A Combinatorial Perspective on the Noncommutative Symmetric Functions

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Abstract

The noncommutative symmetric functions **NSym** were first defined abstractly by Gelfand et al. in 1995 as the free associative algebra generated by noncommuting indeterminates $\{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ that were taken as a noncommutative analogue of the elementary symmetric functions. The resulting space was thus a variation on the traditional symmetric functions Λ. Giving noncommutative analogues of generating function relations for other bases of Λ allowed Gelfand et al. to define additional bases of **NSym** and then determine change-of-basis formulas using quasideterminants. In this paper, we aim for a self-contained exposition that expresses these bases concretely as functions in infinitely many noncommuting variables and avoids quasideterminants. With the exposition out of the way, we look at the noncommutative analogues of two different interpretations of change of basis in Λ: both as a product of a minimal number of matrices, mimicking Macdonald's exposition of Λ in *Symmetric Functions and Hall Polynomials*, and as statistics on brick tabloids, as in work by Eğecioğlu and Remmel, 1990.

Mathematics Subject Classifications: 05E05

1 Introduction

We define three well-known vector spaces: the symmetric functions, Λ, the quasisymmetric functions, QSym, and lastly, the noncommutative symmetric functions, **NSym**, the focus of this work. The noncommutative symmetric functions were first defined in [7] by Gelfand et al., with a definition inspired by the well-known Fundamental Theorem of Symmetric Polynomials. Let $\mathbb{N}$ be the set of all nonnegative integers.

Theorem 1. *Every symmetric function can be written uniquely as a polynomial in the elementary symmetric functions* $\{e_n\}_{n\in\mathbb{N}}$.

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Inherent to the statement is the fact that the elementary symmetric functions commute with each other. The noncommutative symmetric functions were originally defined by Gelfand et al. in [7], and answer the following question.

> "What happens if the elementary symmetric functions are replaced by a non-commutative multiplicative basis?"

Formally, Gelfand et al. define:

Definition 2. [[7], section 3] The ring of **noncommutative symmetric functions** is the free associative algebra

$$\mathbf{NSym} = \langle \boldsymbol{e}_n \rangle_{n \in \mathbb{Z}^+},$$

where $\{\boldsymbol{e}_n\}_{n\in\mathbb{Z}^+}$ is a set of algebraically independent indeterminates which do not commute. By convention, we occasionally also use $\boldsymbol{e}_0$ for 1, to parallel the symmetric function literature, although one should be careful, since $\boldsymbol{e}_0$ commutes with the remainder. The symbol $\boldsymbol{e}_n$ is the n^{th} **noncommutative elementary symmetric function**.

Gelfand et al. proceed to define additional bases by generalizing the relations between generating series of various well-known bases of Λ. They then determine explicit change-of-basis formulas between the resulting bases, using their previous work on quasideterminants as a key tool. Finally, they derive a concrete realization of several of their newly defined bases in terms of a set of noncommuting variables in Section 7 of their work, but do not give an explicit characterization of the resulting space as in Definition 18 below. Thus their presentation, while inherently logical to its inspiration, is the reverse of the traditional presentation of the symmetric functions in well-known texts in this area (e.g., Stanley, Chapter 7 in [24], or Macdonald [17]). Such texts start with concrete realizations of these spaces as subspaces of the space of power series in infinitely many commuting variables, define explicit bases in those variables, and then derive relations from those definitions. This convention continues in texts which cover the noncommutative symmetric functions. For example, Luoto et al. in [16] give a friendly, well-written introduction to Λ, QSym, and **NSym**. They define the first two as subspaces of the space of formal power series in infinitely many variables, but define **NSym** using Definition 2 above.

Our first goal is an elementary (pun intended) exposition of the noncommutative symmetric functions, following traditional presentations of the symmetric functions by emphasizing their realization in terms of a noncommuting set of variables, then deriving the defining relations of Gelfand et al., emphasizing their similarity to the well-studied symmetric functions, all while avoiding quasideterminants.

Then in Section 6 and Section 7, we turn to new results, where we will show that many of the change-of-basis matrices between well-known bases of **NSym** that were derived in Gelfand et al. can also be interpreted to generalize well-studied change-of-basis results in Λ. Particularly, we will give a natural generalization of brick tabloids, introduced by Eğecioğlu and Remmel in [6] to unify combinatorial interpretations for a number of basis transitions in Λ, and consider a generalization of the commuting diagram Macdonald gives in [17] to express change of basis as a product of a few key transition matrices.

1.1 Additional References

Previous sources introduce bases of **NSym** as formal power series in noncommuting variables, first among them Gelfand et al. in Section 7 of [7], where they are shown to span an isomorphic space to their more abstractly defined free associative algebra. Other papers in the same series, as for example [12], introduce the bases in noncommuting variables first. Huang [11] is perhaps the closest to this work; on page 16 **NSym** is defined as the span of the noncommutative complete homogeneous symmetric functions, given in terms of noncommuting variables as a fact (and citing [7] for the equivalence). The noncommutative ribbon and elementary symmetric functions are also given in terms of the same set of variables in the same work, citing [7]. Other sources prove indirectly that the free associative algebra defined in [7] can be realized in terms of noncommuting variables: the equivalence is given indirectly in [9] in Corollary 8.1.14 and is implicit in the proof of Proposition 6.2 in Meliot's text [20].

Definition 18 and Corollary 51 below, which give an explicit characterization of when a polynomial in n noncommuting variables is in **NSym** (and show that this characterization gives an isomorphic space to that of Gelfand et al. in [7]) are implicit in section 5 of [7], where the authors give a natural isomorphism between Solomon's descent algebra and **NSym**, defined by sending the formal sum of permutations with a given descent set to the ribbon Schur basis defined below. The isomorphism follow easily from previous work showing the duality of the quasisymmetric functions and Solomon's decent algebra in [18] and well summarized and further explored in [21].

1.2 Notation

Before we begin, we offer a few brief remarks on notation. In order to distinguish between commuting and noncommuting variables, let $X = (x_1, x_2, \dots)$ give an infinite sequence of commuting variables and $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots)$ give an infinite sequence of noncommuting variables. This paper follows the convention of [16]; variable names for bases are deliberately reused across Λ, **NSym**, and QSym. Since this can occasionally cause confusion, we will use "standard" (lowercase) type for bases of Λ, bold type for bases of **NSym**, and capitalization for bases of QSym to make distinguishing them as easy as possible. Repeatedly, we use $\mathbb{1}_{\mathcal{A}}$, the indicator function that is 1 if statement $\mathcal{A}$ is true and is 0 if $\mathcal{A}$ is false.

Finally, throughout we give citations to theorems, indicating where they are stated in [7]. With few noted exceptions, the proofs in [7] are distinct, and in many places definitions and theorems are reversed from the presentation below, due to the differences in what we take to be the definition of the space.

2 Partitions and Compositions

The vector spaces discussed here have bases that are naturally indexed by either partitions or compositions, so we begin there.

Definition 3. An infinite sequence of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ is a **weak integer composition**, or simply a **weak composition**, of n if $\sum \alpha_i = n$, denoted $\alpha \models_w n$. The α_i are the **parts** of α, and the **size** of α is n, written $|\alpha| = n$.

Definition 4. A **strong integer composition** of positive integer n, or often just simply a **composition** of n, is a finite sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$ summing to n, denoted $\alpha \models n$. In this case, $\ell(\alpha)$ is the **length** of α, and α has **last part** $\alpha_{\ell(\alpha)}$. The length of the empty composition is taken to be 0.

We also write strong(α) to denote the composition attained by removing all zeros from the weak composition α. There is a classical bijection between strong compositions of n and subsets of $[n-1] = \{1, 2, \dots, n-1\}$. In particular, if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$, we say that

$$\mathrm{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell(\alpha)-1}\}.$$

There are three well-known involutions on the set of strong compositions (and their associated sets). Consider $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)}) \models n$ and its associated set, $\mathrm{set}(\alpha) \subseteq [n-1]$.

Definition 5. The **reverse** of α is $\alpha^r = (\alpha_{\ell(\alpha)}, \alpha_{\ell(\alpha)-1}, \dots, \alpha_1)$.

Definition 6. If $A \subseteq [n-1]$, let $A^c = [n-1] \setminus A$, the complement of the set A in $[n-1]$. Then the **complement** of α is the composition $\alpha^c = \mathrm{set}^{-1}(\mathrm{set}(\alpha)^c)$.

Definition 7. The **transpose** of α, written α^t, is the composition obtained by applying the two previous involutions: $\alpha^t = (\alpha^r)^c = (\alpha^c)^r$.

It is not hard to check that composing any two of these distinct maps yields the third. We will also repeatedly use the fact that for any (nonempty) strong composition α, it is true that $\ell(\alpha) + \ell(\alpha^c) - 1 = |\alpha|$.

Example 8. If $\alpha = (2, 3, 2, 1) \models 8$, then

- $\alpha^r = (1, 2, 3, 2)$;
- $\alpha^c = \mathrm{set}^{-1}(\mathrm{set}(\alpha)^c) = \mathrm{set}^{-1}(\{2, 5, 7\}^c) = \mathrm{set}^{-1}(\{1, 3, 4, 6\}) = (1, 2, 1, 2, 2)$;
- $\alpha^t = (\alpha^c)^r = (1, 2, 1, 2, 2)^r = (2, 2, 1, 2, 1)$.

Also see that $\ell(\alpha) + \ell(\alpha^c) - 1 = 4 + 5 - 1 = 8 = |\alpha|$.

Definition 9. If $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)})$ and $\beta = (\beta_1, \dots, \beta_{\ell(\beta)})$ are strong compositions of n, then $\boldsymbol{\beta}$ **is a refinement of** $\boldsymbol{\alpha}$, or $\boldsymbol{\beta}$ **refines** $\boldsymbol{\alpha}$, denoted $\beta \preceq \alpha$, if there exists an integer sequence $0 = j_0 < j_1 < j_2 < \cdots < j_{\ell(\alpha)} = \ell(\beta)$ such that for each $i \in \{1, 2, \dots, \ell(\alpha)\}$,

$$\alpha_i = \beta_{j_{i-1}+1} + \beta_{j_{i-1}+2} + \cdots + \beta_{j_i}.$$

That is, each part of α can be obtained by summing consecutive parts of β. Equivalently, $\beta \preceq \alpha$ if and only if $\mathrm{set}(\beta) \supseteq \mathrm{set}(\alpha)$. In this context, let $\beta^{(i)}$ denote the subcomposition of β which sums to α_i for $i = 1, \dots, \ell(\alpha)$ so that $|\beta^{(i)}| = \alpha_i$.

We note that the direction of refinement ($\preceq$) is not consistent across the literature in this area, and in particular, this work uses the opposite convention of [7] and follows that of [16].

Example 10. Both $\alpha = (1, 6, 3, 4) \models 13$ and $\beta = (1, 3, 1, 2, 2, 1, 1, 3) \models 13$. See that $\beta \preceq \alpha$ with $\beta^{(1)} = (1)$, $\beta^{(2)} = (3, 1, 2)$, $\beta^{(3)} = (2, 1)$, and $\beta^{(4)} = (1, 3)$.

Since strong compositions of n are in bijection with subsets of $[n-1]$, they inherit the same Möbius function, μ. We will repeatedly use the following basic facts:

- For $S, T \subseteq [n-1]$, $\mu(S, T) = (-1)^{|S|-|T|}$.
- For $S, T \subseteq [n-1]$, $\sum_{S \subseteq U \subseteq T} \mu(U, T) = \mathbb{1}_{S=T}$.
- (Möbius Inversion.) If K is a commutative ring and $f, g : [n-1] \to K$, then

$$g(T) = \sum_{S \subseteq T} f(S) \quad \text{for all } T \subseteq [n-1]$$

  if and only if

$$f(T) = \sum_{S \subseteq T} g(S)\mu(S, T) \quad \text{for all } T \subseteq [n-1].$$

Definition 11. The **sort** of α, $\text{sort}(\alpha)$, is the composition obtained by rewriting α in weakly decreasing order.

Definition 12. An **integer partition** of n, or simply a **partition** of n, is a composition λ of n for which $\text{sort}(\lambda) = \lambda$, denoted $\lambda \vdash n$.

Where the order of the parts of a composition α is immaterial, such as when α is a partition, we may write $\alpha = (1^{m_1(\alpha)} 2^{m_2(\alpha)} \cdots n^{m_n(\alpha)})$, where $m_i(\alpha)$ gives the number of parts of size i occurring in α. We may also use partial exponentiation to write certain compositions, e.g., $(1^k, n-k)$ will represent $(1, 1, \ldots, 1, n-k) \models n$.

3 Three Rings

We consider three (graded) rings in this paper: the ring of symmetric functions, Λ, the ring of quasisymmetric functions, QSym, and the ring of noncommutative symmetric functions, **NSym**. Our goal in this section is to briefly introduce all three spaces, with an emphasis on **NSym**. The unfamiliar reader may wish to consult [17] or Chapter 7 of [24] to learn more about the fundamentals of symmetric function theory, and [16] for an introduction to quasisymmetric functions. There is not, to our knowledge, a well-known text covering noncommutative symmetric functions in detail. A series of papers exploring noncommutative symmetric functions and their significance include [7], [12], [5], [13], and [14]. These comprise perhaps the best known introduction to this area.

Notation. Let X denote an infinite sequence of commuting variables $(x_1, x_2, \dots)$. Given a composition $\alpha = (\alpha_1, \alpha_2, \dots)$, we write x^α to denote the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. If $f \in \mathbb{C}[[x_1, x_2, \dots]]$ is a power series of bounded total degree, we write $f|_{x^\alpha}$ to denote the coefficient of x^α in the expansion of f into monomials. Any such f is **homogeneous of degree n** if each monomial x^α appearing in f is such that $|\alpha| = n$.

Definition 13. Let $f \in \mathbb{C}[[x_1, x_2, \dots]]$ be a power series of bounded total degree. Then f is **symmetric** if for all integers $k > 0$, all compositions $\alpha = (\alpha_1, \dots, \alpha_k)$, and all lists $(i_1, i_2, \dots, i_k)$ of distinct positive integers,

$$f|_{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}} = f|_{x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}}.$$

Let Λ^n be the set of all symmetric functions homogeneous of degree n. Then the ring of symmetric functions is

$$\Lambda = \bigoplus_{n \geqslant 0} \Lambda^n.$$

Example 14. The power series

$$x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + \cdots + x_1 x_2^2 + x_2^2 x_3 + x_2^2 x_4 + \cdots \in \Lambda^3.$$

Definition 15. Similarly, f is **quasisymmetric** if for all integers $k > 0$, all compositions $(\alpha_1, \dots, \alpha_k)$, and all increasing lists $i_1 < i_2 < \cdots < i_k$ of distinct positive integers,

$$f|_{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}} = f|_{x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}}.$$

Let QSym^n be the set of all quasisymmetric functions homogeneous of degree n. Then the ring of quasisymmetric functions is

$$\mathrm{QSym} = \bigoplus_{n \geqslant 0} \mathrm{QSym}^n.$$

Example 16. Both of the following power series are quasisymmetric functions in QSym^3.

$$x_1^2 x_2 + x_1^2 x_3 + \cdots + x_2^2 x_3 + x_2^2 x_4 + \cdots$$

$$x_1 x_2^2 + x_1 x_3^2 + \cdots + x_2 x_3^2 + x_2 x_4^2 + \cdots$$

Bases for Λ^n are generally indexed by partitions of n, while bases for QSym^n are indexed by strong compositions of n (or subsets of $[n-1]$).

Definition 17. Let $I = (i_1, i_2, \dots, i_k)$ be a sequence of nonnegative integers. Then the **descent set** of I is

$$\mathrm{des}(I) = \{j \mid i_j > i_{j+1}\}.$$

Definition 18. Let **NSym** be the subset of $\mathbb{C}[[\boldsymbol{x}_1, \boldsymbol{x}_2, \dots]]$ of all power series f of bounded total degree with the following property: for all integers $k > 0$ and all sequences of positive integers $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ such that $\mathrm{des}(I) = \mathrm{des}(J)$,

$$f|_{\boldsymbol{x}_{i_1} \cdots \boldsymbol{x}_{i_k}} = f|_{\boldsymbol{x}_{j_1} \cdots \boldsymbol{x}_{j_k}}.$$

Example 19. We give the all the terms below which involve only the variables $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ for a particular element of $\mathbf{NSym}^3$:

$$\boldsymbol{x}_1^3 + \boldsymbol{x}_1^2\boldsymbol{x}_2 - \boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_1 + \boldsymbol{x}_1\boldsymbol{x}_2^2 + 2\boldsymbol{x}_2\boldsymbol{x}_1^2 + 2\boldsymbol{x}_2\boldsymbol{x}_1\boldsymbol{x}_2 - \boldsymbol{x}_2^2\boldsymbol{x}_1 + \boldsymbol{x}_2^3 + \cdots$$

Below, in Corollary 51, we will show that the set of noncommutative elementary symmetric functions freely generate this space, giving a concrete realization of the space originally defined more abstractly by Gelfand et al. in [7]. Although this characterization is immediate from work in [7], as we will see below, the literature does not appear to include an explicit characterization of the overall space as above to our knowledge.

Remark. For any $n \in \mathbb{N}$, the elements of **NSym** are constant under the standard Lascoux-Schützenberger involution paired parenthesis action of $\mathfrak{S}_n$, studied for example by Lascoux and Leclerc in [15], and thus satisfy a reasonable analogue of being "symmetric."

This well-known algorithm acts on words and specifies a way to exchange the number of i's and $(i+1)$'s in a word. In particular, if there are a i's and b $(i+1)$'s in a word w, θ_i acts on w by first picturing each $(i+1)$ as a left parenthesis '(', and each i as a right parenthesis ')'. If k pairs of parentheses can be paired, they correspond to k i's and k $(i+1)$'s which remain unchanged in $\theta_i(w)$, while the remaining $a-k$ i's must occur to the left of the $b-k$ unpaired $(i+1)$'s in w. In these $a+b-2k$ spaces, place $b-k$ i's followed by $a-k$ $(i+1)$'s. This action forms an involution, and satisfies the Moore-Coxeter relations of the symmetric group. When one starts with the word of a semistandard Young tableau, the image under the action is also the word of a semistandard Young tableau. Thus the action can be used to show directly that the (skew) Schur functions, defined using the semistandard Young tableaux definition, are symmetric.

See [7], Proposition 7.17 for the proof that elements of **NSym** are fixed by the action, although it is immediate from the fact that one of the bases of **NSym** is a noncommutative version of a subset of the skew Schur functions sometimes referred to as the ribbon Schur functions. (See just after Example 44.) Not all power series f of bounded total degree in $\mathbb{C}[[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots]]$ which are closed under this action are in **NSym**, however, so one should be careful not to take this as a definition. (See Example 20 below.)

It is also worth noting that the noncommutative symmetric functions defined here are completely distinct from "Symmetric Functions in Noncommuting Variables," developed more recently, which are defined by being fixed under the more standard $\mathfrak{S}_n$ action which simply permutes the indices of the noncommutative variables. (See Rosas and Sagan [22].)

Example 20. For example, $f = \sum_{i\geqslant 1} \boldsymbol{x}_i^2$ is fixed under the standard action of the symmetric group, and thus is a symmetric function in noncommuting variables. It is also closed under the paired parentheses action, and yet, it is not in **NSym**, since, for example the coefficient of $\boldsymbol{x}_1^2$ is not equal to the coefficient of $\boldsymbol{x}_1\boldsymbol{x}_2$ in f, although the descent set of $(1,1)$ is the same as the descent set of $(1,2)$.

4 Bases of Λ^n and QSym^n

There are a number of well-known bases for each of the three spaces defined above. In this section, we will briefly cover some of the relevant ones for Λ^n and QSym^n, before moving on to a more careful study of the bases of $\mathbf{NSym}^n$ and their relations.

4.1 Well-Known Bases of Λ^n

Definition 21. For $\lambda \vdash n$, the **monomial symmetric function** (associated to $\lambda \vdash n$) is

$$m_\lambda = \sum_{\substack{\alpha \models_w |\lambda| \\ \mathrm{sort}(\alpha)=\lambda}} x^\alpha.$$

The monomial symmetric function associated to $\lambda \vdash n$ is minimal in the following sense: if $f \in \Lambda$ such that $f|_{x^\lambda} = 1$, then the support of f contains the support of m_λ, i.e.,

$$\{\alpha \models_w n : f|_{x^\alpha} \neq 0\} \supseteq \{\alpha \models_w n : m_\lambda|_{x^\alpha} \neq 0\}.$$

It is easy to see $\{m_\lambda\}_{\lambda \vdash n}$ is a basis of Λ^n.

Definition 22. For $n \in \mathbb{N}$, the n^{th} **elementary symmetric function** is

$$e_n = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

with $e_0 = 1$. Then, the elementary symmetric function associated to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ is defined multiplicatively: $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$. The Fundamental Theorem of Symmetric Polynomials gives that $\{e_\lambda\}_{\lambda \vdash n}$ is a basis for Λ^n.

Definition 23. For $n \in \mathbb{N}$, the (n^{th}) **complete homogeneous symmetric function** is

$$h_n = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

with $h_0 = 1$. The complete homogeneous symmetric functions are also defined multiplicatively: $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$. The set $\{h_\lambda\}_{\lambda \vdash n}$ is also a basis for Λ^n.

Definition 24. For $n \in \mathbb{N}$, the n^{th} **power sum symmetric function** is

$$p_n = \sum_{i \in \mathbb{Z}^+} x_i^n,$$

with $p_0 = 1$. Once more, $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}$, and $\{p_\lambda\}_{\lambda \vdash n}$ is a basis for Λ^n.

One additional important basis of Λ, the Schur functions, can be defined using semistandard Young tableaux, which we describe next.

Definition 25. The **Young diagram of shape $\boldsymbol{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \vdash n}$** is the left-justified array of n boxes with λ_i boxes in its i^{th} row from the bottom (adopting French notation), $1 \leqslant i \leqslant \ell(\lambda)$. In particular, we assume the box in the i^{th} row and j^{th} column has its upper-right-hand corner at the integer lattice point (i, j).

Definition 26. A **filling** of a Young diagram of shape $\lambda \vdash n$ is one where each of its boxes is filled with a positive integer. The resulting filled Young diagram is called a **Young tableau**.

If in Young tableau T the integers are both weakly increasing in the rows from left-to-right and strictly increasing in the columns from bottom-to-top, we call T a **semi-standard Young tableau**. The set of all semi-standard Young tableaux of shape λ is denoted $\mathrm{SSYT}(\lambda)$. A **standard Young tableau** is a semistandard Young tableau in which each of the integers $\{1, 2, \dots, |\lambda|\}$ occurs exactly once.

Definition 27. The **type** or **content** of a Young tableau T, $\mathrm{type}(T)$, is the composition $(m_1, m_2, \dots, m_n) \models_w |\lambda|$ where m_i is the number of i's appearing in T. So in Example 30 below, $\mathrm{type}(T) = (2, 2, 3)$.

We can now define the final well-known basis of Λ, which interpolates between the elementary and complete homogeneous symmetric functions.

Definition 28. The **Schur function** associated to partition $\lambda \vdash n$ is

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{type}(T)}.$$

Definition 29. Let λ be a partition of n. If the Young diagram of shape λ is flipped about the southwest-northeast diagonal, the resulting Young diagram is of shape λ^t, the **transpose** or **conjugate** partition of λ. Note that $(\lambda^t)^t = \lambda$.

Example 30. The following Young tableau is semistandard, but not standard.

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 3 & \multicolumn{3}{c}{} \\ \cline{1-2} 2 & 3 & \multicolumn{2}{c}{} \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array} \quad \in \quad \mathrm{SSYT}(4, 2, 1).$$

The conjugate partition of $\lambda = (4, 2, 1)$ is $\lambda^t = (3, 2, 1, 1)$.

4.1.1 Involutions and the Hall inner product on Λ

Since $\langle e_n \rangle_{n \in \mathbb{N}}$ generates all of Λ, the following defines a homomorphism on Λ:

Definition 31. Define the endomorphism $\omega : \Lambda \to \Lambda$ by setting $\omega(e_n) = h_n$ for all $n \in \mathbb{N}$.

Then we have the following well-known theorem:

Theorem 32. *For any partition λ,*

$$\begin{aligned} \omega(e_\lambda) &= h_\lambda; \\ \omega(h_\lambda) &= e_\lambda; \\ \omega(p_\lambda) &= (-1)^{|\lambda| - \ell(\lambda)} p_\lambda; \\ \omega(s_\lambda) &= s_{\lambda^t}. \end{aligned}$$

By the first two lines, ω is an involution, and since $\{m_\lambda\}$ is a basis of Λ, the set $\{\omega(m_\lambda)\}$ must form a basis of Λ as well.

Definition 33. The **forgotten symmetric function** associated to λ is defined to be $f_\lambda = \omega(m_\lambda)$.[1] Generally, the forgotten symmetric functions are defined indirectly via the map ω or the Hall inner product and duality.

Definition 34. The **Hall inner product**, $\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \to \mathbb{C}$, is determined by

$$\langle m_\lambda, h_\mu \rangle = \mathbb{1}_{\lambda=\mu}. \tag{1}$$

Theorem 35. *The forgotten symmetric functions are dual to the elementary symmetric functions, and the Schur functions form an orthonormal basis of* Λ *under the Hall inner product. That is,*

$$\langle f_\lambda, e_\mu \rangle = \langle s_\lambda, s_\mu \rangle = \mathbb{1}_{\lambda=\mu}.$$

Furthermore, it is also true that

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \cdot \mathbb{1}_{\lambda=\mu},$$

where

$$z_\lambda = \prod_{i=1}^{\ell(\lambda)} i^{m_i(\lambda)} m_i(\lambda)!$$

is the well-known combinatorial coefficient that measures the size of the centralizer of any symmetric group element having cycle type λ.

It can be also be shown that ω is an isometry with respect to the Hall inner product, i.e., for any $g, g' \in \Lambda$,

$$\langle \omega(g), \omega(g') \rangle = \langle g, g' \rangle.$$

Figure 1 on the next page summarizes the results we have provided pertaining to the relationships between the various bases of Λ under ω and $\langle \cdot, \cdot \rangle$.

4.2 Well-Known Bases of QSym

As mentioned above, bases for QSym are indexed by strong compositions of n or subsets of $[n-1]$. A first basis for QSym is the following:

Definition 36. For any strong composition α, the **monomial quasisymmetric function** (associated to $\alpha \models n$) is

$$M_\alpha = \sum_{\substack{\beta \models_w |\alpha| \\ \text{strong}(\beta)=\alpha}} x^\beta.$$

[1] The forgotten symmetric functions are sometimes defined by $f_\lambda = (-1)^{|\lambda|}\omega(m_\lambda)$. (See Doubilet [4]).

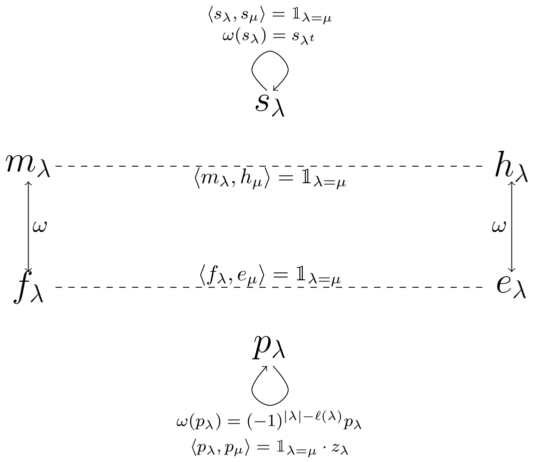


Figure 1: Images of ω and Self-Duality of Λ

The monomial quasisymmetric function associated to $\alpha \models n$ is minimal in the following sense: if $f \in \mathrm{QSym}$ such that $f|_{x^\alpha} = 1$, then the support of f contains the support of M_α, i.e.,

$$\{\beta \models_w n : f|_{x^\beta} \neq 0\} \supseteq \{\beta \models_w n : M_\alpha|_{x^\beta} \neq 0\}.$$

For any $\lambda \vdash n$,

$$m_\lambda = \sum_{\substack{\alpha \models n \\ \mathrm{sort}(\alpha)=\lambda}} M_\alpha.$$

Another well-known basis of QSym^n is **Gessel's fundamental basis**, so called since it was defined by Gessel in [8]. It is one of several bases of quasisymmetric functions considered a generalization of the Schur symmetric function basis.

Definition 37. For $\alpha \models n$, the **Gessel Fundamental quasisymmetric function** associated to α is

$$F_\alpha = \sum_{\substack{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \\ k \in \mathrm{set}(\alpha) \Rightarrow i_k < i_{k+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

There are two other commonly studied quasisymmetric analogues of the Schur functions: the quasisymmetric Schur functions, first defined by Haglund et al. in [10], and the dual immaculate functions, defined by duality to the immaculate noncommutative symmetric functions by Berg et al. in [3]. Images of these bases under the automorphisms below are also sometimes studied. We also mention for completeness that there are two quasisymmetric analogues of the power sums, which were first defined indirectly as dual bases, as described in the next section.

There are not one, but three natural analogues of ω defined on QSym. As in [16], define ρ, ψ, ω : QSym $\to$ QSym, all automorphisms, by

$$\rho(F_\alpha) = F_{\alpha^r}; \tag{2}$$
$$\psi(F_\alpha) = F_{\alpha^c}; \tag{3}$$
$$\omega(F_\alpha) = F_{\alpha^t}. \tag{4}$$

The names for these maps are not at all uniform in the literature, so here, as elsewhere, we follow the convention of [16].

5 Bases for NSym

We begin with the basis for the space with the minimal number of terms, the noncommutative ribbon Schur functions.

Definition 38. The **noncommutative ribbon Schur function** associated to $\alpha \models n$ is

$$\boldsymbol{r}_\alpha = \sum_{\substack{I=(i_1,i_2,\dots,i_n)\in(\mathbb{Z}^+)^n \text{ s.t.} \\ \operatorname{des}(I)=\operatorname{set}(\alpha)}} \boldsymbol{x}_{i_1}\boldsymbol{x}_{i_2}\cdots\boldsymbol{x}_{i_n}.$$

Theorem 39 ([7], section 4.4)**.** *$\{\boldsymbol{r}_\alpha\}_{\alpha\models n}$ is a basis for $\boldsymbol{NSym}^n$.*

Proof. It is clear from the definition of the space that **NSym** is the span of the noncommutative ribbon Schur functions. To see they are independent, note that for every sequence of positive integers $I = (i_1, i_2, \dots, i_n)$, the monomial $\boldsymbol{x}_{i_1}\boldsymbol{x}_{i_2}\cdots\boldsymbol{x}_{i_n}$ occurs uniquely with positive coefficient in exactly one $\boldsymbol{r}_\alpha$; in particular, where $\alpha = \operatorname{set}^{-1}(\operatorname{des}(I))$. □

The following theorem also follows easily from the definition.

Theorem 40. *For strong compositions $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_k)$,*

$$\boldsymbol{r}_\alpha\boldsymbol{r}_\beta = \boldsymbol{r}_{(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_k)} + \boldsymbol{r}_{(\alpha_1,\dots,\alpha_{n-1},\alpha_n+\beta_1,\beta_2,\dots,\beta_k)}$$

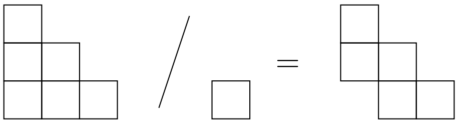
The noncommutative ribbon Schur function $\boldsymbol{r}_\alpha$ is a minimal noncommutive symmetric function in the following sense: if $\boldsymbol{f} \in$ **NSym** such that $\boldsymbol{f}|_{(\boldsymbol{x}_{\ell(\alpha)})^{\alpha_1}(\boldsymbol{x}_{\ell(\alpha)-1})^{\alpha_2}\cdots(\boldsymbol{x}_1)^{\alpha_{\ell(\alpha)}}} = 1$, then the support of $\boldsymbol{f}$ contains the support of r_α. That is, for all n,

$$\{(i_1, i_2, \dots, i_n) : \boldsymbol{f}|_{\boldsymbol{x}_{i_1}\boldsymbol{x}_{i_2}\cdots\boldsymbol{x}_{i_n}} \neq 0\} \supseteq \{(i_1, i_2, \dots, i_n) : \boldsymbol{r}_\alpha|_{\boldsymbol{x}_{i_1}\boldsymbol{x}_{i_2}\cdots\boldsymbol{x}_{i_n}} \neq 0\}.$$

While the noncommutative ribbon Schur functions share a minimality condition with the monomial symmetric functions and the monomial quasisymmetric functions, they, like the fundamental quasisymmetric functions, are usually considered an analogue of the Schur functions for a number of reasons.

Definition 41. A **skew Young diagram** associated to λ/μ is realized by taking a Young diagram, say of shape λ, and removing μ, a Young diagram sitting inside it. If the resulting skew Young diagram is connected and contains no 2×2 boxes, it is called a **ribbon Young diagram**.

Example 42. The skew Young diagram $(3, 2, 1)/(1)$ is a ribbon Young diagram since it has no 2×2 boxes, as seen below.

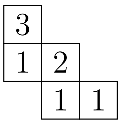


To stay consistent with [7], say that this is the ribbon of **shape** $(1, 2, 2)$, the lengths of the resulting rows when read from *top-to-bottom*.

It is from these combinatorial objects that the basis above gets its name. If the noncommutative ribbon Schur function $\boldsymbol{r}_\alpha$ is expanded in the noncommuting variables $\boldsymbol{x}$, the indices appearing on the monomials appearing (left-to-right) in $\boldsymbol{r}_\alpha$ fill in ribbons of shape α to yield **ribbon tableaux** of shape α.

Definition 43. A **ribbon tableau of shape** $\boldsymbol{\alpha}$ is a filling of the ribbon of shape α with positive integers that weakly increase across rows, left-to-right, and increase along columns, bottom-to-top. That is, ribbon tableaux are just semistandard Young tableaux of ribbon shape.

Example 44. Let $\alpha = (1, 2, 2)$. Then one of infinitely many ribbon tableaux of shape α is the one corresponding to the monomial $\boldsymbol{x}_3\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_1^2$, depicted below.



Definition 45. Let $\chi : \mathbb{C}[[\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots]] \to \mathbb{C}[[x_1, x_2, \cdots]]$ be the "forgetful" function that sends the noncommutative variables to their commutative analogues: i.e. $\chi(\boldsymbol{x}_i) = x_i$.

Then $\chi(\boldsymbol{r}_\alpha) = r_\alpha$, the commutative ribbon function α; it is also the skew Schur function of ribbon shape α. The set $\{r_\alpha\}_{\alpha \vDash n}$ forms a spanning set of Λ^n, since for $\lambda \vdash n$, $h_\lambda = \sum_{\mathrm{set}(\alpha) \subseteq \mathrm{set}(\lambda)} r_\alpha$ and thus $\chi(\mathbf{NSym}) = \Lambda$.

With a generalization of the Hall inner product defined in [7], QSym and **NSym** are dual spaces. Define $\langle \cdot, \cdot \rangle : \mathrm{QSym} \times \mathbf{NSym} \to \mathbb{C}$, where

$$\langle F_\alpha, \boldsymbol{r}_\beta \rangle = \mathbb{1}_{\alpha=\beta}. \tag{5}$$

The combined results of Gessel in [8] and Malvenuto and Reutenauer in [18] imply that QSym and **NSym** are dual with respect to this inner product, as first observed in [7].

There are three natural involutions f on **NSym** that when composed with the forgetful map χ give $\chi \circ f = \omega \circ \chi$ that correspond to the three involutions on QSym mentioned above.

Definition 46. Let $\boldsymbol{\rho}, \boldsymbol{\psi}, \boldsymbol{\omega} : \mathbf{NSym} \to \mathbf{NSym}$ be linear transformations that satisfy the following:

$$\begin{aligned} \boldsymbol{\rho}(\boldsymbol{r}_\alpha) &= \boldsymbol{r}_{\alpha^r} && (6) \\ \boldsymbol{\psi}(\boldsymbol{r}_\alpha) &= \boldsymbol{r}_{\alpha^c} && (7) \\ \boldsymbol{\omega}(\boldsymbol{r}_\alpha) &= \boldsymbol{r}_{\alpha^t} && (8) \end{aligned}$$

Here again we adopt the notation as in [16] and we note, just as they did, that $\boldsymbol{\psi}$ is an automorphism, and that $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$ are anti-automorphisms. The following theorem follows easily from Theorem 40 and the above definitions.

Theorem 47. *For any strong compositions α and β,*

$$\begin{gathered} \boldsymbol{\rho}(\boldsymbol{r}_\alpha \boldsymbol{r}_\beta) = \boldsymbol{\rho}(\boldsymbol{r}_\beta)\boldsymbol{\rho}(\boldsymbol{r}_\alpha); \\ \boldsymbol{\psi}(\boldsymbol{r}_\alpha \boldsymbol{r}_\beta) = \boldsymbol{\psi}(\boldsymbol{r}_\alpha)\boldsymbol{\psi}(\boldsymbol{r}_\beta); \\ \boldsymbol{\omega}(\boldsymbol{r}_\alpha \boldsymbol{r}_\beta) = \boldsymbol{\omega}(\boldsymbol{r}_\beta)\boldsymbol{\omega}(\boldsymbol{r}_\alpha). \end{gathered}$$

For any $f \in \mathrm{QSym}$ and $\boldsymbol{g} \in \mathbf{NSym}$,

$$\langle f, \boldsymbol{g} \rangle = \langle \rho(f), \boldsymbol{\rho}(\boldsymbol{g}) \rangle = \langle \psi(f), \boldsymbol{\psi}(\boldsymbol{g}) \rangle = \langle \omega(f), \boldsymbol{\omega}(\boldsymbol{g}) \rangle. \qquad (9)$$

Definition 48. Let $\boldsymbol{e}_0 = 1$, $\boldsymbol{h}_0 = 1$, and for any positive integer n, define

$$\boldsymbol{e}_n = \sum_{i_1 > i_2 > \cdots > i_n \geqslant 1} \boldsymbol{x}_{i_1} \boldsymbol{x}_{i_2} \cdots \boldsymbol{x}_{i_n}; \qquad \boldsymbol{h}_n = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n} \boldsymbol{x}_{i_1} \boldsymbol{x}_{i_2} \cdots \boldsymbol{x}_{i_n}. \qquad (10)$$

Then $\boldsymbol{e}_n$ is the n^{th} **noncommutative elementary symmetric function** and $\boldsymbol{h}_n$ is the n^{th} **noncommutative homogeneous complete symmetric function**. For any $\alpha \models n$, let

$$\boldsymbol{e}_\alpha = \boldsymbol{e}_{\alpha_1} \boldsymbol{e}_{\alpha_2} \cdots \boldsymbol{e}_{\alpha_{\ell(\alpha)}} \text{ and } \boldsymbol{h}_\alpha = \boldsymbol{h}_{\alpha_1} \boldsymbol{h}_{\alpha_2} \cdots \boldsymbol{h}_{\alpha_{\ell(\alpha)}}.$$

It is easy to see the following from the definitions of $\boldsymbol{h}_n$, $\boldsymbol{e}_n$, and $\boldsymbol{r}_\alpha$.

Theorem 49. *For all $n \geqslant 0$, and any $\alpha \models n$,*

$$\chi(\boldsymbol{e}_n) = e_n, \ \chi(\boldsymbol{h}_n) = h_n, \text{ and } \chi(\boldsymbol{r}_\alpha) = r_\alpha.$$

Theorem 50 ([7], section 4.4 and 4.7)**.**

$$\boldsymbol{h}_\alpha = \sum_{\beta \succeq \alpha} \boldsymbol{r}_\beta \qquad \boldsymbol{r}_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} \boldsymbol{h}_\beta \qquad (11)$$

$$\boldsymbol{e}_\alpha = \sum_{\beta^c \succeq \alpha} \boldsymbol{r}_\beta \qquad \boldsymbol{r}_\alpha = \sum_{\beta \succeq \alpha^c} (-1)^{\ell(\alpha^c) - \ell(\beta)} \boldsymbol{e}_\beta \qquad (12)$$

Proof. For $\alpha \models n$, and any $I = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n$,

$$\boldsymbol{h}_\alpha\big|_{\boldsymbol{x}_{i_1}\boldsymbol{x}_{i_2}\cdots\boldsymbol{x}_{i_n}} = \mathbb{1}_{\mathrm{set}(\alpha)\supseteq \mathrm{des}(I)},$$

giving the left-hand side of (11). Similarly,

$$\boldsymbol{e}_\alpha\big|_{\boldsymbol{x}_{i_1}\boldsymbol{x}_{i_2}\cdots\boldsymbol{x}_{i_n}} = \mathbb{1}_{\mathrm{set}(\alpha)^c\subseteq \mathrm{des}(I)},$$

giving the left-hand side of (12). Then, as observed in [7] the equations on the right are an application of Möbius Inversion, applied to the Boolean algebra. □

Corollary 51 ([7], p. 16)**.** *Both* $\{\boldsymbol{e}_\alpha\}_{\alpha\models n}$ *and* $\{\boldsymbol{h}_\alpha\}_{\alpha\models n}$ *are bases of* $\boldsymbol{NSym}^n$. *In particular,* $\boldsymbol{NSym}$ *is generated freely by* $\{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ *or* $\{\boldsymbol{h}_n\}_{n\in\mathbb{N}}$.

Thus Definition 18 above, which gives **NSym** as a subspace of $\mathbb{C}[[\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots]]$, defines a space which is isomorphic to the more abstractly defined space of Gelfand et al. as described in Definition 2, justifying the use of the same name. Moreover,

Corollary 52 ([7], p. 15 and p. 19)**.** *For any strong compositions* α *and* β*,*

$$\boldsymbol{\rho}(\boldsymbol{h}_\alpha) = \boldsymbol{h}_{\alpha^r}, \qquad \boldsymbol{\psi}(\boldsymbol{h}_\alpha) = \boldsymbol{e}_\alpha, \qquad \boldsymbol{\omega}(\boldsymbol{h}_\alpha) = \boldsymbol{e}_{\alpha^r}, \tag{13}$$

and

$$\langle M_\alpha, \boldsymbol{h}_\beta\rangle = \mathbb{1}_{\alpha=\beta}. \tag{14}$$

Theorem 53 ([7], Section 4.1)**.** *For* $n \geqslant 1$*, and any* $\beta \models n$*,*

$$\boldsymbol{h}_\beta = \sum_{\alpha\preceq\beta} (-1)^{(\ell(\alpha)-|\beta|)} \boldsymbol{e}_\alpha.$$

Proof.

$$\begin{aligned}
\sum_{\alpha\preceq\beta} (-1)^{(\ell(\alpha)-|\beta|)} \boldsymbol{e}_\alpha &= \sum_{\alpha\succeq\beta^c} (-1)^{(\ell(\alpha)-1)} \boldsymbol{e}_{\alpha^c} && \alpha \to \alpha^c \\
&= \sum_{\alpha\succeq\beta^c} (-1)^{(\ell(\alpha)-1)} \sum_{\gamma\preceq\alpha} r_\gamma && \text{by (12)} \\
&= \sum_{\gamma\models|\beta|} \boldsymbol{r}_\gamma \sum_{\substack{\alpha\succeq\gamma \\ \alpha\succeq\beta^c}} (-1)^{(\ell(\alpha)-1)} \\
&= \sum_{\gamma\models|\beta|} \boldsymbol{r}_\gamma \sum_{\mathrm{set}(\alpha)\subseteq \mathrm{set}(\gamma)\cap \mathrm{set}(\beta)^c} (-1)^{|\,\mathrm{set}(\alpha))|} \\
&= \sum_{\gamma\succeq\beta} \boldsymbol{r}_\gamma && \text{by Möbius function properties} \\
&= \boldsymbol{h}_\beta. && \text{by (11)}
\end{aligned}$$

The second-to-last equality follows from properties of the Möbius function on the Boolean algebra, since the sum will be nonzero unless $\mathrm{set}(\gamma) \cap \mathrm{set}(\beta)^c = \emptyset$. □

As in Λ, the generating series of the noncommutative elementary and complete homogeneous symmetric functions are particularly nice:

$$\mathbf{E}(t) = \sum_{n\in\mathbb{N}} \boldsymbol{e}_n t^n = \overleftarrow{\prod_{i\geqslant 1}}(1+\boldsymbol{x}_i t) \quad \text{and} \quad \mathbf{H}(t) = \sum_{n\in\mathbb{N}} \boldsymbol{h}_n t^n = \overrightarrow{\prod_{i\geqslant 1}} \frac{1}{(1-\boldsymbol{x}_i t)}. \tag{15}$$

Here, we must take t to be a formal variable which commutes with $\boldsymbol{x}_i$ for all i. It is easy to see that, as in the commutative case,

$$\mathbf{E}(-t)\mathbf{H}(t) = 1 = \mathbf{H}(t)\mathbf{E}(-t). \tag{16}$$

This is taken as the defining relation for the noncommutative homogeneous basis in [7]. As observed there, with this relation it is immediate from comparing the coefficients of t^n in (16) that

Theorem 54 ([7], Proposition 3.3)**.** *For any positive integer n,*

$$\sum_{i=0}^{n}(-1)^{n-i}\boldsymbol{e}_i\boldsymbol{h}_{n-i} = 0 = \sum_{i=0}^{n}(-1)^{n-i}\boldsymbol{h}_i\boldsymbol{e}_{n-i}. \tag{17}$$

Last, we turn our attention to the noncommutative power sums. The reader expecting a similarly simple definition of $\boldsymbol{p}_n$, analogous to that of $\boldsymbol{e}_n$ and $\boldsymbol{h}_n$, will be disappointed. In particular, as mentioned above for $k > 1$,

$$\sum_{i\geqslant 1} \boldsymbol{x}_i^k \notin \mathbf{NSym}.$$

Thus, if one wishes to define $\boldsymbol{p}_n \in \mathbf{NSym}$ such that $\chi(\boldsymbol{p}_n) = p_n$, it must be that $\boldsymbol{p}_n$ has both positive and negative terms, when written as a sum of monomials. There is not a unique such $\boldsymbol{p}_n$ in $\mathbf{NSym}$; Gelfand et al. in [7] define two noncommutative analogues of the power sums, $\boldsymbol{\phi}_n$ and $\boldsymbol{\psi}_n$. While they originally define these bases based on their relation to the noncommutative complete homogeneous basis, we begin with their expansion in the noncommutative ribbon basis, since this allows us to easily read off their definition in terms of monomials.

Definition 55. Let $\boldsymbol{\psi}_0 = 1$, $\boldsymbol{\phi}_0 = 1$, and for $n \geqslant 1$, let

$$\boldsymbol{\psi}_n = \sum_{I=(i_1,\dots,i_n)\in A_n} (-1)^{k(I)-1}\boldsymbol{x}_{i_1}\boldsymbol{x}_{i_2}\cdots\boldsymbol{x}_{i_n} \tag{18}$$

$$= \sum_{k=0}^{n-1}(-1)^k \boldsymbol{r}_{1^k(n-k)} \tag{19}$$

and

$$\boldsymbol{\phi}_n = \sum_{I=(i_1,i_2,\dots,i_n)\in(\mathbb{Z}^+)^n} \frac{(-1)^{|\operatorname{des}(I)|}}{\binom{n-1}{|\operatorname{des}(I)|}}\boldsymbol{x}_{i_1}\boldsymbol{x}_{i_2}\cdots\boldsymbol{x}_{i_n} \tag{20}$$

$$= \sum_{\alpha\models n} \frac{(-1)^{\ell(\alpha)-1}}{\binom{n-1}{\ell(\alpha)-1}}\boldsymbol{r}_\alpha, \tag{21}$$

where

$$A_n = \left\{ (i_1, \ldots, i_n) \in (\mathbb{Z}^+)^n \;\middle|\; \exists k \text{ s.t. } 1 \leqslant k \leqslant n \text{ and } i_1 > i_2 > \cdots > i_{k-1} > i_k \leqslant i_{k+1} \leqslant \cdots \leqslant i_n \right\},$$

and where $k(I)$ is the unique k satisfying the condition required for $I = (i_1, \ldots, i_n) \in A_n$.

For any strong composition $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\boldsymbol{\psi}_\alpha = \boldsymbol{\psi}_{\alpha_1}\boldsymbol{\psi}_{\alpha_2}\cdots$ and $\boldsymbol{\phi}_\alpha = \boldsymbol{\phi}_{\alpha_1}\boldsymbol{\phi}_{\alpha_2}\cdots$. Then $\{\boldsymbol{\psi}_\alpha\}_{\alpha \models n}$ and $\{\boldsymbol{\phi}_\alpha\}_{\alpha \models n}$ are the **noncommutative power sums of the 1st** and **2nd kinds** (respectively).

Example 56.

$$\begin{aligned}\psi_3 = {} & \boldsymbol{x}_1^3 + \boldsymbol{x}_1^2\boldsymbol{x}_2 + \boldsymbol{x}_1^2\boldsymbol{x}_3 + \boldsymbol{x}_1\boldsymbol{x}_2^2 + \boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3 + \boldsymbol{x}_1\boldsymbol{x}_3^2 - \boldsymbol{x}_2\boldsymbol{x}_1^2 - \boldsymbol{x}_2\boldsymbol{x}_1\boldsymbol{x}_2 \\ & - \boldsymbol{x}_2\boldsymbol{x}_1\boldsymbol{x}_3 + \boldsymbol{x}_2^3 + \boldsymbol{x}_2^2\boldsymbol{x}_3 + \boldsymbol{x}_2\boldsymbol{x}_3^2 - \boldsymbol{x}_3\boldsymbol{x}_1^2 - \boldsymbol{x}_3\boldsymbol{x}_1\boldsymbol{x}_2 - \boldsymbol{x}_3\boldsymbol{x}_1\boldsymbol{x}_3 \\ & + \boldsymbol{x}_3\boldsymbol{x}_2\boldsymbol{x}_1 - \boldsymbol{x}_3\boldsymbol{x}_2^2 - \boldsymbol{x}_3\boldsymbol{x}_2\boldsymbol{x}_3 + \boldsymbol{x}_3^3 + \cdots\end{aligned}$$

and

$$\begin{aligned}\phi_3 = {} & \boldsymbol{x}_1^3 + \boldsymbol{x}_1^2\boldsymbol{x}_2 + \boldsymbol{x}_1^2\boldsymbol{x}_3 - \frac{1}{2}\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_1 + \boldsymbol{x}_1\boldsymbol{x}_2^2 + \boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3 - \frac{1}{2}\boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{x}_1 - \frac{1}{2}\boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{x}_2 + \boldsymbol{x}_1\boldsymbol{x}_3^2 \\ & - \frac{1}{2}\boldsymbol{x}_2\boldsymbol{x}_1^2 - \frac{1}{2}\boldsymbol{x}_2\boldsymbol{x}_1\boldsymbol{x}_2 - \frac{1}{2}\boldsymbol{x}_2\boldsymbol{x}_1\boldsymbol{x}_3 - \frac{1}{2}\boldsymbol{x}_2^2\boldsymbol{x}_1 + \boldsymbol{x}_2^3 + \boldsymbol{x}_2^2\boldsymbol{x}_3 - \frac{1}{2}\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_1 \\ & - \frac{1}{2}\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_2 + \boldsymbol{x}_2\boldsymbol{x}_3^2 - \frac{1}{2}\boldsymbol{x}_3\boldsymbol{x}_1^2 - \frac{1}{2}\boldsymbol{x}_3\boldsymbol{x}_1\boldsymbol{x}_2 - \frac{1}{2}\boldsymbol{x}_3\boldsymbol{x}_1\boldsymbol{x}_3 + \boldsymbol{x}_3\boldsymbol{x}_2\boldsymbol{x}_1 \\ & - \frac{1}{2}\boldsymbol{x}_3\boldsymbol{x}_2^2 - \frac{1}{2}\boldsymbol{x}_3\boldsymbol{x}_2\boldsymbol{x}_3 - \frac{1}{2}\boldsymbol{x}_3^2\boldsymbol{x}_1 - \frac{1}{2}\boldsymbol{x}_3^2\boldsymbol{x}_2 + \boldsymbol{x}_3^3 + \cdots\end{aligned}$$

Several of the places in which symmetric power sums pay a key role do not have analogues with any of the noncommutative power sums. Their relation to other bases in terms of generating functions is less straightforward; see the discussion after the proof of Theorem 61 below. From an algebraic perspective, one important role of the symmetric power sum basis is as they appear in the definition of the Frobenius character map (as is explained, for example, in [23]), where they encode the characters of representations of the symmetric group in Λ. There is a natural analogue of the Frobenius character map which encodes representations of the 0-Hecke algebra using **NSym**, as explained in [13]. However, there is no similar character defined on representations of the 0-Hecke algebra, so the map is defined quite differently, and none of the noncommutative power sums appear. Similarly, the symmetric power sums play an important role in defining or simplifying plethysm in the commuting variables, but are not present in the noncommutative story in the same way as explored in [12].

Using the standard inner product, Definition 55 indirectly defines bases of QSym dual to the two kinds of noncommutative power sum symmetric functions. These **quasisymmetric power sums**, $\{\psi_\alpha\}$ and $\{\phi_\alpha\}$, were explored in detail by Ballantine et al. in [1], and satisfy

$$\langle \psi_\alpha, \boldsymbol{\psi}_\beta \rangle = \langle \phi_\alpha, \boldsymbol{\phi}_\beta \rangle = z_\alpha \cdot \mathbb{1}_{\alpha=\beta}.$$

Here, and elsewhere, for any strong composition α, $z_\alpha = z_{\mathrm{sort}(\alpha)}$, the coefficient seen before.

While the images under the remaining involutions are not as straightforward, it is easy to see from the expansions of the noncommutative power sums of the first and second kinds in the noncommutative ribbon basis that we have the following theorem.

Theorem 57 ([7], Section 3)**.** *For any strong composition α,*

$$\boldsymbol{\omega}(\boldsymbol{\psi}_\alpha) = (-1)^{|\alpha|-\ell(\alpha)}\boldsymbol{\psi}_{\alpha^r}; \tag{22}$$

$$\boldsymbol{\omega}(\boldsymbol{\phi}_\alpha) = (-1)^{|\alpha|-\ell(\alpha)}\boldsymbol{\phi}_{\alpha^r}; \tag{23}$$

$$\boldsymbol{\psi}(\boldsymbol{\phi}_\alpha) = (-1)^{|\alpha|-\ell(\alpha)}\boldsymbol{\phi}_{\alpha}; \tag{24}$$

$$\boldsymbol{\rho}(\boldsymbol{\phi}_\alpha) = \boldsymbol{\phi}_{\alpha^r}. \tag{25}$$

These allow us to focus only on change of basis going forward in the noncommutative complete symmetric function basis, since expansion in the elementary symmetric functions will follow from applying the $\boldsymbol{\omega}$ map to each side.

Theorem 58 ([7], Section 4.2)**.** *For n a postive integer,*

$$\boldsymbol{\psi}_n = \sum_{\beta \models n} (-1)^{1+\ell(\beta)} \beta_{\ell(\beta)} \boldsymbol{h}_\beta.$$

Proof.

$$\begin{aligned}
\boldsymbol{\psi}_n &= \sum_{k=0}^{n-1} (-1)^k \boldsymbol{r}_{1^k(n-k)} && \text{by Definition 55} \\
&= \sum_{k=0}^{n-1} \sum_{\beta \succeq 1^k(n-k)} (-1)^{1+\ell(\beta)} \boldsymbol{h}_\beta && \text{by (11)} \\
&= \sum_{\beta \models n} (-1)^{1+\ell(\beta)} \boldsymbol{h}_\beta \sum_{k=0}^{n-1} \mathbb{1}_{\beta \succeq (1^k, n-k)} \\
&= \sum_{\beta \models n} (-1)^{1+\ell(\beta)} \boldsymbol{h}_\beta \sum_{k=0}^{n-1} \mathbb{1}_{\beta_{\ell(\beta)} \geqslant n-k} \\
&= \sum_{\beta \models n} (-1)^{1+\ell(\beta)} \beta_{\ell(\beta)} \boldsymbol{h}_\beta.
\end{aligned}$$

□

Before giving an analogous result for the $\boldsymbol{\phi}_n$, we first need the following lemma.

Lemma 59. *Let n and c be nonnegative integers. Then,*

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{n+c}{k+c}} = \frac{n+c+1}{c+1}.$$

Proof. By induction on n. The base case $n = 0$ is trivially true. Then

$$\begin{aligned}\sum_{k=0}^{n+1} \frac{\binom{n+1}{k}}{\binom{n+1+c}{k+c}} &= 1 + \sum_{k=0}^{n} \frac{\left(\frac{n+1}{n+1-k}\right)\binom{n}{k}}{\left(\frac{n+1+c}{n+1-k}\right)\binom{n+c}{k+c}} \\ &= 1 + \left(\frac{n+1}{n+1+c}\right) \sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{n+c}{k+c}} \\ &\overset{\text{I.H.}}{=} 1 + \left(\frac{n+1}{n+1+c}\right)\left(\frac{n+1+c}{c+1}\right) \\ &= \frac{(n+1)+c+1}{c+1}.\end{aligned}$$

□

Theorem 60 ([7], Section 4.3)**.** *If n is a positive integer,*

$$\boldsymbol{\phi}_n = \sum_{\beta \models n} (-1)^{\ell(\beta)+1} \frac{n}{\ell(\beta)} \boldsymbol{h}_\beta.$$

Proof.

$$\begin{aligned}\boldsymbol{\phi}_n &= \sum_{\alpha \models n} \frac{(-1)^{\ell(\alpha)-1}}{\binom{n-1}{\ell(\alpha)-1}} \boldsymbol{r}_\alpha && \text{by Definition 55} \\ &= \sum_{\alpha \models n} \frac{(-1)^{\ell(\alpha)-1}}{\binom{n-1}{\ell(\alpha)-1}} \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \boldsymbol{h}_\beta && \text{by (11)} \\ &= \sum_{\beta \models n} \boldsymbol{h}_\beta (-1)^{\ell(\beta)+1} \sum_{\alpha \preceq \beta} \frac{1}{\binom{n-1}{\ell(\alpha)-1}} \\ &= \sum_{\beta \models n} \boldsymbol{h}_\beta (-1)^{\ell(\beta)+1} \sum_{k=1}^{n} \frac{1}{\binom{n-1}{k-1}} \sum_{\alpha \preceq \beta} \mathbb{1}_{\ell(\alpha)=k} \\ &= \sum_{\beta \models n} \boldsymbol{h}_\beta (-1)^{\ell(\beta)+1} \sum_{k=\ell(\beta)}^{n} \frac{\binom{n-\ell(\beta)}{k-\ell(\beta)}}{\binom{n-1}{k-1}} \\ &= \sum_{\beta \models n} \boldsymbol{h}_\beta (-1)^{\ell(\beta)+1} \sum_{k=0}^{n-\ell(\beta)} \frac{\binom{n-\ell(\beta)}{k}}{\binom{n-1}{k+\ell(\beta)-1}} \\ &= \sum_{\beta \models n} (-1)^{\ell(\beta)+1} \frac{n}{\ell(\beta)} \boldsymbol{h}_\beta. && \text{by Lemma 59}\end{aligned}$$

□

Theorem 61 ([7], Proposition 3.3)**.** *For $n \geqslant 1$,*

$$\sum_{i=0}^{n-1} \boldsymbol{h}_i \boldsymbol{\psi}_{n-i} = n\boldsymbol{h}_n.$$

Proof.

$$\begin{aligned}\sum_{i=0}^{n-1} \boldsymbol{h}_i \boldsymbol{\psi}_{n-i} &= \boldsymbol{\psi}_n + \sum_{i=1}^{n-1} \boldsymbol{h}_i \boldsymbol{\psi}_{n-i} \\ &= \sum_{\beta \models n} (-1)^{1+\ell(\beta)} \beta_{\ell(\beta)} \boldsymbol{h}_\beta + \sum_{i=1}^{n-1} \boldsymbol{h}_i \sum_{\beta \models n-i} (-1)^{1+\ell(\beta)} \beta_{\ell(\beta)} \boldsymbol{h}_\beta \\ &= \sum_{\beta \models n} (-1)^{1+\ell(\beta)} \beta_{\ell(\beta)} \boldsymbol{h}_\beta + \sum_{\substack{\beta \models n \\ \ell(\beta) \geqslant 2}} (-1)^{(\ell(\beta))} \beta_{\ell(\beta)} \boldsymbol{h}_\beta \\ &= n\boldsymbol{h}_n. \end{aligned}$$

□

There are other equally natural choices for an analogue of the power sums. Gelfand et al. chose these based on two generating series relations on the symmetric functions, whose analogues below are each satisfied by only one of the noncommutative power sum symmetric functions.

Theorem 62 ([7], Section 3.1)**.** *Let*

$$\boldsymbol{\psi}(t) = \sum_{n \in \mathbb{Z}^+} \frac{\boldsymbol{\psi}_n}{n} t^n \text{ and } \boldsymbol{\phi}(t) = \sum_{n \in \mathbb{Z}^+} \frac{\boldsymbol{\phi}_n}{n} t^n.$$

Then

$$\frac{d}{dt}(\boldsymbol{H}(t)) = \boldsymbol{H}(t) \frac{d}{dt} \boldsymbol{\psi}(t), \tag{26}$$

and

$$\boldsymbol{H}(t) = \exp(\boldsymbol{\phi}(t)), \tag{27}$$

or equivalently

$$\boldsymbol{\phi}(t) = \log \left(1 + \sum_{k \geqslant 1} \boldsymbol{h}_k t^k \right). \tag{28}$$

Proof. As explained briefly in [7], Equation (26) follows immediately from Theorem 61 and equation (27) is only slightly less straightforward:

$$\begin{aligned}\log\left(1+\sum_{k\geqslant 1}\boldsymbol{h}_k t^k\right) &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\sum_{k\geqslant 1}\boldsymbol{h}_k t^k\right)^n \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{\ell(\alpha)=n}\boldsymbol{h}_{\alpha_1}\boldsymbol{h}_{\alpha_2}\cdots\boldsymbol{h}_{\alpha_n}t^{\sum_i\alpha_i} \\ &= \sum_{s=1}^{\infty}\frac{t^s}{s}\sum_{\beta\models s}(-1)^{\ell(\beta)+1}\frac{s}{\ell(\beta)}\boldsymbol{h}_\beta \\ &= \sum_{s=1}^{\infty}\frac{t^s}{s}\boldsymbol{\phi}_s,\end{aligned}$$

where the last equality follows from Theorem 60. □

Note that while the analogous differential equation in commuting variables,

$$\frac{d}{dt}(H(t)) = H(t)P(t),$$

defines a unique basis of power sums from the basis of homogeneous complete symmetric functions, there are many equivalent ways to write the same relationship that yield distinct noncommutative analogues. (To see one additional easy example, reversing the order of the right-hand side of (26) would yield a different basis than $\{\boldsymbol{\psi}_n\}$, which Gelfand et al. do not name, as the result is sufficiently similar as to not be interesting.)

Corollary 63 ([7], Section 3.1)**.** *For n a nonnegative integer,*

$$\chi(\boldsymbol{\psi}_n) = p_n \text{ and } \chi(\boldsymbol{\phi}_n) = p_n.$$

Theorem 64 ([7], Section 4.2)**.** *For n a nonnegative integer,*

$$\boldsymbol{h}_n = \sum_{\beta\models n}\frac{1}{\prod_{i=1}^{\ell(\beta)}\sum_{j=1}^{i}\beta_j}\boldsymbol{\psi}_\beta.$$

Proof. By strong induction on n, with the base case of $n=0$ being trivial:

$$\begin{aligned}\boldsymbol{h}_n &= \frac{1}{n}\sum_{i=0}^{n-1}\boldsymbol{h}_i\boldsymbol{\psi}_{n-i} \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\sum_{\beta\models i}\frac{1}{\prod_{k=1}^{\ell(\beta)}\sum_{j=1}^{k}\beta_j}\boldsymbol{\psi}_\beta\boldsymbol{\psi}_{n-i} \\ &= \sum_{\gamma\models n}\frac{1}{\prod_{i=1}^{\ell(\gamma)}\sum_{j=1}^{i}\gamma_j}\boldsymbol{\psi}_\gamma,\end{aligned}$$

where $\gamma = (\beta, n-i)$. □

Theorem 65 ([7], Section 4.3)**.** *For n a nonnegative integer,*

$$\boldsymbol{h}_n = \sum_{\beta \models n} \frac{1}{\ell(\beta)! \prod_{i=1}^{\ell(\beta)} \beta_i} \boldsymbol{\phi}_\beta.$$

Proof. By induction on n, with the base case of $n = 0$ being trivial. We simultaneously prove both the statement, and its image under the $\boldsymbol{\omega}$ map:

$$\boldsymbol{e}_n = \sum_{\beta \models n} \frac{1}{\ell(\beta)! \prod_{i=1}^{\ell(\beta)} \beta_i} (-1)^{|\beta| - \ell(\beta)} \boldsymbol{\phi}_\beta.$$

Then

$$\begin{aligned}
\boldsymbol{h}_n &= \sum_{i=0}^{n-1} (-1)^{n-i-1} \boldsymbol{h}_i \boldsymbol{e}_{n-i} && \text{by (17)} \\
&= \sum_{i=0}^{n-1} (-1)^{n-i-1} \sum_{\beta \models i} \left(\frac{1}{\ell(\beta)! \prod_{i=1}^{\ell(\beta)} \beta_i} \boldsymbol{\phi}_\beta \right) \\
&\qquad \left(\sum_{\gamma \models n-i} \frac{1}{\ell(\gamma)! \prod_{i=1}^{\ell(\gamma)} \gamma_i} (-1)^{|\gamma| - \ell(\gamma)} \boldsymbol{\phi}_\gamma \right) && \text{by I.H.} \\
&= \sum_{\delta \models n} \boldsymbol{\phi}_\delta \frac{1}{\ell(\delta)! \prod_{i=1}^{\ell(\delta)} \delta_i} \sum_{j=1}^{\ell(\delta)} (-1)^{j-1} \binom{\ell(\delta)}{j} && \delta = (\beta, \gamma) \text{ and } j = \ell(\beta) \\
&= \sum_{\delta \models n} \frac{1}{\ell(\delta)! \prod_{i=1}^{\ell(\delta)} \delta_i} \boldsymbol{\phi}_\delta. && \text{by Binomial Theorem}
\end{aligned}$$

Applying the $\boldsymbol{\omega}$ map again completes the induction. □

Corollary 66 (Gelfand et al. [7])**.** *Both $\{\boldsymbol{\psi}_\alpha\}_{\alpha \models n}$ and $\{\boldsymbol{\phi}_\alpha\}_{\alpha \models n}$ are bases of $\boldsymbol{NSym}^n$.*

A number of the reoccurring statistics above are given names in [7] and repeated here. They need to be generalized to cover extending the above results to the multiplicative bases.

Definition 67. Let $\beta = (\beta_1, \beta_2, \ldots, \beta_{\ell(\beta)})$ be a strong composition. Denote the **last part** of β by

$$\mathrm{lp}(\beta) = \beta_{\ell(\beta)}.$$

Also say β's **product of partial sums**, **product**, and **special product**, respectively, are

$$\pi_u(\beta) = \prod_{i=1}^{\ell(\beta)} \sum_{k=1}^{i} \beta_k = \beta_1(\beta_1 + \beta_2) \cdots (\beta_1 + \beta_2 + \cdots + \beta_{\ell(\beta)}); \tag{29}$$

$$\prod \beta = \beta_1 \beta_2 \cdots \beta_{\ell(\beta)}; \tag{30}$$

$$\mathrm{sp}(\beta) = \ell(\beta)! \prod \beta. \tag{31}$$

Recall that if $\beta \preceq \alpha$, $\beta^{(i)}$ is the subcomposition of β which sums to α_i for $i = 1, \ldots, \ell(\alpha)$. Extend the above definitions to refinements $\beta \preceq \alpha$, with $|\beta^{(i)}| = \alpha_i$, $i \in [\ell(\alpha)]$:

$$\mathrm{lp}(\beta, \alpha) = \prod_{i=1}^{\ell(\alpha)} \mathrm{lp}(\beta^{(i)}) \quad (32) \qquad \pi_u(\beta, \alpha) = \prod_{i=1}^{\ell(\alpha)} \pi_u(\beta^{(i)}) \quad (34)$$

$$\ell(\beta, \alpha) = \prod_{i=1}^{\ell(\alpha)} \ell(\beta^{(i)}) \quad (33) \qquad \mathrm{sp}(\beta, \alpha) = \prod_{i=1}^{\ell(\alpha)} \mathrm{sp}(\beta^{(i)}) \quad (35)$$

In [7], all of the following change-of-basis equations were established.

Theorem 68. *For $\alpha \models n$,*

$$\boldsymbol{h}_\alpha = \sum_{\beta \preceq \alpha} (-1)^{|\alpha| - \ell(\beta)} \boldsymbol{e}_\beta; \qquad \boldsymbol{e}_\alpha = \sum_{\beta \preceq \alpha} (-1)^{|\alpha| - \ell(\beta)} \boldsymbol{h}_\beta \tag{36}$$

$$\boldsymbol{h}_\alpha = \sum_{\beta \succeq \alpha} \boldsymbol{r}_\beta \qquad \boldsymbol{r}_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} \boldsymbol{h}_\beta \tag{37}$$

$$\boldsymbol{e}_\alpha = \sum_{\beta^t \succeq \alpha^r} \boldsymbol{r}_\beta \qquad \boldsymbol{r}_\alpha = \sum_{\beta^r \succeq \alpha^t} (-1)^{\ell(\alpha^t) - \ell(\beta)} \boldsymbol{e}_\beta \tag{38}$$

$$\boldsymbol{h}_\alpha = \sum_{\beta \preceq \alpha} \frac{1}{\pi_u(\beta, \alpha)} \boldsymbol{\psi}_\beta \qquad \boldsymbol{\psi}_\alpha = \sum_{\beta \preceq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} \, \mathrm{lp}(\beta, \alpha) \boldsymbol{h}_\beta \tag{39}$$

$$\boldsymbol{h}_\alpha = \sum_{\beta \preceq \alpha} \frac{1}{\mathrm{sp}(\beta, \alpha)} \boldsymbol{\phi}_\beta \qquad \boldsymbol{\phi}_\alpha = \sum_{\beta \preceq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} \frac{\prod \alpha}{\ell(\beta, \alpha)} \boldsymbol{h}_\beta \tag{40}$$

$$\boldsymbol{e}_\alpha = \sum_{\beta \preceq \alpha} \frac{(-1)^{|\alpha| - \ell(\beta)}}{\pi_u(\beta^r, \alpha^r)} \boldsymbol{\psi}_\beta \qquad \boldsymbol{\psi}_\alpha = \sum_{\beta \preceq \alpha} (-1)^{|\alpha| - \ell(\beta)} \, \mathrm{lp}(\beta^r, \alpha^r) \boldsymbol{e}_\beta \tag{41}$$

$$\boldsymbol{e}_\alpha = \sum_{\beta \preceq \alpha} \frac{(-1)^{|\alpha| - \ell(\beta)}}{\mathrm{sp}(\beta, \alpha)} \boldsymbol{\phi}_\beta \qquad \boldsymbol{\phi}_\alpha = \sum_{\beta \preceq \alpha} (-1)^{|\alpha| - \ell(\beta)} \frac{\prod \alpha}{\ell(\beta, \alpha)} \boldsymbol{e}_\beta \tag{42}$$

Proof. Line (36) comes from Theorem 53 and an application of the $\boldsymbol{\psi}$ map (see line (13)). Lines (37) and (38) were shown in Theorem 50. The remaining lines follow from Theorems 60, 64, and 65, the multiplicative definitions of $\boldsymbol{h}_\alpha$, $\boldsymbol{e}_\alpha$, $\boldsymbol{\phi}_\alpha$, and $\boldsymbol{\psi}_\alpha$, and applications of the $\boldsymbol{\omega}$ map (see Theorem 57). □

There are two remaining explicit change-of-basis formulas in [7], both of which follow from the product formula for $\boldsymbol{r}_\alpha$, Theorem 40 above. We need the following definition.

Definition 69. Let α and β be strong compositions of n. Let $\gamma = \text{set}^{-1}(\text{set}(\alpha) \cup \text{set}(\beta))$. Then $\gamma \preceq \beta$, so we can let $\gamma = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(k)})$ give the subsequences such that $|\gamma^{(j)}| = \beta_j$. Then the **ribbon decomposition of $\boldsymbol{\alpha}$ with respect to $\boldsymbol{\beta}$** is

$$\text{rd}(\alpha, \beta) = (\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(k)}).$$

Furthermore, let

$$\text{psr}(\alpha, \beta) = (-1)^{|\gamma^{(1)}|-1}(-1)^{|\gamma^{(2)}|-1} \cdots (-1)^{|\gamma^{(k)}|-1} \mathbb{1}_{\gamma^{(1)} \text{ is a hook}} \mathbb{1}_{\gamma^{(2)} \text{ is a hook}} \cdots \mathbb{1}_{\gamma^{(k)} \text{ is a hook}}$$

and

$$\text{phr}(\alpha, \beta) = \frac{(-1)^{\ell(\gamma^{(1)})-1}}{\binom{n-1}{\ell(\gamma^{(1)})-1}} \cdots \frac{(-1)^{\ell(\gamma^{(k)})-1}}{\binom{n-1}{\ell(\gamma^{(k)})-1}}.$$

Example 70. Let $\alpha = (1, 3, 2, 4, 4)$ and $\beta = (4, 3, 5, 2)$. Then $\gamma = (1, 3, 2, 1, 3, 2, 2)$ and we have a ribbon decomposition of α with respect to β is

$$\text{rd}(\alpha, \beta) = ((1, 3), (2, 1), (3, 2), (2))$$

As first observed by Gelfand et al., it is easy to see from Definition 55 that

Theorem 71 ([7], Prop. 4.23 and Prop. 4.27)**.** *Let α be a strong composition. Then*

$$\boldsymbol{\psi}_\alpha = \sum_{\beta \models n} \text{psr}(\beta, \alpha) \boldsymbol{r}_\alpha \text{ and } \boldsymbol{\phi}_\alpha = \sum_{\beta \models n} \text{phr}(\beta, \alpha) \boldsymbol{r}_\alpha.$$

Remark. Omitted here is the work in [7] towards change of basis between $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, which did not result in as nice of change-of-basis formulas. The interested reader should consult Section 4.10 in [7], where there is a somewhat more complex formula for $\boldsymbol{\phi}_n$ in terms of $\{\boldsymbol{\psi}_\alpha\}_{\alpha \models n}$.

Note. In [16], the authors mention that $\{\omega(M_\alpha)\}$ yields the dual basis to $\{\boldsymbol{e}_\alpha\}$. For our purposes later, it will be most natural to utilize ψ to define a quasisymmetric analogue to the forgotten symmetric functions. While the result is not quite a dual to $\{\boldsymbol{e}_\alpha\}$, the resulting basis still restricts to the forgotten symmetric functions under the forgetful map. Therefore, define the **forgotten quasisymmetric function** (associated to α) to be

$$\text{For}_\alpha = \psi(M_\alpha).$$

Then, by duality, the following corresponding equations can be established.

Corollary 72. *For* $\alpha \models n$,

$$F_\alpha = \sum_{\beta \preceq \alpha} M_\beta \qquad M_\alpha = \sum_{\beta \preceq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} F_\beta \tag{43}$$

$$F_\alpha = \sum_{\beta^r \preceq \alpha^t} \mathrm{For}_\beta \qquad \mathrm{For}_\alpha = \sum_{\beta^t \preceq \alpha^r} (-1)^{\ell(\beta^t)-\ell(\alpha)} F_\beta \tag{44}$$

$$\mathrm{For}_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha)-|\beta|} M_\beta \qquad M_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha)-|\beta|} \mathrm{For}_\beta\,. \tag{45}$$

$$\psi_\alpha = z_\alpha \sum_{\beta \succeq \alpha} \frac{1}{\pi_u(\alpha,\beta)} M_\beta \qquad M_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \mathrm{lp}(\alpha,\beta) \frac{\psi_\beta}{z_\beta} \tag{46}$$

$$\phi_\alpha = z_\alpha \sum_{\beta \succeq \alpha} \frac{1}{sp(\alpha,\beta)} M_\beta \qquad M_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \frac{\prod \beta}{\ell(\alpha,\beta)} \frac{\phi_\beta}{z_\beta} \tag{47}$$

$$\psi_\alpha = z_\alpha \sum_{\beta \succeq \alpha} \frac{(-1)^{|\beta|-\ell(\alpha)}}{\pi_u(\alpha^r,\beta^r)} \mathrm{For}_\beta \qquad \mathrm{For}_\alpha = \sum_{\beta \succeq \alpha} (-1)^{|\beta|-\ell(\alpha)} \mathrm{lp}(\alpha^r,\beta^r) \frac{\psi_\beta}{z_\beta} \tag{48}$$

$$\phi_\alpha = z_\alpha \sum_{\beta \succeq \alpha} \frac{(-1)^{|\beta|-\ell(\alpha)}}{sp(\alpha,\beta)} \mathrm{For}_\beta \qquad \mathrm{For}_\alpha = \sum_{\beta \succeq \alpha} (-1)^{|\beta|-\ell(\alpha)} \frac{\prod \beta}{\ell(\alpha,\beta)} \frac{\phi_\beta}{z_\beta} \tag{49}$$

Figure 2, on the next page, gives a diagram analogous to Figure 1 depicting the results from this section. Horizontal dashed segments indicate duality once again, unless marked by an 'r' to denote reversing the order. Note the vertical edges labeled ψ and $\boldsymbol{\psi}$ indicate that a basis element indexed by α is sent to its counterpart (also indexed by α) in the other basis. All other edges, outside set braces, indicate basis elements are most often not sent to their exact counterparts in the image set. Suppressed are four loops that would be labeled with either ρ or $\boldsymbol{\rho}$, as well as the combinatorial coefficients z_α at the bottom, for readability.

We end by discussing two ways in which a number of the change-of-basis results in Λ have been condensed, and show analogous results in the dual bases of QSym and **NSym**.

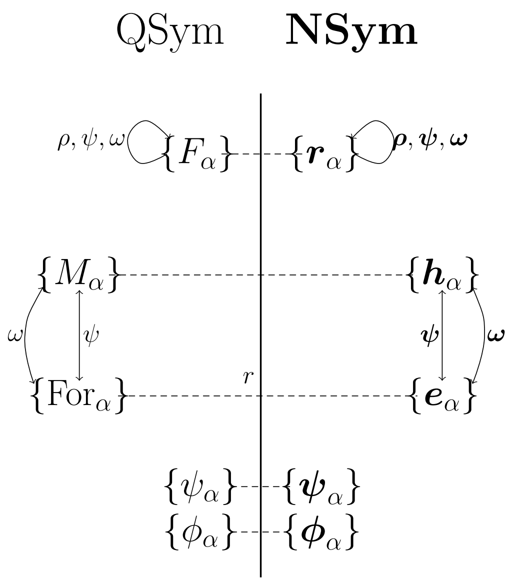


Figure 2: Automorphisms and Duality of QSym and **NSym**

6 Relations between transition matrices

A well-known diagram in Macdonald [17], p.105, reproduced in Figure 3 (on the next page), summarizes the relationship between various transitions matrices in Λ. Our next goal is to reproduce a similar diagram for QSym and **NSym**.

Notation. Let $b = \{b_\beta\}_{\beta \in B}$ and $a = \{a_\alpha\}_{\alpha \in A}$ be bases of some vector space V with ordered indexing sets B and A, respectively. Then the **change-of-basis matrix from $\{b_\beta\}_{\beta \in B}$ to $\{a_\alpha\}_{\alpha \in A}$** is the matrix $\mathrm{M}(a,b)$ for which

$$[a_\alpha]_{\alpha \in A} = \mathrm{M}(a,b)[b_\beta]_{\beta \in B}.$$

Here, $[a_\alpha]_{\alpha \in A}$ and $[b_\beta]_{\beta \in B}$ denote the column vectors of all the basis elements a_α and b_β, respectively.

Let n be a positive integer and let $p(n)$ be the number of integer partitions of n. Then, in the context of Λ^n, Macdonald defines the following $p(n) \times p(n)$ matrices:

- $K = M(s,m) = [K_{\lambda,\mu}]_{\lambda,\mu}$, where $K_{\lambda,\mu}$ is the **Kostka number** $K_{\lambda,\mu}$, the number of semi-standard Young tableaux of shape $\lambda \vdash n$ and type μ;
- $J = (\mathbb{1}_{\lambda = \mu^t})_{\mu,\lambda}$;
- $z = (z_\lambda \cdot \mathbb{1}_{\lambda=\mu})_{\mu,\lambda}$;
- $L = L(p,m) = [L_{\lambda,\mu}]_{\lambda,\mu}$, where $L_{\lambda,\mu} = \left| \left\{ f : [\ell(\lambda)] \to \mathbb{Z}^+ \;\middle|\; \mu = \left(\sum_{f(j)=i} \lambda_j \right)_{i=1}^{\infty} \right\} \right|$. (See the language of (6.9) in [17].)

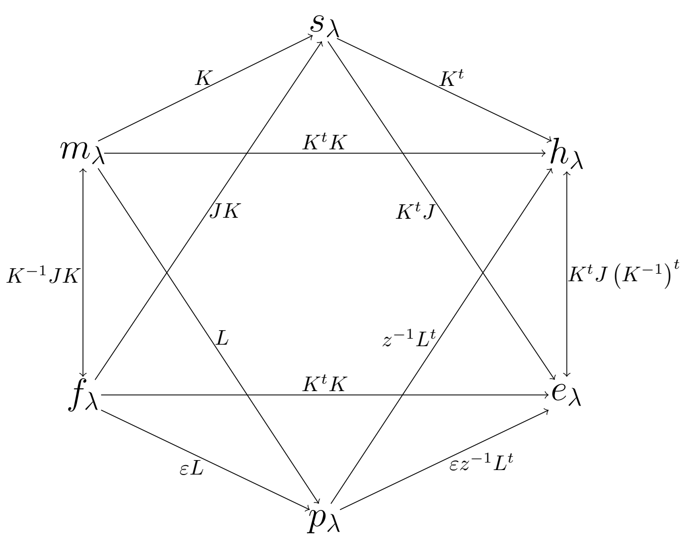


Figure 3: Matrix Change-of-Basis Expressions in Λ

6.1 Change-of-Basis Expressions in **NSym** and QSym

In this subsection, we generalize the results on matrix equations in [17] to the spaces of QSym and **NSym**. The proofs in this section are quite similar to the originals in [17], with only minor additional complexity from the multiple isometries and power sum bases in QSym and **NSym**.

It is helpful to recall three well-known facts about general change of basis, and two additional facts pertaining to our involutions:

(I) For any bases $\{a_\alpha\}$, $\{b_\alpha\}$, and $\{d_\alpha\}$ in a fixed vector space, $\mathrm{M}(d,b)\mathrm{M}(b,a) = \mathrm{M}(d,a)$;

(II) For any bases $\{a_\alpha\}$, and $\{b_\alpha\}$, $\mathrm{M}(b,a) = \mathrm{M}(a,b)^{-1}$.

(III) For any bases $\{A_\alpha\}, \{B_\alpha\} \subset$ QSym, and their respective dual bases $\{\boldsymbol{a}_\alpha\}, \{\boldsymbol{b}_\alpha\} \subset$ **NSym**, $\mathrm{M}(A,B) = \mathrm{M}(\boldsymbol{b},\boldsymbol{a})^t$.

(IV) For any bases $\{A_\alpha\}, \{B_\alpha\} \subset$ QSym, $\mathrm{M}(A,B) = \mathrm{M}(\psi(A),\psi(B)) = \mathrm{M}(\rho(A),\rho(B)) = \mathrm{M}(\omega(A),\omega(B))$;

(V) For any bases $\{\boldsymbol{a}_\alpha\}, \{\boldsymbol{b}_\alpha\} \subset$ **NSym**, $\mathrm{M}(\boldsymbol{a},\boldsymbol{b}) = \mathrm{M}(\boldsymbol{\psi}(\boldsymbol{a}),\boldsymbol{\psi}(\boldsymbol{b})) = \mathrm{M}(\boldsymbol{\rho}(\boldsymbol{a}),\boldsymbol{\rho}(\boldsymbol{b})) = \mathrm{M}(\boldsymbol{\omega}(\boldsymbol{a}),\boldsymbol{\omega}(\boldsymbol{b}))$.

With these at our disposal, we imitate the approach from Chapter 6 of [17].

In the classical case, the matrix of Kostka numbers is defined, $K = \mathrm{M}(s, m)$. Now let n be a positive integer and define the quasisymmetric analogue to K indexed by the $2^{n-1} \times 2^{n-1}$ strong compositions of n (see (43)),

$$\mathcal{K} = \mathrm{M}(F, M) = (\mathbb{1}_{\beta \preceq \alpha})_{\alpha,\beta}\,, \tag{50}$$

where $\{F_\alpha\}$ and $\{M_\alpha\}$ are the Gessel Fundamental and monomial quasisymmetric function bases, respectively. Reusing the same labels, similarly define the analogous matrices

- $\varepsilon = (\varepsilon(\alpha) \cdot \mathbb{1}_{\alpha=\beta})_{\alpha,\beta} = \left((-1)^{|\alpha|-\ell(\alpha)} \cdot \mathbb{1}_{\alpha=\beta}\right)_{\alpha,\beta}$ and
- $z = (z_\alpha \cdot \mathbb{1}_{\alpha=\beta})_{\alpha,\beta}$.

Then, by (II) and (III), and from lines (37) and (43), we immediately have the following change-of-basis matrices as well.

$$\mathcal{K}^{-1} = \mathrm{M}(M, F) = \left((-1)^{\ell(\beta)-\ell(\alpha)} \cdot \mathbb{1}_{\beta \preceq \alpha}\right)_{\alpha,\beta} = \varepsilon \mathcal{K} \varepsilon \tag{51}$$

$$(\mathcal{K}^t)^{-1} = \mathrm{M}(\boldsymbol{r}, \boldsymbol{h}) = \left((-1)^{\ell(\alpha)-\ell(\beta)} \cdot \mathbb{1}_{\alpha \preceq \beta}\right)_{\alpha,\beta} = \varepsilon \mathcal{K}^t \varepsilon \tag{52}$$

$$\mathcal{K}^t = \mathrm{M}(\boldsymbol{h}, \boldsymbol{r}) = (\mathbb{1}_{\alpha \succeq \beta})_{\alpha,\beta} \tag{53}$$

Here, $\{\boldsymbol{r}_\alpha\}$ and $\{\boldsymbol{h}_\alpha\}$ are the noncommutative ribbon and noncommutative complete homogeneous symmetric function bases of **NSym**, respectively. Note that in contrast to Λ, we do not need to use a separate matrix $\mathcal{K}^{-1}$ (and note that $\varepsilon^{-1} = \varepsilon$).

Remark. There are several change-of-basis matrices in Λ that do not make sense with the defined bases for **NSym** and QSym. For example $\mathrm{M}(h, m)$ is well defined, but $\{\boldsymbol{h}_\alpha\}_{\alpha \models n}$ and $\{M_\alpha\}_{\alpha \models n}$ are in different spaces and thus there is no analogous matrix here. In principle, one could define a new basis based on these change-of-basis results; for example, we could define $\{\boldsymbol{m}_\alpha\}_{\alpha \models n} \subset \mathbf{NSym}$ so that $\mathrm{M}(\boldsymbol{m}, \boldsymbol{h}) = \mathcal{K}^t \mathcal{K}$. The result is consistent (so we can define $\boldsymbol{m}$ by its relation to any of the other bases and get the same basis), but a bit algebraically uninteresting. Because of duality, this definition would yield

$$\boldsymbol{m}_\alpha = \sum_{\beta \preceq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} \boldsymbol{r}_\beta,$$

which is not far from

$$M_\alpha = \sum_{\beta \preceq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} F_\beta.$$

In some cases, where the hypothetical resulting basis is either $\boldsymbol{r}$- (or F)-positive, it would encode some representation via one of the two Frobenius maps of Krob and Thibon in [13], which encode modules of the type A 0-Hecke Algebra. See [19] for additional details.

Remark. Once again, our definition of $\boldsymbol{m}_\alpha$ does not depend on which basis we "start from" in Λ, as indicated by the three equalities above.

To continue, we would like to imitate the usage of the matrix J from Chapter 6 of [17]. With three analogs to the involution ω to choose from, there are three natural generalizations in each of QSym and **NSym**.

Definition 73. Let $J_f = (J_f)_{\alpha,\beta} = (\mathbb{1}_{f(F_\alpha)=F_\beta})_{\alpha,\beta}$ for $f = \psi, \rho$, or ω.

Remark. Equivalently, $J_f = (J_f)_{\alpha,\beta} = (\mathbb{1}_{f(\boldsymbol{r}_\alpha)=\boldsymbol{r}_\beta})_{\alpha,\beta}$ for any of $f = \boldsymbol{\psi}, \boldsymbol{\rho}$, or $\boldsymbol{\omega}$. Furthermore, since all of the maps f under consideration are involutions, it follows that J_f^2 is the identity matrix, and thus $J_f^{-1} = J_f$, just as we saw in the classical case. Since J_f is a permutation matrix, it must also be orthogonal, and thus its own transpose, $J_f^t = J_f$.

As we alluded to when defining the forgotten quasisymmetric functions, it turns out most useful to us will be J_ψ, corresponding to the involutions $\psi : \mathrm{QSym} \to \mathrm{QSym}$ and $\boldsymbol{\psi} : \mathbf{NSym} \to \mathbf{NSym}$, which respectively preserve the indexing on the relevant bases (see Figure 3.1). In fact, if we choose to order the integer compositions which index these matrices J with any ordering consistent with reverse lexicographic ordering, the matrix J_ψ has the particularly simple form with 1s on the anti diagonal and 0s elsewhere.

Our first result shows that each of the remaining permissible relations from the table on page 101 of [17] generalize appropriately to the spaces of QSym (or **NSym**), with J_ψ in place of J.

Theorem 74. *We have the following change-of-basis relations.*

(i) $M(M, \mathrm{For}) = M(\mathrm{For}, M) = \varepsilon\mathcal{K}\varepsilon J_\psi\mathcal{K}$;

(ii) $M(\mathrm{For}, F) = \varepsilon\mathcal{K}\varepsilon J_\psi$;

(iii) $M(F, \mathrm{For}) = J_\psi\mathcal{K}$;

(iv) $M(\boldsymbol{h}, \boldsymbol{e}) = M(\boldsymbol{e}, \boldsymbol{h}) = \mathcal{K}^t J_\psi\varepsilon\mathcal{K}^t\varepsilon$;

(v) $M(\boldsymbol{e}, \boldsymbol{r}) = \mathcal{K}^t J_\psi$;

(vi) $M(\boldsymbol{r}, \boldsymbol{e}) = J_\psi\varepsilon\mathcal{K}^t\varepsilon$.

Proof. Utilizing (52), (I), and (V), we can immediately establish *(vi)*:

$$\mathrm{M}(\boldsymbol{r}, \boldsymbol{e}) = \mathrm{M}(\boldsymbol{\psi r}, \boldsymbol{\psi e}) = \mathrm{M}(\boldsymbol{\psi r}, \boldsymbol{h}) = \mathrm{M}(\boldsymbol{\psi r}, \boldsymbol{r})\mathrm{M}(\boldsymbol{r}, \boldsymbol{h}) = J_\psi\varepsilon\mathcal{K}^t\varepsilon.$$

Then (II) gives *(v)*:

$$\mathrm{M}(\boldsymbol{e}, \boldsymbol{r}) = \mathrm{M}(\boldsymbol{r}, \boldsymbol{e})^{-1} = \left(J_\psi(\mathcal{K}^t)^{-1}\right)^{-1} = \mathcal{K}^t J_\psi.$$

With (52), (53), *(vi)*, and *(v)*, we may use (I) again to establish *(iv)*:

$$\begin{aligned} \mathrm{M}(\boldsymbol{e}, \boldsymbol{h}) &= \mathrm{M}(\boldsymbol{e}, \boldsymbol{r})\mathrm{M}(\boldsymbol{r}, \boldsymbol{h}) \\ &= \mathcal{K}^t J_\psi(\mathcal{K}^t)^{-1} \\ &= \mathrm{M}(\boldsymbol{h}, \boldsymbol{r})\mathrm{M}(\boldsymbol{r}, \boldsymbol{e}) \\ &= \mathrm{M}(\boldsymbol{h}, \boldsymbol{e}). \end{aligned}$$

Recalling that $\langle F_\alpha, \boldsymbol{r}_\beta\rangle = \langle \mathrm{For}_\alpha, \boldsymbol{e}_\beta\rangle = \mathbb{1}_{\alpha=\beta}$, (III) along with *(v)* establish *(iii)*:

$$\mathrm{M}(F, \mathrm{For}) = \mathrm{M}(\mathbf{e}, \mathbf{r})^t = (\mathcal{K}^t J_\psi)^t = J_\psi\mathcal{K}.$$

By (II) once again, *(iii)*, and (53), we get *(ii)*:

$$\mathrm{M}(\mathrm{For}, F) = \mathrm{M}(F, \mathrm{For})^{-1} = (J_\psi \mathcal{K})^{-1} = \mathcal{K}^{-1} J_\psi = \varepsilon \mathcal{K} \epsilon J_\psi.$$

Lastly, with (50), (51), and (I) once again, we finally establish *(i)*:

$$\begin{aligned} \mathrm{M}(\mathrm{For}, M) &= \mathrm{M}(\mathrm{For}, F)\mathrm{M}(F, M) \\ &= \mathcal{K}^{-1} J_\psi \mathcal{K} \\ &= \mathrm{M}(M, F)\mathrm{M}(F, \mathrm{For}) \\ &= \mathrm{M}(M, \mathrm{For}), \end{aligned}$$

concluding the proof. □

These provide all of the change-of-basis matrices between all the bases considered thus far, excluding both kinds of power sums in each of QSym and **NSym**. We now aim to generalize the results on the power sums in Λ.

Definition 75. Let

$$\mathcal{L}_\phi = \mathrm{M}(\phi, M); \tag{54}$$

$$\mathcal{L}_\psi = \mathrm{M}(\psi, M), \tag{55}$$

where $\{\phi_\alpha\}$ and $\{\psi_\alpha\}$ are the quasisymmetric power sum bases of the second and first kinds, respectively.

The results of the following lemma can be found in [1], derived from [7], and can be seen from Theorem 57 and duality.

Lemma 76. *In* QSym*, for any α, both $\psi(\boldsymbol{\phi}_\alpha) = \varepsilon(\alpha)\phi_\alpha$ and $\omega(\psi_\alpha) = \varepsilon(\alpha)\psi_{\alpha^r}$.*

The reversals on the bases appearing in the expansions (41) and (48) will force J_ρ to appear in several of the expressions involving the power sums of the first kind we give in the next theorem, adding some complexity that does not appear in the classical case.

Theorem 77. *In* QSym*,*

$$M(\phi, \mathrm{For}) = \varepsilon \mathcal{L}_\phi \quad \text{and} \quad M(\psi, \mathrm{For}) = \varepsilon \mathcal{L}_\psi J_\rho;$$

$$M(\phi, F) = \mathcal{L}_\phi \varepsilon \mathcal{K} \varepsilon \quad \text{and} \quad M(\phi, F) = \mathcal{L}_\phi \varepsilon \mathcal{K} \varepsilon.$$

In ***NSym****,*

$$M(\boldsymbol{h}, \boldsymbol{\phi}) = z^{-1} \mathcal{L}_\phi^t \quad \text{and} \quad M(\boldsymbol{h}, \boldsymbol{\psi}) = z^{-1} \mathcal{L}_\psi^t;$$

$$M(\boldsymbol{e}, \boldsymbol{\phi}) = \varepsilon z^{-1} \mathcal{L}_\phi^t \quad \text{and} \quad M(\boldsymbol{e}, \boldsymbol{\psi}) = \varepsilon z^{-1} J_\rho \mathcal{L}_\psi^t;$$

$$M(\boldsymbol{r}, \boldsymbol{\phi}) = z^{-1} \mathcal{L}_\phi \varepsilon \mathcal{K} \varepsilon \quad \text{and} \quad M(\boldsymbol{r}, \boldsymbol{\psi}) = z^{-1} \mathcal{L}_\psi \varepsilon \mathcal{K} \varepsilon.$$

Proof. By Lemma 57 and (IV),

$$\mathrm{M}(\phi, \mathrm{For}) = \mathrm{M}(\psi(\phi), \psi(\mathrm{For})) = \mathrm{M}(\varepsilon\phi, M) = \varepsilon \cdot \mathrm{M}(\phi, M) = \varepsilon\mathcal{L}_\phi.$$

Similarly, but also with use of (I) and (IV),

$$\begin{aligned}\mathrm{M}(\psi, \mathrm{For}) &= \mathrm{M}(\psi, \omega(\mathrm{For}))\mathrm{M}(\omega(\mathrm{For}), \mathrm{For}) \\ &= \mathrm{M}(\omega(\psi), \omega^2(\mathrm{For}))J_\rho \\ &= \mathrm{M}(\varepsilon\psi, \mathrm{For})J_\rho \\ &= \varepsilon\mathcal{L}_\psi J_\rho.\end{aligned}$$

By (I),

$$\mathrm{M}(\phi, F) = \mathrm{M}(\phi, M)\mathrm{M}(M, F) = \mathcal{L}_\phi\mathcal{K}^{-1} = \mathcal{L}_\phi\varepsilon\mathcal{K}\varepsilon;$$
$$\mathrm{M}(\psi, F) = \mathrm{M}(\psi, M)\mathrm{M}(M, F) = \mathcal{L}_\psi\mathcal{K}^{-1} = \mathcal{L}_\psi\varepsilon\mathcal{K}\varepsilon.$$

Next, by (III),

$$\mathrm{M}(\boldsymbol{h}, \boldsymbol{\phi}) = \mathrm{M}(z^{-1}\phi, M)^t = z^{-1}\mathcal{L}_\phi^t;$$
$$\mathrm{M}(\boldsymbol{h}, \boldsymbol{\psi}) = \mathrm{M}(z^{-1}\psi, M)^t = z^{-1}\mathcal{L}_\psi^t.$$

Similarly,

$$\mathrm{M}(\boldsymbol{e}, \boldsymbol{\phi}) = \mathrm{M}(z^{-1}\phi, \mathrm{For})^t = z^{-1}\varepsilon\mathcal{L}_\phi^t;$$
$$\mathrm{M}(\boldsymbol{e}, \boldsymbol{\psi}) = \mathrm{M}(z^{-1}\psi, \mathrm{For})^t = \varepsilon z^{-1}J_\rho\mathcal{L}_\psi^t.$$

Lastly, by (III) and (I),

$$\mathrm{M}(\boldsymbol{r}, \boldsymbol{\phi}) = \mathrm{M}(z^{-1}\phi, F)^t = z^{-1}(\mathrm{M}(\phi, M)\mathrm{M}(M, F))^t = z^{-1}(\mathcal{K}^t)^{-1}\mathcal{L}_\phi^t = z^{-1}\varepsilon\mathcal{K}^t\varepsilon\mathcal{L}_\phi^t;$$

$$\mathrm{M}(\boldsymbol{r}, \boldsymbol{\psi}) = \mathrm{M}(z^{-1}\psi, F)^t = z^{-1}(\mathrm{M}(\psi, M)\mathrm{M}(M, F))^t = z^{-1}(\mathcal{K}^t)^{-1}\mathcal{L}_\psi^t = z^{-1}\varepsilon\mathcal{K}^t\varepsilon\mathcal{L}_\psi^t. \quad \square$$

Remark. We may also give the change-of-basis relations between the two kinds of power sums in both spaces as matrix products, but they would each involve either the inverse of $\mathcal{L}_\phi$ or $\mathcal{L}_\psi$. For example, in QSym,

$$\mathrm{M}(\phi, \psi) = \mathrm{M}(\phi, M)\mathrm{M}(M, \psi) = \mathcal{L}_\phi\mathcal{L}_\psi^{-1}.$$

We provide a figure similar to Figure 3 (you guessed it, on the next page) depicting the results from this subsection. Just as in Figure 3, an arrow from an element from the basis $\{b_\alpha\}$ to $\{a_\alpha\}$ is labeled with $\mathrm{M}(a, b)$. We suppress several edges, including the repetitive ones to/from the power sums of the two kinds. The hatted terms $\hat{J}_\rho$ along the bottom edges correspond only to the power sum bases of the first kind, ψ and $\boldsymbol{\psi}$.

7 Combinatorial Models for Change of Basis

Many of the statistics occurring in the change-of-basis matrices in **NSym** are natural generalizations of the statistics on brick tabloids found in [2, 6], which give combinatorial descriptions of the transition matrices in Λ. In this section we look at generalizing this work to change of basis in QSym and **NSym**.

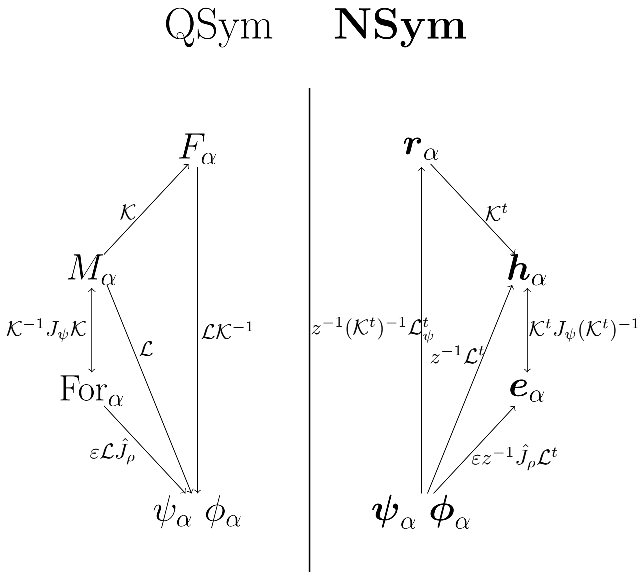


Figure 4: Matrix Change-of-Basis Expressions in QSym and **NSym**

7.1 Combinatorial Models in Λ

Before we summarize the results of Eğecioğlu and Remmel [6], we note that we have taken the liberty of adjusting some of the authors' notation and conventions. As examples, they choose integer partitions to be written in weakly *increasing* order, and their change-of-basis matrices multiply on the right of row vectors; both of these conventions are opposite of the presentation in this work. Our goal is to generalize the following theorem, which gives a unified combinatorial model for most of the change-of-basis matrices in Λ discussed above. Let n be a positive integer.

Theorem 78 (Eğecioğlu and Remmel, [6])**.** *For $\lambda \vdash n$,*

$$e_\lambda = \sum_{T \in B_\lambda} (-1)^{|\lambda| - \ell(\mathrm{type}(T))} h_{\mathrm{type(T)}} \tag{56}$$

$$h_\lambda = \sum_{T \in B_\lambda} (-1)^{|\lambda| - \ell(\mathrm{type}(T))} e_{\mathrm{type}(T)} \tag{57}$$

$$m_\lambda = \sum_{T \in B^\lambda} (-1)^{|\,\mathrm{shape}(T)| - \ell(\lambda)} f_{\mathrm{shape}(T)} \tag{58}$$

$$f_\lambda = \sum_{T \in B^\lambda} (-1)^{|\,\mathrm{shape}(T)| - \ell(\lambda)} m_{\mathrm{shape}(T)} \tag{59}$$

$$p_\lambda = \sum_{T \in B_\lambda} (-1)^{|\lambda| - \ell(\mathrm{type}(T))} w(B_\lambda^{\mathrm{type}(T)}) e_{\mathrm{type}(T)} \tag{60}$$

$$p_\lambda = \sum_{T\in B_\lambda} (-1)^{\ell(\lambda)-\ell(\text{type}(T))} w(B_\lambda^{\text{type}(T)}) h_{\text{type}(T)} \tag{61}$$

$$f_\lambda = \sum_{T\in B^\lambda} (-1)^{|\lambda|-\ell(\text{type}(T))} \frac{w(B^\lambda_{\text{shape}(T)})}{z_{\text{shape}(T)}} p_{\text{shape}(T)} \tag{62}$$

$$m_\lambda = \sum_{T\in B_\lambda} (-1)^{\ell(\lambda)-\ell(\text{type}(T))} \frac{w(B_\lambda^{\text{type}(T)})}{z_{\text{type}(T)}} p_{\text{type}(T)} \tag{63}$$

$$p_\lambda = \sum_{T\in B^\lambda} |OB^\lambda_{\text{shape}(T)}| m_{\text{shape}(T)} \tag{64}$$

$$p_\lambda = \sum_{T\in B^\lambda} (-1)^{|\,\text{type}(T)|-\ell(\lambda)} |OB^\lambda_{\text{shape}(T)}| f_{\text{shape}(T)} \tag{65}$$

$$h_\lambda = \sum_{T\in B_\lambda} \frac{|OB_\lambda^{\text{type}(T)}|}{z_{\text{type}(T)}} p_{\text{type(T)}} \tag{66}$$

$$e_\lambda = \sum_{T\in B_\lambda} (-1)^{|\lambda|-\ell(\text{type}(T))} \frac{|OB_\lambda^{\text{type}(T)}|}{z_{\text{type}(T)}} p_{\text{type(T)}} \tag{67}$$

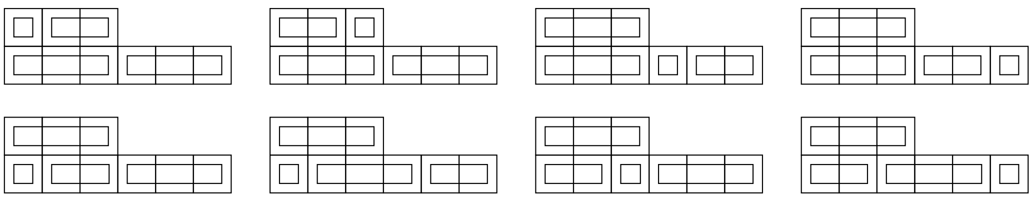
To understand the theorem, the following definitions are necessary.

Definition 79 (Eğecioğlu and Remmel [6]). A **brick** b of length k is a horizontal strip of k boxes (a Young diagram of shape (k)). If $\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}) \vdash n$, **associate the set of bricks $\{b_1, b_2, \ldots, b_{\ell(\mu)}\}$ with μ** if brick $|b_i| = \mu_i$ for each $i \in [\ell(\mu)]$. Then, T is a **μ-brick tabloid of shape λ** if T gives a filling of the Young diagram of shape $\lambda \vdash n$ with the set of bricks associated to μ such that
(i) each brick b_i covers exactly μ_i boxes in a single row of the diagram of shape λ;
(ii) no two bricks overlap.

Say that B_λ is the set of all possible brick tabloids of shape λ, B^μ is the set of all possible μ-brick tabloids (with **type** μ), and let B^μ_λ denote the set of all μ-brick tabloids of shape λ.

Note. In the above definition, it is important to note that bricks of the same size are indistinguishable.

Example 80. Below are the eight $(3,3,2,1)$-brick tabloids of shape $(6,3)$.



Note the 8 in parentheses in the equation below, attained from (56) with $\lambda = (6,3)$:

$$\mathrm{M}(e,h)_{(3,3,2,1),(6,3)} = (-1)^{|(6,3)|-\ell((3,3,2,1))} |B^{(3,3,2,1)}_{(6,3)}| = (-1)^{9-4}(8) = -8.$$

Definition 81. Define a weight function on brick tabloids, $wt : T \to \mathbb{Z}^+$, as follows. Let $\mu, \lambda \vdash n$ and let T be any μ-brick tabloid of shape λ. If $B(T) = \{b_i\}_{i \in [\ell(\mu)]}$ is the set of bricks associated to T, let $B_r(T) \subseteq B(T)$ be the subset of $\ell(\lambda)$ bricks that appear at the rightmost ends of the rows in T. Then, the **weight** of the brick tabloid T is

$$wt(T) = \prod_{b \in B_r(T)} |b|.$$

The weight of the entire set of μ-brick tabloids of shape λ is the sum of their weights,

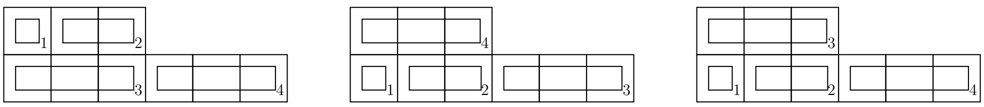
$$w(B^{\mu}_{\lambda}) = \sum_{T \in B^{\mu}_{\lambda}} wt(T).$$

Example 82. The brick tabloids from Example 80, in reading order, have weights $6, 3, 6, 3, 9, 6, 9$, and 3, respectively. Thus $w(B^{(3,3,2,1)}_{(6,3)}) = 45$. From (60), we have

$$\mathrm{M}(p, e)_{(3,3,2,1),(6,3)} = (-1)^{|(6,3)| - \ell((3,3,2,1))} |B^{(3,3,2,1)}_{(6,3)}| = (-1)^{9-4}(45) = -45.$$

Definition 83. Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)})$ and associated set of bricks $\{b_i\}_{i \in [\ell(\mu)]}$ ($|b_i| = \mu_i$), index each brick b_i with the subscript $\ell(\mu) - i + 1$, $i \in [\ell(\mu)]$. That is, index the bricks from smallest to largest with the integers $1, 2, \ldots, \ell(\mu)$. Then, an **ordered $\boldsymbol{\mu}$-brick tabloid of shape $\boldsymbol{\lambda}$** is a μ-brick tabloid of shape λ filled with associated *indexed* bricks such that in each row, the subscripts on the bricks increase from left-to-right. Denote the set of all μ-ordered brick tabloids of shape λ by OB^{μ}_{λ}.

Example 84. Below are the three $(3, 3, 2, 1)$-ordered brick tabloids of shape $(6, 3)$.



Applying (64), we have

$$\mathrm{M}(p, m)_{(6,3),(3,3,2,1)} = |OB^{(3,3,2,1)}_{(6,3)}| = 3.$$

7.2 Brick Walls and Change of Basis in QSym and NSym.

This section generalizes the concepts in Eğecioğlu and Remmel [6] to QSym and **NSym**, where the statistics from the change-of-basis equations in Gelfand et al. [7] are often very natural extensions of the original statistics on brick tabloids. These first four equations easily generalize the first four equations in Theorem 78. Let n be a positive integer.

Theorem 85. *For $\alpha \models n$,*

$$\boldsymbol{e}_{\alpha} = \sum_{W \in \mathcal{W}_{\alpha}} (-1)^{\ell(\mathrm{type}(W)) - |W|} \boldsymbol{h}_{\mathrm{type}(W)} \tag{68}$$

$$\boldsymbol{h}_\alpha = \sum_{W \in \mathcal{W}_\alpha} (-1)^{\ell(\mathrm{type}(W)) - |W|} \boldsymbol{e}_{\mathrm{type}(W)} \tag{69}$$

$$\mathrm{For}_\alpha = \sum_{W \in \mathcal{W}^\alpha} (-1)^{\ell(\mathrm{type}(W)) - |W|} M_{\mathrm{sh}(W)} \tag{70}$$

$$M_\alpha = \sum_{W \in \mathcal{W}^\alpha} (-1)^{\ell(\mathrm{type}(W)) - |W|} \mathrm{For}_{\mathrm{sh}(W)} \tag{71}$$

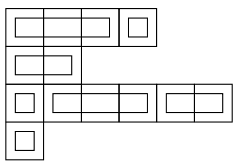
We need the following generalizations of brick tabloids, referred to here as walls, which are simpler in the this case.

Definition 86. If $\beta \preceq \alpha$ with $\beta^{(i)} \models \alpha_i$, for all $i \in [\ell(\alpha)]$, let the *ordered* set of bricks $B = (b_1, \ldots, b_{\ell(\beta)})$ be associated to β (where $|b_j| = \beta_j$ for $j \in [\ell(\beta)]$). Then, the (unique) **$\boldsymbol{\beta}$-wall of shape $\boldsymbol{\alpha}$**, or **$\boldsymbol{\alpha\beta}$-wall**, is the filling of the bricks from B into the Young diagram of shape α in order from left-to-right, bottom-up (adopting French notation).

It is clear that the $\alpha\beta$-wall exists if and only if $\beta \preceq \alpha$. (See Definition 9.) For some integers $0 < j_1 < \cdots < j_{\ell(\alpha)} = \ell(\beta)$, the equations below correspond to courses in the $\alpha\beta$-wall. (A "course" is a continuous horizontal stretch of bricks (or stone) laid to build a wall.)

$$\begin{aligned}
\alpha_{\ell(\alpha)} &= \beta_{j_{\ell(\alpha)-1}+1} + \beta_{j_{\ell(\alpha)-1}+2} + \cdots + \beta_{\ell(\beta)};\\
&\vdots\\
\alpha_2 &= \beta_{j_1+1} + \beta_{j_1+2} + \cdots + \beta_{j_2};\\
\alpha_1 &= \beta_1 + \beta_2 + \cdots + \beta_{j_1}.
\end{aligned}$$

Example 87. Let $\alpha = (1,6,2,4)$ and let $\beta = (1,1,3,2,2,3,1)$. Then $\beta \preceq \alpha$, so the $(1,6,2,4)(1,1,3,2,2,3,1)$-wall exists, and is shown below.



Definition 88. Say the $\alpha\beta$-wall has **shape** $\mathrm{sh}(W) = \alpha$, **size** $|W| = |\alpha|$, and **type** $\mathrm{type}(W) = \beta$. For a fixed composition α, let $\mathcal{W}_\alpha$ denote the set of all walls of shape α. Similarly, for composition β, let $\mathcal{W}^\beta$ be the set of all walls of type β. Clearly there are one-to-one correspondences between walls and compositions $\mathcal{W}_\alpha \leftrightarrow \{\beta \mid \beta \preceq \alpha\}$ and $\mathcal{W}^\beta \leftrightarrow \{\alpha \mid \alpha \succeq \beta\}$.

Although Eğecioğlu and Remmel do not mention the Schur functions in their work, the corresponding bases $\{F_\alpha\}_{\alpha \models n}$ and $\{\boldsymbol{r}_\alpha\}_{\alpha \models n}$ fit nicely here into the same framework:

Theorem 89. *For $\alpha \models n$,*

$$\boldsymbol{h}_\alpha = \sum_{W \in \mathcal{W}^\alpha} \boldsymbol{r}_{\mathrm{sh}(W)} \tag{72}$$

$$\boldsymbol{r}_\alpha = \sum_{W \in \mathcal{W}^\alpha} (-1)^{\ell(\mathrm{sh}(W))-\ell(\mathrm{type}(W))} \boldsymbol{h}_{\mathrm{sh}(W)} \tag{73}$$

$$\boldsymbol{e}_\alpha = \sum_{W \in \mathcal{W}^\alpha} \boldsymbol{r}_{\mathrm{sh}(W)^c} \tag{74}$$

$$\boldsymbol{r}_\alpha = \sum_{W \in \mathcal{W}^{\alpha^c}} (-1)^{\ell(\mathrm{sh}(W))-\ell(\mathrm{type}(W))} \boldsymbol{e}_{\mathrm{sh}(W)} \tag{75}$$

$$F_\alpha = \sum_{W \in \mathcal{W}_\alpha} M_{\mathrm{type}(W)} \tag{76}$$

$$M_\alpha = \sum_{W \in \mathcal{W}_\alpha} (-1)^{\ell(\mathrm{type}(W))-\ell(\mathrm{sh}(W))} F_{\mathrm{type}(W)} \tag{77}$$

$$F_\alpha = \sum_{W \in \mathcal{W}_{\alpha^c}} \mathrm{For}_{\mathrm{type}(W)} \tag{78}$$

$$\mathrm{For}_\alpha = \sum_{W \in \mathcal{W}_\alpha} (-1)^{\ell(\mathrm{type}(W))-\ell(\mathrm{sh}(W))} F_{\mathrm{type}(W)^c} \tag{79}$$

In order to establish combinatorial versions of their other equations, we must define several more statistics on walls. Three of them, below, are imitations of the weight function defined in [6] utilizing (32) (taken from [7]).

Definition 90. If $\beta \preceq \alpha$ with $\beta^{(i)} \models \alpha_i$ for all $i \in [\ell(\alpha)]$, and W is the $\alpha\beta$-wall, say the **last parts product** and **first parts product** of W are

$$\mathrm{lp}(W) \;=\; \mathrm{lp}(\beta, \alpha) = \prod_{i=1}^{\ell(\alpha)} \beta_{j_i}; \tag{80}$$

$$\mathrm{fp}(W) \;=\; \mathrm{lp}(\beta^r, \alpha^r) = \prod_{i=0}^{\ell(\alpha)-1} \beta_{j_i+1}. \tag{81}$$

Thus the statistic $\mathrm{lp}(W)$ (respectively $\mathrm{fp}(W)$) gives the product of the sizes of the bricks at the right (respectively left) ends of the rows in W.

A less obvious replacement for the weight function in this context is required to cover the power sums of the second kind.

Definition 91. If W is an α, β-wall, let $\mathrm{pb}(W)$ give the product of the number of bricks in each row (or course).

Example 92. For the wall W in Example 87, $\mathrm{pb}(W) = 1 \cdot 3 \cdot 1 \cdot 2 = 6$.

Theorem 93. *For $\alpha \models n$,*

$$\boldsymbol{\psi}_\alpha \;=\; \sum_{W \in \mathcal{W}_\alpha} (-1)^{\ell(\mathrm{type}(W))-\ell(\mathrm{sh}(W))}\, \mathrm{lp}(W) \boldsymbol{h}_{\mathrm{type}(W)}$$

$$\boldsymbol{\psi}_\alpha \;=\; \sum_{W \in \mathcal{W}_\alpha} (-1)^{\ell(\mathrm{type}(W))-\ell(\mathrm{sh}(W))}\, \mathrm{fp}(W) \boldsymbol{e}_{\mathrm{type}(W)}$$

$$
\begin{aligned}
M_\alpha &= \sum_{W \in \mathcal{W}^\alpha} (-1)^{\ell(\mathrm{type}(W)) - \ell(\mathrm{sh}(W))} \, \mathrm{lp}(W) \frac{\psi_{\mathrm{sh}(W)}}{z_{\mathrm{sh}(W)}} \\
\mathrm{For}_\alpha &= \sum_{W \in \mathcal{W}^\alpha} (-1)^{\ell(\mathrm{type}(W)) - \ell(\mathrm{sh}(W))} \, \mathrm{fp}(W) \frac{\psi_{\mathrm{sh}(W)}}{z_{\mathrm{sh}(W)}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\phi}_\alpha &= \sum_{W \in \mathcal{W}_\alpha} (-1)^{\ell(\mathrm{type}(W)) - \ell(\mathrm{sh}(W))} \frac{\prod \mathrm{sh}(W)}{\mathrm{pb}(W)} \boldsymbol{h}_{\mathrm{type}(W)} \\
\boldsymbol{\phi}_\alpha &= \sum_{W \in \mathcal{W}_\alpha} (-1)^{|\mathrm{sh}(W)| - \ell(\mathrm{type}(W))} \frac{\prod \mathrm{sh}(W)}{\mathrm{pb}(W)} \boldsymbol{e}_{\mathrm{type}(W)} \\
M_\alpha &= \sum_{W \in \mathcal{W}^\alpha} (-1)^{\ell(\mathrm{type}(W)) - \ell(\mathrm{sh}(W))} \frac{\prod \mathrm{sh}(W)}{\mathrm{pb}(W)} \frac{\phi_{\mathrm{sh}(W)}}{z_{\mathrm{sh}(W)}} \\
\mathrm{For}_\alpha &= \sum_{W \in \mathcal{W}^\alpha} (-1)^{|\mathrm{sh}(W)| - \ell(\mathrm{type}(W))} \frac{\prod \mathrm{sh}(W)}{\mathrm{pb}(W)} \frac{\phi_{\mathrm{sh}(W)}}{z_{\mathrm{sh}(W)}}
\end{aligned}
$$

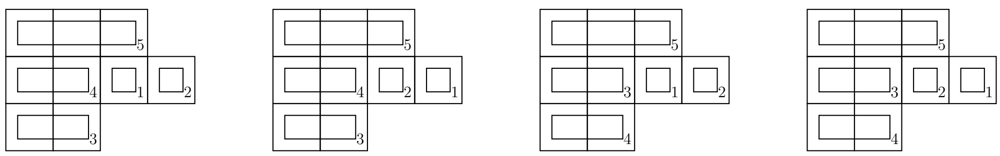
Proof. These simply give combinatorial translations of the right-hand equations in the last four lines in both Theorem 68 and Corollary 72. □

Next, we imitate the alteration made on brick tabloids to give ordered brick tabloids in [6]. Before we give the definition, we remark that there are always (weakly) fewer μ-ordered brick tabloids of shape λ than there are brick tabloids of the same shape and type. (Consult Examples 80 and 84.) The opposite will be true of our analogous objects, next.

Definition 94. Let $\beta \preceq \alpha$ and let W be the $\alpha\beta$-wall. Then, a ***β*-indexed wall of shape *α***, or **indexed *αβ*-wall**, is an indexing of the bricks in W (associated to β) in order of increasing size with the integers from $[\ell(\beta)]$.

Thus, in an indexed wall, bricks of the same size are distinguishable.

Example 95. There are four indexed $(2,4,3)(2,2,1,1,3)$-walls, shown below.



Note that there is no alteration on the order in which the bricks associated to β are laid to build (an indexed) $\alpha\beta$-wall, unlike in the case of ordered brick tabloids. (Compare with Examples 80 and 84 once more).

Definition 96. Let $\mathcal{IW}_\alpha$ denote the set of all ordered walls of shape α, let $\mathcal{IW}^\beta$ denote the set of all ordered walls of type β, and let $\mathcal{IW}_\alpha^\beta$ be the set of indexed $\alpha\beta$-walls.

Theorem 97. *For any fixed strong compositions $\beta \preceq \alpha$,*

$$|\mathcal{IW}_\alpha^\beta| = m_1(\beta)!m_2(\beta)! \cdots m_n(\beta)!.$$

Proof. Since the bricks in the $\alpha\beta$-wall must be indexed in order of increasing size, there are $m_i(\beta)!$ ways to index the bricks of size i for each $i \in [n]$. □

Definition 98. If W is an indexed wall of shape α and type β, let $\mathrm{fb}(W)$ give the product of the factorial of the number of bricks in each row.

$$\mathrm{fb}(W) = \prod_{i=1}^{\ell(\alpha)} \ell(\beta^{(i)})!$$

Example 99. For the wall W in Example 87, $\mathrm{fb}(W) = 1! \cdot 3! \cdot 1! \cdot 2! = 12$.

Theorem 100. *For $\alpha \models n$,*

$$\begin{aligned}
\boldsymbol{h}_\alpha &= \sum_{W \in \mathcal{IW}_\alpha} \frac{1}{\mathrm{fb}(W)} \frac{\boldsymbol{\phi}_{\mathrm{type}(W)}}{z_{\mathrm{type}(W)}} \\
\boldsymbol{e}_\alpha &= \sum_{W \in \mathcal{IW}_\alpha} \frac{(-1)^{|\,\mathrm{sh}(W)| - \ell(\mathrm{type}(W))}}{\mathrm{fb}(W)} \frac{\boldsymbol{\phi}_{\mathrm{type}(W)}}{z_{\mathrm{type}(W)}} \\
\phi_\alpha &= \sum_{W \in \mathcal{IW}^\alpha} \frac{1}{\mathrm{fb}(W)} M_{\mathrm{sh}(W)} \qquad (82) \\
\phi_\alpha &= \sum_{W \in \mathcal{IW}^\alpha} \frac{(-1)^{|\,\mathrm{sh}(W)| - \ell(\mathrm{type}(W))}}{\mathrm{fb}(W)} \mathrm{For}_{\mathrm{sh}(W)}
\end{aligned}$$

The (incredibly) attentive reader will notice that to this point we have not given combinatorial interpretations for a number of change-of-basis equations involving $\{\psi_\alpha\}$. While it is possible to give such an interpretation, there does not appear to be an analogue of indexed walls that is natural and simplifies their presentation from the original in [7], so we omit them.

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