^* = 0$.

Proof. Use the forms in the table above Lemma 34. □

Corollary 71. *Assume that* $2 \leqslant i \leqslant D - 1$ *and* $\gamma^* \neq 0$. *Then* Ex_y^-, $E\hat{x}$, $E\hat{y}$ *are linearly independent. Moreover,*

$$\frac{\zeta_i(x, y, *) - z_i^-}{\theta_1^* - \theta_2^*} > 0.$$

Proof. By Proposition 68 and Lemmas 69, 70. □

12 When are $Ex_y^+, E\hat{x}, E\hat{y}$ linearly dependent

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. Pick $1 \leqslant i \leqslant D - 1$ and $x, y \in X$ at distance $\partial(x, y) = i$. In this section, our goal is to obtain a necessary and sufficient condition for the vectors Ex_y^+, $E\hat{x}$, $E\hat{y}$ to be linearly dependent.

By Lemmas 4, 5 the vectors $E\hat{x}, E\hat{y}$ are linearly independent.

Lemma 72. *We have*

$$|X|\langle Ex_y^+, E\hat{x}\rangle = b_i\theta_1^*, \qquad |X|\langle Ex_y^+, E\hat{y}\rangle = b_i\theta_{i+1}^*.$$

Proof. By Lemma 2(i) and the construction. □

Lemma 73. *For* $z \in \Gamma(x) \cap \Gamma_{i+1}(y)$ *we have*

$$|X|\langle Ex_y^+, E\hat{z}\rangle = \theta_0^* + \big(a_1 - \zeta_{i+1}(z, y, x)\big)\theta_1^* + \big(b_i - 1 - a_1 + \zeta_{i+1}(z, y, x)\big)\theta_2^*. \tag{28}$$

Proof. By construction, $|\Gamma(x) \cap \Gamma_{i+1}(y)| = b_i$. Also by construction, any two distinct vertices in $\Gamma(x) \cap \Gamma_{i+1}(y)$ are at distance 1 or 2 in Γ. By Lemma 50, the vertex z is adjacent to exactly $a_1 - \zeta_{i+1}(z, y, x)$ vertices in $\Gamma(x) \cap \Gamma_{i+1}(y)$. The result follows in view of Lemma 2(i). □

Lemma 74. *We have*

$$|X|b_i^{-1}\|Ex_y^+\|^2 = \theta_0^* + \big(a_1 - \zeta_{i+1}(*, y, x)\big)\theta_1^* + \big(b_i - 1 - a_1 + \zeta_{i+1}(*, y, x)\big)\theta_2^*.$$

Proof. Compute the average of (28) over all $z \in \Gamma(x) \cap \Gamma_{i+1}(y)$. Evaluate the resulting equation using Definitions 13, 51. □

For notational convenience, define

$$R_i = b_i \frac{\theta^*_0\theta^*_1 - \theta^*_{i+1}\theta^*_i}{\theta^{*2}_0 - \theta^{*2}_i}, \qquad S_i = b_i \frac{\theta^*_0\theta^*_{i+1} - \theta^*_1\theta^*_i}{\theta^{*2}_0 - \theta^{*2}_i}. \tag{29}$$

Lemma 75. *For $R, S \in \mathbb{R}$ the following are equivalent:*

(i) $Ex^+_y - RE\hat{x} - SE\hat{y}$ *is orthogonal to each of* $E\hat{x}$, $E\hat{y}$*;*

(ii) *both*

$$b_i\theta^*_1 = R\theta^*_0 + S\theta^*_i, \qquad b_i\theta^*_{i+1} = R\theta^*_i + S\theta^*_0.$$

(iii) $R = R_i$ *and* $S = S_i$.

Proof. Similar to the proof of Lemma 63. □

Definition 76. Define the real number

$$z^+_{i+1} = \frac{\theta^*_0 + a_1\theta^*_1 + (b_i - 1 - a_1)\theta^*_2 - R_i\theta^*_1 - S_i\theta^*_{i+1}}{\theta^*_1 - \theta^*_2}.$$

Lemma 77. *We have*

$$\|Ex^+_y - R_iE\hat{x} - S_iE\hat{y}\|^2 = |X|^{-1}b_i(\theta^*_1 - \theta^*_2)\big(z^+_{i+1} - \zeta_{i+1}(*, y, x)\big). \tag{30}$$

Proof. By Lemmas 72, 74, 75 each side of (30) is equal to

$$\langle Ex^+_y - R_iE\hat{x} - S_iE\hat{y}, Ex^+_y\rangle.$$ □

Lemma 78. *We have*

$$\frac{z^+_{i+1} - \zeta_{i+1}(*, y, x)}{\theta^*_1 - \theta^*_2} \geqslant 0. \tag{31}$$

Proof. By Lemma 77. □

Lemma 79. *The following hold.*

(i) *Assume that* $\theta^*_1 > \theta^*_2$. *Then* $\zeta_{i+1}(*, y, x) \leqslant z^+_{i+1}$.

(ii) *Assume that* $\theta^*_1 < \theta^*_2$. *Then* $\zeta_{i+1}(*, y, x) \geqslant z^+_{i+1}$.

Proof. By Lemma 78. □

Proposition 80. *The following are equivalent:*

(i) *equality holds in* (31)*;*

(ii) $Ex_y^+ = R_i E\hat{x} + S_i E\hat{y}$;

(iii) *the vectors* Ex_y^+, $E\hat{x}$, $E\hat{y}$ *are linearly dependent;*

(iv) $\zeta_{i+1}(z, y, x) = z_{i+1}^+$ *for all* $z \in \Gamma(x) \cap \Gamma_{i+1}(y)$;

(v) $\zeta_{i+1}(*, y, x) = z_{i+1}^+$.

Assume that (i)–(v) *hold. Then* z_{i+1}^+ *is an integer and* $0 \leqslant z_{i+1}^+ \leqslant a_1$.

Proof. Similar to the proof of Proposition 68. □

Lemma 81. *Referring to Proposition 80, assume that the equivalent conditions* (i)–(v) *hold. Then for all integers* j $(0 \leqslant j \leqslant i)$ *we have*

$$\theta_{j+1}^* = \theta_j^* \frac{\theta_0^*\theta_1^* - \theta_{i+1}^*\theta_i^*}{\theta_0^{*2} - \theta_i^{*2}} + \theta_{i-j}^* \frac{\theta_0^*\theta_{i+1}^* - \theta_1^*\theta_i^*}{\theta_0^{*2} - \theta_i^{*2}}. \tag{32}$$

Proof. Pick $w \in X$ such that $\partial(x, w) = j$ and $\partial(y, w) = i - j$. Take the inner product of $E\hat{w}$ with each term in $Ex_y^+ = R_i E\hat{x} + S_i E\hat{y}$. Evaluate the resulting equation using Lemma 2(i) and (29). □

Recall the parameter γ^* from (19).

Lemma 82. *Assume that* $2 \leqslant i \leqslant D - 1$. *Then the following are equivalent:*

(i) *the equation* (32) *holds for all integers* j $(0 \leqslant j \leqslant i)$;

(ii) $\gamma^* = 0$.

Proof. Use the forms in the table above Lemma 34. □

Corollary 83. *Assume that* $2 \leqslant i \leqslant D - 1$ *and* $\gamma^* \neq 0$. *Then* Ex_y^+, $E\hat{x}$, $E\hat{y}$ *are linearly independent. Moreover,*

$$\frac{z_{i+1}^+ - \zeta_{i+1}(*, y, x)}{\theta_1^* - \theta_2^*} > 0.$$

Proof. By Proposition 80 and Lemmas 81, 82. □

13 When are $Ex_y^-, Ex_y^+, E\hat{x}, E\hat{y}$ linearly dependent

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. Pick $2 \leqslant i \leqslant D - 1$ and $x, y \in X$ at distance $\partial(x, y) = i$. In this section, our goal is to obtain a necessary and sufficient condition for the vectors $Ex_y^-, Ex_y^+, E\hat{x}, E\hat{y}$ to be linearly dependent.

By Lemmas 4, 5 the vectors $E\hat{x}, E\hat{y}$ are linearly independent. Recall the scalars r_i, s_i from (23). By Lemma 63 the vector $Ex_y^- - r_i E\hat{x} - s_i E\hat{y}$ is orthogonal to each of $E\hat{x}, E\hat{y}$. Recall the scalars R_i, S_i from (29). By Lemma 75 the vector $Ex_y^+ - R_i E\hat{x} - S_i E\hat{y}$ is orthogonal to each of $E\hat{x}, E\hat{y}$.

Lemma 84. *The following are equivalent:*

(i) *the vectors* $Ex_y^-, Ex_y^+, E\hat{x}, E\hat{y}$ *are linearly dependent;*

(ii) *the vectors* $Ex_y^- - r_iE\hat{x} - s_iE\hat{y}$ *and* $Ex_y^+ - R_iE\hat{x} - S_iE\hat{y}$ *are linearly dependent.*

Proof. By the comments above the lemma statement. □

Next we compute the matrix of inner products for $Ex_y^- - r_iE\hat{x} - s_iE\hat{y}$ and $Ex_y^+ - R_iE\hat{x} - S_iE\hat{y}$. We computed $\|Ex_y^- - r_iE\hat{x} - s_iE\hat{y}\|^2$ in Lemma 65. We computed $\|Ex_y^+ - R_iE\hat{x} - S_iE\hat{y}\|^2$ in Lemma 77.

Lemma 85. *We have*

$$\begin{aligned}
&|X|\langle Ex_y^- - r_iE\hat{x} - s_iE\hat{y}, Ex_y^+ - R_iE\hat{x} - S_iE\hat{y}\rangle \\
&\quad = c_ib_i\theta_2^* - r_ib_i\theta_1^* - s_ib_i\theta_{i+1}^* = c_ib_i\theta_2^* - R_ic_i\theta_1^* - S_ic_i\theta_{i-1}^* \\
&\quad = c_ib_i\left(\theta_2^* - \frac{\theta_0^*\theta_1^{*2} - \theta_1^*\theta_{i-1}^*\theta_i^* + \theta_0^*\theta_{i-1}^*\theta_{i+1}^* - \theta_1^*\theta_i^*\theta_{i+1}^*}{\theta_0^{*2} - \theta_i^{*2}}\right).
\end{aligned}$$

Proof. Use (23) and Lemma 63, or else (29) and Lemma 75. □

Lemma 86. *We have*

$$\theta_2^* - \frac{\theta_0^*\theta_1^{*2} - \theta_1^*\theta_{i-1}^*\theta_i^* + \theta_0^*\theta_{i-1}^*\theta_{i+1}^* - \theta_1^*\theta_i^*\theta_{i+1}^*}{\theta_0^{*2} - \theta_i^{*2}} = \gamma^*\frac{\theta_i^* - \theta_1^*}{\theta_i^* + \theta_0^*}. \tag{33}$$

Proof. Use the forms in the table above Lemma 34. □

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\|Ex_y^- - r_iE\hat{x} - s_iE\hat{y}\|^2\|Ex_y^+ - R_iE\hat{x} - S_iE\hat{y}\|^2 \\
&\qquad \geqslant \langle Ex_y^- - r_iE\hat{x} - s_iE\hat{y}, Ex_y^+ - R_iE\hat{x} - S_iE\hat{y}\rangle^2.
\end{aligned} \tag{34}$$

Equality holds in (34) if and only if the following vectors are linearly dependent:

$$Ex_y^- - r_iE\hat{x} - s_iE\hat{y}, \qquad\qquad Ex_y^+ - R_iE\hat{x} - S_iE\hat{y}.$$

Lemma 87. *We have*

$$\big(\zeta_i(x,y,*) - z_i^-\big)\big(z_{i+1}^+ - \zeta_{i+1}(*,y,x)\big) \geqslant c_ib_i\left(\frac{\gamma^*}{\theta_1^* - \theta_2^*}\,\frac{\theta_i^* - \theta_1^*}{\theta_i^* + \theta_0^*}\right)^2. \tag{35}$$

Proof. Evaluate the terms in (34) using Lemmas 65, 77, 85, 86. □

Proposition 88. *The following* (i)–(iii) *are equivalent:*

(i) *equality holds in* (35)*;*

(ii) *the vectors* $Ex_y^-, Ex_y^+, E\hat{x}, E\hat{y}$ *are linearly dependent;*

(iii) *the vectors* $Ex_y^- - r_i E\hat{x} - s_i E\hat{y}$ *and* $Ex_y^+ - R_i E\hat{x} - S_i E\hat{y}$ *are linearly dependent.*

Proof. By Lemma 84 and the comments above Lemma 87. $\square$

Assume for the moment that the equivalent conditions (i)–(iii) hold in Proposition 88. Our next goal is to find the dependency between the vectors in part (iii).

Until further notice, assume that $\gamma^* = 0$. By Lemmas 85, 86 the following vectors are orthogonal:

$$Ex_y^- - r_i E\hat{x} - s_i E\hat{y}, \qquad Ex_y^+ - R_i E\hat{x} - S_i E\hat{y}. \tag{36}$$

The inequality (35) becomes

$$\big(\zeta_i(x, y, *) - z_i^-\big)\big(z_{i+1}^+ - \zeta_{i+1}(*, y, x)\big) \geqslant 0.$$

Lemma 89. *Assume that* $\gamma^* = 0$*, and the equivalent conditions* (i)–(iii) *hold in Proposition 88. Then at least one of the vectors* (36) *is equal to zero.*

Proof. The vectors (36) are orthogonal and linearly dependent. $\square$

Lemma 90. *Assume that* $\gamma^* = 0$*, and the equivalent conditions* (i)–(iii) *hold in Proposition 88. Then*

$$Ex_y^-, Ex_y^+ \in \operatorname{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}.$$

Proof. By (11) and Lemma 89. $\square$

For the rest of this section, assume that $\gamma^* \neq 0$. By Corollary 71 the vectors $Ex_y^-, E\hat{x}, E\hat{y}$ are linearly independent. By Corollary 83 the vectors $Ex_y^+, E\hat{x}, E\hat{y}$ are linearly independent.

Lemma 91. *Assume that* $\gamma^* \neq 0$*, and the equivalent conditions* (i)–(iii) *hold in Proposition 88. Then*

$$Ex_y^- - r_i E\hat{x} - s_i E\hat{y} = \lambda_i\big(Ex_y^+ - R_i E\hat{x} - S_i E\hat{y}\big), \tag{37}$$

where

$$\frac{\lambda_i}{\theta_1^* - \theta_2^*} = \frac{\zeta_i(x, y, *) - z_i^-}{\theta_i^* - \theta_1^*}\,\frac{\theta_i^* + \theta_0^*}{\gamma^* b_i}, \qquad \frac{\lambda_i^{-1}}{\theta_1^* - \theta_2^*} = \frac{z_{i+1}^+ - \zeta_{i+1}(*, y, x)}{\theta_i^* - \theta_1^*}\,\frac{\theta_i^* + \theta_0^*}{\gamma^* c_i}.$$

Proof. By Proposition 88(ii) there exists $\lambda_i \in \mathbb{R}$ such that (37) holds. To obtain λ_i, take the inner product of each side of (37) with $Ex_y^- - r_i E\hat{x} - s_i E\hat{y}$ or $Ex_y^+ - R_i E\hat{x} - S_i E\hat{y}$. $\square$

Lemma 92. *Assume that* $\gamma^* \neq 0$*, and the equivalent conditions* (i)–(iii) *hold in Proposition 88. Assume that the scalar* λ_i *from Lemma 91 satisfies*

$$\lambda_i \neq \frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}.$$

Then

$$Ex_y^-, Ex_y^+ \in \operatorname{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}.$$

Proof. By (11) and (37). □

Lemma 93. *Assume that* $\gamma^* \neq 0$, *and the equivalent conditions* (i)–(iii) *hold in Proposition 88.*

(i) *For all* $z \in \Gamma(x) \cap \Gamma_{i-1}(y)$,

$$\zeta_i(x, y, z) = \zeta_i(x, y, *).$$

(ii) *For all* $z \in \Gamma(x) \cap \Gamma_{i+1}(y)$,

$$\zeta_{i+1}(z, y, x) = \zeta_{i+1}(*, y, x).$$

Proof. (i) Take the inner product of $E\hat{z}$ with each side of (37). Evaluate the result using Lemmas 2, 61 and Definition 64.

(ii) Take the inner product of $E\hat{z}$ with each side of (37). Evaluate the result using Lemmas 2, 73 and Definition 76. □

By a *μ-graph* for Γ we mean the subgraph induced on $\Gamma(u) \cap \Gamma(v)$, where $u, v \in X$ are at distance $\partial(u, v) = 2$.

Corollary 94. *Assume that* $\gamma^* \neq 0$ *and the set* $\{E\hat{x} | x \in X\}$ *is Norton-balanced. Then every μ-graph of* Γ *is regular.*

Proof. Set $i = 2$ in Lemma 93(i), and interpret the result using Lemma 46. □

14 When Γ is reinforced

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. Throughout this section, we assume that Γ is reinforced in the sense of Definition 57. Under this assumption, we will describe the main results of the previous three sections. We will treat separately the cases $\gamma^* = 0$ and $\gamma^* \neq 0$.

Throughout this section, fix $2 \leqslant i \leqslant D - 1$ and $x, y \in X$ at distance $\partial(x, y) = i$. Recall the scalars r_i, s_i from (23). By Lemma 63 the vector $Ex_y^- - r_i E\hat{x} - s_i E\hat{y}$ is orthogonal to each of $E\hat{x}, E\hat{y}$. Recall the scalars R_i, S_i from (29). By Lemma 75 the vector $Ex_y^+ - R_i E\hat{x} - S_i E\hat{y}$ is orthogonal to each of $E\hat{x}, E\hat{y}$.

First assume that $\gamma^* = 0$. The following vectors are orthogonal:

$$Ex_y^- - r_i E\hat{x} - s_i E\hat{y}, \qquad\qquad Ex_y^+ - R_i E\hat{x} - S_i E\hat{y}.$$

We have

$$\frac{z_i - z_i^-}{\theta_1^* - \theta_2^*} \geqslant 0,$$

with equality iff $Ex_y^- = r_i E\hat{x} + s_i E\hat{y}$ iff Ex_y^-, $E\hat{x}$, $E\hat{y}$ are linearly dependent. We also have

$$\frac{z_{i+1}^+ - z_{i+1}}{\theta_1^* - \theta_2^*} \geqslant 0,$$

with equality iff $Ex_y^+ = R_i E\hat{x} + S_i E\hat{y}$ iff Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent. We have

$$(z_i - z_i^-)(z_{i+1}^+ - z_{i+1}) \geqslant 0, \tag{38}$$

with equality iff Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent iff the following are linearly dependent:

$$Ex_y^- - r_i E\hat{x} - s_i E\hat{y}, \qquad Ex_y^+ - R_i E\hat{x} - S_i E\hat{y}. \tag{39}$$

In this case, at least one of (39) is zero and

$$Ex_y^-, Ex_y^+ \in \mathrm{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\}.$$

For the rest of this section, assume that $\gamma^* \neq 0$. We have

$$\frac{z_i - z_i^-}{\theta_1^* - \theta_2^*} > 0, \qquad \frac{z_{i+1}^+ - z_{i+1}}{\theta_1^* - \theta_2^*} > 0.$$

The vectors Ex_y^-, $E\hat{x}$, $E\hat{y}$ are linearly independent and the vectors Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly independent. We have

$$\bigl(z_i - z_i^-\bigr)\bigl(z_{i+1}^+ - z_{i+1}\bigr) \geqslant c_i b_i \left(\frac{\gamma^*}{\theta_1^* - \theta_2^*} \, \frac{\theta_i^* - \theta_1^*}{\theta_i^* + \theta_0^*} \right)^2, \tag{40}$$

with equality iff Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent iff the following are linearly dependent:

$$Ex_y^- - r_i E\hat{x} - s_i E\hat{y}, \qquad Ex_y^+ - R_i E\hat{x} - S_i E\hat{y}.$$

In this case

$$Ex_y^- - r_i E\hat{x} - s_i E\hat{y} = \lambda_i \bigl(Ex_y^+ - R_i E\hat{x} - S_i E\hat{y}\bigr), \tag{41}$$

where

$$\frac{\lambda_i}{\theta_1^* - \theta_2^*} = \frac{z_i - z_i^-}{\theta_i^* - \theta_1^*} \, \frac{\theta_i^* + \theta_0^*}{\gamma^* b_i}, \qquad \frac{\lambda_i^{-1}}{\theta_1^* - \theta_2^*} = \frac{z_{i+1}^+ - z_{i+1}}{\theta_i^* - \theta_1^*} \, \frac{\theta_i^* + \theta_0^*}{\gamma^* c_i}. \tag{42}$$

Also in this case, we have

$$Ex_y^-, Ex_y^+ \in \mathrm{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\},$$

provided that

$$\lambda_i \neq \frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}.$$

15 The polynomials $\Phi_i(\lambda)$

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. In this section, we introduce some polynomials $\Phi_i(\lambda)$ and use them to determine when the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Recall the scalars α_i, β_i from Lemma 41.

Definition 95. Let λ denote an indeterminate. For $2 \leqslant i \leqslant D-1$ define a polynomial

$$\Phi_i(\lambda) = u_i\lambda^2 + v_i\lambda + w_i,$$

where

$$\begin{aligned} u_i &= -\alpha_i\alpha_{i+1}, \\ v_i &= \alpha_i\big(z^+_{i+1} - a_1\beta_{i+1}\big) - \alpha_{i+1}\big(a_1\beta_i - z^-_i\big), \\ w_i &= \big(a_1\beta_i - z^-_i\big)\big(z^+_{i+1} - a_1\beta_{i+1}\big) - c_ib_i\left(\frac{\gamma^*}{\theta^*_1 - \theta^*_2}\frac{\theta^*_i - \theta^*_1}{\theta^*_i + \theta^*_0}\right)^2. \end{aligned}$$

The next result indicates why the polynomials $\Phi_i(\lambda)$ are of interest.

Lemma 96. *For $2 \leqslant i \leqslant D-1$,*

$$\Phi_i(z_2) = \big(z_i - z^-_i\big)\big(z^+_{i+1} - z_{i+1}\big) - c_ib_i\left(\frac{\gamma^*}{\theta^*_1 - \theta^*_2}\frac{\theta^*_i - \theta^*_1}{\theta^*_i + \theta^*_0}\right)^2. \tag{43}$$

Proof. By Lemma 41 we have

$$z_i = z_2\alpha_i + a_1\beta_i, \qquad z_{i+1} = z_2\alpha_{i+1} + a_1\beta_{i+1}.$$

Using these equations we eliminate z_i, z_{i+1} from the right-hand side of (43), and evaluate the result using Definition 95. □

Proposition 97. *Assume that Γ is reinforced. Then for $2 \leqslant i \leqslant D-1$,*

$$\Phi_i(z_2) \geqslant 0. \tag{44}$$

Moreover, the following are equivalent:

(i) *equality holds in* (44)*;*

(ii) *for all $x, y \in X$ at distance $\partial(x, y) = i$, the vectors Ex^-_y, Ex^+_y, $E\hat{x}$, $E\hat{y}$ are linearly dependent;*

(iii) *there exists $x, y \in X$ at distance $\partial(x, y) = i$ such that Ex^-_y, Ex^+_y, $E\hat{x}$, $E\hat{y}$ are linearly dependent.*

Proof. First assume that $\gamma^* = 0$. The inequality (44) follows from (38) and Lemma 96. The equivalence of (i)–(iii) follows from the discussion below (38). Next assume that $\gamma^* \neq 0$. The inequality (44) follows from (40) and Lemma 96. The equivalence of (i)–(iii) follows from the discussion below (40). □

Corollary 98. *Assume that* Γ *is reinforced. Then the following are equivalent:*

(i) *for all* $x, y \in X$ *the vectors* Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ *are linearly dependent;*

(ii) $\Phi_i(z_2) = 0$ *for* $2 \leqslant i \leqslant D-1$.

Proof. By Proposition 97(i),(ii) and since Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent for all $x, y \in X$ with $\partial(x, y) \in \{0, 1, D\}$. □

Next, we describe the Norton-balanced condition in terms of the polynomials $\Phi_i(\lambda)$, under the assumption that Γ is reinforced. We will treat separately the cases $\gamma^* = 0$ and $\gamma^* \neq 0$.

Proposition 99. *Assume that* $\gamma^* = 0$ *and* Γ *is reinforced. Then the following are equivalent:*

(i) *the set* $\{E\hat{x} | x \in X\}$ *is Norton-balanced;*

(ii) $\Phi_i(z_2) = 0$ *for* $2 \leqslant i \leqslant D-1$.

Proof. (i) $\Rightarrow$ (ii) By Lemma 32 and Corollary 98.
(ii) $\Rightarrow$ (i) By Corollary 98 and the comment below (39). □

Proposition 100. *Assume that* $\gamma^* \neq 0$ *and* Γ *is reinforced. Then the following are equivalent:*

(i) *the set* $\{E\hat{x} | x \in X\}$ *is Norton-balanced;*

(ii) *for* $2 \leqslant i \leqslant D-1$ *both*

$$\Phi_i(z_2) = 0, \qquad\qquad \lambda_i \neq \frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}, \tag{45}$$

where λ_i *is from* (41).

Proof. (i) $\Rightarrow$ (ii) We have $\Phi_i(z_2) = 0$ by Lemma 32 and Corollary 98. To verify the inequality on the right in (45), we assume that $\lambda_i = (\theta_i^* - \theta_{i+1}^*)/(\theta_{i-1}^* - \theta_i^*)$ and get a contradiction. Pick $x, y \in X$ at distance $\partial(x, y) = i$. Combining (11), (41) we find $E\hat{x} \star E\hat{y} \in \mathrm{Span}\{E\hat{x}, E\hat{y}\}$. By Lemma 23,

$$Ex_y^-, Ex_y^+ \in \mathrm{Span}\{E\hat{x}, E\hat{y}, E\hat{x} \star E\hat{y}\} = \mathrm{Span}\{E\hat{x}, E\hat{y}\}.$$

This contradicts Corollary 71 and Corollary 83. Therefore $\lambda_i \neq (\theta_i^* - \theta_{i+1}^*)/(\theta_{i-1}^* - \theta_i^*)$.
(ii) $\Rightarrow$ (i) By Lemma 23, Corollary 98, and the comment below (42). □

Motivated by Propositions 99 and 100, we next consider how the polynomial $\Phi_i(\lambda)$ depends on i for $2 \leqslant i \leqslant D-1$.

Lemma 101. *If $\beta \neq -2$ then $u_i \neq 0$ for $2 \leqslant i \leqslant D-1$. If $\beta = -2$ then $u_i = 0$ for $2 \leqslant i \leqslant D-1$.*

Proof. By Lemma 43 and Definition 95, along with the fact that $\{\theta^*_j\}_{j=0}^D$ are mutually distinct. □

Lemma 102. *The rank of the following matrix is at most 2:*

$$\begin{pmatrix} u_2 & u_3 & u_4 & \cdots & u_{D-1} \\ v_2 & v_3 & v_4 & \cdots & v_{D-1} \\ w_2 & w_3 & w_4 & \cdots & w_{D-1} \end{pmatrix}.$$

Proof. It suffices to show that for $2 \leqslant h < i < j \leqslant D-1$,

$$\det \begin{pmatrix} u_h & u_i & u_j \\ v_h & v_i & v_j \\ w_h & w_i & w_j \end{pmatrix} = 0. \tag{46}$$

First assume that $\beta = -2$. Then (46) holds since the top row is zero by Lemma 101. Next assume that $\beta \neq -2$. To verify (46) in this case, we refer to the table above Remark 35. We verify (46) for each subcase such that $\beta \neq -2$. For each of these subcases, the verification of (46) is done by evaluating the matrix entries using Definition 95 and the data in [45, Section 20]. □

For $2 \leqslant i \leqslant D-1$, by a *root* of $\Phi_i(\lambda)$ we mean a scalar $\xi \in \mathbb{C}$ such that $\Phi_i(\xi) = 0$. As we investigate these roots, we will treat separately the cases $\beta \neq -2$ and $\beta = -2$.

Lemma 103. *Assume that $\beta \neq -2$. Then for $2 \leqslant i, j \leqslant D-1$ the following hold.*

(i) *Assume that $\Phi_i(\lambda)$, $\Phi_j(\lambda)$ have no roots in common. Then*

$$(u_i w_j - u_j w_i)^2 \neq (v_i w_j - v_j w_i)(u_i v_j - u_j v_i).$$

(ii) *Assume that $\Phi_i(\lambda)$, $\Phi_j(\lambda)$ have a root in common, and $\Phi_i(\lambda)$, $\Phi_j(\lambda)$ are linearly independent. Then*

$$(u_i w_j - u_j w_i)^2 = (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

and $u_i v_j - u_j v_i \neq 0$. The common root is

$$\frac{w_i u_j - w_j u_i}{u_i v_j - u_j v_i}.$$

(iii) *Assume that $\Phi_i(\lambda)$, $\Phi_j(\lambda)$ are linearly dependent. Then both*

$$u_i v_j - u_j v_i = 0, \qquad u_i w_j - u_j w_i = 0.$$

Proof. Write

$$\Phi_i(\lambda) = u_i(\lambda - r)(\lambda - s), \qquad \Phi_j(\lambda) = u_j(\lambda - R)(\lambda - S).$$

We have

$$v_i = -u_i(r+s), \qquad w_i = u_i rs, \qquad v_j = -u_j(R+S), \qquad w_j = u_j RS.$$

We obtain

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i) = u_i^2 u_j^2 (r-R)(r-S)(s-R)(s-S)$$

and

$$u_i v_j - u_j v_i = u_i u_j (r + s - R - S), \qquad w_i u_j - w_j u_i = u_i u_j (rs - RS).$$

Using these comments we routinely obtain the result. □

Lemma 104. *Assume that $\beta \neq -2$. Then for $2 \leqslant h < i < j \leqslant D-1$ the following are equivalent:*

(i) *any two of $\Phi_h(\lambda), \Phi_i(\lambda), \Phi_j(\lambda)$ have a root in common;*

(ii) *there exists $\xi \in \mathbb{C}$ such that $\Phi_h(\xi) = \Phi_i(\xi) = \Phi_j(\xi) = 0$.*

Proof. (i) ⇒ (ii) We assume that (ii) is false, and get a contradiction. There exist mutually distinct $r, s, t \in \mathbb{C}$ such that

$$\Phi_h(\lambda) = u_h(\lambda - r)(\lambda - s), \qquad \Phi_i(\lambda) = u_i(\lambda - s)(\lambda - t), \qquad \Phi_j(\lambda) = u_j(\lambda - t)(\lambda - r).$$

Using these forms we obtain

$$\det \begin{pmatrix} u_h & u_i & u_j \\ v_h & v_i & v_j \\ w_h & w_i & w_j \end{pmatrix} = u_h u_i u_j (r-s)(s-t)(t-r) \neq 0.$$

This contradicts Lemma 102.

(ii) ⇒ (i) Clear. □

Proposition 105. *Assume that $\beta \neq -2$. Then the following are equivalent:*

(i) *there exists $\xi \in \mathbb{C}$ such that $\Phi_i(\xi) = 0$ for $2 \leqslant i \leqslant D-1$;*

(ii) *for $2 \leqslant i, j \leqslant D-1$,*

$$(u_i w_j - u_j w_i)^2 = (v_i w_j - v_j w_i)(u_i v_j - u_j v_i).$$

Proof. (i) ⇒ (ii) By Lemma 103.

(ii) ⇒ (i) By Lemmas 103, 104. □

Next, we examine condition (ii) of Proposition 105. Under the assumption that $\beta \neq -2$, we compute the scalars

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i) \qquad (2 \leqslant i, j \leqslant D-1).$$

Recall the subcases listed in the table above Remark 35. For each subcase such that $\beta \neq -2$, we will do the above computation using the data in [45, Section 20].

Proposition 106. *Assume the given Q-polynomial structure has q-Racah type. Then for* $2 \leqslant i, j \leqslant D-1$ *the scalar*

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

is equal to

$$a_1^*(r_1^2 - s)(r_2^2 - s)(r_3^2 - s)$$

times

$$s + s^* - q^{-1}r_1 - q^{-1}r_2 + r_3 + r_1 r_2 - q r_2 r_3 - q r_3 r_1$$

times

$$s + s^* - q^{-1}r_2 - q^{-1}r_3 + r_1 + r_2 r_3 - q r_3 r_1 - q r_1 r_2$$

times

$$s + s^* - q^{-1}r_3 - q^{-1}r_1 + r_2 + r_3 r_1 - q r_1 r_2 - q r_2 r_3$$

times

$$\frac{u_i^2 u_j^2 (\theta_i^* - \theta_j^*)^2 h^4 h^*}{(\theta_i^* + \theta_0^*)^2 (\theta_j^* + \theta_0^*)^2} \, \frac{1 - q^4 s}{(1 - q^2 s)^3} \, \frac{(1 - q^3 s^*)^4}{(1 - q^4 s^*)^8} \, \frac{q^{10}(q-1)^7}{s},$$

where $r_3 = q^{-D-1}$.

Proof. Use the data in [45, Example 20.1]. □

Remark 107. Referring to Proposition 106,

$$s + s^* - q^{-1}r_1 - q^{-1}r_2 + r_3 + r_1 r_2 - q r_2 r_3 - q r_3 r_1 = \frac{a_D^*(\theta_0 - \theta_1)(\theta_{D-1} - \theta_D)}{h h^* (q-1)^2 (\theta_0 - \theta_D)}.$$

Proposition 108. *Assume the given Q-polynomial structure has q-Hahn type. Then for* $2 \leqslant i, j \leqslant D-1$ *the scalar*

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

is equal to

$$-a_1^* r^2 r_3^2 (s^* - q^{-1}r + r_3 - qrr_3)(s^* - q^{-1}r_3 + r - qrr_3)(s^* - q^{-1}r_3 - q^{-1}r + rr_3)$$

times

$$\frac{u_i^2 u_j^2 (\theta_i^* - \theta_j^*)^2 h^4 h^*}{(\theta_i^* + \theta_0^*)^2 (\theta_j^* + \theta_0^*)^2} \frac{(1 - q^3 s^*)^4 q^{10} (q-1)^7}{(1 - q^4 s^*)^8},$$

where $r_3 = q^{-D-1}$.

Proof. Use the data in [45, Example 20.2]. □

Remark 109. With reference to Proposition 108,

$$s^* - q^{-1}r + r_3 - qrr_3 = \frac{a_D^* (\theta_0 - \theta_1)(\theta_{D-1} - \theta_D)}{hh^*(q-1)^2(\theta_0 - \theta_D)}.$$

Proposition 110. *Assume the given Q-polynomial structure has dual q-Hahn type. Then for* $2 \leqslant i, j \leqslant D-1$ *the scalar*

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

is equal to

$$-a_1^* (r^2 - s)(r_3^2 - s)(s - q^{-1}r + r_3 - qrr_3)(s - q^{-1}r_3 + r - qrr_3)(s - q^{-1}r_3 - q^{-1}r + rr_3)$$

times

$$\frac{u_i^2 u_j^2 (\theta_i^* - \theta_j^*)^2 h^4 h^*}{(\theta_i^* + \theta_0^*)^2 (\theta_j^* + \theta_0^*)^2} \frac{(1 - q^4 s) q^{10} (q-1)^7}{(1 - q^2 s)^3},$$

where $r_3 = q^{-D-1}$.

Proof. Use the data in [45, Example 20.3]. □

Remark 111. With reference to Proposition 110,

$$s - q^{-1}r + r_3 - qrr_3 = \frac{a_D^* (\theta_0 - \theta_1)(\theta_{D-1} - \theta_D)}{hh^*(q-1)^2(\theta_0 - \theta_D)}.$$

Proposition 112. *Assume the given Q-polynomial structure has q-Krawtchouk type. Then for* $2 \leqslant i, j \leqslant D-1$ *the scalar*

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

is equal to 0.

Proof. Use the data in [45, Example 20.5]. □

Proposition 113. *Assume the given Q-polynomial structure has affine q-Krawtchouk type. Then for $2 \leqslant i, j \leqslant D-1$ the scalar*

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

is equal to

$$-a_1^* r^2 r_3^2 (-q^{-1} r + r_3 - q r r_3)(-q^{-1} r_3 + r - q r r_3)(-q^{-1} r_3 - q^{-1} r + r r_3)$$

times

$$\frac{u_i^2 u_j^2 (\theta_i^* - \theta_j^*)^2 h^4 h^*}{(\theta_i^* + \theta_0^*)^2 (\theta_j^* + \theta_0^*)^2} \, q^{10} (q-1)^7,$$

where $r_3 = q^{-D-1}$.

Proof. Use the data in [45, Example 20.6]. □

Remark 114. Referring to Proposition 113,

$$-q^{-1} r + r_3 - q r r_3 = \frac{a_D^* (\theta_0 - \theta_1)(\theta_{D-1} - \theta_D)}{h h^* (q-1)^2 (\theta_0 - \theta_D)}.$$

Proposition 115. *Assume the given Q-polynomial structure has dual q-Krawtchouk type. Then for $2 \leqslant i, j \leqslant D-1$ the scalar*

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

is equal to

$$a_1^* s^2 (r_3^2 - s)(s + r_3)(s - q^{-1} r_3)(s - q^{-1} r_3)$$

times

$$\frac{u_i^2 u_j^2 (\theta_i^* - \theta_j^*)^2 h^4 h^*}{(\theta_i^* + \theta_0^*)^2 (\theta_j^* + \theta_0^*)^2} \, \frac{1 - q^4 s}{(1 - q^2 s)^3} \, \frac{q^{10} (q-1)^7}{s},$$

where $r_3 = q^{-D-1}$.

Proof. Use the data in [45, Example 20.7]. □

Remark 116. Referring to Proposition 115,

$$s + r_3 = \frac{a_D^* (\theta_0 - \theta_1)(\theta_{D-1} - \theta_D)}{h h^* (q-1)^2 (\theta_0 - \theta_D)}.$$

Proposition 117. *Assume the given Q-polynomial structure has Racah type. Then for* $2 \leqslant i, j \leqslant D-1$ *the scalar*

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

is equal to

$$a_1^*(2r_1 - s)(2r_2 - s)(2r_3 - s)$$

times

$$(2r_1 r_2 - 2r_3 - 2 - ss^*)(2r_2 r_3 - 2r_1 - 2 - ss^*)(2r_3 r_1 - 2r_2 - 2 - ss^*)$$

times

$$\frac{u_i^2 u_j^2 (\theta_i^* - \theta_j^*)^2 h^4 h^*}{(\theta_i^* + \theta_0^*)^2 (\theta_j^* + \theta_0^*)^2} \frac{s+4}{(s+2)^3} \frac{(s^*+3)^4}{(s^*+4)^8},$$

where $r_3 = -D-1$.

Proof. Use the data in [45, Example 20.8]. □

Remark 118. Referring to Proposition 117,

$$2r_1 r_2 - 2r_3 - 2 - ss^* = \frac{a_D^*(\theta_0 - \theta_1)(\theta_{D-1} - \theta_D)}{hh^*(\theta_0 - \theta_D)}.$$

Proposition 119. *Assume the given Q-polynomial structure has Hahn type. Then for* $2 \leqslant i, j \leqslant D-1$ *the scalar*

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

is equal to

$$-a_1^*(2r - s^*)(2r_3 - s^*)$$

times

$$\frac{u_i^2 u_j^2 (\theta_i^* - \theta_j^*)^2 s^4 h^*}{(\theta_i^* + \theta_0^*)^2 (\theta_j^* + \theta_0^*)^2} \frac{(s^*+2)(s^*+3)^4}{(s^*+4)^8},$$

where $r_3 = -D-1$.

Proof. Use the data in [45, Example 20.9]. □

Remark 120. Referring to Proposition 119,

$$2r - s^* = \frac{a_D^*(\theta_0 - \theta_1)(\theta_{D-1} - \theta_D)}{sh^*(\theta_0 - \theta_D)}.$$

Proposition 121. *Assume the given Q-polynomial structure has dual Hahn type. Then for $2 \leqslant i, j \leqslant D-1$ the scalar*

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

is equal to zero.

Proof. Use the data in [45, Example 20.10]. □

Proposition 122. *Assume the given Q-polynomial structure has Krawtchouk type. Then for $2 \leqslant i, j \leqslant D-1$ the scalar*

$$(u_i w_j - u_j w_i)^2 - (v_i w_j - v_j w_i)(u_i v_j - u_j v_i)$$

is equal to zero.

Proof. Use the data in [45, Example 20.11]. □

We have been discussing the case $\beta \neq -2$. Next, we discuss the case $\beta = -2$. This case is called Bannai/Ito type.

Assume that $\beta = -2$. Pick $2 \leqslant i \leqslant D-1$ and consider the polynomial $\Phi_i(\lambda)$. We have $u_i = 0$ by Lemma 101, so $\Phi_i(\lambda) = v_i \lambda + w_i$. We will show that $w_i = 0$.

Proposition 123. *Assume that $\beta = -2$. Then $w_i = 0$ and $\Phi_i(0) = 0$ for $2 \leqslant i \leqslant D-1$.*

Proof. We invoke the classification [37, Theorem 2]. There are three solutions for Γ: the Odd graph O_{D+1}; the Hamming graph $H(D,2)$ with D even; and the folded cube $\tilde{H}(2D+1,2)$. The graphs O_{D+1} and $\tilde{H}(2D+1,2)$ are almost bipartite, and $H(D,2)$ is bipartite. For each solution Γ the set $\{E\hat{x} | x \in X\}$ is Norton-balanced by Proposition 27. Each solution Γ is distance-transitive, and hence reinforced by Lemma 58. By these comments and Propositions 99, 100 we have $\Phi_i(z_2) = 0$ for $2 \leqslant i \leqslant D-1$. Each solution Γ has $a_1 = 0$, so Γ is kite-free. Consequently $z_2 = 0$, so $\Phi_i(0) = 0$ for $2 \leqslant i \leqslant D-1$. Observe that $w_i = \Phi_i(0) = 0$ for $2 \leqslant i \leqslant D-1$. □

A detailed discussion of O_{D+1}, $H(D,2)$, $\tilde{H}(2D+1,2)$ can be found in Sections 18, 23, 25 respectively.

16 The DC condition

We continue to discuss the Q-polynomial distance-regular graph $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ.

Definition 124. We say that E is a *dependency candidate* (or *DC*) whenever there exists $\xi \in \mathbb{C}$ such that $\Phi_i(\xi) = 0$ for $2 \leqslant i \leqslant D-1$.

Remark 125. Referring to Definition 124, E being DC is a condition on the intersection numbers of Γ.

Lemma 126. *Assume that* Γ *is reinforced. Assume that for all* $x, y \in X$ *the vectors* Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ *are linearly dependent. Then* E *is DC.*

Proof. By Corollary 98 and Definition 124. □

Lemma 127. *Assume that* Γ *is reinforced. Assume that the set* $\{E\hat{x} | x \in X\}$ *is Norton-balanced. Then* E *is DC.*

Proof. By Propositions 99, 100 and Definition 124. □

We now give our main result about DC. In this result we refer to the data in [45, Section 20].

Theorem 128. *For* $D \geqslant 4$ *the following* (i), (ii) *hold.*

(i) *Assume that the type of* E *is included in the table below. Then* E *is DC iff at least one of the listed scalars is zero.*

type of E	E is DC iff at least one of these scalars is zero
q-Racah	$a_1^*, \quad r_1^2 - s, \quad r_2^2 - s, \quad r_3^2 - s,$
	$s + s^* - q^{-1}r_1 - q^{-1}r_2 + r_3 + r_1r_2 - qr_2r_3 - qr_3r_1,$
	$s + s^* - q^{-1}r_2 - q^{-1}r_3 + r_1 + r_2r_3 - qr_3r_1 - qr_1r_2,$
	$s + s^* - q^{-1}r_3 - q^{-1}r_1 + r_2 + r_3r_1 - qr_1r_2 - qr_2r_3$
q-Hahn	$a_1^*, \quad s^* - q^{-1}r + r_3 - qrr_3,$
	$s^* - q^{-1}r_3 + r - qrr_3, \quad s^* - q^{-1}r_3 - q^{-1}r + rr_3$
dual q-Hahn	$a_1^*, \quad r^2 - s, \quad r_3^2 - s, \quad s - q^{-1}r + r_3 - qrr_3,$
	$s - q^{-1}r_3 + r - qrr_3, \quad s - q^{-1}r_3 - q^{-1}r + rr_3$
affine q-Krawtchouk	$a_1^*, \quad -q^{-1}r + r_3 - qrr_3,$
	$-q^{-1}r_3 + r - qrr_3, \quad -q^{-1}r_3 - q^{-1}r + rr_3$
dual q-Krawtchouk	$a_1^*, \quad r_3^2 - s, \quad s + r_3, \quad s - q^{-1}r_3$
Racah	$a_1^*, \quad 2r_1 - s, \quad 2r_2 - s, \quad 2r_3 - s,$
	$2r_1r_2 - 2r_3 - 2 - ss^*, \quad 2r_2r_3 - 2r_1 - 2 - ss^*,$
	$2r_3r_1 - 2r_2 - 2 - ss^*$
Hahn	$a_1^*, \quad 2r - s^*, \quad 2r_3 - s^*$

(ii) *Assume that the type of* E *is* q*-Krawtchouk or dual Hahn or Krawtchouk or Bannai/Ito. Then* E *is DC.*

Proof. (i) For Propositions 106, 108, 110, 113, 115, 117, 119, examine the factorization in the proposition statement. For each factorization, consider which factors could be zero. Some of the factors are nonzero because of the inequalities in [43, Section 5]. The remaining factors are listed in the above table.

(ii) By Propositions 112, 121, 122, 123. □

The book [2, Chapter 6.4] gives a list of the known infinite families of Q-polynomial distance-regular graphs with unbounded diameter. For each listed graph, every Q-polynomial structure is described. In Sections 17–29, we will examine these Q-polynomial structures. For each listed graph $\Gamma = (X, \mathcal{R})$ and each Q-polynomial primitive idempotent E of Γ, we will determine if the set $\{E\hat{x} | x \in X\}$ is Norton-balanced or not. We will also determine if E is DC or not. Considerable supporting data will be given, using the notation of [45, Section 20]. We obtained this supporting data using Sections 11–16; the computations are routine and omitted. For the rest of the paper, the integer D is assumed to be at least 3.

17 Example: the Johnson graph

Example 129. (See [2, Chapter 6.4], [40, Example 6.1(1)].) The *Johnson graph* $J(N, D)$ $(N \geqslant 2D)$ has vertex set X consisting of the subsets of $\{1, 2, \ldots, N\}$ that have cardinality D. Vertices $x, y \in X$ are adjacent whenever $|x \cap y| = D - 1$. The graph $J(N, D)$ is distance-regular with diameter D and intersection numbers

$$c_i = i^2, \qquad b_i = (D - i)(N - D - i) \qquad (0 \leqslant i \leqslant D).$$

The graph $J(2D, D)$ is an antipodal 2-cover.

Example 130. The graph $J(N, D)$ has a Q-polynomial structure such that

$$\theta_i = (D - i)(N - D - i) - i \qquad (0 \leqslant i \leqslant D),$$
$$\theta^*_i = N - 1 - \frac{iN(N-1)}{D(N-D)} \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has dual Hahn type with

$$r = D - N - 1, \qquad s = -N - 2, \qquad h = 1, \qquad s^* = \frac{N(1-N)}{D(N-D)}.$$

This structure is DC with $\gamma^* = 0$.

For $2 \leqslant i \leqslant D$,

$$\alpha_i = i - 1, \qquad \beta_i = 0,$$
$$r_i = i(i-1), \qquad s_i = i, \qquad z^-_i = 2(i-1).$$

For $1 \leqslant i \leqslant D - 1$,

$$R_i = \frac{(D-i)(N-D-i)(2D^2 - 2DN + iN + N)}{iN - 2DN + 2D^2},$$
$$S_i = \frac{N(D-i)(N-D-i)}{iN - 2DN + 2D^2},$$
$$z^+_{i+1} = \frac{N(N-2i)i}{2DN - 2D^2 - iN}.$$

For $2 \leqslant i \leqslant D-1$,

$$\begin{aligned}
&u_i = -i(i-1), \qquad v_i = \frac{i(i-1)\big(2N(2i-N)+(N-2D)^2\big)}{iN-2DN+2D^2},\\
&w_i = \frac{2i(i-1)N(N-2i)}{iN-2DN+2D^2},\\
&\Phi_i(\lambda) = u_i(\lambda-\xi)(\lambda-\xi_i), \qquad \xi = 2, \qquad \xi_i = \frac{N(2i-N)}{iN-2DN+2D^2}.
\end{aligned}$$

If $N = 2D$ then $\xi_i = 2$.

Lemma 131. *For $J(N, D)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover $z_i = 2(i-1)$ for $2 \leqslant i \leqslant D$.*

Proof. By combinatorial counting. □

Lemma 132. *We refer to Example 130 and write $E = E_1$. Pick distinct $x, y \in X$ and write $i = \partial(x, y)$. For $2 \leqslant i \leqslant D$,*

$$Ex_y^- = i(i-1)E\hat{x} + iE\hat{y}. \tag{47}$$

For $1 \leqslant i \leqslant D-1$ and $N = 2D$,

$$Ex_y^+ = (D-i)(D-i-1)E\hat{x} + (i-D)E\hat{y}. \tag{48}$$

In any case, the set $\{E\hat{x} | x \in X\}$ is Norton-balanced.

Proof. First assume that $N > 2D$. Then Γ is not an antipodal 2-cover. To get (47), use Proposition 68 and $z_i^- = z_i$. Next assume that $N = 2D$. Then Γ is an antipodal 2-cover. To get (47) for $1 \leqslant i \leqslant D-1$, use Proposition 68 and $z_i^- = z_i$. To get (47) for $i = D$, use $E\hat{x} + E\hat{y} = 0$ and $x_y^- = A\hat{x}$ and $\theta_1 = D(D-2)$. To get (48), use Proposition 80 and $z_{i+1}^+ = z_{i+1}$. Next assume that $N \geqslant 2D$. It follows from Lemma 24 and (47) that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. □

18 Example: the Odd graph

Example 133. (See [2, Chapter 6.4], [40, Example 6.1(2)].) The *Odd graph* O_{D+1} has vertex set X consisting of the D-element subsets of the set $\{1, 2, \ldots, 2D+1\}$. Vertices $x, y \in X$ are adjacent whenever they are disjoint. The graph O_{D+1} is distance-regular with diameter D and intersection numbers

$$\begin{aligned}
c_i &= \frac{2i+1-(-1)^i}{4} && (1 \leqslant i \leqslant D),\\
b_i &= D + \frac{3-2i+(-1)^i}{4} && (0 \leqslant i \leqslant D-1).
\end{aligned}$$

The graph O_{D+1} is almost bipartite.

Example 134. The graph O_{D+1} has a Q-polynomial structure such that

$$\theta_i = (-1)^i(D-i+1) \qquad (0 \leqslant i \leqslant D),$$
$$\theta^*_i = \frac{(-1)^i(4D^2-4iD+4D-2i+1)-1}{2(D+1)} \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has Bannai/Ito type with

$$r_1 = -D-1, \qquad r_2 = -2D-3, \qquad s = 2D+3,$$
$$s^* = 2D+2, \qquad h = -1/2, \qquad h^* = -\frac{2D+1}{2(D+1)}.$$

This structure is DC with

$$\gamma^* = -\frac{2}{D+1} \neq 0.$$

For $2 \leqslant i \leqslant D$,

$$\alpha_i = \frac{D-1}{2}\,\frac{(-1)^i+1}{D-i+1}, \qquad \beta_i = \frac{3-2i+(-1)^i}{4(D-i+1)},$$
$$r_i = \frac{1}{2}\,\frac{2i+1-(-1)^i}{(-1)^i(2D-2i+1)-2D-1} \times \frac{8iD^2-4i^2D-8D^2+12iD-2i^2-8D+4i-1+(-1)^i}{(-1)^i(4D^2-4iD+4D-2i+1)+4D^2+4D-1},$$
$$s_i = D\frac{2i+1-(-1)^i}{(-1)^i(2D-2i+1)-2D-1} \times \frac{(-1)^i(4D-2i+3)+1}{(-1)^i(4D^2-4iD+4D-2i+1)+4D^2+4D-1},$$
$$z^-_i = -\frac{1}{D-1}\,\frac{1}{(-1)^i(2D-2i+1)-2D-1} \times \frac{(-1)^i\big((2D-i)^2+4D-3i+1\big)+(2D-i)(4iD-2i^2-2D+5i-2)+i-1}{(-1)^i(4D^2-4iD+4D-2i+1)+4D^2+4D-1}.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = \frac{1}{2}\,\frac{4D-2i+3+(-1)^i}{(-1)^i(2D-2i+1)-2D-1} \times \frac{8iD^2-4i^2D+4iD-2i^2+1-(-1)^i}{(-1)^i(4D^2-4iD+4D-2i+1)+4D^2+4D-1},$$
$$S_i = D\frac{1-(-1)^i(2i+1)}{(-1)^i(2D-2i+1)-2D-1} \times \frac{4D-2i+3+(-1)^i}{(-1)^i(4D^2-4iD+4D-2i+1)+4D^2+4D-1},$$

$$z^+_{i+1} = \frac{1}{D-1}\,\frac{1}{(-1)^i(2D-2i+1)-2D-1}$$
$$\times \frac{(-1)^i(2D-i^2-i+1)+4i^2D-2i^3+i^2-2D+i-1}{(-1)^i(4D^2-4iD+4D-2i+1)+4D^2+4D-1}.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = 0,$$
$$v_i = -\frac{(-1)^i}{2(D-i)(D-i+1)}$$
$$\times \frac{(-1)^i(2D^2-6iD+3i^2+3D-3i+1)+(2iD-i^2-D+i-1)(2D-2i+1)}{(-1)^i(4D^2-4iD+4D-2i+1)+4D^2+4D-1},$$
$$w_i = 0, \qquad \Phi_i(\lambda) = v_i(\lambda-\xi), \qquad \xi = 0.$$

Lemma 135. *For O_{D+1} the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover $z_i = 0$ for $2 \leqslant i \leqslant D$.*

Proof. The graph O_{D+1} is almost bipartite, and hence kite-free. □

Lemma 136. *We refer to Example 134 and write $E = E_1$. The set $\{E\hat{x} | x \in X\}$ is Norton-balanced. For $0 \leqslant i \leqslant D-1$ and $x, y \in X$ at distance $\partial(x, y) = i$,*

$$0 = Ex^-_y + Ex^+_y + DE\hat{x}.$$

Proof. The graph O_{D+1} is almost bipartite and $\theta_1 = -D$. □

19 Example: the Grassmann graph

Example 137. (See [2, Chapter 6.4], [40, Example 6.1(5)].) Let $GF(q)$ denote a finite field with cardinality q. Fix an integer $N \geqslant 2D$, and let U denote a vector space over $GF(q)$ that has dimension N. The Grassmann graph $J_q(N, D)$ has vertex set X consisting of the subspaces of U that have dimension D. Vertices $x, y \in X$ are adjacent whenever $x \cap y$ has dimension $D-1$. The graph $J_q(N, D)$ is distance-regular with diameter D and intersection numbers

$$c_i = \left(\frac{q^i-1}{q-1}\right)^2, \qquad b_i = q\frac{q^D-q^i}{q-1}\frac{q^{N-D}-q^i}{q-1} \qquad (0 \leqslant i \leqslant D).$$

Example 138. The graph $J_q(N, D)$ has a Q-polynomial structure such that

$$\theta_i = q^{1-i}\frac{q^D-q^i}{q-1}\frac{q^{N-D}-q^i}{q-1} - \frac{q^i-1}{q-1} \qquad (0 \leqslant i \leqslant D),$$
$$\theta^*_i = \frac{q^N-q}{q-1} - q^{-i}\frac{q^N-q}{q^D-1}\frac{q^N-1}{q^{N-D}-1}\frac{q^i-1}{q-1} \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has dual q-Hahn type with

$$r = q^{D-N-1}, \qquad s = q^{-N-2}, \qquad h = \frac{q^{N+1}}{(q-1)^2},$$
$$h^* = \frac{q(q^N-1)(q^{N-1}-1)}{(q-1)(q^D-1)(q^{N-D}-1)}.$$

Q-polynomial structure is DC if and only if $N = 2D$ (provided that $D \geqslant 4$). Assume that $N = 2D$. We have

$$\gamma^* = \frac{2(q-1)(q^{2D-1}-1)}{q^D-1} \neq 0.$$

For $2 \leqslant i \leqslant D$,

$$\alpha_i = \frac{q^{i-1}-1}{q-1}, \qquad \beta_i = 0,$$
$$r_i = \frac{q^i-1}{q-1}\,\frac{q^{i-1}-1}{q-1}\,\frac{q^i(q^D-2q-1)+q^{D+1}+q}{q^i(q^D-3)+q^D+1},$$
$$s_i = \frac{q^{i-1}(q^i-1)}{q-1}\,\frac{q^{D+1}+q^D+1-q-2q^i}{q^i(q^D-3)+q^D+1},$$
$$z_i^- = 2q\frac{q^{i-1}-1}{q-1}\,\frac{q^{2i}-q^i(q+3)+q^{D+1}+q^D+1}{q^i(q^D-3)+q^D+1}.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = \left(\frac{q^D-q^i}{q-1}\right)^2 \frac{q^i(q^D-2q-1)+q^D+1}{q^i(q^D-3)+q^D+1},$$
$$S_i = -\frac{2q^i}{q-1}\,\frac{(q^D-q^i)^2}{q^i(q^D-3)+q^D+1},$$
$$z_{i+1}^+ = 2q\frac{q^i-1}{q-1}\,\frac{1+2q^i(q^D-1)-q^{2i}}{q^i(q^D-3)+q^D+1}.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = -\frac{q^i-1}{q-1}\,\frac{q^{i-1}-1}{q-1},$$
$$v_i = 2q\frac{q^i-1}{q-1}\,\frac{q^{i-1}-1}{q-1}\,\frac{q^i(2q^D-q-5)+q^{D+1}+q^D+2}{q^i(q^D-3)+q^D+1},$$
$$w_i = -4q^2\frac{q^i-1}{q-1}\,\frac{q^{i-1}-1}{q-1}\,\frac{q^i(q^D-q-2)+q^{D+1}+1}{q^i(q^D-3)+q^D+1},$$
$$\Phi_i(\lambda) = u_i(\lambda-\xi)(\lambda-\xi_i) \qquad \xi = 2q, \qquad \xi_i = 2q\frac{q^i(q^D-q-2)+q^{D+1}+1}{q^i(q^D-3)+q^D+1}.$$

Lemma 139. *For $J_q(N,D)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = 2q\frac{q^{i-1}-1}{q-1} \qquad\qquad (2 \leqslant i \leqslant D).$$

Proof. By combinatorial counting. □

Lemma 140. *We refer to Example 138 with $N = 2D$. Write $E = E_1$. Then the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Pick distinct $x, y \in X$ and write $i = \partial(x,y)$. For $2 \leqslant i \leqslant D-1$,*

$$0 = Ex_y^- + \frac{q^{i-1}-1}{q^D-q^i}Ex_y^+ - \frac{q^{i-1}-1}{q-1}\frac{q^D-q}{q-1}E\hat{x} - q^{i-1}E\hat{y}. \tag{49}$$

For $i = D$,

$$0 = Ex_y^- - q\left(\frac{q^{D-1}-1}{q-1}\right)^2 E\hat{x} - q^{D-1}E\hat{y}. \tag{50}$$

Proof. To get the first assertion, we use Proposition 100(ii). Pick an integer i ($2 \leqslant i \leqslant D-1$). We verify the conditions in (45). We have $z_2 = \xi$, so $\Phi_i(z_2) = 0$. We have

$$\lambda_i = -\frac{q^{i-1}-1}{q^D-q^i}, \qquad\qquad \frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} = q^{-1}.$$

Therefore

$$\frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} - \lambda_i = \frac{q^{D-1}-1}{q^D-q^i} \neq 0.$$

We have verified the conditions in (45), so the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. The linear dependence (49) is obtained using (41), (42). To obtain (50), use Proposition 68 and $z^-_D = z_D$. □

20 Example: the dual polar graphs

Example 141. (See [2, Chapter 6.4], [40, Example 6.1(6)].) Let U denote a finite vector space with one of the following nondegenerate forms:

name	dim(U)	field	form	e
$B_D(p^n)$	$2D+1$	$GF(p^n)$	quadratic	0
$C_D(p^n)$	$2D$	$GF(p^n)$	symplectic	0
$D_D(p^n)$	$2D$	$GF(p^n)$	quadratic (Witt index D)	-1
${}^2D_{D+1}(p^n)$	$2D+2$	$GF(p^n)$	quadratic (Witt index D)	1
${}^2A_{2D}(p^n)$	$2D+1$	$GF(p^{2n})$	Hermitean	$1/2$
${}^2A_{2D-1}(p^n)$	$2D$	$GF(p^{2n})$	Hermitean	$-1/2$

A subspace of U is called *isotropic* whenever the form vanishes completely on that subspace. In each of the above cases, the dimension of any maximal isotropic subspace is D. The corresponding dual polar graph Γ has vertex set X consisting of the maximal isotropic subspaces of U. Vertices $x, y \in X$ are adjacent whenever $x \cap y$ has dimension $D-1$. The graph Γ is distance-regular with diameter D and intersection numbers

$$c_i = \frac{q^i - 1}{q-1}, \qquad b_i = q^{e+1}\frac{q^D - q^i}{q-1} \qquad (0 \leqslant i \leqslant D),$$

where $q = p^n, p^n, p^n, p^n, p^{2n}, p^{2n}$. The dual polar graph $D_D(q)$ is bipartite.

Example 142. The dual polar graph Γ has a Q-polynomial structure such that

$$\theta_i = q^{e+1}\frac{q^D - 1}{q-1} - \frac{(q^i - 1)(q^{D+e+1-i}+1)}{q-1} \qquad (0 \leqslant i \leqslant D),$$

$$\theta^*_i = \frac{q^{D+e}+q}{q^e+1}\,\frac{q^{-i}(q^{D+e}+1) - q^e - 1}{q-1} \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has dual q-Krawtchouk type with

$$s = -q^{-D-e-2}, \qquad h = \frac{q^{D+e+1}}{q-1}, \qquad h^* = \frac{(q^{D+e}+1)(q^{D+e}+q)}{(q-1)(q^e+1)}.$$

This Q-polynomial structure is DC if and only if $\Gamma = D_D(q)$ (provided that $D \geqslant 4$). For $\Gamma = D_D(q)$ we have the following.

$$\gamma^* = (q-1)(q^{D-2}+1) \neq 0.$$

For $2 \leqslant i \leqslant D$,

$$\alpha_i = \frac{q^{i-1}-1}{q-1}, \qquad \beta_i = 0,$$

$$r_i = \frac{q^{i-1}-1}{q-1}\,\frac{q^i(q^D - q^2 - q - 1) + q^{D+1} + q^2}{q^i(q^D - q - 2) + q^D + q},$$

$$s_i = \frac{(q+1)q^{i-1}(q^D - q^i)}{q^i(q^D - q - 2) + q^D + q},$$

$$z^-_i = -\frac{(q+1)(q^i - q)(q^D - q^i)}{q^i(q^D - q - 2) + q^D + q}.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = \frac{1}{q}\,\frac{q^D - q^i}{q-1}\,\frac{q^i(q^D - q^2 - q - 1) + q^D + q}{q^i(q^D - q - 2) + q^D + q},$$

$$S_i = -\frac{(q+1)q^{i-1}(q^D - q^i)}{q^i(q^D - q - 2) + q^D + q},$$

$$z^+_{i+1} = \frac{(q+1)(q^i - 1)(q^i - q)}{q^i(q^D - q - 2) + q^D + q}.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = -\frac{q^i - 1}{q-1}\,\frac{q^{i-1}-1}{q-1},$$
$$v_i = \frac{(q+1)(q^i-1)(q^{i-1}-1)}{q-1}\,\frac{q^i(q+1)-q^{D+1}-q}{q^i(q^D-q-2)+q^D+q},$$
$$w_i = 0,$$
$$\Phi_i(\lambda) = u_i(\lambda-\xi)(\lambda-\xi_i) \qquad \xi = 0, \qquad \xi_i = (q^2-1)\frac{q^i(q+1)-q^{D+1}-q}{q^i(q^D-q-2)+q^D+q}.$$

Lemma 143. *For a dual polar graph Γ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = 0 \qquad\qquad (2 \leqslant i \leqslant D).$$

Proof. The graph Γ is a regular near polygon [7, Section 6.4] and hence kite-free. □

Lemma 144. *We refer to Example 142 with $\Gamma = D_D(q)$. Write $E = E_1$. Then the set $\{E\hat{x}|x \in X\}$ is Norton-balanced. For $x, y \in X$ we have*

$$0 = Ex_y^- + Ex_y^+ - \frac{q^{D-1}-q}{q-1}E\hat{x}.$$

Proof. The graph $D_D(q)$ is bipartite and $\theta_1 = (q^{D-1}-q)/(q-1)$. □

The dual polar graph ${}^2A_{2D-1}(p^n)$ has a second Q-polynomial structure, which we now describe.

Example 145. (See [2, Chapter 6.4], [40, Example 6.1(7)].) The intersection numbers of ${}^2A_{2D-1}(p^n)$ can be expressed as

$$c_i = \frac{q^{2i}-1}{q^2-1}, \qquad\qquad b_i = -q^{2i+1}\frac{q^{2D-2i}-1}{q^2-1} \qquad\qquad (0 \leqslant i \leqslant D),$$

where $q = -p^n$. The graph ${}^2A_{2D-1}(p^n)$ has a Q-polynomial structure such that

$$\theta_i = \frac{(q^i-1)(q^{2D-i+1}-1)}{q^2-1} - q\frac{q^{2D}-1}{q^2-1} \qquad\qquad (0 \leqslant i \leqslant D),$$
$$\theta_i^* = -q^{-i}\frac{q^{2D}-q}{q-1} \qquad\qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure is almost dual-bipartite. It has dual q-Hahn type with

$$r = -q^{-D-1}, \qquad s = q^{-2D-2}, \qquad h = -\frac{q^{2D+1}}{q^2-1}, \qquad h^* = -\frac{q^{2D}-q}{q-1}.$$

This Q-polynomial structure is DC with $\gamma^* = 0$. For $2 \leqslant i \leqslant D$,

$$\alpha_i = \frac{q^{i-1}-1}{q-1}, \qquad \beta_i = 0,$$
$$r_i = q\frac{q^{2i-2}-1}{q^2-1}, \qquad s_i = q^{i-1}, \qquad z_i^- = 0.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = -\frac{q^{2D}-q^{2i}}{q^2-1}, \qquad S_i = 0, \qquad z_{i+1}^+ = 0.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = -\frac{q^i-1}{q-1}\frac{q^{i-1}-1}{q-1}, \qquad v_i = 0, \qquad w_i = 0,$$
$$\Phi_i(\lambda) = u_i(\lambda-\xi)(\lambda-\xi_i), \qquad \xi = 0, \qquad \xi_i = 0.$$

Lemma 146. *We refer to Example 145 and write $E = E_1$. The set $\{E\hat{x}|x \in X\}$ is Norton-balanced. Pick distinct $x, y \in X$ and write $i = \partial(x,y)$. For $2 \leqslant i \leqslant D$,*

$$Ex_y^- = q\frac{q^{2i-2}-1}{q^2-1}E\hat{x} + q^{i-1}E\hat{y}. \tag{51}$$

For $1 \leqslant i \leqslant D-1$,

$$Ex_y^+ = -\frac{q^{2D}-q^{2i}}{q^2-1}E\hat{x}. \tag{52}$$

Proof. To get (51), use Proposition 68 and $z_i^- = z_i$. To get (52), use Proposition 80 and $z_{i+1}^+ = z_{i+1}$. It follows from (51), (52) that the set $\{E\hat{x}|x \in X\}$ is Norton-balanced. □

21 Example: the halved bipartite dual polar graph

Recall that the dual polar graph $D_D(p^n)$ is bipartite.

Example 147. (See [2, Chapter 6.4], [40, Example 6.1(8)].) The halved graph $\frac{1}{2}D_{2D}(p^n)$ is distance-regular, with diameter D and intersection numbers

$$c_i = \frac{q^i-1}{q-1}\frac{q^{i-\frac{1}{2}}-1}{q^{\frac{1}{2}}-1}, \qquad b_i = \frac{q^D-q^i}{q-1}\frac{q^D-q^{i+\frac{1}{2}}}{q^{\frac{1}{2}}-1} \qquad (0 \leqslant i \leqslant D),$$

where $q = p^{2n}$.

Example 148. The graph $\frac{1}{2}D_{2D}(p^n)$ has a Q-polynomial structure such that

$$\theta_i = q^{\frac{1}{2}} \frac{q^D-1}{q-1} \frac{q^{D-\frac{1}{2}}-1}{q^{\frac{1}{2}}-1} - \frac{q^i-1}{q-1} \frac{q^{2D-i}-1}{q^{\frac{1}{2}}-1} \qquad (0 \leqslant i \leqslant D),$$

$$\theta^*_i = q^{\frac{1}{2}} \frac{q^D-1}{q-1} \frac{q^{2D-1}-q}{q^D-q} - \frac{q^{D-\frac{1}{2}}+1}{q^{i-1}} \frac{q^i-1}{q-1} \frac{q^{2D-1}-q}{q^D-q} \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has dual q-Hahn type with

$$r = q^{-D-\frac{1}{2}}, \qquad s = q^{-2D-1}, \qquad h = \frac{q^{2D}}{(q-1)(q^{\frac{1}{2}}-1)},$$

$$h^* = \frac{(q^{D-\frac{1}{2}}+1)(q^{2D}-q^2)}{(q-1)(q^D-q)}.$$

This Q-polynomial structure is DC and

$$\gamma^* = \frac{(q-1)(q^{\frac{1}{2}}+1)(q^{2D}-q^2)}{q^{\frac{3}{2}}(q^D-q)} \neq 0.$$

For $2 \leqslant i \leqslant D$,

$$\alpha_i = \frac{q^{i-1}-1}{q-1}, \qquad \beta_i = 0,$$

$$r_i = \frac{q^{i-1}-1}{q-1} \frac{q^{i-\frac{1}{2}}-1}{q^{\frac{1}{2}}-1} \frac{q^i(q^{D+\frac{1}{2}}-q^2-q^{\frac{3}{2}}-q^{\frac{1}{2}})+q^{D+\frac{3}{2}}+q^2}{q^i(q^{D+\frac{1}{2}}-q-2q^{\frac{1}{2}})+q^{D+\frac{1}{2}}+q},$$

$$s_i = \frac{q^{i-1}(q^{i-\frac{1}{2}}-1)}{q^{\frac{1}{2}}-1} \frac{q^{D+\frac{3}{2}}+q^{D+\frac{1}{2}}-q^{\frac{3}{2}}+q-q^i(q+q^{\frac{1}{2}})}{q^i(q^{D+\frac{1}{2}}-q-2q^{\frac{1}{2}})+q^{D+\frac{1}{2}}+q},$$

$$z^-_i = (q+q^{\frac{1}{2}}) \frac{q^{i-1}-1}{q^{\frac{1}{2}}-1} \frac{q^{2i}q^{\frac{1}{2}}-q^i(q^{\frac{3}{2}}+q+2q^{\frac{1}{2}})+q^{D+\frac{3}{2}}+q^{D+\frac{1}{2}}+q}{q^i(q^{D+\frac{1}{2}}-q-2q^{\frac{1}{2}})+q^{D+\frac{1}{2}}+q}.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = q^{-\frac{1}{2}} \frac{q^D-q^i}{q-1} \frac{q^{D-\frac{1}{2}}-q^i}{q^{\frac{1}{2}}-1} \frac{q^i(q^{D+\frac{1}{2}}-q^2-q^{\frac{3}{2}}-q^{\frac{1}{2}})+q^{D+\frac{1}{2}}+q}{q^i(q^{D+\frac{1}{2}}-q-2q^{\frac{1}{2}})+q^{D+\frac{1}{2}}+q},$$

$$S_i = -\frac{q^D-q^i}{q^{\frac{1}{2}}-1} \frac{(q^{\frac{1}{2}}+1)q^i(q^{D-\frac{1}{2}}-q^i)}{q^i(q^{D+\frac{1}{2}}-q-2q^{\frac{1}{2}})+q^{D+\frac{1}{2}}+q},$$

$$z^+_{i+1} = \frac{(q^{\frac{1}{2}}+1)(q^i-1)}{q^{\frac{1}{2}}(q^{\frac{1}{2}}-1)} \frac{q^2+q^i(q^{D+\frac{3}{2}}+q^{D+1}-2q^{\frac{3}{2}})-q^{2i}q^{\frac{3}{2}}}{q^i(q^{D+\frac{1}{2}}-q-2q^{\frac{1}{2}})+q^{D+\frac{1}{2}}+q}.$$

For $2 \leqslant i \leqslant D-1$,

$$
\begin{aligned}
u_i &= -\frac{q^i-1}{q-1}\frac{q^{i-1}-1}{q-1},\\
v_i &= \frac{q^i-1}{q-1}\frac{(q^{i-1}-1)(q^{\frac{1}{2}}+1)}{q^{\frac{1}{2}}(q^{\frac{1}{2}}-1)}\frac{q^i(q^{D+\frac{3}{2}}+q^{D+1}-q^{\frac{5}{2}}-q^2-4q^{\frac{3}{2}})+q^{D+\frac{5}{2}}+q^{D+\frac{3}{2}}+2q^2}{q^i(q^{D+\frac{1}{2}}-q-2q^{\frac{1}{2}})+q^{D+\frac{1}{2}}+q},\\
w_i &= -\frac{(q^{\frac{1}{2}}+1)^2(q^i-1)(q^{i-1}-1)}{(q^{\frac{1}{2}}-1)^2}\frac{q^i(q^{D+1}-q^{\frac{5}{2}}-2q^{\frac{3}{2}})+q^{D+\frac{5}{2}}+q^2}{q^i(q^{D+\frac{1}{2}}-q-2q^{\frac{1}{2}})+q^{D+\frac{1}{2}}+q},\\
\Phi_i(\lambda) &= u_i(\lambda-\xi)(\lambda-\xi_i), \qquad\qquad \xi = q^{\frac{1}{2}}(q^{\frac{1}{2}}+1)^2,\\
\xi_i &= \frac{(q^{\frac{1}{2}}+1)^2}{q^{\frac{1}{2}}}\frac{q^i(q^{D+1}-q^{\frac{5}{2}}-2q^{\frac{3}{2}})+q^{D+\frac{5}{2}}+q^2}{q^i(q^{D+\frac{1}{2}}-q-2q^{\frac{1}{2}})+q^{D+\frac{1}{2}}+q}.
\end{aligned}
$$

Lemma 149. *For the graph $\frac{1}{2}D_{2D}(p^n)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = q^{\frac{1}{2}}\frac{(q-1)(q^{i-1}-1)}{(q^{\frac{1}{2}}-1)^2} \qquad\qquad (2 \leqslant i \leqslant D).$$

Proof. By combinatorial counting using [31, Section 5]. □

Lemma 150. *We refer to Example 148 and write $E = E_1$. The set $\{E\hat{x}|x \in X\}$ is Norton-balanced. Pick distinct $x, y \in X$ and write $i = \partial(x,y)$. For $2 \leqslant i \leqslant D-1$,*

$$0 = Ex_y^- + \frac{q^{i-1}-1}{q^{D-\frac{1}{2}}-q^i}Ex_y^+ - \frac{q^{i-1}-1}{q-1}\frac{q^{D-\frac{1}{2}}-q}{q^{\frac{1}{2}}-1}E\hat{x} - q^{i-1}E\hat{y}. \tag{53}$$

For $i = D$,

$$0 = Ex_y^- - \frac{q^{D-1}-1}{q-1}\frac{q^{D-\frac{1}{2}}-q}{q^{\frac{1}{2}}-1}E\hat{x} - q^{D-1}E\hat{y}. \tag{54}$$

Proof. To get the first assertion, we use Proposition 100(ii). Pick an integer i ($2 \leqslant i \leqslant D-1$). We verify the conditions in (45). We have $z_2 = \xi$, so $\Phi_i(z_2) = 0$. We have

$$\lambda_i = \frac{q^{i-1}-1}{q^i - q^{D-\frac{1}{2}}}, \qquad\qquad \frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} = q^{-1}.$$

Therefore

$$\frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} - \lambda_i = \frac{q^{D-1} - q^{\frac{1}{2}}}{q^D - q^{i+\frac{1}{2}}} \neq 0.$$

We have verified the conditions in (45), so the set $\{E\hat{x}|x \in X\}$ is Norton-balanced. The linear dependence (53) is obtained using (41), (42). To obtain (54), use Proposition 68 and $z_D^- = z_D$. □

Example 151. (See[2, Chapter 6.4], [40, Example 6.1(9)].) The halved graph $\frac{1}{2}D_{2D+1}(p^n)$ is distance-regular, with diameter D and intersection numbers

$$c_i = \frac{q^i - 1}{q-1}\,\frac{q^{i-\frac{1}{2}} - 1}{q^{\frac{1}{2}} - 1}, \qquad b_i = q^{\frac{1}{2}}\frac{q^D - q^i}{q-1}\,\frac{q^{D+\frac{1}{2}} - q^i}{q^{\frac{1}{2}} - 1} \qquad (0 \leqslant i \leqslant D),$$

where $q = p^{2n}$.

Example 152. The graph $\frac{1}{2}D_{2D+1}(p^n)$ has a Q-polynomial structure such that

$$\theta_i = q^{\frac{1}{2}}\frac{q^D - 1}{q-1}\,\frac{q^{D+\frac{1}{2}} - 1}{q^{\frac{1}{2}} - 1} - \frac{q^i - 1}{q-1}\,\frac{q^{2D-i+1} - 1}{q^{\frac{1}{2}} - 1} \qquad (0 \leqslant i \leqslant D),$$

$$\theta^*_i = \frac{q^{D+\frac{1}{2}} - 1}{q-1}\,\frac{q^{2D} - q}{q^D - q^{\frac{1}{2}}} - \frac{q^D + 1}{q^i}\,\frac{q^i - 1}{q-1}\,\frac{q^{2D} - q}{q^{D-\frac{1}{2}} - 1} \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has dual q-Hahn type with

$$r = q^{-D-\frac{3}{2}}, \qquad s = q^{-2D-2}, \qquad h = \frac{q^{2D+1}}{(q-1)(q^{\frac{1}{2}} - 1)},$$

$$h^* = \frac{(q^D + 1)(q^{2D} - q)}{(q-1)(q^{D-\frac{1}{2}} - 1)}.$$

This Q-polynomial structure is DC and

$$\gamma^* = \frac{(q-1)(q^{\frac{1}{2}} + 1)(q^{2D-1} - 1)}{q^D - q^{\frac{1}{2}}} \neq 0.$$

For $2 \leqslant i \leqslant D$,

$$\alpha_i = \frac{q^{i-1} - 1}{q-1}, \qquad \beta_i = 0,$$

$$r_i = \frac{q^{i-1} - 1}{q-1}\,\frac{q^{i-\frac{1}{2}} - 1}{q^{\frac{1}{2}} - 1}\,\frac{q^i(q^{D+\frac{1}{2}} - q^{\frac{3}{2}} - q - 1) + q^{D+\frac{3}{2}} + q^{\frac{3}{2}}}{q^i(q^{D+\frac{1}{2}} - q^{\frac{1}{2}} - 2) + q^{D+\frac{1}{2}} + q^{\frac{1}{2}}},$$

$$s_i = \frac{q^{i-1}(q^{i-\frac{1}{2}} - 1)}{q^{\frac{1}{2}} - 1}\,\frac{q^{D+\frac{3}{2}} + q^{D+\frac{1}{2}} - q + q^{\frac{1}{2}} - q^i(q^{\frac{1}{2}} + 1)}{q^i(q^{D+\frac{1}{2}} - q^{\frac{1}{2}} - 2) + q^{D+\frac{1}{2}} + q^{\frac{1}{2}}},$$

$$z^-_i = (q + q^{\frac{1}{2}})\frac{q^{i-1} - 1}{q^{\frac{1}{2}} - 1}\,\frac{q^{2i} - q^i(q + q^{\frac{1}{2}} + 2) + q^{D+\frac{3}{2}} + q^{D+\frac{1}{2}} + q^{\frac{1}{2}}}{q^i(q^{D+\frac{1}{2}} - q^{\frac{1}{2}} - 2) + q^{D+\frac{1}{2}} + q^{\frac{1}{2}}}.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = \frac{q^D - q^i}{q-1}\,\frac{q^D - q^{i-\frac{1}{2}}}{q^{\frac{1}{2}} - 1}\,\frac{q^i(q^{D+\frac{1}{2}} - q^{\frac{3}{2}} - q - 1) + q^{D+\frac{1}{2}} + q^{\frac{1}{2}}}{q^i(q^{D+\frac{1}{2}} - q^{\frac{1}{2}} - 2) + q^{D+\frac{1}{2}} + q^{\frac{1}{2}}},$$

$$S_i = -\frac{q^D - q^i}{q^{\frac{1}{2}} - 1}\,\frac{(q^{\frac{1}{2}} + 1)q^i(q^D - q^{i-\frac{1}{2}})}{q^i(q^{D+\frac{1}{2}} - q^{\frac{1}{2}} - 2) + q^{D+\frac{1}{2}} + q^{\frac{1}{2}}},$$

$$z^+_{i+1} = q^{\frac{1}{2}}\frac{(q^{\frac{1}{2}} + 1)(q^i - 1)}{q^{\frac{1}{2}} - 1}\,\frac{q^{\frac{1}{2}} + q^i(q^{D+\frac{1}{2}} + q^D - 2) - q^{2i}}{q^i(q^{D+\frac{1}{2}} - q^{\frac{1}{2}} - 2) + q^{D+\frac{1}{2}} + q^{\frac{1}{2}}}.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = -\frac{q^i - 1}{q-1}\frac{q^{i-1}-1}{q-1},$$

$$v_i = \frac{q^i-1}{q-1}\frac{(q^{i-1}-1)(q+q^{\frac{1}{2}})}{q^{\frac{1}{2}}-1}\frac{q^i(q^{D+\frac{1}{2}}+q^D-q-q^{\frac{1}{2}}-4)+q^{D+\frac{3}{2}}+q^{D+\frac{1}{2}}+2q^{\frac{1}{2}}}{q^i(q^{D+\frac{1}{2}}-q^{\frac{1}{2}}-2)+q^{D+\frac{1}{2}}+q^{\frac{1}{2}}},$$

$$w_i = -\frac{q(q^{\frac{1}{2}}+1)^2(q^i-1)(q^{i-1}-1)}{(q^{\frac{1}{2}}-1)^2}\frac{q^i(q^D-q-2)+q^{D+\frac{3}{2}}+q^{\frac{1}{2}}}{q^i(q^{D+\frac{1}{2}}-q^{\frac{1}{2}}-2)+q^{D+\frac{1}{2}}+q^{\frac{1}{2}}},$$

$$\Phi_i(\lambda) = u_i(\lambda-\xi)(\lambda-\xi_i), \qquad \xi = q^{\frac{1}{2}}(q^{\frac{1}{2}}+1)^2,$$

$$\xi_i = q^{\frac{1}{2}}(q^{\frac{1}{2}}+1)^2\frac{q^i(q^D-q-2)+q^{D+\frac{3}{2}}+q^{\frac{1}{2}}}{q^i(q^{D+\frac{1}{2}}-q^{\frac{1}{2}}-2)+q^{D+\frac{1}{2}}+q^{\frac{1}{2}}}.$$

Lemma 153. *For the graph $\frac{1}{2}D_{2D+1}(p^n)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = q^{\frac{1}{2}}\frac{(q-1)(q^{i-1}-1)}{(q^{\frac{1}{2}}-1)^2} \qquad (2 \leqslant i \leqslant D).$$

Proof. By combinatorial counting using [31, Section 5]. □

Lemma 154. *We refer to Example 152 and write $E = E_1$. The set $\{E\hat{x} | x \in X\}$ is Norton-balanced. For $2 \leqslant i \leqslant D-1$ and $x, y \in X$ at distance $\partial(x,y) = i$,*

$$0 = Ex_y^- + \frac{q^{i-1}-1}{q^D-q^i}Ex_y^+ - \frac{q^{i-1}-1}{q-1}\frac{q^D-q}{q^{\frac{1}{2}}-1}E\hat{x} - q^{i-1}E\hat{y}. \tag{55}$$

Proof. To get the first assertion, we use Proposition 100(ii). Pick an integer i ($2 \leqslant i \leqslant D-1$). We verify the conditions in (45). We have $z_2 = \xi$, so $\Phi_i(z_2) = 0$. We have

$$\lambda_i = \frac{q^{i-1}-1}{q^i-q^D}, \qquad \frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i} = q^{-1}.$$

Therefore

$$\frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i} - \lambda_i = \frac{q^{D-1}-1}{q^D-q^i} \neq 0.$$

We have verified the conditions in (45), so the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. The linear dependence (55) is obtained using (41), (42). □

22 Example: the Hemmeter graph

Example 155. (See [2, Chapter 6.4], [40, Example 6.1(10)–(12)].) Let $GF(p^n)$ denote a finite field with p odd. The Hemmeter graph $Hem_D(p^n)$ is described in [2, p. 383]; it is distance-regular with diameter D and intersection numbers

$$c_i = \frac{q^i-1}{q-1}, \qquad b_i = \frac{q^D-q^i}{q-1} \qquad (0 \leqslant i \leqslant D),$$

where $q = p^n$. Note that $Hem_D(p^n)$ has the same intersection numbers as $D_D(p^n)$. The graphs $Hem_D(p^n)$ and $D_D(p^n)$ are not isomorphic. The assertions about $D_D(p^n)$ in Example 142 and Lemmas 143, 144 hold for $Hem_D(p^n)$ as well. Note that $Hem_D(p^n)$ is bipartite. The assertions about $\frac{1}{2}D_{2D}(p^n)$ in Section 21 hold for $\frac{1}{2}Hem_{2D}(p^n)$ as well. The assertions about $\frac{1}{2}D_{2D+1}(p^n)$ in Section 21 hold for $\frac{1}{2}Hem_{2D+1}(p^n)$ as well.

23 Example: the Hamming graph

Example 156. (See [2, Chapter 6.4], [40, Example 6.1(13)].) For an integer $N \geqslant 2$, the *Hamming graph* $H(D,N)$ has vertex set X consisting of the D-tuples of elements taken from the set $\{1,2,\dots,N\}$. Vertices $x,y \in X$ are adjacent whenever x,y differ in exactly one coordinate. The graph $H(D,N)$ is distance-regular with diameter D and intersection numbers

$$c_i = i, \qquad b_i = (N-1)(D-i) \qquad (0 \leqslant i \leqslant D).$$

The graph $H(D,2)$ is often called a *D-cube* or *hypercube*. It is bipartite and an antipodal 2-cover.

Example 157. The graph $H(D,N)$ has a Q-polynomial structure such that

$$\theta_i = \theta^*_i = D(N-1) - iN \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has Krawtchouk type, with

$$s = -N, \qquad s^* = -N, \qquad r = N(N-1).$$

This Q-polynomial structure is DC with $\gamma^* = 0$.

Until further notice, assume that $N \geqslant 3$. For $2 \leqslant i \leqslant D$,

$$\alpha_i = i-1, \qquad \beta_i = 0,$$
$$r_i = i-1, \qquad s_i = 1, \qquad z^-_i = 0.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = \frac{(N-1)(D-i)(2DN-iN-N-2D)}{2DN-iN-2D},$$
$$S_i = -\frac{N(N-1)(D-i)}{2DN-iN-2D},$$
$$z^+_{i+1} = \frac{N(N-2)i}{2DN-iN-2D}.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = -i(i-1), \qquad v_i = \frac{i(i-1)N(N-2)}{2DN-iN-2D}, \qquad w_i = 0,$$
$$\Phi_i(\lambda) = u_i(\lambda-\xi)(\lambda-\xi_i), \qquad \xi = 0, \qquad \xi_i = \frac{N(N-2)}{2DN-iN-2D}.$$

We have been assuming that $N \geqslant 3$. From now until the beginning of Lemma 158, we assume that $N = 2$. For $2 \leqslant i \leqslant D$,

$$\alpha_i = i - 1, \qquad \beta_i = 0.$$

For $2 \leqslant i \leqslant D - 1$,

$$r_i = i - 1, \qquad s_i = 1, \qquad z_i^- = 0.$$

For $1 \leqslant i \leqslant D - 1$,

$$R_i = D - i - 1, \qquad S_i = -1, \qquad z_{i+1}^+ = 0.$$

For $2 \leqslant i \leqslant D - 1$,

$$u_i = -i(i-1), \qquad v_i = 0, \qquad w_i = 0,$$
$$\Phi_i(\lambda) = u_i(\lambda - \xi)(\lambda - \xi_i), \qquad \xi = 0, \qquad \xi_i = 0.$$

Lemma 158. *For $H(D, N)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover $z_i = 0$ for $2 \leqslant i \leqslant D$.*

Proof. The graph $H(D, N)$ is a regular near polygon, and hence kite-free. □

Lemma 159. *We refer to Example 157 and write $E = E_1$. Pick distinct $x, y \in X$ and write $i = \partial(x, y)$. For $2 \leqslant i \leqslant D$,*

$$Ex_y^- = (i-1)E\hat{x} + E\hat{y}. \tag{56}$$

For $1 \leqslant i \leqslant D - 1$ and $N = 2$,

$$Ex_y^+ = (D - i - 1)E\hat{x} - E\hat{y}. \tag{57}$$

In any case, the set $\{E\hat{x} | x \in X\}$ is Norton-balanced.

Proof. To get (56), use Proposition 68 and $z_i^- = z_i$. To get (57), use Proposition 80 and $z_{i+1}^+ = z_{i+1}$. It follows from Lemma 24 and (56) that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. □

For D even, the hypercube $H(D, 2)$ has a second Q-polynomial structure that we now describe.

Example 160. (See [2, Chapter 6.4], [40, Example 6.1(14)].) Assume that D is even. The hypercube $H(D, 2)$ has a Q-polynomial structure such that

$$\theta_i = \theta_i^* = (-1)^i(D - 2i) \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has Bannai/Ito type, with

$$r_1 = r_2 = -(D+1)/2, \qquad s = s^* = D + 1, \qquad h = h^* = -1.$$

This Q-polynomial structure is DC with $\gamma^* = 0$.

For $2 \leqslant i \leqslant D$,

$$\alpha_i = \frac{(D-3)\big(1+(-1)^i\big)}{2(-1)^i(D-2i+1)}, \qquad \beta_i = \frac{1-(-1)^i(2i-3)}{2(-1)^i(D-2i+1)}.$$

For $2 \leqslant i \leqslant D-1$,

$$r_i = 1-i, \qquad s_i = (-1)^{i-1}, \qquad z_i^- = 0.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = i+1-D, \qquad S_i = (-1)^i, \qquad z_{i+1}^+ = 0.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = 0, \qquad v_i = 0, \qquad w_i = 0, \qquad \Phi_i(\lambda) = 0.$$

Lemma 161. *We refer to Example 160 and write $E = E_1$. The set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Pick distinct $x, y \in X$ and write $i = \partial(x, y)$. For $2 \leqslant i \leqslant D$,*

$$Ex_y^- = (1-i)E\hat{x} - (-1)^i E\hat{y}. \tag{58}$$

For $1 \leqslant i \leqslant D-1$,

$$Ex_y^+ = (i+1-D)E\hat{x} + (-1)^i E\hat{y}. \tag{59}$$

Proof. To get (58), use Proposition 68 and $z_i^- = z_i$. To get (59), use Proposition 80 and $z_{i+1}^+ = z_{i+1}$. It follows from (58), (59) that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. □

24 Example: the halved hypercube

Recall that the hypercube $H(D, 2)$ is bipartite.

Example 162. (See [2, Chapter 6.4], [40, Example 6.1(15)].) The halved graph $\frac{1}{2}H(2D, 2)$ is distance-regular, with diameter D and intersection numbers

$$c_i = i(2i-1), \qquad b_i = (D-i)(2D-1-2i) \qquad (0 \leqslant i \leqslant D).$$

The graph $\frac{1}{2}H(2D, 2)$ is an antipodal 2-cover.

Example 163. The graph $\frac{1}{2}H(2D, 2)$ has a Q-polynomial structure such that

$$\theta_i = D(2D-1) - 2i(2D-i), \qquad \theta_i^* = 2D - 4i \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has dual Hahn type with

$$r = -D - 1/2, \qquad s = -2D - 1, \qquad s^* = -4, \qquad h = 2.$$

This Q-polynomial structure is DC with $\gamma^* = 0$. For $2 \leqslant i \leqslant D$,

$$\alpha_i = i - 1, \qquad \beta_i = 0.$$

For $2 \leqslant i \leqslant D-1$,

$$r_i = (2i-1)(i-1), \qquad s_i = 2i - 1, \qquad z_i^- = 4(i-1).$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = (2D-2i-1)(D-i-1), \qquad S_i = 2i + 1 - 2D, \qquad z_{i+1}^+ = 4i.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = -i(i-1), \qquad v_i = 8i(i-1), \qquad w_i = -16i(i-1),$$
$$\Phi_i(\lambda) = u_i(\lambda - \xi)(\lambda - \xi_i), \qquad \xi = 4, \qquad \xi_i = 4.$$

Lemma 164. *For the graph $\frac{1}{2}H(2D,2)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = 4(i-1) \qquad (2 \leqslant i \leqslant D).$$

Proof. By combinatorial counting. □

Lemma 165. *We refer to Example 163 and write $E = E_1$. The set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Pick distinct $x, y \in X$ and write $i = \partial(x,y)$. For $2 \leqslant i \leqslant D$,*

$$Ex_y^- = (2i-1)(i-1)E\hat{x} + (2i-1)E\hat{y}. \tag{60}$$

For $1 \leqslant i \leqslant D-1$,

$$Ex_y^+ = (2D-2i-1)(D-i-1)E\hat{x} + (2i+1-2D)E\hat{y}. \tag{61}$$

Proof. To get (60), use Proposition 68 and $z_i^- = z_i$. To get (61), use Proposition 80 and $z_{i+1}^+ = z_{i+1}$. It follows from (60), (61) that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. □

Example 166. (See [2, Chapter 6.4], [40, Example 6.1(16)].) The halved graph $\frac{1}{2}H(2D+1,2)$ is distance-regular, with diameter D and intersection numbers

$$c_i = i(2i-1), \qquad b_i = (D-i)(2D+1-2i) \qquad (0 \leqslant i \leqslant D).$$

Example 167. The graph $\frac{1}{2}H(2D+1,2)$ has a Q-polynomial structure such that

$$\theta_i = D(2D+1) - 2i(2D-i+1), \qquad \theta_i^* = 2D+1-4i \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has dual Hahn type with

$$r = -D - 3/2, \qquad s = -2D-2, \qquad s^* = -4, \qquad h = 2.$$

This Q-polynomial structure is DC with $\gamma^* = 0$. For $2 \leqslant i \leqslant D$,

$$\alpha_i = i - 1, \qquad \beta_i = 0,$$
$$r_i = (2i-1)(i-1), \qquad s_i = 2i - 1, \qquad z_i^- = 4(i-1).$$

For $1 \leqslant i \leqslant D - 1$,

$$R_i = (2D - 2i - 1)(D - i), \qquad S_i = 2(i - D), \qquad z_{i+1}^+ = 4i.$$

For $2 \leqslant i \leqslant D - 1$,

$$u_i = -i(i-1), \qquad v_i = 8i(i-1), \qquad w_i = -16i(i-1),$$
$$\Phi_i(\lambda) = u_i(\lambda - \xi)(\lambda - \xi_i), \qquad \xi = 4, \qquad \xi_i = 4.$$

Lemma 168. *For the graph $\frac{1}{2}H(2D+1, 2)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = 4(i-1) \qquad (2 \leqslant i \leqslant D).$$

Proof. By combinatorial counting. □

Lemma 169. *We refer to Example 167 and write $E = E_1$. The set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Pick distinct $x, y \in X$ and write $i = \partial(x, y)$. For $2 \leqslant i \leqslant D$,*

$$Ex_y^- = (2i-1)(i-1)E\hat{x} + (2i-1)E\hat{y}. \tag{62}$$

For $1 \leqslant i \leqslant D - 1$,

$$Ex_y^+ = (2D - 2i - 1)(D - i)E\hat{x} + 2(i - D)E\hat{y}. \tag{63}$$

Proof. To get (62), use Proposition 68 and $z_i^- = z_i$. To get (63), use Proposition 80 and $z_{i+1}^+ = z_{i+1}$. It follows from (62), (63) that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. □

We now give a second Q-polynomial structure for $\frac{1}{2}H(2D+1, 2)$.

Example 170. (See [2, Chapter 6.4], [40, Example 6.1(18)].) The graph $\frac{1}{2}H(2D+1, 2)$ has a Q-polynomial structure such that

$$\theta_i = \theta_i^* = D(2D+1) - 4i(2D - 2i + 1) \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has Racah type with

$$r_1 = -D/2 - 3/4, \qquad r_2 = -D/2 - 5/4,$$
$$s = s^* = -D - 3/2, \qquad h = h^* = 8.$$

This Q-polynomial structure is DC with $\gamma^* = 16$. For $2 \leqslant i \leqslant D$,

$$\alpha_i = \frac{(i-1)(2D-5)(2D-2i+1)}{(2D-3)(2D-4i+3)},$$
$$\beta_i = -\frac{4(i-1)(i-2)}{(2D-3)(2D-4i+3)},$$
$$r_i = \frac{(i-1)(2i-1)\big(4i^2-2i(2D+3)+2D^2+D\big)}{4i^2-2i(2D+1)+2D^2+D},$$
$$s_i = \frac{(2i-1)\big(2D^2+D-2i(2D-1)\big)}{4i^2-2i(2D+1)+2D^2+D},$$
$$z_i^- = \frac{4(i-1)}{2D-5}\,\frac{4D^3-D-i(16D^2+4D-6)+8i^2(3D-1)-8i^3}{4i^2-2i(2D+1)+2D^2+D}.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = \frac{(D-i)(2D-2i-1)\big(4i^2-2i(2D-1)+2D^2-3D-2\big)}{4i^2-2i(2D+1)+2D^2+D},$$
$$S_i = -\frac{2(D-i)\big(2D^2-D-1-2i(2D-1)\big)}{4i^2-2i(2D+1)+2D^2+D},$$
$$z_{i+1}^+ = \frac{4i}{2D-5}\,\frac{8i^3-8i^2(D+2)+8i(2D+1)+4D^3-12D^2-D-1}{4i^2-2i(2D+1)+2D^2+D}.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = -\frac{i(i-1)(2D-5)^2(2D-2i+1)(2D-2i-1)}{(2D-3)^2(2D-4i+3)(2D-4i-1)},$$
$$v_i = \frac{4i(i-1)(2D-2i+1)(2D-2i-1)}{(2D-3)^2(2D-4i+3)(2D-4i-1)} \times \frac{16D^4-64D^3+80D^2+3-4i(8D^3-20D^2+14D+13)+8i^2(4D^2-12D+13)}{4i^2-2i(2D+1)+2D^2+D},$$
$$w_i = -\frac{16i(i-1)(2D-1)(2D-2i+1)(2D-2i-1)}{(2D-3)^2(2D-4i-1)(2D-4i+3)} \times \frac{4D^3-12D^2+19D-3-2i(4D^2-1)+4i^2(2D-1)}{4i^2-2i(2D+1)+2D^2+D},$$
$$\Phi_i(\lambda) = u_i(\lambda-\xi)(\lambda-\xi_i), \qquad \xi = 4,$$
$$\xi_i = \frac{4(2D-1)}{(2D-5)^2}\,\frac{4D^3-12D^2+19D-3-2i(4D^2-1)+4i^2(2D-1)}{4i^2-2i(2D+1)+2D^2+D}.$$

Lemma 171. *We refer to Example 170 and write $E = E_1$. The set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Pick distinct $x, y \in X$ and write $i = \partial(x,y)$. For $2 \leqslant i \leqslant D-1$,*

$$0 = Ex_y^- + \frac{2(i-1)}{2D-2i-1}Ex_y^+ - (i-1)(2D-5)E\hat{x} + \frac{2D-3}{2i-2D+1}E\hat{y}. \tag{64}$$

For $i = D$,

$$0 = Ex_y^- - (D-1)(2D-5)E\hat{x} + (2D-3)E\hat{y}. \tag{65}$$

Proof. To get the first assertion, we use Proposition 100(ii). Pick an integer i ($2 \leqslant i \leqslant D-1$). We verify the conditions in (45). We have $z_2 = \xi$, so $\Phi_i(z_2) = 0$. We have

$$\lambda_i = \frac{2(i-1)}{2i-2D+1}, \qquad \frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} = \frac{2D-4i-1}{2D-4i+3}.$$

Therefore

$$\frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} - \lambda_i = \frac{(2D-5)(2D-4i+1)}{(2D-2i-1)(2D-4i+3)} \neq 0.$$

We have verified the conditions in (45), so the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. The linear dependence (64) is obtained using (41), (42). To obtain (65), use Proposition 68 and $z_D^- = z_D$. □

25 Example: the folded hypercube

Recall that the hypercube $H(D, 2)$ is an antipodal 2-cover. Its antipodal quotient is called a folded cube.

Example 172. (See [2, Chapter 6.4], [40, Example 6.1(20)].) The folded cube $\tilde{H}(2D, 2)$ is distance-regular, with diameter D and intersection numbers

$$c_i = i \quad (1 \leqslant i \leqslant D-1), \qquad c_D = 2D,$$
$$b_i = 2D - i \quad (0 \leqslant i \leqslant D-1).$$

Example 173. The graph $\tilde{H}(2D, 2)$ has a Q-polynomial structure such that

$$\theta_i = 2D - 4i \quad (0 \leqslant i \leqslant D),$$
$$\theta_i^* = D(2D-1) - 2i(2D-i) \quad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has Hahn type with

$$r = -D - 1/2, \qquad s = -4, \qquad s^* = -2D - 1, \qquad h^* = 2.$$

This Q-polynomial structure is DC with $\gamma^* = 4$. We have

$$\alpha_i = \frac{(2D-3)(i-1)(2D-i)}{2(D-1)(2D-2i+1)} \qquad (2 \leqslant i \leqslant D),$$

$$\beta_i = -\frac{(i-1)(i-2)}{2(D-1)(2D-2i+1)} \qquad (2 \leqslant i \leqslant D),$$

$$r_i = \frac{(i-1)(i^2 - i(2D+1) + 2D^2 - D)}{i^2 - 2iD + 2D^2 - D} \qquad (2 \leqslant i \leqslant D-1), \qquad r_D = 2(D-2),$$

$$s_i = \frac{(2D-1)(D-i)}{i^2 - 2iD + 2D^2 - D} \qquad (2 \leqslant i \leqslant D),$$

$$z_i^- = -\frac{2(i-1)(2D-i)(2D-i-1)}{(2D-3)(i^2 - 2iD + 2D^2 - D)} \qquad (2 \leqslant i \leqslant D-1), \qquad z_D^- = 0.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = \frac{(2D-i-1)(i^2 - i(2D-1) + 2D^2 - 3D)}{i^2 - 2iD + 2D^2 - D},$$
$$S_i = -\frac{(2D-1)(D-i)}{i^2 - 2iD + 2D^2 - D},$$
$$z^+_{i+1} = \frac{2i(i-1)(2D-i-1)}{(2D-3)(i^2 - 2iD + 2D^2 - D)}.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = -\frac{i(i-1)(2D-3)^2(2D-i)(2D-i-1)}{4(D-1)^2(2D-2i+1)(2D-2i-1)},$$
$$v_i = -\frac{2i(i-1)(2D-i)(2D-i-1)(2i^2-4iD+2D^2+D-1)}{(D-1)(2D-2i+1)(2D-2i-1)(i^2-2iD+2D^2-D)},$$
$$w_i = 0,$$
$$\Phi_i(\lambda) = u_i(\lambda - \xi)(\lambda - \xi_i), \qquad \xi = 0,$$
$$\xi_i = -\frac{8(D-1)(2i^2-4iD+2D^2+D-1)}{(2D-3)^2(i^2-2iD+2D^2-D)}.$$

Lemma 174. *For the graph $\tilde{H}(2D,2)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = 0 \qquad (2 \leqslant i \leqslant D).$$

Proof. The graph $\tilde{H}(2D,2)$ is bipartite, and hence kite-free. □

Lemma 175. *We refer to Example 173 and write $E = E_1$. The set $\{E\hat{x}|x \in X\}$ is Norton-balanced. For $x, y \in X$ we have*

$$0 = Ex^-_y + Ex^+_y - 2(D-2)E\hat{x}.$$

Proof. The graph $\tilde{H}(2D,2)$ is bipartite and $\theta_1 = 2(D-2)$. □

Example 176. (See [2, Chapter 6.4], [40, Example 6.1(19)].) The folded cube $\tilde{H}(2D+1,2)$ is distance-regular, with diameter D and intersection numbers

$$c_i = i \qquad (1 \leqslant i \leqslant D),$$
$$b_i = 2D+1-i \qquad (0 \leqslant i \leqslant D-1).$$

Example 177. The graph $\tilde{H}(2D+1,2)$ has a Q-polynomial structure such that

$$\theta_i = 2D+1-4i \qquad (0 \leqslant i \leqslant D),$$
$$\theta^*_i = D(2D+1) - 2i(2D-i+1) \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has Hahn type with

$$r = -D - 3/2, \qquad s = -4, \qquad s^* = -2D - 2, \qquad h^* = 2.$$

This Q-polynomial structure is DC with $\gamma^* = 4$. For $2 \leqslant i \leqslant D$,

$$\begin{aligned}
\alpha_i &= \frac{(D-1)(i-1)(2D-i+1)}{(2D-1)(D-i+1)}, \\
\beta_i &= -\frac{(i-1)(i-2)}{2(2D-1)(D-i+1)}, \\
r_i &= \frac{(i-1)(i^2-2i(D+1)+2D^2+D)}{i^2-i(2D+1)+2D^2+D}, \\
s_i &= \frac{D(2D-2i+1)}{i^2-i(2D+1)+2D^2+D}, \\
z_i^- &= -\frac{(i-1)(2D-i)(2D-i+1)}{(D-1)(i^2-i(2D+1)+2D^2+D)}.
\end{aligned}$$

For $1 \leqslant i \leqslant D-1$,

$$\begin{aligned}
R_i &= \frac{(2D-i)(i^2-2iD+2D^2-D-1)}{i^2-i(2D+1)+2D^2+D}, \\
S_i &= -\frac{D(2D-2i+1)}{i^2-i(2D+1)+2D^2+D}, \\
z_{i+1}^+ &= \frac{i(i-1)(2D-i)}{(D-1)(i^2-i(2D+1)+2D^2+D)}.
\end{aligned}$$

For $2 \leqslant i \leqslant D-1$,

$$\begin{aligned}
u_i &= -\frac{i(i-1)(D-1)^2(2D-i)(2D-i+1)}{(2D-1)^2(D-i)(D-i+1)}, \\
v_i &= -\frac{i(i-1)(2D-i)(2D-i+1)(2i^2-2i(2D+1)+2D^2+3D)}{(D-i)(2D-1)(D-i+1)(i^2-i(2D+1)+2D^2+D)}, \\
w_i &= 0, \\
\Phi_i(\lambda) &= u_i(\lambda-\xi)(\lambda-\xi_i), \qquad \xi = 0, \\
\xi_i &= -\frac{(2D-1)(2i^2-2i(2D+1)+2D^2+3D)}{(D-1)^2(i^2-i(2D+1)+2D^2+D)}.
\end{aligned}$$

Lemma 178. *For the graph $\tilde{H}(2D+1,2)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = 0 \qquad (2 \leqslant i \leqslant D).$$

Proof. The graph $\tilde{H}(2D+1,2)$ is almost bipartite, and hence kite-free. □

Lemma 179. *We refer to Example 177 and write $E = E_1$. The set $\{E\hat{x}|x \in X\}$ is Norton-balanced. For $0 \leqslant i \leqslant D-1$ and $x, y \in X$ at distance $\partial(x,y) = i$,*

$$0 = Ex_y^- + Ex_y^+ + (3-2D)E\hat{x}.$$

Proof. The graph $\tilde{H}(2D+1,2)$ is almost bipartite and $\theta_1 = 2D-3$. □

Example 180. (See [2, Chapter 6.4], [40, Example 6.1(17)].) The graph $\tilde{H}(2D+1,2)$ has a second Q-polynomial structure such that

$$\theta_i = \theta_i^* = (-1)^i(2D-2i+1) \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has Bannai/Ito type, with

$$r_1 = -D-1, \qquad r_2 = -2D-2, \qquad s = s^* = 2D+2, \qquad h = h^* = -1.$$

This Q-polynomial structure is DC with $\gamma^* = 0$.

For $2 \leqslant i \leqslant D$,

$$\alpha_i = \frac{(D-1)\big(1+(-1)^i\big)}{2(-1)^i(D-i+1)}, \qquad \beta_i = \frac{1-(-1)^i(2i-3)}{4(-1)^i(D-i+1)},$$
$$r_i = 1-i, \qquad s_i = (-1)^{i-1}, \qquad z_i^- = 0.$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = i-2D, \qquad S_i = (-1)^i, \qquad z_{i+1}^+ = 0.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = 0, \qquad v_i = 0, \qquad w_i = 0, \qquad \Phi_i(\lambda) = 0.$$

Lemma 181. *We refer to Example 180 and write $E = E_1$. The set $\{E\hat{x}|x \in X\}$ is Norton-balanced. Pick distinct $x, y \in X$ and write $i = \partial(x,y)$. For $2 \leqslant i \leqslant D$,*

$$Ex_y^- = (1-i)E\hat{x} - (-1)^i E\hat{y}. \tag{66}$$

For $1 \leqslant i \leqslant D-1$,

$$Ex_y^+ = (i-2D)E\hat{x} + (-1)^i E\hat{y}. \tag{67}$$

Proof. To get (66), use Proposition 68 and $z_i^- = 0 = z_i$. To get (67), use Proposition 80 and $z_{i+1}^+ = 0 = z_{i+1}$. It follows from (66), (67) that the set $\{E\hat{x}|x \in X\}$ is Norton-balanced. □

26 Example: the folded-half hypercube

Example 182. (See [2, Chapter 6.4], [40, Example 6.1(21)].) The folded-half graph $\frac{1}{2}\tilde{H}(4D,2)$ is distance-regular, with diameter D and intersection numbers

$$c_i = i(2i-1) \qquad (1 \leqslant i \leqslant D-1), \qquad c_D = 2D(2D-1),$$
$$b_i = (2D-i)(4D-2i-1) \qquad (0 \leqslant i \leqslant D-1).$$

Example 183. The graph $\frac{1}{2}\tilde{H}(4D,2)$ has a Q-polynomial structure such that

$$\theta_i = \theta_i^* = 2D(4D-1) - 8i(2D-i) \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has Racah type with

$$r_1 = -D - 1/2, \qquad r_2 = -2D - 1/2, \qquad s = s^* = -2D - 1, \qquad h = h^* = 8.$$

This Q-polynomial structure is DC with $\gamma^* = 16$. We have

$$\alpha_i = \frac{(i-1)(2D-3)(2D-i)}{2(D-1)(2D-2i+1)} \qquad (2 \leqslant i \leqslant D),$$
$$\beta_i = -\frac{(i-1)(i-2)}{2(D-1)(2D-2i+1)} \qquad (2 \leqslant i \leqslant D),$$
$$r_i = \frac{(i-1)(2i-1)\big(2i^2 - 2i(2D+1) + 4D^2 - D\big)}{2i^2 - 4iD + 4D^2 - D} \qquad (2 \leqslant i \leqslant D-1),$$
$$r_D = 2(D-1)(2D-3),$$
$$s_i = \frac{(2i-1)(2i - 4iD + 4D^2 - D)}{2i^2 - 4iD + 4D^2 - D} \qquad (2 \leqslant i \leqslant D-1), \qquad s_D = 2,$$
$$z_i^- = \frac{4(i-1)\big(8D^3 - 6D^2 + D - i(16D^2 - 6D - 1) + i^2(12D-5) - 2i^3\big)}{(2D-3)(2i^2 - 4iD + 4D^2 - D)}$$
$$(2 \leqslant i \leqslant D-1), \qquad z_D^- = 4(D-1).$$

For $1 \leqslant i \leqslant D-1$,

$$R_i = \frac{(2D-i-1)(4D-2i-1)\big(2i^2 - 2i(2D-1) + 4D^2 - 5D\big)}{2i^2 - 4iD + 4D^2 - D},$$
$$S_i = -\frac{(4D-2i-1)(2i - 4iD + 4D^2 - 3D)}{2i^2 - 4iD + 4D^2 - D},$$
$$z_{i+1}^+ = \frac{4i\big(2i^3 - i^2(4D+3) + 8iD + 8D^3 - 18D^2 + 7D - 1\big)}{(2D-3)(2i^2 - 4iD + 4D^2 - D)}.$$

For $2 \leqslant i \leqslant D-1$,

$$u_i = -\frac{i(i-1)(2D-3)^2(2D-i)(2D-i-1)}{4(D-1)^2(2D-2i+1)(2D-2i-1)},$$

$$
\begin{aligned}
v_i &= \frac{2i(i-1)(2D-i)(2D-i-1)}{(D-1)^2(2D-2i+1)(2D-2i-1)} \\
&\quad \times \frac{i^2(8D^2-16D+10)-i(16D^3-32D^2+20D)+16D^4-48D^3+50D^2-17D+2}{2i^2-4iD+4D^2-D}, \\
w_i &= -\frac{4i(i-1)(2D-1)(2D-i)(2D-i-1)}{(D-1)^2(2D-2i-1)(2D-2i+1)} \\
&\quad \times \frac{i^2(4D-2)-i(8D^2-4D)+8D^3-18D^2+17D-4}{2i^2-4iD+4D^2-D}, \\
\Phi_i(\lambda) &= u_i(\lambda-\xi)(\lambda-\xi_i), \qquad \xi = 4, \\
\xi_i &= \frac{4(2D-1)\big(i^2(4D-2)-i(8D^2-4D)+8D^3-18D^2+17D-4\big)}{(2D-3)^2(2i^2-4iD+4D^2-D)}.
\end{aligned}
$$

Lemma 184. *For the graph $\frac{1}{2}\tilde{H}(4D,2)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = 4(i-1) \qquad (2 \leqslant i \leqslant D).$$

Proof. By combinatorial counting. □

Lemma 185. *We refer to Example 183 and write $E = E_1$. The set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Pick distinct $x, y \in X$ and write $i = \partial(x,y)$. For $2 \leqslant i \leqslant D-1$,*

$$0 = Ex_y^- + \frac{i-1}{2D-i-1}Ex_y^+ + 2(i-1)(3-2D)E\hat{x} + \frac{2(D-1)}{i+1-2D}E\hat{y}. \tag{68}$$

For $i = D$,

$$0 = Ex_y^- + 2(D-1)(3-2D)E\hat{x} - 2E\hat{y}. \tag{69}$$

Proof. To get the first assertion, we use Proposition 100(ii). Pick an integer i ($2 \leqslant i \leqslant D-1$). We verify the conditions in (45). We have $z_2 = \xi$, so $\Phi_i(z_2) = 0$. We have

$$\lambda_i = \frac{i-1}{i+1-2D}, \qquad \frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} = \frac{2D-2i-1}{2D-2i+1}.$$

Therefore

$$\frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} - \lambda_i = \frac{2(2D-3)(D-i)}{(2D-i-1)(2D-2i+1)} \neq 0.$$

We have verified the conditions in (45), so the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. The linear dependence (68) is obtained using (41), (42). To obtain (69), use Proposition 68 and $z_D^- = z_D$. □

Example 186. (See [2, Chapter 6.4], [40, Example 6.1(22)].) The folded-half graph $\frac{1}{2}\tilde{H}(4D+2,2)$ is distance-regular, with diameter D and intersection numbers

$$
\begin{aligned}
c_i &= i(2i-1) \qquad (1 \leqslant i \leqslant D), \\
b_i &= (2D-i+1)(4D-2i+1) \qquad (0 \leqslant i \leqslant D-1).
\end{aligned}
$$

Example 187. The graph $\frac{1}{2}\tilde{H}(4D+2,2)$ has a Q-polynomial structure such that

$$\theta_i = \theta_i^* = (2D+1)(4D+1) - 8i(2D-i+1) \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has Racah type with

$$r_1 = -D - 3/2, \qquad r_2 = -2D - 3/2, \qquad s = s^* = -2D - 2, \qquad h = h^* = 8.$$

This Q-polynomial structure is DC with $\gamma^* = 16$.

For $2 \leqslant i \leqslant D$,

$$\begin{aligned}
\alpha_i &= \frac{(i-1)(D-1)(2D-i+1)}{(2D-1)(D-i+1)}, \\
\beta_i &= -\frac{(i-1)(i-2)}{2(2D-1)(D-i+1)}, \\
r_i &= \frac{(i-1)(2i-1)\big(4i^2 - 8i(D+1) + 8D^2 + 6D + 1\big)}{4i^2 - 4i(2D+1) + 8D^2 + 6D + 1}, \\
s_i &= \frac{(2i-1)(8D^2 + 6D + 1 - 8iD)}{4i^2 - 4i(2D+1) + 8D^2 + 6D + 1}, \\
z_i^- &= \frac{4(i-1)\big(8D^3 + 6D^2 + D - i(16D^2 + 10D) + i^2(12D+1) - 2i^3\big)}{(D-1)\big(4i^2 - 4i(2D+1) + 8D^2 + 6D + 1\big)}.
\end{aligned}$$

For $1 \leqslant i \leqslant D-1$,

$$\begin{aligned}
R_i &= \frac{(2D-i)(4D-2i+1)(4i^2 - 8iD + 8D^2 - 2D - 3)}{4i^2 - 4i(2D+1) + 8D^2 + 6D + 1}, \\
S_i &= -\frac{(4D-2i+1)(8D^2 + 2D - 1 - 8iD)}{4i^2 - 4i(2D+1) + 8D^2 + 6D + 1}, \\
z_{i+1}^+ &= \frac{4i}{D-1} \, \frac{2i^3 - i^2(4D+5) + 4i(2D+1) + 8D^3 - 6D^2 - 5D - 1}{4i^2 - 4i(2D+1) + 8D^2 + 6D + 1}.
\end{aligned}$$

For $2 \leqslant i \leqslant D-1$,

$$\begin{aligned}
u_i &= -\frac{i(i-1)(D-1)^2(2D-i)(2D-i+1)}{(2D-1)^2(D-i)(D-i+1)}, \\
v_i &= \frac{4i(i-1)(2D-i)(2D-i+1)}{(2D-1)^2(D-i)(D-i+1)} \\
&\quad \times \frac{16D^4 - 16D^3 + 2D^2 + 5D + 1 - 4i(4D^3 - 2D^2 + 1) + 4i^2(2D^2 - 2D + 1)}{4i^2 - 4i(2D+1) + 8D^2 + 6D + 1}, \\
w_i &= -\frac{16Di(i-1)(2D-i)(2D-i+1)}{(2D-1)^2(D-i)(D-i+1)} \\
&\quad \times \frac{4i^2D - 4iD(2D+1) + 8D^3 - 6D^2 + 5D + 1}{4i^2 - 4i(2D+1) + 8D^2 + 6D + 1},
\end{aligned}$$

$$\Phi_i(\lambda) = u_i(\lambda-\xi)(\lambda-\xi_i), \qquad \xi = 4,$$
$$\xi_i = \frac{4D}{(D-1)^2}\,\frac{4i^2D - 4iD(2D+1) + 8D^3 - 6D^2 + 5D + 1}{4i^2 - 4i(2D+1) + 8D^2 + 6D + 1}.$$

Lemma 188. *For the graph $\frac{1}{2}\tilde{H}(4D+2,2)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = 4(i-1) \qquad (2 \leqslant i \leqslant D).$$

Proof. By combinatorial counting. □

Lemma 189. *We refer to Example 187 and write $E = E_1$. The set $\{E\hat{x} | x \in X\}$ is Norton-balanced. For $2 \leqslant i \leqslant D-1$ and $x, y \in X$ at distance $i = \partial(x,y)$,*

$$0 = Ex_y^- + \frac{i-1}{2D-i}Ex_y^+ - 4(D-1)(i-1)E\hat{x} + \frac{2D-1}{i-2D}E\hat{y}. \tag{70}$$

Proof. To get the first assertion, we use Proposition 100(ii). Pick an integer i $(2 \leqslant i \leqslant D-1)$. We verify the conditions in (45). We have $z_2 = \xi$, so $\Phi_i(z_2) = 0$. We have

$$\lambda_i = \frac{i-1}{i-2D}, \qquad \frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} = \frac{D-i}{D-i+1}.$$

Therefore

$$\frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} - \lambda_i = \frac{(D-1)(2D-2i+1)}{(2D-i)(D-i+1)} \neq 0.$$

We have verified the conditions in (45), so the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. The linear dependence (70) is obtained using (41), (42). □

27 Example: the Hermitean forms graph

Example 190. (See [2, Chapter 6.4], [40, Note 6.2].) let $GF(p^n)$ denote a finite field. The Hermitean forms graph $Her_D(p^n)$ is distance-regular with diameter D and intersection numbers

$$c_i = q^{i-1}\frac{q^i-1}{q-1}, \qquad b_i = -\frac{q^{2D}-q^{2i}}{q-1} \qquad (0 \leqslant i \leqslant D),$$

where $q = -p^n$.

Example 191. The graph $Her_D(p^n)$ has a Q-polynomial structure such that

$$\theta_i = \theta^*_i = -\frac{q^{2D-i}-1}{q-1} \qquad (0 \leqslant i \leqslant D).$$

This Q-polynomial structure has affine q-Krawtchouk type, with

$$r = -q^{-D-1}, \qquad h = -\frac{q^{2D}}{q-1}, \qquad h^* = -\frac{q^{2D}}{q-1}.$$

This Q-polynomial structure is DC iff $a_1^* = 0$ iff $a_1 = 0$ iff $p^n = 2$ iff $q = -2$ (provided that $D \geqslant 4$). Assume that $q = -2$. We have

$$\gamma^* = q^{-1} - 1 \neq 0.$$

For $2 \leqslant i \leqslant D$,

$$\begin{aligned}
\alpha_i &= \frac{q^{i-1}-1}{q-1}, \qquad \beta_i = 0,\\
r_i &= q^{i-1}\frac{q^{i-1}-1}{q-1}\,\frac{q^{2D+i}+q^{2D+1}+q^i}{q^{2D+i}+q^{2D}-2q^i},\\
s_i &= q^{2i-2}\frac{q^{2D+1}+q^{2D}-q^i-q}{q^{2D+i}+q^{2D}-2q^i},\\
z_i^- &= (q^{i-1}-1)\frac{q^{2D+1}+q^{2D}+q^{2i}}{q^{2D+i}+q^{2D}-2q^i}.
\end{aligned}$$

For $1 \leqslant i \leqslant D-1$,

$$\begin{aligned}
R_i &= -\frac{1}{q}\,\frac{q^{2D}-q^{2i}}{q-1}\,\frac{q^{2D+i}+q^{2D}+q^i}{q^{2D+i}+q^{2D}-2q^i},\\
S_i &= q^{i-1}\frac{q^{2D}-q^{2i}}{q^{2D+i}+q^{2D}-2q^i},\\
z_{i+1}^+ &= -\frac{q^i(q^i-1)(q^i-q)}{q^{2D+i}+q^{2D}-2q^i}.
\end{aligned}$$

For $2 \leqslant i \leqslant D-1$,

$$\begin{aligned}
u_i &= -\frac{q^i-1}{q-1}\,\frac{q^{i-1}-1}{q-1},\\
v_i &= \frac{(q^i-1)(q^{i-1}-1)}{q-1}\,\frac{q^{2D+1}+q^{2D}-2q^i}{q^{2D+i}+q^{2D}-2q^i} \qquad w_i = 0,\\
\Phi_i(\lambda) &= u_i(\lambda-\xi)(\lambda-\xi_i), \qquad \xi = 0, \qquad \xi_i = (q-1)\frac{q^{2D+1}+q^{2D}-2q^i}{q^{2D+i}+q^{2D}-2q^i}.
\end{aligned}$$

Lemma 192. *For $Her_D(p^n)$ the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Moreover*

$$z_i = 0 \qquad (2 \leqslant i \leqslant D).$$

Proof. The graph $Her_D(p^n)$ is kite-free by [41, Theorem 2.12]. □

Lemma 193. *We refer to Example 191 with $q = -2$. Write $E = E_1$. The set $\{E\hat{x}|x \in X\}$ is not Norton-balanced. However the following linear dependencies hold. Pick distinct $x, y \in X$ and write $i = \partial(x, y)$. For $1 \leqslant i \leqslant D-1$,*

$$0 = Ex_y^- - q^{-1}Ex_y^+ + \frac{1}{q^2}\frac{q^i - q^{2D}}{q-1}E\hat{x} + q^{i-2}E\hat{y}. \tag{71}$$

For $i = D$,

$$0 = Ex_y^- - q^{D-2}\frac{q^D - 1}{q-1}E\hat{x} + q^{D-2}E\hat{y}. \tag{72}$$

Proof. The first assertion follows from Lemma 33(i). To obtain (71), use (41), (42) and

$$\lambda_i = q^{-1} \qquad (2 \leqslant i \leqslant D-1).$$

To obtain (72), use Proposition 68 and $z_D^- = 0 = z_D$. Alternatively, (71) and (72) follow from Lemma 33(ii). □

28 Example: the Doob graphs

In this section, we will discuss the Doob graphs and their relationship to the Hamming graphs. In this discussion the Shrikhande graph makes an appearance. The Shrikhande graph is distance-regular with diameter 2; it has the same intersection numbers as the Hamming graph $H(2, 4)$. The Shrikhande graph is not isomorphic to $H(2, 4)$, because the Shrikhande graph has a 2-kite and $H(2, 4)$ does not. See [2, Example 2.10] for more information about the Shrikhande graph.

Example 194. (See [2, Chapter 6.4], [7, p. 262].) By a *Doob graph*, we mean a Cartesian product of graphs, with each factor isomorphic to the Shrikhande graph or the complete graph K_4. We require that in the Cartesian product, at least one factor is isomorphic to the Shrikhande graph. Let Γ denote a Doob graph with diameter D. We have $D = 2n+m$, where n (resp. m) is the number of factors isomorphic to the Shrikhande graph (resp. K_4). The graph Γ is distance-regular and has the same intersection numbers as $H(D, 4)$. However Γ is not isomorphic to $H(D, 4)$. Both $H(D, 4)$ and Γ have a Q-polynomial structure such that

$$\theta_i = \theta_i^* = 3D - 4i \qquad (0 \leqslant i \leqslant D).$$

Every assertion about $H(D, 4)$ in Example 157 holds for Γ. In particular, the Q-polynomial structure for Γ is DC and $\gamma^* = 0$. Moreover

$$z_i^- = 0, \qquad z_{i+1}^+ = \frac{4i}{3D - 2i} \qquad (2 \leqslant i \leqslant D-1). \tag{73}$$

Lemma 195. *Assume that $\Gamma = (X, \mathcal{R})$ is a Doob graph, and write $E = E_1$. Then the set $\{E\hat{x}|x \in X\}$ is not Norton-balanced.*

Proof. We assume that the set $\{E\hat{x}|x \in X\}$ is Norton-balanced, and get a contradiction. There exists a subset $S \subseteq X$ such that (i) the subgraph of Γ induced on S is isomorphic to the Shrikhande graph; (ii) for all $x, y \in S$ and all $z \in X\backslash S$, $\partial(x,z)+\partial(y,z) \geqslant \partial(x,y)+2$. Since the Shrikhande graph has a 2-kite, there exist $x, y \in S$ at distance $\partial(x,y) = 2$ such that $\Gamma(x) \cap \Gamma(y)$ contains an edge. By Definition 44, $\zeta_2(x,y,*) = 1$. By Definition 51 and the construction, $\zeta_3(*,y,x) = 0$. We have $\zeta_2(x,y,*) \neq z_2^-$, so $Ex_y^- \neq r_2E\hat{x} + s_2E\hat{y}$ by Proposition 68. We have $\zeta_3(*,y,x) \neq z_3^+$, so $Ex_y^+ \neq R_2E\hat{x} + S_2E\hat{y}$ by Proposition 80. The vectors $Ex_y^-, Ex_y^+, E\hat{x}, E\hat{y}$ are linearly dependent by Lemma 32. This contradicts Lemma 89, and the result follows. □

29 Further examples

Below we list some Q-polynomial distance-regular graphs $\Gamma = (X, \mathcal{R})$ with diameter $D \geqslant 4$. We describe the Q-polynomial structure of Γ, using the data in [2, Chapter 6.4] and the notation of [45, Section 20]. In each case (i)–(vi) below, the Q-polynomial structure is not DC because the condition in Theorem 128(i) is violated. In each case, $\gamma^* \neq 0$. In each case the set $\{E\hat{x}|x \in X\}$ is not Norton-balanced, where E is the Q-polynomial primitive idempotent of Γ attached to the given Q-polynomial structure. For the cases (i)–(iv) this Norton-balanced assertion follows from Lemma 127, because Γ is distance-transitive and therefore reinforced. For case (vi) the assertion follows from Corollary 94 and the fact that Γ has a non-regular μ-graph [16, 1]. For case (v) the assertion is proved in Lemma 201 below. Let $GF(q)$ denote a finite field.

(i) The folded graph $\tilde{J}(4D, 2D)$ has Racah type with

$$r_1 = -D - 1/2, \qquad r_2 = -2D - 1, \qquad s = -2D - 3/2, \qquad s^* = -2D - 1.$$

(ii) The folded graph $\tilde{J}(4D+2, 2D+1)$ has Racah type with

$$r_1 = -D - 3/2, \qquad r_2 = -2D - 2, \qquad s = -2D - 5/2, \qquad s^* = -2D - 2.$$

(iii) The bilinear forms graph $H_q(D, N)$ $(N \geqslant D)$ has affine q-Krawtchouk type with

$$r = q^{-N-1}.$$

(iv) The alternating forms graph $Alt_q(N)$ $(D = \lfloor N/2 \rfloor)$ has affine q-Krawtchouk type with

$$r = q^{-D-\frac{1}{2}} \text{ (if } N \text{ is even)}, \qquad r = q^{-D-\frac{3}{2}} \text{ (if } N \text{ is odd)}.$$

(v) The quadratic forms graph $Quad_q(N)$ $(D = \lfloor (N+1)/2 \rfloor)$ has affine q-Krawtchouk type with

$$r = q^{-D-\frac{3}{2}} \text{ (if } N \text{ is even)}, \qquad r = q^{-D-\frac{1}{2}} \text{ (if } N \text{ is odd)}.$$

(vi) The twisted Grassmann graph ${}^2J_q(2D+1, D)$ has dual q-Hahn type with

$$r = q^{-D-2}, \qquad s = q^{-2D-3}.$$

This graph has the same intersection numbers as $J_q(2D+1, D)$.

For the rest of this section, our goal is to show that the graph from item (v) is not Norton-balanced. To reach the goal, we will derive some preliminary results that apply to more general Q-polynomial distance-regular graphs. We will be discussing Lemma 91, which involves two vertices x, y and a scalar λ_i. Going forward, the scalar λ_i will be denoted by $\lambda_i(x, y)$ in order to emphasize that it might depend on x, y as well as i.

Assumption 1. Let $\Gamma = (X, \mathcal{R})$ denote a Q-polynomial distance-regular graph with diameter $D \geqslant 4$. Let E denote a Q-polynomial primitive idempotent of Γ with $\gamma^* \neq 0$. Assume that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced.

Lemma 196. *With reference to Assumption 1, pick an integer i $(3 \leqslant i \leqslant D-1)$ and $x, y, z \in X$ such that*

$$\partial(x, y) = i, \qquad \partial(x, z) = 1, \qquad \partial(y, z) = i - 1.$$

Then

$$\begin{aligned} \zeta_i(x, y, z) = \zeta_i(x, y, *) &= z_i^- + \lambda_i(x, y) \frac{\gamma^* b_i}{\theta_i^* + \theta_0^*} \frac{\theta_i^* - \theta_1^*}{\theta_1^* - \theta_2^*} \\ &= \zeta_i(*, y, z) = z_i^+ - \frac{1}{\lambda_{i-1}(z, y)} \frac{\gamma^* c_{i-1}}{\theta_{i-1}^* + \theta_0^*} \frac{\theta_{i-1}^* - \theta_1^*}{\theta_1^* - \theta_2^*}. \end{aligned}$$

Proof. To get the first two equalities, apply Lemmas 32, 91, 93(i) to x, y. To get the last two equalities, apply Lemmas 32, 91 and 93(ii) to z, y. □

Definition 197. With reference to Assumption 1, pick $y \in X$ and an integer $n \geqslant 0$. A path $\{x_i\}_{i=0}^n$ in Γ is called *raising/lowering with respect to y* whenever the following (i), (ii) hold:

(i) $2 \leqslant \partial(x_i, y) \leqslant D - 1$ for $0 \leqslant i \leqslant n$;

(ii) $\partial(x_{i-1}, y) \neq \partial(x_i, y)$ for $1 \leqslant i \leqslant n$.

Lemma 198. *With reference to Assumption 1, pick $y \in X$. Pick an integer i $(2 \leqslant i \leqslant D-1)$ and $x, x' \in \Gamma_i(y)$. Assume that x, x' are connected by a path that is raising/lowering with respect to y. Then $\lambda_i(x, y) = \lambda_i(x', y)$.*

Proof. Routine using Lemma 196 and Definition 197. □

Lemma 199. *With reference to Assumption 1, pick $y \in X$. Pick an integer i $(2 \leqslant i \leqslant D-1)$ and adjacent $x, x' \in \Gamma_i(y)$. Then*

$$\begin{aligned}
&|\Gamma(x) \cap \Gamma(x') \cap \Gamma_{i-1}(y)| + \frac{c_i\theta_2^* - r_i\theta_1^* - s_i\theta_i^*}{\theta_1^* - \theta_2^*} \\
&= \lambda_i(x,y)\left(|\Gamma(x) \cap \Gamma(x') \cap \Gamma_{i+1}(y)| + \frac{b_i\theta_2^* - R_i\theta_1^* - S_i\theta_i^*}{\theta_1^* - \theta_2^*}\right) \\
&= \lambda_i(x',y)\left(|\Gamma(x) \cap \Gamma(x') \cap \Gamma_{i+1}(y)| + \frac{b_i\theta_2^* - R_i\theta_1^* - S_i\theta_i^*}{\theta_1^* - \theta_2^*}\right).
\end{aligned}$$

Proof. To get the first equality, take the inner product of $E\hat{x}'$ with each side of (37), and evaluate the result using Lemma 2(i). To get the second equality, apply the first equality with x, x' interchanged. □

Lemma 200. *With reference to Assumption 1, pick $y \in X$. Pick an integer i $(2 \leqslant i \leqslant D-1)$ and adjacent $x, x' \in \Gamma_i(y)$ such that $\lambda_i(x,y) \neq \lambda_i(x',y)$. Then the following* (i)–(iv) *hold:*

(i) $|\Gamma(x) \cap \Gamma(x') \cap \Gamma_{i-1}(y)| + \dfrac{c_i\theta_2^* - r_i\theta_1^* - s_i\theta_i^*}{\theta_1^* - \theta_2^*} = 0$;

(ii) $|\Gamma(x) \cap \Gamma(x') \cap \Gamma_{i+1}(y)| + \dfrac{b_i\theta_2^* - R_i\theta_1^* - S_i\theta_i^*}{\theta_1^* - \theta_2^*} = 0$;

(iii) $\Gamma(x) \cap \Gamma(x') \cap \Gamma_{i-1}(y) = \emptyset$ *if* $3 \leqslant i \leqslant D-1$;

(iv) $\Gamma(x) \cap \Gamma(x') \cap \Gamma_{i+1}(y) = \emptyset$ *if* $2 \leqslant i \leqslant D-2$.

Proof. (i), (ii) By Lemma 199.

(iii) Assume that $3 \leqslant i \leqslant D-1$ and $\Gamma(x) \cap \Gamma(x') \cap \Gamma_{i-1}(y) \neq \emptyset$. There exists $z \in \Gamma(x) \cap \Gamma(x') \cap \Gamma_{i-1}(y)$. The sequence x, z, x' is a path in Γ that is lowering/raising with respect to y, forcing $\lambda_i(x,y) = \lambda_i(x',y)$ by Lemma 198. This is a contradiction.

(iv) Assume that $2 \leqslant i \leqslant D-2$ and $\Gamma(x) \cap \Gamma(x') \cap \Gamma_{i+1}(y) \neq \emptyset$. There exists $z \in \Gamma(x) \cap \Gamma(x') \cap \Gamma_{i+1}(y)$. The sequence x, z, x' is a path in Γ that is lowering/raising with respect to y, forcing $\lambda_i(x,y) = \lambda_i(x',y)$ by Lemma 198. This is a contradiction. □

We return our attention to the graph Γ from item (v) above.

Lemma 201. *Assume that Γ is from item* (v), *with $D \geqslant 4$. Then the set $\{E\hat{x} | x \in X\}$ is not Norton-balanced.*

Proof. We assume that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced, and get a contradiction. To obtain the contradiction, we show that the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Until further notice, fix $y \in X$. Our first step is to show that $\lambda_2(x,y)$ is independent of x for all $x \in \Gamma_2(y)$. To this end, we define a set of vertices $\Delta = \Delta(y)$ by $\Delta = \cup_{i=0}^{2}\Gamma_i(y)$. We consider the subgraph of Γ induced on Δ. In the subgraph Δ, each vertex is connected to y by a path of length at most 2. Therefore the subgraph Δ is connected. One checks that the subgraph Δ has diameter 4. Let ∂_Δ denote the distance function for the subgraph Δ. Suppose that there exists a pair of vertices $x, x' \in \Gamma_2(y)$ such that $\lambda_2(x, y) \neq \lambda_2(x', y)$. Of all such pairs of vertices, choose a pair x, x' such that $\partial_\Delta(x, x')$ is minimal. By construction $1 \leqslant \partial_\Delta(x, x') \leqslant 4$. We now examine the cases.

Case $\boldsymbol{\partial_\Delta(x, x') = 1}$. We have $\partial(x, x') = 1$. Setting $i = 2$ in Lemma 200(ii),(iv) we obtain $\Gamma(x) \cap \Gamma(x') \cap \Gamma_3(y) = \emptyset$ and

$$b_2\theta_2^* - R_2\theta_1^* - S_2\theta_2^* = 0.$$

Using the data in [45, Example 20.6], we obtain for N even:

$$b_2\theta_2^* - R_2\theta_1^* - S_2\theta_2^* = -\frac{(q^{D+\frac{1}{2}} + q^D - 1)(q^D - q^2)(q^{D+\frac{1}{2}} - q^2)q^{2D+\frac{1}{2}}}{q^{2D+2+\frac{1}{2}} + q^{2D+\frac{1}{2}} - 2q^{D+2+\frac{1}{2}} - 2q^{D+2} + 2q^2} \neq 0,$$

and for N odd:

$$b_2\theta_2^* - R_2\theta_1^* - S_2\theta_2^* = -\frac{(q^{D-\frac{1}{2}} + q^D - 1)(q^D - q^2)(q^{D-\frac{1}{2}} - q^2)q^{2D-\frac{1}{2}}}{q^{2D+2-\frac{1}{2}} + q^{2D-\frac{1}{2}} - 2q^{D+2-\frac{1}{2}} - 2q^{D+2} + 2q^2} \neq 0.$$

This is a contradiction.

Case $\boldsymbol{\partial_\Delta(x, x') = 2}$. We have $\partial(x, x') = 2$. By the triangle inequality,

$$\Gamma(x) \cap \Gamma(x') = \cup_{i=1}^{3}\Big(\Gamma(x) \cap \Gamma(x') \cap \Gamma_i(y)\Big).$$

The set $\Gamma(x) \cap \Gamma(x') \cap \Gamma_3(y)$ must be empty, because if it contains a vertex z then the sequence x, z, x' is a path in Γ that is raising/lowering with respect to y, contradicting Lemma 198. The set $\Gamma(x) \cap \Gamma(x') \cap \Gamma_2(y)$ must be empty, because if it contains a vertex z then x, z, x' is a path in the subgraph Δ, forcing $\lambda_2(x, y) = \lambda_2(z, y) = \lambda_2(x', y)$ by the minimality of $\partial_\Delta(x, x')$. By the above comments,

$$\Gamma(x) \cap \Gamma(x') = \Gamma(x) \cap \Gamma(x') \cap \Gamma(y).$$

Note that

$$|\Gamma(x) \cap \Gamma(x') \cap \Gamma(y)| = |\Gamma(x) \cap \Gamma(x')| = c_2.$$

We have

$$\Gamma(x) \cap \Gamma(y) = \Gamma(x') \cap \Gamma(y) = \Gamma(x) \cap \Gamma(x') \cap \Gamma(y),$$

because the first two sets have cardinality c_2 and contain the third set. By Lemma 45, the scalar $\zeta_2(x, y, *)$ is the average valency of the induced subgraph $\Gamma(x) \cap \Gamma(y)$. Similarly, $\zeta_2(x', y, *)$ is the average valency of the induced subgraph $\Gamma(x') \cap \Gamma(y)$. These subgraphs coincide, so $\zeta_2(x, y, *) = \zeta_2(x', y, *)$. By this and Lemma 91 (with $i = 2$), we obtain $\lambda_2(x, y) = \lambda_2(x', y)$. This is a contradiction.

Case $\boldsymbol{\partial_\Delta(x, x') = 3}$. In the subgraph Δ, the vertices x, x' are connected by a path of length 3. Denote such a path by x, z, z', x'. The vertices x, z' are not adjacent; otherwise x, z', x' is a path in Δ, contradicting $\partial_\Delta(x, x') = 3$. Similarly, the vertices x', z are not adjacent. The vertex z is contained in Δ and adjacent to x, so $z \in \Gamma(y) \cup \Gamma_2(y)$. If $z \in \Gamma_2(y)$ then $\lambda_2(x, y) = \lambda_2(z, y) = \lambda_2(x', y)$ by the minimality of $\partial_\Delta(x, x')$. This is a contradiction, so $z \in \Gamma(y)$. We have $z \in \Gamma(x) \cap \Gamma(y)$. Similarly, $z' \in \Gamma(x') \cap \Gamma(y)$. These comments apply to every choice of z, z'. For each choice of z there are c_2 choices for z', and these are all contained in $\Gamma(x') \cap \Gamma(y)$. For each choice of z' there are c_2 choices for z, and these are all contained in $\Gamma(x) \cap \Gamma(y)$. We have $|\Gamma(x) \cap \Gamma(y)| = c_2$ and $|\Gamma(x') \cap \Gamma(y)| = c_2$. By these comments, every vertex in $\Gamma(x) \cap \Gamma(y)$ is adjacent to every vertex in $\Gamma(x') \cap \Gamma(y)$. Pick $z \in \Gamma(x) \cap \Gamma(y)$ and $w' \in \Gamma(x') \cap \Gamma_3(y)$. By construction $\partial(z, w') \in \{2, 3\}$. Suppose for the moment that $\partial(z, w') = 2$. Then there exists $v \in \Gamma(z) \cap \Gamma(w')$. By construction $v \in \Gamma_2(y)$. We have $\partial_\Delta(x, v) \leqslant 2$ since x, z, v is a path in Δ. Therefore, $\lambda_2(x, y) = \lambda_2(v, y)$ by the minimality of $\partial_\Delta(x, x')$. We have $\lambda_2(v, y) = \lambda_2(x', y)$ by Lemma 198 and since v, w', x' is a path in Γ that is lowering/raising with respect to y. By these comments, $\lambda_2(x, y) = \lambda_2(v, y) = \lambda_2(x', y)$ for a contradiction. We have shown that $\partial(z, w') \neq 2$, so $\partial(z, w') = 3$. It follows that in the graph Γ, every vertex in $\Gamma(x) \cap \Gamma(y)$ is at distance 3 from every vertex in $\Gamma(x') \cap \Gamma_3(y)$. Similarly, in the graph Γ every vertex in $\Gamma(x') \cap \Gamma(y)$ is at distance 3 from every vertex in $\Gamma(x) \cap \Gamma_3(y)$. Pick $z' \in \Gamma(x') \cap \Gamma(y)$. Take the inner product of $E\hat{z}'$ with each side of (37), and evaluate the result using Lemma 2(i); this yields

$$c_2\theta_1^* - r_2\theta_2^* - s_2\theta_1^* = \lambda_2(x, y)\big(b_2\theta_3^* - R_2\theta_2^* - S_2\theta_1^*\big).$$

Interchanging the roles of x, x' we obtain

$$c_2\theta_1^* - r_2\theta_2^* - s_2\theta_1^* = \lambda_2(x', y)\big(b_2\theta_3^* - R_2\theta_2^* - S_2\theta_1^*\big).$$

By these comments and $\lambda_2(x, y) \neq \lambda_2(x', y)$, we obtain

$$c_2\theta_1^* - r_2\theta_2^* - s_2\theta_1^* = 0, \qquad\qquad b_2\theta_3^* - R_2\theta_2^* - S_2\theta_1^* = 0.$$

Using the data in [45, Example 20.6], we obtain

$$b_2\theta_3^* - R_2\theta_2^* - S_2\theta_1^* = b_2\theta_2^* - R_2\theta_1^* - S_2\theta_2^* \neq 0.$$

This is a contradiction.

Case $\boldsymbol{\partial_\Delta(x, x') = 4}$. Pick $z \in \Gamma(x) \cap \Gamma(y)$ and $z' \in \Gamma(x') \cap \Gamma(y)$. Note that $\partial_\Delta(z, z') = 2 = \partial(z, z')$. Since $c_2 > 1$, there exists $u \in \Gamma(z) \cap \Gamma(z')$ with $u \neq y$. By construction $u \in \Gamma(y) \cup \Gamma_2(y)$, so $u \in \Delta$. The sequence x, z, u is a path in Δ. The vertices x, u are not adjacent; otherwise x, u, z', x' is a path in Δ of length 3. By these comments, $\partial_\Delta(x, u) = 2$. Similarly, $\partial_\Delta(x', u) = 2$. Suppose for the moment that $u \in \Gamma_2(y)$. Then $\lambda_2(x, y) = \lambda_2(u, y) = \lambda_2(x', y)$ by the minimality of $\partial_\Delta(x, x')$. This is a contradiction, so $u \in \Gamma(y)$. Note that $|\Gamma(u) \cap \Gamma_2(y)| = b_1$, so $\Gamma(u) \cap \Gamma_2(y) \neq \emptyset$. Pick $v \in \Gamma(u) \cap \Gamma_2(y)$. In the graph Δ, the sequence x, z, u, v is path, so $\partial_\Delta(x, v) \leqslant 3$. Also in the graph Δ, the sequence x', z', u, v is path, so $\partial_\Delta(x', v) \leqslant 3$. Now $\lambda_2(x, y) = \lambda_2(v, y) = \lambda_2(x', y)$ by the minimality of $\partial_\Delta(x, x')$. This is a contradiction.

Conclusion. We have shown that there does not exist a pair of vertices $x, x' \in \Gamma_2(y)$ such that $\lambda_2(x, y) \neq \lambda_2(x', y)$. Consequently $\lambda_2(x, y)$ is independent of x for all $x \in \Gamma_2(y)$; we call this common value the λ_2-value of y. Until now, the vertex y has been fixed. Next, we let y vary. Pick any $x, y \in X$ at distance $\partial(x, y) = 2$. We have $\zeta_2(x, y, *) = \zeta_2(y, x, *)$ because each side is equal to the average valency of the subgraph induced on $\Gamma(x) \cap \Gamma(y)$. By this and Lemma 91 (with $i = 2$) we obtain $\lambda_2(x, y) = \lambda_2(y, x)$. Therefore x, y have the same λ_2-value. By this and since Γ is not bipartite, we find that every vertex in X has the same λ_2-value. This means that $\lambda_2(x, y)$ is independent of x, y for all $x, y \in X$ with $\partial(x, y) = 2$. By this and Lemma 196, for $2 \leqslant i \leqslant D - 1$ the scalar $\lambda_i(x, y)$ is independent of x, y for all $x, y \in X$ with $\partial(x, y) = i$. By this and Lemmas 91, 93, the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. On one hand, Γ is reinforced by Lemma 59, so E is DC by Lemma 127. On the other hand, E is not DC because the condition in Theorem 128(i) is violated. This is a contradiction, so the set $\{E\hat{x} | x \in X\}$ is not Norton-balanced. □

30 When Γ affords a spin model

We are done discussing the known infinite families of Q-polynomial distance-regular graphs with unbounded diameter. There is one more family of Q-polynomial distance-regular graphs that we would like to discuss; members of this family afford a spin model [9, 12, 13, 14, 25, 33, 34]. Very few examples are known; see [13, Section 9] or [33, Section 15].

Throughout this section, the following assumptions and notation are in effect. Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter $D \geqslant 3$. Assume that Γ affords a spin model in the sense of [33, Definition 11.1]. By [33, Lemma 11.4] there exists an ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of Γ that is formally self-dual in the sense of [33, Definition 10.1]. By [33, Lemma 10.2] the ordering $\{E_i\}_{i=0}^D$ is Q-polynomial; to avoid trivialities we assume that this ordering has q-Racah type [45, Example 20.1]. Write $E = E_1$.

We discuss some cases. Throughout this paragraph, assume that Γ is bipartite or almost bipartite. The given Q-polynomial structure is formally self-dual, so E is dual-bipartite or almost dual-bipartite. The set $\{E\hat{x} | x \in X\}$ is strongly balanced by Lemma 21. The set $\{E\hat{x} | x \in X\}$ is Norton-balanced by Definitions 20, 22. The kite function ζ_i is constant for $2 \leqslant i \leqslant D$, because Γ has no kites. The graph Γ is reinforced by Lemma 59, so E is DC in view of Lemma 127.

For the rest of this section, assume that Γ is not bipartite and not almost bipartite. By [9, Remark 6.10] or [34, Remark 7.4], [34, Appendix 18] the parameters q, r_1, r_2, s, s^* from [45, Example 20.1] satisfy

$$r_1 = -q^{-1}\eta, \qquad r_2 = -\eta^3 q^{D-2}, \qquad s = s^* = r_1^2$$

for an appropriate $\eta \in \mathbb{C}$. By [33, Lemma 12.2] and [34, Remark 7.4], we have

$$q^i \neq 1 \qquad (1 \leqslant i \leqslant D), \qquad\qquad q^i\eta^2 \neq 1 \qquad (0 \leqslant i \leqslant 2D-2),$$
$$q^i\eta^3 \neq -1 \qquad (D-1 \leqslant i \leqslant 2D-2).$$

The intersection numbers of Γ are given in [9, Theorems 6.6, 6.8] and [9, Corollary 6.9]. By [9, Corollary 6.9] and since Γ is not bipartite, $q^{D-1}\eta^2 \neq -1$. By [9, Corollary 6.9] and since Γ is not almost bipartite, $q^D\eta \neq 1$. The primitive idempotent E is DC by Theorem 128(i) and $s = r_1^2$. By (19) and [45, Example 20.1],

$$\gamma^* = \frac{(q-1)(q\eta^2-1)(q^{D-1}\eta^2+1)(q^D\eta-1)}{q\eta(q^D\eta^2-1)(q^{D-1}\eta+1)} \neq 0.$$

By [34, Remark 15.6] the kite function ζ_i is constant for $2 \leqslant i \leqslant D$. Therefore Γ is reinforced. By [34, Remark 7.4] and [34, Lemma 15.7],

$$z_i = -\frac{(q^i-q)(q\eta^2-1)(q^{D-1}\eta^2+1)(q^D\eta-1)}{(q^i\eta-q)(q^D\eta^2-1)(q^{D-1}\eta+1)(q\eta-1)} \qquad (2 \leqslant i \leqslant D). \tag{74}$$

Lemma 202. *With the above notation, the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Pick distinct $x, y \in X$ and write $i = \partial(x, y)$. There is a linear dependence with the following terms and coefficients:*

$0 =$

term	coefficient
Ex_y^-	1
Ex_y^+	$\frac{\eta(q^i-q)(q^i\eta-1)}{(q^i\eta-q)(q^i\eta^2-1)}$
$E\hat{x}$	$-\frac{q^{D-1}(\eta-1)(q\eta+1)(q\eta^2-1)(q^i-q)}{(q-1)(q^i\eta-q)(q^{D-1}\eta+1)(q^D\eta^2-1)}$
$E\hat{y}$	$-\frac{q^i(\eta-1)(q\eta^2-1)}{(q^i\eta^2-1)(q^i\eta-q)}$

Proof. We first verify the linear dependence in the above table. For $2 \leqslant i \leqslant D-1$ we compute the polynomial $\Phi_i(\lambda)$ using Definition 95 along with Lemma 41, Definitions 64, 76 and the data in [45, Example 20.1]. We check using (74) that $\Phi_i(z_2) = 0$. By this and Corollary 98, the vectors Ex_y^-, Ex_y^+, $E\hat{x}$, $E\hat{y}$ are linearly dependent. The coefficients in this linear dependence are found using (41), (42). This yields the linear dependence in the above table, for $2 \leqslant i \leqslant D-1$. Next, assume that $i = D$. Using Definition 64 and (74), we obtain $z_D^- = z_D$. This and Proposition 68 yield the linear dependence in the above table for $i = D$. For $i = 1$ the linear dependence in the above table holds vacuously. We have shown that the linear dependence in the above table holds in every case. Next, we verify that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. We will use Proposition 100(ii). We mentioned earlier that $\gamma^* \neq 0$ and Γ is reinforced. Pick an integer i $(2 \leqslant i \leqslant D-1)$. We verify the conditions in (45). The condition $\Phi_i(z_2) = 0$ is already verified. Referring to the above table, in our calculation of the Ex_y^+ coefficient we found that the scalar λ_i from (41), (42) is given by

$$\lambda_i = -\frac{\eta(q^i-q)(q^i\eta-1)}{(q^i\eta-q)(q^i\eta^2-1)}.$$

By the data in [45, Example 20.1],

$$\frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} = \frac{q^{2i}\eta^2 - 1}{q(q^{2i-2}\eta^2 - 1)}.$$

We have

$$\frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} - \lambda_i = \frac{(q\eta + 1)(q^i\eta - 1)(q^{2i-1}\eta^2 - 1)}{q(q^i\eta^2 - 1)(q^{2i-2}\eta^2 - 1)} \neq 0.$$

We have verified the conditions in (45), so the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. □

31 Directions for future research

In this section, we give some suggestions for future research.

Problem 203. Classify up to isomorphism the Q-polynomial distance-regular graphs with diameter $D \geqslant 3$ that are Norton-balanced.

Problem 204. Classify up to isomorphism the distance-regular graphs with diameter $D \geqslant 4$ that have a Q-polynomial primitive idempotent that is DC.

Problem 205. Classify up to isomorphism the distance-regular graphs with diameter $D \geqslant 3$ that have a Q-polynomial structure such that $\gamma^* = 0$.

Conjecture 206. Let $\Gamma = (X, \mathcal{R})$ denote a Q-polynomial distance-regular graph with diameter $D \geqslant 3$. Let E denote a Q-polynomial primitive idempotent of Γ. Assume that the set $\{E\hat{x} | x \in X\}$ is Norton-balanced. Then the kite function ζ_i is constant for $2 \leqslant i \leqslant D$.

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