Maximum spread of $K_{s,t}$-minor-free graphs

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Abstract

The spread of a graph G is the difference between the largest and smallest eigenvalue of the adjacency matrix of G. In this paper, we consider the family of graphs which contain no $K_{s,t}$-minor. We show that for any $t \geqslant s \geqslant 2$ and sufficiently large n, there is an integer ξ_t such that the extremal n-vertex $K_{s,t}$-minor-free graph attaining the maximum spread is the graph obtained by joining a graph L on $(s-1)$ vertices to the disjoint union of $\lfloor \frac{2n+\xi_t}{3t} \rfloor$ copies of K_t and $n-s+1-t\lfloor \frac{2n+\xi_t}{3t} \rfloor$ isolated vertices. Furthermore, we give an explicit formula for ξ_t and an explicit description for the graph L for $t \geqslant \frac{3}{2}(s-3) + \frac{4}{s-1}$.

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1 Introduction

Given a square matrix M, the *spread* of M, denoted by $S(M)$, is defined as $S(M) := \max_{i,j} |\lambda_i - \lambda_j|$, where the maximum is taken over all pairs of eigenvalues of M, so that $S(M)$ is the diameter of the spectrum of M. Given a graph $G = (V, E)$ on n vertices, the *spread* of G, denoted by $S(G)$, is defined as the spread of the adjacency matrix $A(G)$ of G. The adjacency matrix $A(G)$ is the $n \times n$ matrix with rows and columns indexed by the vertices of G such that for every pair of vertices $u, v \in V(G)$, $(A(G))_{uv} = 1$ if $uv \in E(G)$ and $(A(G))_{uv} = 0$ otherwise. Since $A(G)$ is a real symmetric matrix, its eigenvalues are all real numbers. Let $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$ be the eigenvalues of $A(G)$, where λ_1 is called the *spectral radius* of G. Then $S(G) = \lambda_1 - \lambda_n$.

The systematic study of the spread of graphs was initiated by Gregory, Hershkowitz, and Kirkland [13]. One of the central focuses of this area is to find the maximum or minimum spread over a fixed family of graphs and characterize the extremal graphs. The maximum-spread graph over the family of all n-vertex graphs was recently determined for sufficiently large n by Breen, Riasanovsky, Tait and Urschel [3], building on much prior work [2, 24, 26, 27, 31]. Other problems of such extremal flavor have been investigated for trees [1], graphs with few cycles [11, 22, 33], the family of bipartite graphs [3], graphs

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with a given matching number [16], girth [32], or size [15], outerplanar graphs [12, 17] and planar graphs [17]. We note that the spreads of other matrices associated with a graph have also been extensively studied (see e.g. references in [12, 6, 8]).

Given two graphs G and H, the *join* of G and H, denoted by $G \vee H$, is the graph obtained from the disjoint union of G and H by connecting every vertex of G with every vertex of H. Let P_k denote the path on k vertices. Given two graphs G and H, let $G \cup H$ denote the disjoint union of G and H. Given a graph G and a positive integer k, we use kG to denote the disjoint union of k copies of G. Given $v \subseteq V(G)$, let $N_G(v)$ denote the set of neighbors of v in G, and let $d_G(v)$ denote the degree of v in G, i.e., $d_G(v) = |N_G(v)|$. Given $S \subseteq V(G)$, define $N_G(S)$ as $N_G(S) = \cup_{v \in S}(N_G(v) \backslash S)$. We may ignore the subscript G when there is no ambiguity. A graph H is called a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. A graph G is called *H-minor-free* if H is not a minor of G.

There has been extensive work on finding the maximum spectral radius of $K_{s,t}$-minor-free graphs. Let $\mathcal{G}_{s,t}(n)$ denote the family of all $K_{s,t}$-minor-free graphs on n vertices. Nikiforov [21] proved an upper bound for the maximum spectral radius of a $K_{2,t}$-minor-free graph. Nikiforov showed that this bound is tight for graphs with a sufficiently large number of vertices n with $n \equiv 1 \pmod t$ and determined the extremal graph in these cases. Tait [28] extended Nikiforov's result by proving an upper bound on the maximum spectral radius of $K_{s,t}$-minor-free graphs, and determined the extremal graphs when $n \equiv s-1 \pmod t$ and n is sufficiently large. Recently, Zhai and Lin [36] completely determined the $K_{s,t}$-minor-free graph with maximum spectral radius for a sufficiently large number of vertices n and all $t \geqslant s \geqslant 2$.

In [18], the authors determined the maximum-spread $K_{2,t}$-minor-free graph for sufficiently large n for all $t \geqslant 2$. In this follow-up paper, we determine the structure of the maximum-spread $K_{s,t}$-minor-free graph on n vertices for sufficiently large n and for all $t \geqslant s \geqslant 2$.

Theorem 1. *For $t \geqslant s \geqslant 2$ and n sufficiently large, the graph(s) that maximizes the spread over the family of $K_{s,t}$-minor-free graphs on n vertices has the following form*

$$L_{max} \vee (\ell_0 K_t \cup (n - s + 1 - t\ell_0)\, P_1)$$

where

1. *L_{max} is a graph on $s-1$ vertices which maximizes a function $\psi(L)$ (over all graphs L on $s-1$ vertices) as follows:*

$$\psi(L) = 3 \sum_{v \in V(L)} d_L^2(v) - \frac{2}{s-1} \left(\sum_{v \in V(L)} d_L(v) \right)^2 - (t-1) \sum_{v \in V(L)} d_L(v). \qquad (1)$$

2. *$\ell_0 = \left(\frac{2}{3t} - \frac{2|E(L_{max})|}{3t(t-1)(s-1)} \right)(n - s + 1) + O(n^{\epsilon})$ for any $\epsilon > 0$.*

In particular, we have

$$\max_{G\in\mathcal{G}_{s,t}(n)} S(G) = 2\sqrt{(s-1)(n-s+1)} + \frac{(t-1)^2+\psi(L_{max})/(s-1)}{3\sqrt{(s-1)(n-s+1)}} + O\left(\frac{1}{n^{3/2}}\right). \quad (2)$$

We call a pair (s,t) *admissible* if $L_{max} = (s-1)K_1$, i.e., $\psi(L) \leqslant 0$ and $\psi(L) = 0$ only if $L = (s-1)K_1$. We determine the value of ℓ_0 when (s,t) is admissible and thus determine the precise extremal graph(s) for these cases.

Theorem 2. *Let s and t be integers with $t \geqslant s \geqslant 2$, and suppose that the pair (s,t) is admissible. For n sufficiently large, the maximum spread over the family of $K_{s,t}$-minor-free graphs on n vertices is achieved by*

$$(s-1)K_1 \vee (\ell_0 K_t \cup (n-s+1-t\ell_0)\, P_1)\,.$$

Here ℓ_0 is the nearest integer(s) of $\ell_1 := \frac{2}{3t}\left(n-s+1-\frac{(t-1)^2}{9(s-1)}\right)$. In particular, the extremal graph is unique when ℓ_1 is not a half-integer. Otherwise, there are two extremal graphs.

Furthermore, we determine all of the admissible pairs (s,t).

Theorem 3. *A pair (s,t) with $s \leqslant t$ is admissible if and only if $t \geqslant \frac{3}{2}(s-3) + \frac{4}{s-1}$.*

The smallest non-admissible pair is $(s,t) = (8,8)$.

Our paper is organized as follows. In Section 2, we recall some useful lemmas and prove that in any maximum-spread $K_{s,t}$-minor-free graph G, there are $(s-1)$ vertices $u_1,\ldots,u_{s-1}$ which are adjacent to all other vertices in G. In Section 3, we show that $G-\{u_1,\ldots,u_{s-1}\}$ is a disjoint union of cliques on t vertices and isolated vertices and complete the proofs of Theorems 1, 2, and 3. The non-admissible cases are more complicated and will be handled in a sequel.

2 Notation and Lemmas

Let G be a graph which attains the maximum spread among all n-vertex $K_{s,t}$-minor-free graphs, and $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of $A(G)$. We first recall the following result by Mader [19].

Theorem 4 ([19])**.** *For every positive integer t, there exists a constant C_t such that every graph with average degree at least C_t contains a K_t minor.*

Corollary 5. *Let s and t be positive integers with $s \leqslant t$. There exists a constant C_0 such that for any $K_{s,t}$-minor-free graph G on $n > 0$ vertices,*

$$|E(G)| \leqslant C_0 n.$$

Kostochka and Prince [14] gave a better upper bound on the maximum number of edges in a $K_{s,t}$-minor-free graph when t is sufficiently large compared to s.

Theorem 6. *[14] Let $t \geqslant (180s\log_2 s)^{1+6s\log_2 s}$ be a positive integer, and G be a graph on $n \geqslant s+t$ vertices with no $K_{s,t}$ minor. Then*

$$|E(G)| \leqslant \frac{t+3s}{2}(n-s+1).$$

We mention here that in the case of $s=2$, Chudnovsky, Reed and Seymour [7] showed a tight upper bound $|E(G)| \leqslant \frac{1}{2}(t+1)(n-1)$ for the number of edges in a $K_{2,t}$-minor-free graph G for any $t \geqslant 2$, which extends an earlier result of Myers [20].

We also need the following theorem by Thomason [30] on the number of edges of $K_{s,t}$-minor-free bipartite graphs.

Theorem 7. *[30] Let G be a bipartite graph with at least $(s-1)n+4^{s+1}s!tm$ edges, where $n,m>0$ are the sizes of the two parts of G. Then G has a $K_{s,t}$-minor.*

Corollary 8. *Suppose G is a bipartite graph on n vertices such that one part has at most $c\sqrt{n}$ vertices for some fixed constant $c>0$. If G is $K_{1,t}$-minor-free, then $|E(G)| < 16tc\sqrt{n}$.*

As a first step towards proving Theorem 1, we want to show that G must contain $K_{s-1,n-s+1}$ as a subgraph. We recall the result of Tait [28] on the maximum spectral radius of $K_{s,t}$-minor-free graphs.

Theorem 9. *[28] Let $t \geqslant s \geqslant 2$ and let G be a graph of order n with no $K_{s,t}$ minor. For sufficiently large n, the spectral radius $\lambda_1(G)$ satisfies*

$$\lambda_1(G) \leqslant \frac{s+t-3+\sqrt{(t-s+1)^2+4(s-1)(n-s+1)}}{2},$$

with equality if and only if $n \equiv s-1 \pmod{t}$ and $G = K_{s-1} \vee \lfloor n/t \rfloor K_t$.

We first give some upper and lower bounds on $\lambda_1(G)$ and $|\lambda_n(G)|$ when n is sufficiently large. We use known expressions for the eigenvalues of a join of two regular graphs [4, pg.19].

Lemma 10. *[4] Let G and H be regular graphs with degrees k and ℓ respectively. Suppose that $|V(G)| = m$ and $|V(H)| = n$. Then, the characteristic polynomial of $G \vee H$ is $p_{G\vee H}(t) = ((t-k)(t-\ell)-mn)\frac{p_G(t)p_H(t)}{(t-k)(t-\ell)}$. In particular, if the eigenvalues of G are $k = \lambda_1 \geqslant \ldots \geqslant \lambda_m$ and the eigenvalues of H are $\ell = \mu_1 \geqslant \ldots \geqslant \mu_n$, then the eigenvalues of $G \vee H$ are $\{\lambda_i : 2 \leqslant i \leqslant m\} \cup \{\mu_j : 2 \leqslant j \leqslant n\} \cup \{x : (x-k)(x-\ell)-mn=0\}$.*

We will apply Lemma 10 to the graph $(s-1)K_1 \vee qK_t$ where $q = \lfloor (n-s+1)/t \rfloor$. Let $a_0 = (s-1)(n-s+1)$.

Lemma 11. *We have*

$$\sqrt{a_0} - \frac{s+t-3}{2} - O\left(\frac{1}{\sqrt{n}}\right) \leqslant |\lambda_n| \leqslant \lambda_1 \leqslant \sqrt{a_0} + \frac{s+t-3}{2} + O\left(\frac{1}{\sqrt{n}}\right). \tag{3}$$

Proof. The upper bound of λ_1 is due to Theorem 9. Now let us prove the lower bound.

By Lemma 10, for sufficiently large n, $\lambda_1((s-1)K_1 \vee qK_t)$ and $\lambda_n((s-1)K_1 \vee qK_t)$ are the roots of the equation

$$\lambda(\lambda - (t-1)) - (s-1)qt = 0.$$

Thus, we have

$$\begin{aligned}
\lambda_1((s-1)K_1 \vee qK_t) &= \frac{t-1+\sqrt{(t-1)^2+4(s-1)qt}}{2},\\
\lambda_n((s-1)K_1 \vee qK_t) &= \frac{t-1-\sqrt{(t-1)^2+4(s-1)qt}}{2}.
\end{aligned}$$

Thus $S((s-1)K_1 \vee qK_t) = \sqrt{(t-1)^2+4(s-1)qt}$. Let $q = \lfloor (n-s+1)/t \rfloor$. By the eigenvalue interlacing theorem, we then have

$$\begin{aligned}
S(G) &\geqslant \sqrt{(t-1)^2+4(s-1)qt}\\
&\geqslant \sqrt{4(s-1)(n-s+1)+(t-1)^2-4(s-1)(t-1)}\\
&= 2\sqrt{a_0} + O\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\lambda_n(G)| &= S(G) - \lambda_1(G)\\
&\geqslant 2\sqrt{a_0} + O\left(\frac{1}{\sqrt{n}}\right) - \left(\sqrt{a_0} + \frac{s+t-3}{2} + O\left(\frac{1}{\sqrt{n}}\right)\right)\\
&= \sqrt{a_0} - \frac{s+t-3}{2} - O\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

□

For the rest of this paper, let $\mathbf{x}$ and $\mathbf{z}$ be the eigenvectors of $A(G)$ corresponding to the eigenvalues λ_1 and λ_n respectively. For convenience, let $\mathbf{x}$ and $\mathbf{z}$ be indexed by the vertices of G. By the Perron-Frobenius theorem, we may assume that all entries of $\mathbf{x}$ are positive. We also assume that $\mathbf{x}$ and $\mathbf{z}$ are normalized so that the maximum absolute values of the entries of $\mathbf{x}$ and $\mathbf{z}$ are equal to 1, and so there are vertices u_0 and w_0 with $\mathbf{x}_{u_0} = |\mathbf{z}_{w_0}| = 1$.

Let $V_+ = \{v\colon \mathbf{z}_v > 0\}$, $V_0 = \{v\colon \mathbf{z}_v = 0\}$, and $V_- = \{v\colon \mathbf{z}_v < 0\}$. Since $\mathbf{z}$ is a non-zero vector, at least one of V_+ and V_- is non-empty. By considering the eigen-equations of $\lambda_n \sum_{v\in V_+} \mathbf{z}_v$ or $\lambda_n \sum_{v\in V_-} \mathbf{z}_v$, we obtain that both V_+ and V_- are non-empty. For any vertex subset S, we define the *volume* of S, denoted by $\mathrm{Vol}(S)$, as $\mathrm{Vol}(S) = \sum_{v\in S} |\mathbf{z}_v|$. In the following lemmas, we use the bounds of λ_n to deduce some information on V_+, V_- and V_0.

Lemma 12. *We have*

$$\mathrm{Vol}(V(G)) = O(\sqrt{n}).$$

Proof. For any vertex $v \in V(G)$, we have

$$d(v) \geqslant |\sum_{y \in N(v)} \mathbf{z}_y| = |\lambda_n||\mathbf{z}_v|.$$

Applying Theorem 4 and Corollary 5, we have

$$|\lambda_n|\mathrm{Vol}(V) = \sum_{v \in V(G)} |\lambda_n||\mathbf{z}_v| \leqslant \sum_{v \in V(G)} d(v) = O(n).$$

By Lemma 11, $|\lambda_n| \geqslant \sqrt{(n-s+1)} - \frac{s+t-3}{2} - O\left(\frac{1}{\sqrt{n}}\right)$. We thus have $\mathrm{Vol}(V) = O(\sqrt{n})$. $\square$

Without loss of generality, we assume $|V_+| \leqslant \frac{n}{2}$.

Lemma 13. *We have*

$$\mathrm{Vol}(V_+) = O(1).$$

Proof. Let $\epsilon > 0$ be a small constant depending on s and t to be chosen later. Define two sets L and S as follows:

$$L = \{v \in V_+ : |N(v) \cap V_-| \geqslant \epsilon n\},$$

and $S = V_+ \setminus L$. Let $C = 4^{s+1} s! t$. By Theorem 7, we have

$$|L| \leqslant \frac{E(L, V_-)}{\epsilon n} \leqslant \frac{Cn}{\epsilon n} = \frac{C}{\epsilon}. \tag{4}$$

We then have that

$$\begin{aligned}
\lambda_n^2 \mathrm{Vol}(S) &= \lambda_n^2 \sum_{v \in S} \mathbf{z}_v \\
&= \lambda_n \sum_{v \in S} \sum_{u \in N(v)} \mathbf{z}_u \\
&\leqslant \sum_{v \in S} \sum_{u \in N(v) \cap V_-} \lambda_n \mathbf{z}_u \\
&\leqslant \sum_{v \in S} \sum_{u \in N(v) \cap V_-} \sum_{y \in V_+ \cap N(u)} \mathbf{z}_y \\
&= \sum_{y \in V_+} \mathbf{z}_y |E(S, N(y) \cap V_-)| \\
&= \sum_{y \in L} \mathbf{z}_y |E(S, N(y) \cap V_-)| + \sum_{y \in S} \mathbf{z}_y |E(S, N(y) \cap V_-)|. \qquad (5)
\end{aligned}$$

We apply the following estimation. For $y \in L$, we have

$$|E(S, N(y) \cap V_-)| \leqslant |E(S, V_-)| \leqslant Cn. \tag{6}$$

For $y \in S$, by Theorem 7, we have

$$|E(S, N(y) \cap V_-)| \leqslant (s-1)|S| + C\epsilon n. \tag{7}$$

Now we apply the assumption that $|V_+| \leqslant \frac{n}{2}$. We have

$$|E(S, N(y) \cap V_-)| \leqslant (s-1)\frac{n}{2} + C\epsilon n. \tag{8}$$

Plugging Equations (6) and (8) into Equation (5), we get

$$\lambda_n^2 \mathrm{Vol}(S) \leqslant \mathrm{Vol}(L)Cn + \mathrm{Vol}(S)\left((s-1)\frac{n}{2} + C\epsilon n\right). \tag{9}$$

By Lemma 11, $|\lambda_n| \geqslant \sqrt{(s-1)(n-s+1)} - \frac{s+t-3}{2} - O\left(\frac{1}{\sqrt{n}}\right)$. Set $\epsilon = \frac{s-1}{6C}$. We have that for sufficiently large n,

$$\lambda_n^2 - \left((s-1)\frac{n}{2} + C\epsilon n\right) > \frac{(s-1)n}{4}. \tag{10}$$

Combining Equations (9) and (10) and solving $\mathrm{Vol}(S)$, we get

$$\mathrm{Vol}(S) \leqslant \frac{4C}{s-1}\mathrm{Vol}(L). \tag{11}$$

This implies

$$\begin{aligned}
\mathrm{Vol}(V_+) &\leqslant \left(1 + \frac{4C}{s-1}\right)\mathrm{Vol}(L) \\
&\leqslant \left(1 + \frac{4C}{s-1}\right)|L| \\
&\leqslant \left(1 + \frac{4C}{s-1}\right)\frac{C}{\epsilon} \\
&= O(1).
\end{aligned}$$

At the last step, we apply Inequality (4). The proof of this lemma is thus finished. ☐

Corollary 14. *For any $v \in V_-$, we have*

$$|\mathbf{z}_v| = O\left(\frac{1}{\sqrt{n}}\right).$$

In particular, $w_0 \in V_+$ and $|N(w_0) \cap V_-| = \Omega(n)$.

Proof. For any $v \in V_-$, we have

$$|\lambda_n||\mathbf{z}_v| = \lambda_n \mathbf{z}_v \leqslant \sum_{y \in N(v) \cap V_+} \mathbf{z}_y \leqslant \mathrm{Vol}(V_+) = O(1).$$

This implies $\mathbf{z}_v = O\left(\frac{1}{\sqrt{n}}\right)$. In particular, we have $w_0 \in V_+$. Thus $\mathbf{z}_{w_0} = 1$. We then obtain that

$$\begin{aligned}
\lambda_n^2 &= \lambda_n^2 \mathbf{z}_{w_0} \\
&\leqslant \lambda_n \sum_{v \in N(w_0) \cap V_-} \mathbf{z}_v \\
&\leqslant |N(w_0) \cap V_-| \cdot \mathbf{z}_{w_0} + \sum_{y \in V_+ \setminus \{w_0\}} \mathbf{z}_y |N(y) \cap N(w_0) \cap V_-| \\
&\leqslant |N(w_0) \cap V_-| \mathrm{Vol}(V_+).
\end{aligned}$$

Since $\mathrm{Vol}(V_+) = O(1)$ and $\lambda_n^2 \geqslant (s-1-o(1))n$, we have $|N(w_0) \cap V_-| = \Omega(n)$. $\square$

Lemma 15. *We have $|V_-| \geqslant n - O(\sqrt{n})$ and $|V_+| = O(\sqrt{n})$.*

Proof. We define L now as follows. Let

$$L = \{v \in V_+ \colon |N(v) \cap V_-| \geqslant C_1\sqrt{n}\},$$

where C_1 is some big constant chosen later. Let $S = V_+ \setminus L$. We have

$$|L| \leqslant \frac{E(L, V_-)}{C_1\sqrt{n}} \leqslant \frac{Cn}{C_1\sqrt{n}} = \frac{C}{C_1}\sqrt{n}. \tag{12}$$

By Corollary 14, we have $w_0 \in L$. In particular, $\mathrm{Vol}(L) \geqslant 1$.

Similar to Inequality (5), we have

$$\begin{aligned}
\lambda_n^2 \mathrm{Vol}(L) &\leqslant \sum_{y \in V_+} \mathbf{z}_y |E(L, N(y) \cap V_-)| \\
&= \sum_{y \in L} \mathbf{z}_y |E(L, N(y) \cap V_-)| + \sum_{y \in S} \mathbf{z}_y |E(L, N(y) \cap V_-)|.
\end{aligned} \tag{13}$$

We apply the following estimation. For $y \in L$, we have

$$|E(L, N(y) \cap V_-)| \leqslant |E(L, V_-)| \leqslant (s-1)|V_-| + C|L|. \tag{14}$$

For $y \in S$, we have

$$|E(L, N(y) \cap V_-)| \leqslant (s-1)|L| + CC_1\sqrt{n}. \tag{15}$$

Combining Equations (13), (14), and (15), we get

$$\lambda_n^2 \mathrm{Vol}(L) \leqslant \mathrm{Vol}(L)\left((s-1)|V_-| + C|L|\right) + \mathrm{Vol}(S)\left((s-1)|L| + CC_1\sqrt{n}\right). \tag{16}$$

Equivalently, we have

$$\begin{aligned}
|V_-| &\geqslant \frac{\lambda_n^2}{s-1} - C|L| - \frac{\mathrm{Vol}(S)}{\mathrm{Vol}(L)}\left((s-1)|L| + CC_1\sqrt{n}\right) \\
&\geqslant n - C'\sqrt{n}
\end{aligned}$$

for some sufficiently large constant C'. Here we apply Inequality (12) that $\mathrm{Vol}(S) \leqslant \mathrm{Vol}(V_+) = O(1)$ and $\mathrm{Vol}(L) \geqslant 1$. Thus, we have $|V_+| = O(\sqrt{n})$. $\square$

Lemma 16. *There exist some constant C_2 and $s-1$ vertices $u_1, \ldots, u_{s-1}$ satisfying*

(i) For any $1 \leqslant i \leqslant s-1$, we have $d(u_i) \geqslant n - C_2\sqrt{n}$.

(ii) For any vertex $v \notin \{u_1, \ldots, u_{s-1}\}$, we have $d(v) \leqslant sC_2\sqrt{n}$.

Proof. This time we define L as follows:

$$L = \{v \in V_+ \colon |N(v) \cap V_-| \geqslant n - C_2\sqrt{n}\},$$

where C_2 is some big constant chosen later, and let $S = V_+ \setminus L$.

We first claim that $|L| \leqslant s-1$. Otherwise, there exist s vertices $u_1, \ldots, u_s \in L$. We have

$$\bigcap_{i=1}^{s} (N(u_i) \cap V_-) \geqslant |V_-| - sC_2\sqrt{n} > t,$$

when n is sufficiently large. Therefore, G contains a subgraph $K_{s,t}$, giving a contradiction. Hence $|L| \leqslant s-1$.

Now let us consider $\lambda_n^2 \mathrm{Vol}(V_+)$. By Lemma 15, we know that $|V_+| \leqslant C'\sqrt{n}$ for some constant C'. As before, we have

$$\begin{aligned} \lambda_n^2 \mathrm{Vol}(V_+) &\leqslant \sum_{y \in V_+} \mathbf{z}_y |E(V_+, N(y) \cap V_-)| \\ &= \sum_{y \in L} \mathbf{z}_y |E(V_+, N(y) \cap V_-)| + \sum_{y \in S} \mathbf{z}_y |E(V_+, N(y) \cap V_-)|. \end{aligned} \tag{17}$$

We apply the following estimation. We let $C = 4^{s+1} s! t$. For $y \in S$, we have

$$|E(V_+, N(y) \cap V_-)| \leqslant (s-1)|N(y) \cap V_-| + C|V_+| \leqslant (s-1)(n - C_2\sqrt{n}) + CC'\sqrt{n}. \tag{18}$$

For $y \in L$, we have

$$|E(V_+, N(y) \cap V_-)| \leqslant |E(V_+, V_-)| \leqslant (s-1)n + CC'\sqrt{n}. \tag{19}$$

Plugging Equations (18) and (19) into Equation (17), we get

$$\begin{aligned} \lambda_n^2 \mathrm{Vol}(V_+) &\leqslant \mathrm{Vol}(S)\left((s-1)(n - C_2\sqrt{n}) + CC'\sqrt{n}\right) + \mathrm{Vol}(L)\left((s-1)n + CC'\sqrt{n})\right) \\ &= \mathrm{Vol}(V_+)\left((s-1)n + CC'\sqrt{n})\right) - \mathrm{Vol}(S)(s-1)C_2\sqrt{n}. \end{aligned} \tag{20}$$

Applying the lower bound of $|\lambda_n|$ in Lemma 11, we conclude

$$\mathrm{Vol}(S) \leqslant \frac{CC' + (s-1)(s+t-3) + O(1)}{(s-1)C_2} \mathrm{Vol}(V_+). \tag{21}$$

Choose C_2 large enough such that $\frac{CC'+(s-1)(s+t-3)+O(1)}{(s-1)C_2} \leqslant \frac{1}{s^2}$ and $C_2\sqrt{n} - |V_+| \geqslant t$ (recall that $|V_+| = O(\sqrt{n})$ by Lemma 15). We then have that

$$\mathrm{Vol}(S) \leqslant \frac{1}{s^2} \mathrm{Vol}(V_+).$$

This implies

$$\mathrm{Vol}(S) \leqslant \frac{1}{s^2-1}\mathrm{Vol}(L).$$

Since $\mathrm{Vol}(L) \leqslant |L| \leqslant s-1$, we get

$$\mathrm{Vol}(S) \leqslant \frac{s-1}{s^2-1} = \frac{1}{s+1}.$$

Now we do the similar calculation for $\mathrm{Vol}(L)$. We have

$$\begin{aligned}\lambda_n^2\mathrm{Vol}(L) &\leqslant \sum_{y\in V_+} \mathbf{z}_y|E(L, N(y)\cap V_-)| \\ &= \sum_{y\in L} \mathbf{z}_y|E(L, N(y)\cap V_-)| + \sum_{y\in S} \mathbf{z}_y|E(L, N(y)\cap V_-)|. \end{aligned} \tag{22}$$

We apply the following estimation. For $y \in S$, we have

$$|E(L, N(y)\cap V_-)| \leqslant (s-1)|N(y)\cap V_-)| + C|L| \leqslant (s-1)(n - C_2\sqrt{n}) + C(s-1). \tag{23}$$

For $y \in L$, we have

$$|E(L, N(y)\cap V_-)| \leqslant |E(L, V_-)| \leqslant |L|n. \tag{24}$$

Plugging Equations (23) and (24) into Equation (22), we get

$$\lambda_n^2\mathrm{Vol}(L) \leqslant \mathrm{Vol}(S)\left((s-1)(n - C_2\sqrt{n}) + C(s-1)\right) + \mathrm{Vol}(L)|L|n. \tag{25}$$

Since $w_0 \in L$, we have $\mathrm{Vol}(L) \geqslant 1$. We then obtain that

$$\begin{aligned}|L| &\geqslant \frac{\lambda_n^2}{n} - \frac{1}{(s^2-1)n}\left((s-1)(n - C_2\sqrt{n}) + C(s-1)\right) \\ &\geqslant s - 1 - \frac{1}{s+1} + o(1).\end{aligned}$$

Since $|L|$ is an integer, we have

$$|L| \geqslant s-1.$$

Together with the upper bound in Inequality (12), we get $|L| = s-1$.

Now we could write $L = \{u_1, \ldots, u_{s-1}\}$. We then have that

$$\left|\bigcap_{i=1}^{s-1}(N(u_i)\cap V_-)\right| \geqslant |V_-| - (s-1)C_2\sqrt{n}. \tag{26}$$

Now we claim that for any vertex $v \notin L$, $d(v) \leqslant sC_2\sqrt{n}$. Otherwise, since C_2 is chosen such that $C_2\sqrt{n} - |V_+| \geqslant t$, we then have

$$\left|N(v) \cap \left(\bigcap_{i=1}^{s-1}(N(u_i)\cap V_-)\right)\right| \geqslant sC_2\sqrt{n} - |V_+| - (s-1)C_2\sqrt{n} \geqslant C_2\sqrt{n} - |V_+| \geqslant t,$$

which implies that $L \cup \{v\}$ and t of their common neighbors form a $K_{s,t}$ in G, giving a contradiction. Thus, $d_v \leqslant sC_2\sqrt{n}$ for any $v \notin L$. □

Lemma 17. *We have*

(i) For any $1 \leqslant i \leqslant s-1$, $\mathbf{z}_{u_i} = 1 - O\left(\frac{1}{\sqrt{n}}\right)$.

(ii) For any vertex $v \notin \{u_1, \ldots, u_{s-1}\}$, *we have* $|\mathbf{z}_v| = O(\frac{1}{\sqrt{n}})$.

Proof. We will prove (ii) first. Let C_2 be the same constant obtained from Lemma 16. Let $L = \{v \in V_+ \colon |N(v) \cap V_-| \geqslant n - C_2\sqrt{n}\}$, and $S = V_+ \setminus L$. By Corollary 14, we know that for every $v \in V^-$, $|\mathbf{z}_v| = O(\frac{1}{\sqrt{n}})$. Thus it suffices to show that for every $v \in S$, $|\mathbf{z}_v| = O(\frac{1}{\sqrt{n}})$. Indeed, for every $v \in S$, we have that

$$\begin{aligned}
|\lambda_n|^2 \mathbf{z}_v &\leqslant |\lambda_n| \sum_{u \in N(v) \cap V_-} |\mathbf{z}_u| \\
&\leqslant \sum_{u \in N(v) \cap V_-} \sum_{y \in N(u) \cap V_+} \mathbf{z}_y \\
&= \sum_{y \in V_+} \mathbf{z}_y \cdot |N(v) \cap N(y) \cap V_-| \\
&\leqslant sC_2 \cdot \sum_{y \in V_+} \mathbf{z}_y \\
&\leqslant sC_2 \cdot O(1).
\end{aligned}$$

Thus, $\mathbf{z}_v = O\left(\frac{1}{n}\right)$. This completes the proof of (ii).

Finally, we estimate $\mathbf{z}_{u_i}$ for $1 \leqslant i \leqslant s-1$. By previous lemmas, we know that $w_0 \in \{u_1, \ldots, u_{s-1}\}$. From the eigen-equations, we obtain that for each u_i $(1 \leqslant i \leqslant s-1)$,

$$|\lambda_n|(\mathbf{z}_{w_0} - \mathbf{z}_{u_i}) = -\sum_{u \in N(w_0) \setminus N(u_i)} \mathbf{z}_u + \sum_{u \in N(u_i) \setminus N(w_0)} \mathbf{z}_u \tag{27}$$

$$\leqslant \sum_{u \in (N(w_0) \setminus N(u_i)) \cap V_-} |\mathbf{z}_u| + \sum_{u \in (N(u_i) \setminus N(w_0)) \cap V_+} \mathbf{z}_u \tag{28}$$

$$\leqslant \sum_{u \in (N(w_0) \setminus N(u_i)) \cap V_-} |\mathbf{z}_u| + O(1) \tag{29}$$

$$\leqslant C_2\sqrt{n} \cdot O\left(\frac{1}{\sqrt{n}}\right) + O(1) \tag{30}$$

$$= O(1). \tag{31}$$

Therefore, we have $\mathbf{z}_{u_i} \geqslant 1 - O\left(\frac{1}{\sqrt{n}}\right)$ since $\mathbf{z}_{w_0} = 1$ and $\mathbf{z}_{w_0} - \mathbf{z}_{u_i} = O(\frac{1}{\sqrt{n}})$. □

Recall that we let $L := \{u_1, u_2, \cdots, u_{s-1}\}$. Let $V' = \{v \in V(G) \backslash L : |N(v) \cap L| = s-1\}$ and let $V'' = V(G) \backslash (L \cup V')$. We have the following lemma on the structure of G.

Lemma 18. *We have the following properties.*

(i) $|V'| \geqslant n-(s-1)C_2\sqrt{n}$.

(ii) *For any* $v \in V(G)\backslash L$, $|N(v)\cap V'| \leqslant t-1$.

(iii) *In* $H = G[V(G)\backslash L]$, *for any vertex* $v \in V(H)$, $|N_H(N_H(v))\cap V'| \leqslant t^2$.

Proof. By Lemma 16, $\min\limits_{u\in L} d(u) \geqslant n - C_2\sqrt{n}$. It follows that $|V'| \geqslant n-(s-1)C_2\sqrt{n} \geqslant t$. For any $v \in V(G)\backslash L$, v has at most $t-1$ neighbors in V', otherwise, $L\cup\{v\}$ and t of their common neighbors in V' would form a $K_{s,t}$ in G.

Now for any $v \in V(G)\backslash L$, we claim that $|N_H(N_H(v))\cap V'| \leqslant t^2$. Indeed, suppose not, then by (ii) and the Pigeonhole principle, there exist t vertex-disjoint 2-vertex paths from v to t distinct vertices in V'. But then it is not hard to see that $L\cup\{v\}$ and these t distinct vertices would form a $K_{s,t}$ minor, giving a contradiction. $\square$

Lemma 19. *We have*

(i) *For any* $1 \leqslant i \leqslant s-1$, $\mathbf{x}_{u_i} = 1 - O\left(\frac{1}{\sqrt{n}}\right)$.

(ii) *For any vertex* $v \notin \{u_1,\ldots,u_{s-1}\}$, *we have* $\mathbf{x}_v = O(\frac{1}{\sqrt{n}})$.

Proof. Let us prove (ii) first. For any vertex $v \notin \{u_1,\ldots,u_{s-1}\}$, by the eigen-equations, we have that

$$\begin{aligned}
\lambda_1^2\mathbf{x}_v &= \lambda_1\sum_{u\in N(v)}\mathbf{x}_u\\
&= \lambda_1\left(\sum_{u\in N(v)\cap V'}\mathbf{x}_u + \sum_{u\in N(v)\cap L}\mathbf{x}_u + \sum_{u\in N(v)\cap V''}\mathbf{x}_u\right)\\
&\leqslant \lambda_1\left((t-1)+(s-1)+\sum_{u\in N(v)\cap V''}\mathbf{x}_u\right)\\
&= (t+s-2)\lambda_1 + \sum_{u\in N(v)\cap V''}\lambda_1\mathbf{x}_u\\
&= (t+s-2)\lambda_1 + \sum_{u\in N(v)\cap V''}\sum_{w\in N(u)}\mathbf{x}_w\\
&= (t+s-2)\lambda_1 + \sum_{u\in N(v)\cap V''}\left(\sum_{w\in N(u)\cap L}\mathbf{x}_w + \sum_{w\in N(u)\cap V'}\mathbf{x}_w + \sum_{w\in N(u)\cap V''}\mathbf{x}_w\right)\\
&\leqslant (t+s-2)\lambda_1 + (s-1)|V''| + t^2 + \sum_{u\in N(v)\cap V''}\sum_{w\in N(u)\cap V''}\mathbf{x}_w\\
&\leqslant (t+s-2)\lambda_1 + (s-1)|V''| + t^2 + 2|E(G[V''])|\\
&\leqslant (t+s-2)\lambda_1 + (s-1)(sC_2\sqrt{n}) + t^2 + O(\sqrt{n})
\end{aligned}$$

$$= O(\sqrt{n}).$$

It follows that $\mathbf{x}_v = O(\frac{1}{\sqrt{n}})$.

Now we will prove (i). Recall that u_0 is a vertex such that $\mathbf{x}_{u_0} = 1$. Thus $u_0 \in L$. Let u_i be an arbitrary vertex in $L \backslash \{u_0\}$.

If u_0u_i is not an edge of G, then we have

$$\begin{aligned}
\lambda_1 |\mathbf{x}_{u_0} - \mathbf{x}_{u_i}| &\leqslant \sum_{v \in V''} \mathbf{x}_v + \sum_{v \in L} \mathbf{x}_v \\
&\leqslant |V''| \cdot O\left(\frac{1}{\sqrt{n}}\right) + (s-1) \\
&= O(1).
\end{aligned}$$

If u_0u_i is an edge of G, we have

$$\begin{aligned}
(\lambda_1 - 1) |\mathbf{x}_{u_0} - \mathbf{x}_{u_i}| &\leqslant \sum_{v \in V''} \mathbf{x}_v + \sum_{v \in L} \mathbf{x}_v \\
&\leqslant |V''| \cdot O\left(\frac{1}{\sqrt{n}}\right) + (s-1) \\
&= O(1).
\end{aligned}$$

In both cases, we have

$$|\mathbf{x}_{u_0} - \mathbf{x}_{u_i}| = O\left(\frac{1}{\sqrt{n}}\right).$$

It follows that $\mathbf{x}_{u_i} = 1 - O\left(\frac{1}{\sqrt{n}}\right)$ for any $i \in [s-1]$. ☐

Now we are ready to show that G has $s-1$ vertices that are connected to each of the rest of the $n-s+1$ vertices.

Lemma 20. *G contains the subgraph $K_{s-1,n-s+1}$.*

Proof. Let $\mathbf{x}$ and $\mathbf{z}$ be the eigenvectors associated with λ_1 and λ_n respectively. Assume that $\mathbf{x}$ and $\mathbf{z}$ are both normalized such that the largest entries of them in absolute value are 1. By Lemma 16, there exist $s-1$ vertices $L = \{u_1, u_2, \cdots, u_{s-1}\}$ such that for every $v \in L$, $d(v) \geqslant n - C_2\sqrt{n}$ and for every $v \notin L$, $d(v) \leqslant sC_2\sqrt{n}$. Recall that $V' = \{v \in V(G) \backslash L : |N(v) \cap L| = s-1\}$ and $V'' = V(G) \backslash (L \cup V')$.

To prove the lemma, it suffices to show that V'' is empty. Suppose otherwise that V'' is not empty. Note that V'' induces a $K_{s,t}$-minor-free graph, and by Theorem 4 and its corollary, we know that there exists some constant C_0 and some vertex $v_0 \in V''$ such that $d_{G[V'']}(v_0) \leqslant C_0$. Moreover, observe that v_0 has at most $(t-1)$ neighbors in V', as otherwise $L \cup \{v_0\}$ and t of their common neighbors would form a $K_{s,t}$ in G.

Let G' be obtained from G by removing all the edges of G incident with v_0 and adding an edge from v_0 to every vertex of L, so that $E(G') = E(G - v_0) \cup \{v_0u_1, v_0u_2, \cdots, v_0u_{s-1}\}$. Observe G' is still $K_{s,t}$-minor-free.

We claim that $\lambda_n(G') < \lambda_n(G)$. Indeed, consider the vector $\tilde{\mathbf{z}}$ such that $\tilde{\mathbf{z}}_u = \mathbf{z}_u$ for $u \neq v_0$ and $\tilde{\mathbf{z}}_{v_0} = -|\mathbf{z}_{v_0}|$. Then

$$\begin{aligned}
\tilde{\mathbf{z}}' A(G')\tilde{\mathbf{z}} &= \sum_{uv\in E(G-v_0)} 2\mathbf{z}_u\mathbf{z}_v + 2\tilde{\mathbf{z}}_{v_0}\cdot \mathrm{Vol}(L) \\
&= \sum_{uv\in E(G)} 2\mathbf{z}_u\mathbf{z}_v - 2\sum_{y\sim v_0}\mathbf{z}_y\mathbf{z}_{v_0} - 2|\mathbf{z}_{v_0}|\mathrm{Vol}(L) \\
&\leqslant \mathbf{z}'A(G)\mathbf{z} + 2\sum_{y\sim v_0}|\mathbf{z}_y\mathbf{z}_{v_0}| - 2|\mathbf{z}_{v_0}|\cdot \mathrm{Vol}(L) \\
&\leqslant \mathbf{z}'A(G)\mathbf{z} + 2\cdot(t+C_0)\cdot O\left(\frac{1}{\sqrt{n}}\right)\cdot|\mathbf{z}_{v_0}| - 2\left(1 - O\left(\frac{1}{\sqrt{n}}\right)\right)|\mathbf{z}_{v_0}| \\
&< \mathbf{z}'A(G)\mathbf{z}.
\end{aligned}$$

Similarly, we claim that $\lambda_1(G') > \lambda_1(G)$. Indeed,

$$\begin{aligned}
\mathbf{x}'\mathbf{x}\lambda_1(G') &= \mathbf{x}'A(G')\mathbf{x} \\
&= \mathbf{x}'A(G)\mathbf{x} - 2\sum_{y\sim v_0}\mathbf{x}_y\mathbf{x}_{v_0} + 2\mathbf{x}_{v_0}\mathrm{Vol}(L) \\
&\geqslant \mathbf{x}'\mathbf{x}\lambda_1(G) - 2\cdot(t+C_0)\cdot O\left(\frac{1}{\sqrt{n}}\right)\cdot \mathbf{x}_{v_0} + 2\left(1 - O\left(\frac{1}{\sqrt{n}}\right)\right)\mathbf{x}_{v_0} \\
&> \mathbf{x}'\mathbf{x}\lambda_1(G).
\end{aligned}$$

Hence we have $S(G') = \lambda_1(G') - \lambda_n(G') > \lambda_1(G) - \lambda_n(G) = S(G)$, giving a contradiction. ☐

3 Proof of Theorem 1

By Lemma 20, a maximum-spread $K_{s,t}$-minor-free graph G contains a complete bipartite subgraph $K_{s-1,n-s+1}$. We denote the part of $s-1$ vertices by L and the other part of $n-s+1$ vertices by R. Let α be a normalized eigenvector corresponding to an eigenvalue λ of the adjacency matrix of G. Let A_L (or A_R) be the adjacency matrix of the induced subgraph $G[L]$ (or $G[R]$) respectively.

Let α_L (respectively, α_R) denote the restriction of α to L (respectively, R). The following lemma computes the vectors α_L and α_R.

Lemma 21. *If $|\lambda| > t-1$, then*

$$\alpha_R = (\mathbf{1}'\alpha_L)\sum_{k=0}^{\infty}\lambda^{-(k+1)}A_R^k\mathbf{1}, \tag{32}$$

$$\alpha_L = (\mathbf{1}'\alpha_R)\sum_{k=0}^{\infty}\lambda^{-(k+1)}A_L^k\mathbf{1}. \tag{33}$$

Proof. Note that since G is $K_{s,t}$-minor-free and every vertex in L is adjacent to every vertex in R, it follows that $G[R]$ is $K_{1,t}$-minor-free, and thus the maximum degree of $G[R]$ is at most $t-1$. For n sufficiently large, both $\lambda_1(G)$ and $|\lambda_n(G)|$ are greater than $t-1$. Hence when restricting the coordinates of $A(G)\alpha$ to R, we have that

$$A_R\alpha_R + (\mathbf{1}'\alpha_L)\mathbf{1} = \lambda\alpha_R. \tag{34}$$

It then follows that

$$\begin{aligned}
\alpha_R &= (\mathbf{1}'\alpha_L)(\lambda I - A_R)^{-1}\mathbf{1} \\
&= (\mathbf{1}'\alpha_L)\lambda^{-1}(I - \lambda^{-1}A_R)^{-1}\mathbf{1} \\
&= (\mathbf{1}'\alpha_L)\lambda^{-1}\sum_{k=0}^{\infty}(\lambda^{-1}A_R)^k\mathbf{1} \\
&= (\mathbf{1}'\alpha_L)\sum_{k=0}^{\infty}\lambda^{-(k+1)}A_R^k\mathbf{1}. \qquad (35)
\end{aligned}$$

Here we use the assumption that $|\lambda| > t-1 \geqslant \lambda_1(A_R)$ so that the infinite series converges. Similarly, we have

$$\alpha_L = (\mathbf{1}'\alpha_R)\sum_{k=0}^{\infty}\lambda^{-(k+1)}A_L^k\mathbf{1}.$$

□

Lemma 22. *Both λ_1 and λ_n satisfy the following equation.*

$$\lambda^2 = \left(\sum_{k=0}^{\infty}\lambda^{-k}\mathbf{1}'A_L^k\mathbf{1}\right)\cdot\left(\sum_{k=0}^{\infty}\lambda^{-k}\mathbf{1}'A_R^k\mathbf{1}\right). \tag{36}$$

Proof. From Equations (32) and (33), we have

$$\mathbf{1}'\alpha_R = (\mathbf{1}'\alpha_L)\sum_{k=0}^{\infty}\mathbf{1}'\lambda^{-(k+1)}A_R^k\mathbf{1}, \tag{37}$$

$$\mathbf{1}'\alpha_L = (\mathbf{1}'\alpha_R)\sum_{k=0}^{\infty}\mathbf{1}'\lambda^{-(k+1)}A_L^k\mathbf{1}. \tag{38}$$

Thus

$$\mathbf{1}'\alpha_R = (\mathbf{1}'\alpha_R)\left(\sum_{k=0}^{\infty}\mathbf{1}'\lambda^{-(k+1)}A_L^k\mathbf{1}\right)\cdot\left(\sum_{k=0}^{\infty}\lambda^{-(k+1)}\mathbf{1}'A_R^k\mathbf{1}\right). \tag{39}$$

Since $\mathbf{1}'\alpha_R > 0$, equation (36) is obtained by canceling $\mathbf{1}'\alpha_R$. □

For $k = 1, 2, 3\ldots$, let $l_k = \mathbf{1}'A_L^k\mathbf{1}$, $r_k = \mathbf{1}'A_R^k\mathbf{1}$, and $a_k = \sum_{j=0}^{k} l_j r_{k-j}$. Then Equation (36) can be written as:

$$\lambda^2 = \sum_{k=0}^{\infty}a_k\lambda^{-k}. \tag{40}$$

In particular, we have

$$\begin{aligned} l_0 &= s-1; && (41)\\ l_1 &= 2|E(G[L])|; && (42)\\ r_0 &= n-s+1; && (43)\\ r_1 &= 2|E(G[R])|; && (44)\\ a_0 &= l_0 r_0 = (s-1)(n-s+1), && (45)\\ a_1 &= l_0 r_1 + l_1 r_0. && (46) \end{aligned}$$

Lemma 23. *We have the following estimation on the spread of G:*

$$S(G) = 2\sqrt{a_0} + \frac{2c_2}{\sqrt{a_0}} + \frac{2c_4}{a_0^{3/2}} + \frac{2c_6}{a_0^{5/2}} + O\left(a_0^{-7/2}\right). \tag{47}$$

Here

$$a_0 = (s-1)(n-s+1) \tag{48}$$

$$c_2 = -\frac{3}{8}\left(\frac{a_1}{a_0}\right)^2 + \frac{1}{2}\frac{a_2}{a_0}, \tag{49}$$

$$c_4 = -\frac{105}{128}\left(\frac{a_1}{a_0}\right)^4 + \frac{35}{16}\left(\frac{a_1}{a_0}\right)^2\frac{a_2}{a_0} - \frac{5}{8}\left(\frac{a_2}{a_0}\right)^2 - \frac{5}{4}\frac{a_1}{a_0}\frac{a_3}{a_0} + \frac{1}{2}\frac{a_4}{a_0} \tag{50}$$

$$\begin{aligned} c_6 = &-\frac{3003}{1024}\left(\frac{a_1}{a_0}\right)^6 + \frac{3003}{256}\left(\frac{a_1}{a_0}\right)^4\frac{a_2}{a_0} - \frac{693}{64}\left(\frac{a_1}{a_0}\right)^2\left(\frac{a_2}{a_0}\right)^2 + \frac{21}{16}\left(\frac{a_2}{a_0}\right)^3\\ &- \frac{21}{32}\left(11\left(\frac{a_1}{a_0}\right)^3 - 12\left(\frac{a_1}{a_0}\right)\left(\frac{a_2}{a_0}\right)\right)\left(\frac{a_3}{a_0}\right) - \frac{7}{8}\left(\frac{a_3}{a_0}\right)^2\\ &+ \frac{7}{16}\left(9\left(\frac{a_1}{a_0}\right)^2 - 4\frac{a_2}{a_0}\right)\frac{a_4}{a_0} - \frac{7}{4}\frac{a_1}{a_0}\frac{a_5}{a_0} + \frac{1}{2}\frac{a_6}{a_0}. \end{aligned} \tag{51}$$

Proof. Recall that by (40), we have that for $\lambda \in \{\lambda_1, \lambda_n\}$,

$$\lambda^2 = a_0 + \sum_{k=1}^{\infty} a_k \lambda^{-k}.$$

By the main lemma in the appendix of [17], λ has the following series expansion:

$$\lambda_1 = \sqrt{a_0} + c_1 + \frac{c_2}{\sqrt{a_0}} + \frac{c_3}{a_0} + \frac{c_4}{a_0^{3/2}} + \frac{c_5}{a_0^2} + \frac{c_6}{a_0^{5/2}} + O\left(a_0^{-7/2}\right).$$

Similarly,

$$\lambda_n = -\sqrt{a_0} + c_1 - \frac{c_2}{\sqrt{a_0}} + \frac{c_3}{a_0} - \frac{c_4}{a_0^{3/2}} + \frac{c_5}{a_0^2} - \frac{c_6}{a_0^{5/2}} + O\left(a_0^{-7/2}\right).$$

Using SageMath, we get that c_2, c_4, c_6 are the values in Equations (49), (50), (51) respectively. It follows that

$$S(G) = \lambda_1 - \lambda_n = 2\sqrt{a_0} + \frac{2c_2}{\sqrt{a_0}} + \frac{2c_4}{a_0^{3/2}} + \frac{2c_6}{a_0^{5/2}} + O\left(a_0^{-7/2}\right).$$

□

Proof of Theorem 1. Recall that by Lemma 23, we have the following estimation of the spread of G:

$$S(G) = 2\sqrt{a_0} + \frac{2c_2}{\sqrt{a_0}} + \frac{2c_4}{(n-1)^{3/2}} + \frac{2c_6}{(n-1)^{5/2}} + O\left(n^{-7/2}\right). \tag{52}$$

where c_2, c_4 and c_6 are as in Lemma 23.

Since G is $K_{s,t}$-minor free, $G[R]$ is $K_{1,t}$-minor free. Thus the maximum degree of $G[R]$ is at most $t-1$. In particular, $r_2 \leqslant (t-1)r_1$. All c_i's are bounded by constants depending on t. Note that

$$\begin{aligned}
c_2 &= -\frac{3}{8}\left(\frac{a_1}{a_0}\right)^2 + \frac{1}{2}\frac{a_2}{a_0} \\
&= -\frac{3}{8}\left(\frac{l_1 r_0 + l_0 r_1}{l_0 r_0}\right)^2 + \frac{1}{2}\frac{l_2 r_0 + l_1 r_1 + l_0 r_2}{r_0 l_0} \\
&= -\frac{3}{8}\left(\frac{l_1}{l_0} + \frac{r_1}{r_0}\right)^2 + \frac{1}{2}\left(\frac{l_2}{l_0} + \frac{l_1}{l_0}\frac{r_1}{r_0} + \frac{r_2}{r_0}\right) \\
&= -\frac{3}{8}\left(\frac{l_1}{3l_0} + \frac{r_1}{r_0}\right)^2 + \frac{l_2}{2l_0} - \frac{l_1^2}{3l_0^2} + \frac{r_2}{2r_0} \\
&= \frac{(t-1)^2}{6} - \frac{3}{8}\left(\frac{l_1}{3l_0} + \frac{r_1}{r_0} - \frac{2}{3}(t-1)\right)^2 + \frac{l_2}{2l_0} - \frac{(t-1)l_1}{6l_0} - \frac{l_1^2}{3l_0^2} + \frac{r_2 - (t-1)r_1}{2r_0} \\
&= \frac{(t-1)^2}{6} - \frac{3}{8}\left(\frac{l_1}{3l_0} + \frac{r_1}{r_0} - \frac{2}{3}(t-1)\right)^2 + \frac{\psi(G[L])}{6l_0} + \frac{r_2 - (t-1)r_1}{2r_0} \\
&\leqslant \frac{(t-1)^2}{6} + \frac{\psi(L_{max})}{6l_0}.
\end{aligned}$$

At the last step, the equality holds only if

1. $\psi(L) = \psi(L_{max})$,
2. $r_2 = (t-1)r_1$,
3. $\frac{l_1}{3l_0} + \frac{r_1}{r_0} - \frac{2}{3}(t-1) = 0$.

Thus, we have

$$S(G) \leqslant 2\sqrt{a_0} + \frac{(t-1)^2 + \psi(L_{max})/(s-1)}{3\sqrt{a_0}} + O\left(\frac{1}{n^{3/2}}\right).$$

This upper bound is asymptotically tight. Consider

$$G_0 = L_{max} \vee (\ell_0 K_t \cup (n-s+1-t\ell_0)\, P_1)$$

where ℓ_0 is an integer such that $\frac{l_1}{3l_0} + \frac{r_1}{r_0} - \frac{2}{3}(t-1)$ is close to zero. Thus

$$S(G_0) = 2\sqrt{a_0} + \frac{(t-1)^2 + \psi(L_{max})/(s-1)}{3\sqrt{a_0}} + O\left(\frac{1}{n^{3/2}}\right).$$

Claim 24. $G[L] = L_{max}$.

Proof. Otherwise, we have $\psi(G[L]) < \psi(L_{max})$. Also, by the definition of ψ, we have that $\psi(G[L]) \leqslant \psi(L_{max}) - \frac{1}{s-1}$. It then follows that for sufficiently large n,

$$\begin{aligned} S(G) &\leqslant 2\sqrt{a_0} + \frac{(t-1)^2 + \psi(G[L])/(s-1)}{3\sqrt{a_0}} + O\left(\frac{1}{n^{3/2}}\right) \\ &< 2\sqrt{a_0} + \frac{(t-1)^2 + \psi(L_{max})/(s-1)}{3\sqrt{a_0}} + O\left(\frac{1}{n^{3/2}}\right) \\ &= S(G_0), \end{aligned}$$

giving a contradiction. □

Claim 25. *There is a constant C such that the value of $\frac{l_1}{3l_0} + \frac{r_1}{r_0}$ that maximizes $S(G)$ lies in the interval $\left(\frac{2}{3}(t-1) - Cn^{-1/2}, \frac{2}{3}(t-1) + Cn^{-1/2}\right)$.*

Proof. Otherwise, for any $\frac{l_1}{3l_0} + \frac{r_1}{r_0}$ not in this interval (where C is chosen later), we have

$$\begin{aligned} c_2 &\leqslant \frac{(t-1)^2}{6} - \frac{3}{8}C^2 n^{-1} + \frac{\psi(L_{max})}{6l_0} + \frac{r_2 - (t-1)r_1}{2r_0} \\ &\leqslant \frac{(t-1)^2}{6} - \frac{3}{8}C^2 n^{-1} + \frac{\psi(L_{max})}{6l_0}. \end{aligned}$$

This implies that

$$S(G) - S(G_0) \leqslant -\frac{\frac{3}{4}C^2 n^{-1}}{\sqrt{a_0}} + O\left(\frac{1}{n^{3/2}}\right) < 0,$$

giving a contradiction when C is chosen to be large enough such that $-\frac{\frac{3}{4}C^2 n^{-1}}{\sqrt{a_0}} + O\left(\frac{1}{n^{3/2}}\right) < 0$. □

From now on, we assume that $\frac{l_1}{3l_0} + \frac{r_1}{r_0} \in \left(\frac{2}{3}(t-1) - Cn^{1/2}, \frac{2}{3}(t-1) + Cn^{1/2}\right)$.

Claim 26. *There is a constant C_2 such that the value of r_2 lies in the interval $[(t-1)r_1 - C_2, (t-1)r_1]$.*

Proof. Otherwise, we assume $r_2 < (t-1)r_1 - C_2$ for some C_2 chosen later. We then have

$$S(G) \leqslant 2\sqrt{a_0} + \frac{(t-1)^2 + \psi(L_{max})/(s-1)}{3\sqrt{a_0}} - \frac{C_2}{r_0\sqrt{a_0}} + O\left(\frac{1}{n^{3/2}}\right) < S(G_0),$$

when C_2 is chosen to be sufficiently large, giving a contradiction. □

Claim 27. *For $i \geqslant 3$, we have $r_i \in [(t-1)^{i-1}(r_1 - (i-1)C_2), r_1(t-1)^{i-1}]$.*

Proof. Let R' be the set of vertices in R such that its degree is in the interval $[1, t-2]$. We have

$$C_2 \geqslant (t-1)r_1 - r_2 = \sum_{v\in R'}(t-1-d(v))d(v) \geqslant (t-2)|R'|.$$

This implies

$$|R'| \leqslant \frac{C_2}{t-2}.$$

We have

$$(t-1)r_{i-1} - r_i \leqslant (t-2)|R'|(t-1)^{i-1} \leqslant C_2(t-1)^{i-1}. \tag{53}$$

Thus,

$$\begin{aligned} r_i &\geqslant (t-1)r_{i-1} - C_2(t-1)^{i-1} \\ &\geqslant (t-1)((t-1)r_{i-2} - C_2(t-1)^{i-2}) - C_2(t-1)^{i-1} && \text{by induction} \\ &= (t-1)^2 r_{i-2} - 2C_2(t-1)^{i-1} \\ &\geqslant \cdots \\ &\geqslant (t-1)^{i-1}r_1 - (i-1)C_2(t-1)^{i-1}. \end{aligned}$$

□

Claim 28. $r_2 = (t-1)r_1$ *and* $\frac{l_1}{3l_0} + \frac{r_1}{r_0} - \frac{2}{3}(t-1) = O(n^{-(1-\epsilon)})$ *for any given* $\epsilon > 0$.

Proof. Assume that $\frac{l_1}{3l_0} + \frac{r_1}{r_0} - \frac{2}{3}(t-1) = A$, and $r_2 = (t-1)r_1 - B$, where $A \in [-Cn^{-1/2}, Cn^{-1/2}]$ and $0 \leqslant B \leqslant C_2$. It follows that

$$c_2(G) - c_2(G_0) = O\left(n^{-2}\right) - \frac{3A^2}{8} - \frac{B}{2r_0},$$

and

$$c_4(G) - c_4(G_0) = O(n^{-1/2}).$$

Thus

$$S(G) - S(G_0) = 2\frac{c_2(G) - c_2(G_0)}{\sqrt{a_0}} + 2\frac{c_4(G) - c_4(G_0)}{a_0^{3/2}} + \left(a_0^{-5/2}\right)$$

$$\leqslant 2\frac{O\left(n^{-2}\right)-\frac{3A^2}{8}-\frac{B}{2r_0}}{\sqrt{a_0}}+2\frac{O(n^{-1/2})}{a_0^{3/2}}+\left(a_0^{-5/2}\right).$$

This implies $B=0$ and $A=O(n^{-3/4})$.

Notice that $A=O(n^{-3/4})$ implies $c_4(G)-c_4(G_0)=O(n^{-1/4})$, which implies $A=O(n^{-7/8})$. Iterate this process finitely many times. We get $A=O(n^{-(1-\epsilon)})$ for any given $\epsilon>0$. □

Claim 29. *$G[R]$ is the union of vertex disjoint K_ts and isolated vertices.*

Proof. Recall that $r_1=\mathbf{1}'A_R\mathbf{1}=\sum_{i\in R}d_{G[R]}(i)=2|E(G[R])|$, and $r_2=\mathbf{1}'A_R^2\mathbf{1}=\sum_{i\in R}d_{G[R]}(i)^2$. By Claim 28, we have that

$$\sum_{i\in R}d_{G[R]}(i)^2=(t-1)\sum_{i\in R}d_{G[R]}(i).$$

Since $d_{G[R]}(v)\leqslant t-1$ for every $v\in R$, it follows that $G[R]$ is the disjoint union of $(t-1)$-regular graphs and isolated vertices. Let K be an arbitrary non-trivial component of $G[R]$. We will show that K is a clique on t vertices.

For any $u,v\in V(K)$ with $uv\notin E(K)$, we claim that $|N_K(u)\cap N_K(v)|\geqslant t-2$. Otherwise, $|N_K(u)\backslash N_K(v)|\geqslant 2$ and $|N_K(v)\backslash N_K(u)|\geqslant 2$. Pick an arbitrary vertex $w\in N_K(u)\cap N_K(v)$ and contract uw and wv, we then obtain a $K_{1,t}$-minor in K, and thus a $K_{s,t}$-minor in G. Similarly, for any $u,v\in V(K)$ with $uv\in E(K)$, we have $|N_K(u)\cap N_K(v)|\geqslant t-3$.

We claim now that for any $u,v\in V(K)$, $|N_K(u)\cap N_K(v)|\leqslant t-2$. Suppose otherwise that there exist vertices $u,v\in V(K)$ such that $|N_K(u)\cap N_K(v)|=t-1$. Let w be an arbitrary vertex in L. Note that when n is sufficiently large, we could find a length-two path from w to each vertex in $L\backslash\{w\}$ using distinct vertices in $R\backslash V(K)$ as the internal vertices of these paths. It follows that $(L\backslash\{w\})\cup\{u,v\}$ and $(N_K(u)\cap N_K(v))\cup\{w\}$ would form a $K_{s,t}$-minor in G.

Hence, we have that for any $u,v\in V(K)$ with $uv\notin E(K)$, $|N_K(u)\cap N_K(v)|=t-2$. It then follows that there exist $u',v'\in V(K)$ such that $u'\in N_K(u)\backslash N_K(v)$ and $v'\in N_K(v)\backslash N_K(u)$. We claim that $u'v'\notin E(K)$. Indeed, if $u'v'\in E(K)$, we could contract $v'u'$ into w' and obtain a $K_{s,t}$ minor the same way as above.

Now note that since $u'v\notin E(K)$, we have $|N_K(u')\cap N_K(v)|=t-2$. It follows that $N_K(u')\cap N_K(v)=N_K(u)\cap N_K(v)$. Similarly, $N_K(v')\cap N_K(u)=N_K(u)\cap N_K(v)$. We will now analyze $N_K(u)\cap N_K(v)$.

Let $G_1=G[N_K(u)\cap N_K(v)]$. Note that for each vertex $w\in V(G_1)$, w must have at most two non-neighbors in G_1, otherwise $|N_K(u)\cap N_K(w)|\leqslant t-4$, giving a contradiction. Moreover, each vertex $w\in V(G_1)$ has at least two non-neighbors in G_1, otherwise $d_K(w)\geqslant t-4+4=t$, giving a contradiction. It follows that each vertex in G_1 has exactly two non-neighbors in G_1.

Now, if G_1 is a clique, we could easily find a $K_{1,t}$ in K (by identifying one of the vertices in $N(u)\cap N(v)$ as the center). Hence together with L, we have a $K_{s,t}$-minor in G.

Otherwise, we find $a, b \in G_1$ such that $ab \notin E(K)$. Since $|N_K(a) \cap N_K(b)| = t-2$ and each of a and b has exactly one non-neighbor in G_1, we then obtain that $|N_{G_1}(a) \cap N_{G_1}(b)| = t-6$, and there exist $a', b' \in V(G_1)$ such that $a' \in N_{G_1}(a) \backslash N_{G_1}(b)$ and $b' \in N_{G_1}(b) \backslash N_{G_1}(a)$. Similar to before, we have $a'b' \notin E(K)$, and a', b' is each adjacent to $N_{G_1}(a) \cap N_{G_1}(b)$. Repeat this process, eventually, this process has to terminate, and we will have a $K_{1,t}$-minor in K, thus a $K_{s,t}$ minor in G. $\square$

This completes the proof of Theorem 1. $\square$

We now determine the maximum spread $K_{s,t}$-minor-free graphs for all admissible pairs (s, t).

Proof of Theorem 2 . Since (s, t) is admissible, we have $G[L] = (s-1)K_1$. We only need to consider the graph $G_\ell = (s-1)K_1 \vee (\ell K_t \cup (n-s+1-\ell t)P_1)$. We have $l_0 = (s-1)$ and $l_i = 0$ for $i \geqslant 1$. We have $r_0 = (n-s+1)$, and $r_i = \ell t(t-1)^i$ for each $i \geqslant 1$.

Now we apply Lemma 22 to simplify the equation satisfied by both λ_1 and λ_n. Equation (36) can be simplified as

$$\begin{aligned}
\lambda^2 &= \left(\sum_{k=0}^{\infty} \lambda^{-k} \mathbf{1}' A_L^k \mathbf{1}\right) \cdot \left(\sum_{k=0}^{\infty} \lambda^{-k} \mathbf{1}' A_R^k \mathbf{1}\right) \\
&= (s-1)\left(n-s+1+\sum_{k=1}^{\infty} \lambda^{-k} \ell t(t-1)^k\right) \\
&= (s-1)\left(n-s+1+\frac{\ell t \frac{t-1}{\lambda}}{1-\frac{t-1}{\lambda}}\right) \\
&= \frac{(s-1)\left((n-s+1)\lambda-(n-s+1-\ell t)(t-1)\right)}{\lambda-(t-1)}.
\end{aligned}$$

Simplifying it, we get the following cubic equation:

$$\lambda^3-(t-1)\lambda^2-(s-1)(n-s+1)\lambda+(s-1)(t-1)(n-s+1-\ell t)=0. \tag{54}$$

Now let $\lambda = x + \frac{t-1}{3}$. We get the following reduced cubic equation $\phi(x) = 0$.

$$x^3 - px + q = 0, \tag{55}$$

where $p = (s-1)(n-s+1) + \frac{1}{3}(t-1)^2$ and $q = (s-1)(t-1)\left(\frac{2}{3}(n-s+1) - \ell t\right) - \frac{2}{27}(t-1)^3$. Since $\phi(x)$ has at least two real roots, we know from number theory that $p^3 \geqslant \frac{27}{4}q^2$.

We now need a lemma on the spread of a cubic polynomial. If f is a cubic polynomial with three real roots, then the *spread* $S(f)$ is defined to be the difference between the largest and smallest roots of f.

Lemma 30. *Assume $p^3 > \frac{27}{4}q^2$. Let $S(q)$ (with p fixed) be the spread of the cubic equation*

$$x^3 - px + q = 0. \tag{56}$$

If $2\left(\frac{p}{3}\right)^{3/2} > |q_1| > |q_2|$, then

$$S(q_1) < S(q_2).$$

Before we give the proof of Lemma 30, we complete the proof of Theorem 2 using Lemma 30.

Applying Lemma 30, we conclude that $S(G_\ell) = S(\phi)$ reaches the maximum if and only if $|q|$ reaches the minimum. Let ℓ_1 be the real root of the equation $q = 0$. We have

$$\ell_1 = \frac{2}{3t}\left(n - s + 1 - \frac{(t-1)^2}{9(s-1)}\right).$$

Since q is a linear function on ℓ, the function $|q|$ reaches the minimum at the nearest integer of ℓ_1. This completes the proof of Theorem 2. $\square$

We now give the proof of Lemma 30.

Proof of Lemma 30. Since $p^3 > \frac{27}{4}q^2$, the equation $x^3 - px + q = 0$ has three distinct real roots, say $x_1 > x_2 > x_3$. Observe that $-x_1, -x_2, -x_3$ are the roots of $x^3 - px - q = 0$. Thus, these two cubic polynomials have the same spread. Without loss of generality, we can assume $q \geqslant 0$. Let $\alpha = \frac{1}{3}\arccos(-\frac{q/2}{(p/3)^{3/2}}) \in [\frac{\pi}{6}, \frac{\pi}{3})$. We have

$$\cos(3\alpha) = -\frac{q/2}{(p/3)^{3/2}}.$$

Applying the triple angle cosine formula, we have

$$4\cos^3(\alpha) - 3\cos(\alpha) = -\frac{q/2}{(p/3)^{3/2}}.$$

Plugging $\cos(\alpha) = \frac{x}{2(p/3)^{1/2}}$ and simplifying it, we get

$$x^3 - px + q = 0.$$

Thus $x_1 = 2(p/3)^{1/2}\cos(\alpha)$ is a root of Equation (56). Similarly, we get that $x_2 = 2(p/3)^{1/2}\cos\left(\alpha - \frac{2\pi}{3}\right)$, and $x_3 = 2(p/3)^{1/2}\cos\left(\alpha + \frac{2\pi}{3}\right)$ are also the roots of Equation (56). Since $\alpha \in [\frac{\pi}{6}, \frac{\pi}{3})$, we have

$$\frac{5\pi}{6} \leqslant \alpha + \frac{2\pi}{3} < \pi.$$
$$-\frac{\pi}{2} \leqslant \alpha - \frac{2\pi}{3} < -\frac{\pi}{3}.$$

Therefore

$$x_1 > x_2 > 0 > x_3.$$

In particular, we have

$$\begin{aligned} S(q) &= x_1 - x_3 \\ &= 2(p/3)^{1/2}\left(\cos(\alpha) - \cos(\alpha + \frac{2\pi}{3})\right) \end{aligned}$$

$$
\begin{aligned}
&= 2(p/3)^{1/2} \cdot 2\sin\left(\frac{\pi}{3}\right)\sin\left(\alpha+\frac{\pi}{3}\right) \\
&= 2\sqrt{p}\sin\left(\alpha+\frac{\pi}{3}\right).
\end{aligned}
$$

Since α is an increasing function on q and $S(q)$ is a decreasing function on α, we conclude $S(q)$ is a decreasing function on q. $\square$

We now determine all admissible pairs (s,t).

Proof of Theorem 3. We will first show the 'only if' direction of Theorem 3. Recall that by definition, the pair (s,t) is admissible if $\psi(L) \leqslant 0$ for all graphs L on $s-1$ vertices, and $\psi(L)=0$ only if $L=(s-1)K_1$. For $L=K_{1,s-2}$, we have that

$$
\begin{aligned}
\psi(K_{1,s-2}) &= 3\left((s-2)^2+s-2\right) - \frac{2}{s-1}(2(s-2))^2 - (t-1)2(s-2) \\
&= 3(s-2)(s-1) - \frac{8}{s-1}(s-2)^2 - 2(t-1)(s-2),
\end{aligned}
$$

from which it easily follows that $\psi(K_{1,s-2})>0$ if and only if

$$
t-1 < \frac{3}{2}(s-1) - \frac{4(s-2)}{s-1} \implies t < \frac{3}{2}(s-3) + \frac{4}{s-1}.
$$

Thus we can conclude that if (s,t) is admissible, then $t \geqslant \frac{3}{2}(s-3)+\frac{4}{s-1}$.

Before we show the 'if' direction of Theorem 3, we need two upper bounds on the sum of the squared degrees of a graph due to de Caen [5] and Das [9], respectively.

Theorem 31 (de Caen [5])**.** *Let G be a graph with n vertices, e edges and degrees $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$. Then,*

$$
\sum_{i=1}^{n} d_i^2 \leqslant e\left(\frac{2e}{n-1}+n-2\right).
$$

Theorem 32 (Das [9])**.** *Let G be a graph with n vertices and e edges. Let d_1 and d_n be, respectively, the highest and lowest degrees of G. Then,*

$$
\sum_{i=1}^{n} d_i^2 \leqslant 2e(d_1+d_n) - nd_1d_n.
$$

We first prove a lemma that almost covers the entire range of t using only Theorem 31.

Lemma 33. *If $t \geqslant s$ and $t \geqslant \frac{3}{2}(s-3)+1$, then the pair (s,t) is admissible.*

Proof of Lemma 33. We may assume $s \geqslant 3$. Let L be a graph on $s-1$ vertices with at least one edge. By Theorem 31,

$$
3\sum_{i\in V(L)} d_i^2 \leqslant 3\frac{\sum_{i\in V(L)} d_i}{2}\left(\frac{\sum_{i\in V(L)} d_i}{s-2}+s-3\right)
$$

$$= \frac{3\left(\sum_{i\in V(L)} d_i\right)^2}{2(s-2)} + \frac{3(s-3)\sum_{i\in V(L)} d_i}{2}.$$

Therefore,

$$\psi(L) \leqslant \left(\frac{3}{2(s-2)} - \frac{2}{s-1}\right)\left(\sum_{i\in V(L)} d_i\right)^2 + \left(\frac{3(s-3)}{2} - (t-1)\right)\sum_{i\in V(L)} d_i.$$

It follows that $\psi(L) < 0$, as $\frac{3}{2(s-2)} - \frac{2}{s-1} < 0$ for $s \geqslant 6$, and by assumption $\frac{3(s-3)}{2} - (t-1) \leqslant 0$. For $s \in \{3,4,5\}$, it could be easily checked by hand that $\psi(L) < 0$ for all L on $s-1$ vertices with at least one edge (by computing $\psi(L)$ for all two-vertex, three-vertex and four-vertex graphs L).

This implies that the pair (s,t) is admissible for all $t \geqslant s$ and $t \geqslant \frac{3}{2}(s-3)+1$. $\square$

The only cases missed by Lemma 33 are the following: $s \geqslant 10$ is even and $t = \frac{3}{2}s - 4$. To take care of these cases, we use both Theorem 31 and Theorem 32.

Assume $t = \frac{3}{2}s - 4$, where $s \geqslant 10$ and s is even. As in the proof of Lemma 33, we can use Theorem 31 to bound $\psi(L)$ by

$$\psi(L) \leqslant \left(\frac{3}{2(s-2)} - \frac{2}{s-1}\right)\left(\sum_{i\in V(L)} d_i\right)^2 + \frac{1}{2}\sum_{i\in V(L)} d_i, \tag{57}$$

where L is any graph on $s-1$ vertices with at least one edge. Viewing the right-hand side of (57) as a quadratic polynomial in the variable $\sum_{i\in V(L)} d_i$, we see that the quadratic polynomial has two solutions: one with $\sum_{i\in V(L)} d_i = 0$, and one with

$$\sum_{i\in V(L)} d_i = \frac{\frac{1}{2}}{\frac{2}{s-1} - \frac{3}{2(s-2)}} = \frac{(s-1)(s-2)}{s-5}.$$

Since $\frac{3}{2(s-2)} - \frac{2}{s-1} < 0$, it follows that if

$$\sum_{i\in V(L)} d_i > \frac{(s-1)(s-2)}{s-5},$$

then $\psi(L) < 0$. Thus, assume that $\sum_{i\in V(L)} d_i \leqslant \frac{(s-1)(s-2)}{s-5}$, *i.e.*, that the number of edges e in L is bounded as $e < \frac{1}{2}\frac{(s-1)(s-2)}{s-5}$.

We distinguish two cases: (1) the graph L has at least two isolated vertices; (2) the graph L has no isolated vertices or one isolated vertex. We assume $s \geqslant 12$ as the case $s = 10$ can be directly checked by computer.

Suppose L has at least two isolated vertices. Let L_{-2} be the graph obtained by deleting two of the isolated vertices. Then, $\sum_{i\in V(L)} d_i^2 = \sum_{i\in V(L_{-2})} d_i^2$ and $\sum_{i\in V(L)} d_i =$

$\sum_{i\in V(L_{-2})} d_i$. We use induction on s. So, we assume $s \geqslant 12$, whence by the inductive hypothesis,

$$\begin{aligned}\psi(L_{-2}) = 3 \sum_{i\in V(L_{-2})} d_i^2 - \frac{2}{(s-2)-1}\left(\sum_{i\in V(L_{-2})} d_i\right)^2 \\ - \left(\left(\frac{3}{2}(s-2)-4\right)-1\right)\sum_{i\in V(L_{-2})} d_i \\ < 0.\end{aligned}$$

We show $\psi(L_{-2}) > \psi(L)$. We have

$$\begin{aligned}\psi(L_{-2}) - \psi(L) &= \left(\frac{2}{s-1} - \frac{2}{s-3}\right)\left(\sum_{i\in V(L)} d_i\right)^2 + 3\sum_{i\in V(L)} d_i \\ &= \frac{-4}{(s-1)(s-3)}\left(\sum_{i\in V(L)} d_i\right)^2 + 3\sum_{i\in V(L)} d_i.\end{aligned}$$

Viewing $\psi(L_{-2}) - \psi(L)$ as a quadratic polynomial in $\sum_{i\in V(L)} d_i$, it follows that $\psi(L_{-2}) - \psi(L) > 0$ if $\sum_{i\in V(L)} d_i < \frac{3}{4}(s-1)(s-3)$. Indeed we have that $\sum_{i\in V(L)} d_i \leqslant \frac{(s-1)(s-2)}{s-5} < \frac{3}{4}(s-1)(s-3)$ if $s \geqslant 10$. Therefore, $\psi(L) < \psi(L_{-2}) < 0$.

Now, assume that instead L has at most one isolated vertex. Recall that by our assumption,

$$\sum_{i\in V(L)} d_i \leqslant \frac{(s-1)(s-2)}{s-5} = s + 2 + \frac{12}{s-5}.$$

Without loss of generality, let $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{s-1}$ be the degree sequence of L. We could easily check by hand that $\psi(L) < 0$ for all L with degree sequence of the form $(d_1, 1, \cdots, 1, 1)$, $(d_1, 1, \cdots, 1, 0)$, $(d_1, 2, 1, 1, \cdots, 1, 1)$, or $(d_1, 2, 1, 1, \cdots, 1, 0)$.

Otherwise, we have that $d_1 \leqslant s + 2 + \frac{12}{s-5} - (s-1) = 3 + \frac{12}{s-5}$ if $d_{s-1} = 0$ and similarly $d_1 \leqslant 2 + \frac{12}{s-5}$ if $d_{s-1} = 1$. In either case, $d_1 + d_{s-1} \leqslant 3 + \frac{12}{s-5}$ if $s \geqslant 12$. Since $d_1 + d_{s-1}$ is an integer, we have that $d_1 + d_{s-1} \leqslant 4$ for $s \geqslant 12$. Therefore, by Theorem 32, $\sum_{i\in V(L)} d_i^2 \leqslant 4\sum_{i\in V(L)} d_i$, so

$$\begin{aligned}\psi(L) &= 3\sum_{i\in V(L)} d_i^2 - \frac{2}{s-1}\left(\sum_{i\in V(L)} d_i\right)^2 - \left(\frac{3}{2}s - 5\right)\sum_{i\in V(L)} d_i \\ &\leqslant \left(12 - \left(\frac{3}{2}s - 5\right)\right)\sum_{i\in V(L)} d_i - \frac{2}{s-1}\left(\sum_{i\in V(L)} d_i\right)^2.\end{aligned}$$

But now we see that $\psi(L) < 0$, since $-\frac{2}{s-1}\left(\sum_{i\in V(L)} d_i\right)^2 < 0$ and $17 - \frac{3}{2}s < 0$ if $s \geqslant 12$. This completes the proof of Theorem 3.

$\square$

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