On Commutative Association Schemes and Associated (Directed) Graphs

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Abstract

Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X}$ (not necessarily symmetric), and Γ denote a (strongly) connected (directed) graph with adjacency matrix A. Under the assumption that A belongs to $\mathcal{M}$, we describe the combinatorial structure of Γ. Moreover, we provide an algebraic-combinatorial characterization of Γ when A generates $\mathcal{M}$.

Among else, we show that, if $\mathfrak{X}$ is a commutative 3-class association scheme that is not an amorphic symmetric scheme, then we can always find a (directed) graph Γ such that the adjacency matrix A of Γ generates the Bose–Mesner algebra $\mathcal{M}$ of $\mathfrak{X}$.

Mathematics Subject Classifications: 05E30, 05C75, 05C50.

1 Introduction

In this paper, we study connections between commutative association schemes and (directed) graphs, by considering the following question: when can a commutative association scheme be generated by a (directed) graph? Formal definitions are given in Section 2.

Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$ (note that $\mathcal{M}$ does not need to be symmetric). To give a motivation and an introduction to our problem, in the next few lines, we first show that $\mathcal{M}$ is a monogenic algebra, that is, we show that there always exists a matrix $A \in \mathrm{Mat}_X(\mathbb{C})$ which generates $\mathcal{M}$, i.e., $\mathcal{M} = (\langle A \rangle, +, \cdot)$ (we say that a matrix A *generates* $\mathcal{M}$ if every element in $\mathcal{M}$ can be written as a polynomial in A). Since $\mathcal{M}$ is a space of commutative normal matrices, from a well-known result on commutative sets of normal matrices, there exists a unitary matrix $U \in \mathrm{Mat}_X(\mathbb{C})$ which diagonalizes $\mathcal{M}$: to each $B \in \mathcal{M}$ there corresponds a diagonal matrix $\Lambda \in \mathrm{Mat}_X(\mathbb{C})$ such that $B = U\Lambda\overline{U}^{\top}$, and the diagonal entries of Λ are

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the eigenvalues of B. When B runs through $\mathcal{M}$, the matrix Λ runs through a subalgebra $\mathcal{F}$ of $\mathrm{Mat}_X(\mathbb{C})$ which is isomorphic to $\mathcal{M}$. An explicit isomorphism $\psi : \mathcal{M} \to \mathcal{F}$ is given by $\psi(B) = \overline{U}^\top BU$. Since $\mathcal{M}$ is an algebra of dimension $d+1$, the algebra $\mathcal{F}$ is also of dimension $d+1$. Moreover, there exists a set of diagonal 01-matrices $\{F_i\}_{0\leqslant i\leqslant d}$ which is a basis of $\mathcal{F}$. Then, for arbitrary non-zero pairwise distinct complex scalars α_i $(0 \leqslant i \leqslant d)$, the matrix $F = \sum_{i=0}^{d} \alpha_i F_i$ generates $\mathcal{F}$, i.e., $\mathcal{F} = (\langle F\rangle, +, \cdot)$. Thus, the matrix $A = \psi^{-1}(F)$ generates $\mathcal{M}$ and has $d+1$ distinct eigenvalues. A reader more familiar with the field of association schemes will notice that our claim on the existence of a generator A of $\mathcal{M}$ also follows from the proof of [17, Theorem 2.2]. For a different approach in proving that the Bose–Mesner algebra $\mathcal{M}$ of an arbitrary commutative d-class association scheme $\mathfrak{X}$ (which is not necessarily symmetric) can be generated by A, see Lemma 15 in Subsection 2.5. In this paper, we are interested in the following problem.

Problem 1. When can the Bose–Mesner algebra $\mathcal{M}$ of a commutative d-class association scheme $\mathfrak{X}$ (which is not necessarily symmetric) be generated by a 01-matrix A? In other words, for a given $\mathfrak{X}$, under which combinatorial and algebraic restrictions can we find a 01-matrix A such that $\mathcal{M} = (\langle A\rangle, +, \cdot)$? Moreover, since such a matrix A is the adjacency matrix of some (directed) graph Γ, can we describe the combinatorial structure of Γ? The vice-versa question is also of importance, i.e., what combinatorial structure does a (directed) graph need to have so that its adjacency matrix will generate the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X}$?

In the notation of Ito's paper [30], we are interested in the combinatorial structure of *polynomial* association scheme. The case when $\mathfrak{X}$ is a symmetric association scheme, our problem is connected with quotient-polynomial graphs (undirected graphs which generate symmetric association schemes, see [20, 22]). Recall that in [20] Fiol defined a quotient-polynomial graph a little bit differently, that is, the author defined the quotient-polynomial graph as a graph Γ with vertex set X for which the adjacency matrices of a walk-regular partition of $X \times X$ belong to the adjacency algebra of Γ. Then, the author described some algebraic properties of such graphs and proved that Γ is the connected generating graph of an association scheme $\mathfrak{X}$ if and only if Γ is a quotient-polynomial graph. Following these results, in [22] Fiol and Penjić revisited this topic from another point of view, finding some additional algebraic properties as well as describing the combinatorial structure of quotient-polynomial graphs. In both cited papers [20, 22], the authors studied the case of undirected graphs and with it the case of a symmetric (adjacency) algebra. In this paper, we study commutative association schemes (not necessarily symmetric) and, as a by-product, we also get some interesting results for symmetric association schemes. More precisely we answer the following question: *Is it possible that every symmetric association scheme is generated by some (quotient-polynomial) graph?* (The answer for a 3-class association scheme is given in Theorem 2.)

In the case when $\mathfrak{X}$ is a symmetric 3-class association scheme, by the result of Van Dam in [14, Theorem 5.1] together with our Lemma 15, we get partial answers to questions posted in Problem 1. For the moment, let Γ denote a connected regular graph with 4 distinct eigenvalues and adjacency matrix A. In [14, Theorem 5.1] Van Dam proved that A is one of the adjacency matrices of a 3-class association scheme if and only if two adjacent vertices have a constant number of common neighbors, and the number of common neighbors of any two nonadjacent vertices takes precisely two values. In the same paper [14], the author gave some answers about when and how to use the combinatorial structure of strongly-regular graphs to obtain a 3-class association scheme (see, for example, [14, Proposition 5.2]). In this paper, we fully describe when a 3-class association scheme (not necessarily symmetric) can be generated by a graph.

We are interested in the natural problem of describing the combinatorial structure and algebraic properties of (directed) graphs which will generate commutative association schemes. This provides us with a different approach in finding new association schemes, using the structure of (directed) graphs. For example, if we pick some well-known family of undirected graphs, and give them an orientation on the edges that satisfy some of (if not all) the properties described in this paper, will we get a candidate which generates a commutative association scheme? This paper gives some answers to this question too.

We say that a (directed) graph Γ *generates* a commutative association scheme $\mathfrak{X}$ if and only if the adjacency matrix A of Γ generates the Bose–Mesner algebra $\mathcal{M}$ of $\mathfrak{X}$, and in symbols we write $\mathcal{M} = (\langle A\rangle, +, \cdot)$. Our main results are Theorems 2, 3 and 4.

In Theorem 2, we characterize 3-class amorphic symmetric schemes as the only commutative 3-class association schemes which fails to satisfy the "single-01-matrix generator" property of Problem 1. In other words, except for amorphic symmetric association schemes, every 3-class association scheme can be generated by a 01-matrix A (by a (strongly) connected (directed) graph $\Gamma = \Gamma(A)$ which has 4 distinct eigenvalues).

Theorem 2. *Let $\mathfrak{X}$ denote a 3-class commutative association scheme. If $\mathfrak{X}$ is not an amorphic symmetric scheme, then there exists a (strongly) connected (directed) graph $\Gamma = \Gamma(A)$ such that the following hold.*

(i) *The adjacency matrix A of Γ has exactly 4 distinct eigenvalues.*

(ii) *The adjacency matrix A generates the Bose–Mesner algebra $\mathcal{M}$ of $\mathfrak{X}$.*

Moreover, the scheme $\mathfrak{X}$ is generated by a graph if and only if it is not an amorphic symmetric scheme.

In Theorem 3, we describe the combinatorial structure of a graph which 'lives' in a commutative association scheme. We can say that Corollary 34 of Theorem 3 is in some sense a more general version of the result of Van Dam given in [14, Theorem 5.1] as it includes also non-symmetric commutative 3-class association schemes.

Theorem 3. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$ with adjacency matrices $\{A_i\}_{i=0}^d$ and intersection numbers p^h_{ij} $(0 \leqslant h, i, j \leqslant d)$. Let A denote an arbitrary 01-matrix in $\mathcal{M}$, and consider the (directed) graph $\Gamma = \Gamma(A) = (X, \mathcal{E})$. If Γ is a (strongly) connected (directed) graph, then the following (i)–(iii) hold.*

(i) *For every vertex $x \in X$, the partition $\Pi_x = \left\{\mathcal{P}_i(x) = \{z \mid (A_i)_{xz} = 1\}\right\}_{i=0}^d$ of X is an x-distance-faithful equitable partition with $d+1$ cells.*

(ii) *The structure of the x-distance-faithful intersection diagram of the equitable partition Π_x from* (i) *does not depend on x (i.e., the corresponding parameters of Π_x do not depend on the choice of $x \in X$).*

(iii) *Since $A \in \mathcal{M}$, there exists some nonempty index set Φ such that $A = \sum_{m\in\Phi} A_m$. Then for the equitable partition $\Pi_x = \{\mathcal{P}_0(x), \mathcal{P}_1(x), \ldots, \mathcal{P}_d(x)\}$ from* (i)*, the following holds.*

(a) *For any $z \in \mathcal{P}_i(x)$, the number of vertices in the cell $\mathcal{P}_j(x)$ which are adjacent from the vertex z is equal to $\sum_{m\in\Phi} p^{i^*}_{mj^*}$, or in symbols $|\Gamma_1^{\rightarrow}(z) \cap \mathcal{P}_j(x)| = \sum_{m\in\Phi} p^{i^*}_{mj^*}$, where $\Gamma_1^{\rightarrow}(z) = \{w \mid (z,w) \in \mathcal{E}(\Gamma)\}$, $A_{i^*} = A_i^{\top}$ and $A_{j^*} = A_i^{\top}$.*

(b) *For any $z \in \mathcal{P}_i(x)$, the number of vertices in the cell $\mathcal{P}_j(x)$ which are adjacent to the vertex z is equal to $\sum_{m\in\Phi} p^{i}_{mj}$, or in symbols $|\Gamma_1^{\leftarrow}(z) \cap \mathcal{P}_j(x)| = \sum_{m\in\Phi} p^{i}_{mj}$, where $\Gamma_1^{\leftarrow}(z) = \{w \mid (w,z) \in \mathcal{E}(\Gamma)\}$.*

Theorem 3 is in some sense an extension of the result of Fiol and Penjić given in [22, Theorem 4.1] as it extends to the family of directed graphs; moreover, we described connection between the number of neighbours of an arbitrary vertex and the intersection numbers of a commutative association scheme. With reference to Theorem 3, in Corollary 33, we consider the case when Γ generates symmetric association scheme.

At first glance it seems like the result of Theorem 3 is already known from literature. We did not manage to find something similar explicitly (or implicitly) written in literature. For the case when A is an adjacency matrix A_m (of relation R_m) of an association scheme $\mathfrak{X}$, without any understanding of the structure of a (directed) graph $\Gamma = \Gamma(A_m) = (X, R_m)$, an equitable partition can be derived from definition of $\mathfrak{X}$ (see, for example, [26, Example 2.3]). In Theorem 4, we give one of its applications by characterizing algebraic-combinatorial properties of Γ when Γ generates a commutative association scheme, with one specific restriction. In the period of writing and finalizing this paper, we have also managed to discover two other interesting applications of Theorem 3, see [38, 39].

Theorem 4. *Let $\Gamma = \Gamma(A)$ denote a (strongly) connected (directed) graph with vertex set X, adjacency matrix A, $d+1$ distinct eigenvalues, and adjacency algebra $\mathcal{A} = \mathcal{A}(\Gamma)$. Let*

$$\Delta = \{(i,j) \mid i = \partial(x,y),\ j = \partial(y,x),\ x,y \in X\}$$

(or $\Delta = \{\widetilde{\partial}(x,y) \mid x,y \in X\}$ where $\widetilde{\partial}(x,y) = (\partial(x,y), \partial(y,x))$ denotes the two way distance in Γ). For any $\boldsymbol{i} \in \Delta$ define $R_{\boldsymbol{i}} = \{(x,y) \in X \times X \mid (\partial(x,y), \partial(y,x)) = \boldsymbol{i}\}$ (or, in the notation of the two way distance, for any $\widetilde{i} \in \Delta$ define $R_{\widetilde{i}} = \{(x,y) \in X \times X \mid \widetilde{\partial}(x,y) = \widetilde{i}\}$). If $|\Delta| = d+1$, then the following are equivalent.

(i) *$\mathcal{A}$ is the Bose–Mesner algebra of a commutative d-class association scheme.*

(ii) *$(X, \{R_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\Delta})$ is a commutative d-class association scheme (i.e., Γ is a weakly distance-regular digraph in sense of* Wang *and* Suzuki [48]*).*

(iii) *A is a normal matrix and the number of walks from x to y of every given length $\ell \geqslant 0$ only depends on the distances $\partial(x,y)$ and $\partial(y,x)$ (and does not depend on the choice of the pair (x,y)) (i.e., Γ is a weakly distance-regular digraph in sense of* Comellas et al. [11]*).*

In Theorem 4, we give connections among weakly distance-regular graphs in sense of Comellas et al. [11], weakly distance-regular graphs in sense of Wang and Suzuki [48], and commutative association schemes generated by a 01-matrix A. We explain these two new notations (and their importance) in the next two paragraphs.

A directed graph Γ (of diameter D) is *weakly distance-regular in sense of* Comellas et al. [11] if the number of walks of length ℓ $(0 \leqslant \ell \leqslant D)$ in Γ between two vertices $x, y \in X$ only depends on $h = \partial(x,y)$ (on the distance from x to y). In [11, Theorem 2.2], Comellas et al. provided an algebraic-combinatorial characterization of such graphs. Moreover, in the same paper, the authors proved an equivalence between (i) A is a normal matrix and the set of distance-i matrices $\{A_0, A_1, \ldots, A_D\}$ is a basis of the adjacency algebra $\mathcal{A}$ of Γ; and (ii) there exist numbers b_{ij} $(0 \leqslant i,j \leqslant D)$ such that $|\Gamma_1^{\rightarrow}(y) \cap \Gamma_j^{\rightarrow}(x)| = b_{ij}$, for all $x \in X$, $y \in \Gamma_i^{\rightarrow}(x)$ $(0 \leqslant i,j \leqslant D)$ (see [11, Proposition 2.6]). Note the similarity and difference between our Theorem 4(iii) and the notion of weakly distance-regularity in sense of Comellas et al. [11]. Among else, Comellas et al. [11] studied the spectra of a weakly distance-regular digraph and constructed several examples of such a graph. Our property (iii) of Theorem 4 is restricted property of an open (up to our knowledge) research problem from [11, Subsection 4.3]. Some papers that are related with weakly distance-regular digraphs (in terms of number of walks of certain type) are [12, 16, 25, 34]. We recommend papers [21, 40] for the study of spectrum of a (weakly distance-regular) directed graph.

For the moment, let Γ denote a directed graph with vertex set X, and consider the set Δ and relations $R_{\boldsymbol{i}}$ $(\boldsymbol{i} \in \Delta)$ from Theorem 4. A directed graph Γ is said to be *weakly distance-regular in sense of* Wang *and* Suzuki [48] if and only if $(X, \{R_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \Delta})$ is a $|\Delta|$-class association scheme (for further insights regarding this definition, see paper of Suzuki [43]). In such a case, $\mathfrak{X}(\Gamma)$ is called the *attached scheme* of Γ. In [33, 43, 45, 51, 52], some special families of weakly distance-regular digraphs in sense of Wang and Suzuki of small valency have been classified. Algebraic restrictions on weakly distance-regular digraphs called *thin*, *quasi-thin*, and *thick* were studied in [43, 53, 54]. Other papers that are directly or indirectly involved in the study of weakly distance-regular digraphs in sense of Wang and Suzuki are, for example, [18, 19, 56, 35, 46, 47, 55].

Our paper is organized as follows. In Section 2, we recall basic concepts from algebraic graph theory; in particular Subsection 2.5 is a survey of all well-known properties that we use later in the paper: in this section we explain when a 01-matrix generates the Bose–Mesner algebra of a scheme, and we explicitly (re)prove some results about the adjacency algebra of a (directed) graph, hidden in literature. Our paper then starts from Section 3. In Section 3, we prove Theorem 2. In Section 4, we prove Theorem 3, and we include several interesting corollaries of the claim. In Section 5, we prove Theorem 4, among else by using the combinatorial structure of a (directed) graph, obtained in Theorem 3.

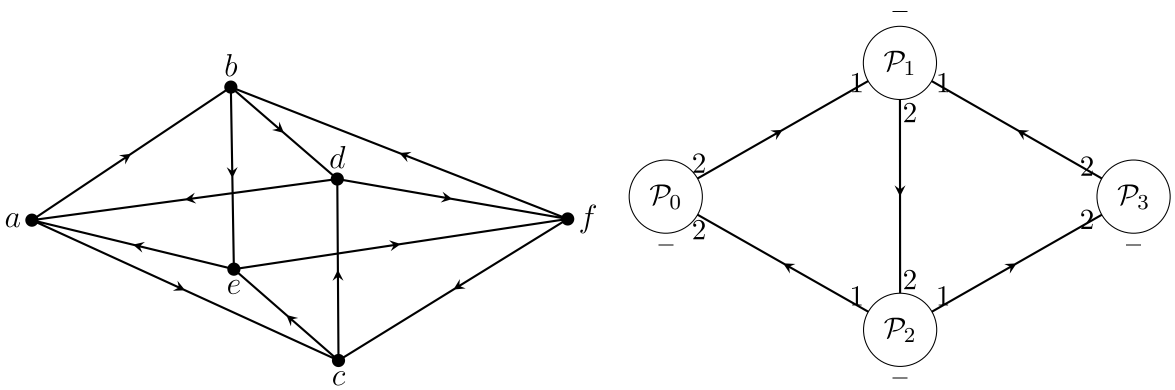

Figure 1: Directed graph Γ (from [24, Example 5.4]) of diameter 3 and the intersection diagram of an equitable distance-faithful partition $\Pi_a = \{\mathcal{P}_0 = \{a\}, \mathcal{P}_1 = \{b, c\}, \mathcal{P}_2 = \{d, e\}, \mathcal{P}_3 = \{f\}\}$ of Γ (around vertex a). The adjacency matrix of this graph generates a commutative 3-class association scheme. Note that $\Gamma_1(a) = \mathcal{P}_1$, $\Gamma_2(a) = \mathcal{P}_2$ and $\Gamma_3(a) = \mathcal{P}_3$.

2 Preliminaries

A *directed graph* with *vertex set* X and *arc set* $\mathcal{E}$ is a pair $\Gamma = (X, \mathcal{E})$ which consists of a finite set $X = X(\Gamma)$ of *vertices* and a set $\mathcal{E} = \mathcal{E}(\Gamma)$ of *arcs* (*directed edges*) between vertices of Γ. As the initial and final vertices of an arc are not necessarily different, the directed graphs may have loops (arcs from a vertex to itself), and multiple arcs, that is, there can be more than one arc from each vertex to any other. If $e = (x, y) \in \mathcal{E}$ is an arc from x to y, then the vertex x (and the arc e) is *adjacent to* the vertex y, and the vertex y (and the arc e) is *adjacent from* x. The *converse directed graph* $\overline{\Gamma}$ is obtained from Γ by reversing the direction of each arc. For a vertex x, let $\Gamma_1^{\leftarrow}(x)$ (and $\Gamma_1^{\rightarrow}(x)$) denote the set of vertices adjacent to (and from) the vertex x, respectively. In another words

$$\Gamma_1^{\leftarrow}(x) = \{z \mid (z, x) \in \mathcal{E}(\Gamma)\} \qquad \text{and} \qquad \Gamma_1^{\rightarrow}(x) = \{z \mid (x, z) \in \mathcal{E}(\Gamma)\}.$$

Two small comments about notations: (i) drawing directed edge from x to z, we have $x \to z$, which yields idea beyond using notation $\Gamma_1^{\rightarrow}(x)$; (ii) drawing directed edge from z to x, we have $x \leftarrow z$ (or $z \to x$), which yields idea beyond using notation $\Gamma_1^{\leftarrow}(x)$. We abbreviate $\Gamma_1(x) = \Gamma_1^{\rightarrow}(x)$. Also, instead of a set of vertices, we can consider a set of arcs: for a vertex y, let $D_1^{\leftarrow}(y)$ (and $D_1^{\rightarrow}(y)$) denote the set of arcs adjacent to (and from) the vertex y, respectively. The number $|D_1^{\rightarrow}(y)|$ we call the *out-degree of* y and is equal to the number of edges leaving y. The number $|D_1^{\leftarrow}(y)|$ we call the *in-degree of* y and is equal to the number of edges going to y. A directed graph Γ is *k-regular* if $|D_1^{\rightarrow}(y)| = |D_1^{\leftarrow}(y)| = k$ for all $y \in X$.

Let $\Gamma = (X, \mathcal{E})$ denote a directed graph. For any two vertices $x, y \in X$, a *directed walk* of length h from x to y is a sequence $[x_0, x_1, x_2, \ldots, x_h]$ ($x_i \in X$, $0 \leqslant i \leqslant h$) such that $x_0 = x$, $x_h = y$, and x_i is adjacent to x_{i+1} (i.e. $x_{i+1} \in \Gamma_1^{\rightarrow}(x_i)$) for $0 \leqslant i \leqslant h - 1$. We say that Γ is *strongly connected* if for any $x, y \in X$ there is a directed walk from x to y. A *closed directed walk* is a directed walk from a vertex to itself. A *directed path* is a directed walk such that all vertices, except the initial and terminal ones, of the directed walk are distinct. A *directed cycle* is a closed directed path.

For any $x, y \in X$, the *distance* between x and y, denoted by $\partial(x, y)$, is the length of a shortest directed path from x to y. The *diameter* $D = D(\Gamma)$ of a strongly connected directed graph Γ is defined to be

$$D = \max\{\partial(y, z) \,|\, y, z \in X\}.$$

For a vertex $x \in X$ and any non-negative integer i not exceeding D, let $\Gamma_i^{\rightarrow}(x)$ (or $\Gamma_i(x)$) denote the subset of vertices in X that are at distance i from x, i.e.,

$$\Gamma_i^{\rightarrow}(x) = \{z \in X \mid \partial(x, z) = i\}.$$

We also define the set $\Gamma_i^{\leftarrow}(x)$ as $\Gamma_i^{\leftarrow}(x) = \{z \in X \mid \partial(z, x) = i\}$. Let $\Gamma_{-1}(x) = \Gamma_{D+1}(x) := \emptyset$. The elements of $\Gamma_1(x) (= \Gamma_1^{\rightarrow}(x))$ are called *neighbors* of x. The *eccentricity* of x, denoted by $\varepsilon = \varepsilon(x)$, is the maximum distance between x and any other vertex of Γ. Note that the diameter of Γ equals $\max\{\varepsilon(x) \mid x \in X\}$.

All undirected graphs in this paper can be understood as directed graphs in which an undirected edge between two vertices x and y represents two arcs, an arc from x to y, and an arc from y to x. In diagrams instead of drawing two arcs we draw one undirected edge between vertices x and y. For a basic introduction to the theory of undirected graphs we refer to [22, Section 2]. With the word *graph* we refer to a finite simple undirected graph.

We say that a graph Γ is *N-partite* if its set of vertices can be decomposed into N disjoint sets such that no two vertices within the same set are adjacent. If $N = 2$ such graphs are called *bipartite*. An *N-partite complete graph* Γ is N-partite graph for which there is an edge between every pair of vertices from different (disjoint) sets.

2.1 Equitable and distance-faithful partition

A *partition* of a (directed) graph Γ is a collection $\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_s\}$ of nonempty subsets of the vertex set X, such that $X = \bigcup_{i=0}^{s} \mathcal{P}_i$ and $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$ for all i, j $(0 \leqslant i, j \leqslant s,\ i \neq j)$. An *equitable partition* of a directed graph Γ is a partition $\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_s\}$ of its vertex set, such that for all integers i, j $(0 \leqslant i, j \leqslant s)$ the following two conditions hold.

(i) The number $d_{ij}^{\rightarrow}$ of neighbors which a vertex in the cell $\mathcal{P}_i$ has in the cell $\mathcal{P}_j$ is independent of the choice of the vertex in $\mathcal{P}_i$ (i.e., for every $y \in \mathcal{P}_i$ we have $|\Gamma_1^{\rightarrow}(y) \cap \mathcal{P}_j| = d_{ij}^{\rightarrow}$).

(ii) The number $d_{ij}^{\leftarrow}$ of vertices from the cell $\mathcal{P}_j$ which are adjacent to a vertex in $\mathcal{P}_i$ is independent of the choice of the vertex in $\mathcal{P}_i$ (i.e., for every $y \in \mathcal{P}_i$ we have $|\sum_{z \in \mathcal{P}_j} |\Gamma_1^{\rightarrow}(z) \cap \{y\}| = |\Gamma_1^{\leftarrow}(y) \cap \mathcal{P}_j| = d_{ij}^{\leftarrow}$).

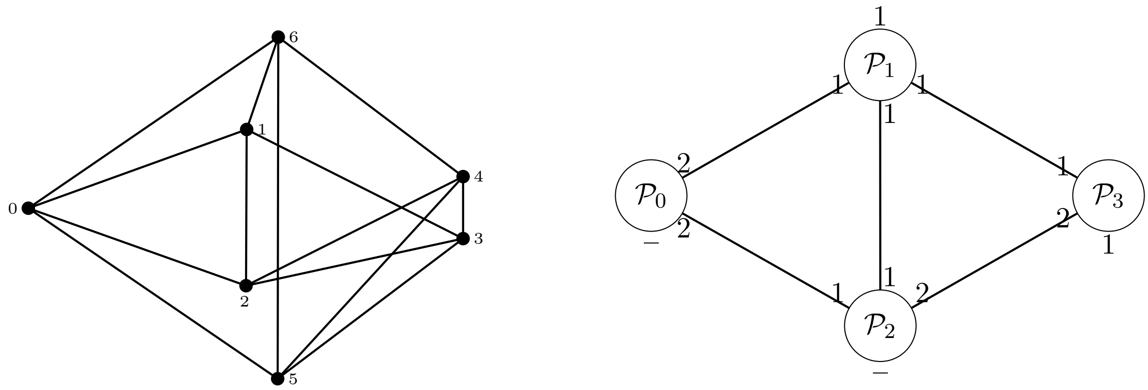

Figure 2: Undirected graph $\Gamma = \mathrm{Cay}(\mathbb{Z}_7; \{1, 2\})$ of diameter 2 and the intersection diagram of an equitable distance-faithful partition of Γ (around vertex 0). The adjacency matrix of this graph generates a symmetric 3-class association scheme.

We call the numbers $d^{\rightarrow}_{ij}$ and $d^{\leftarrow}_{ij}$ $(0 \leqslant i, j \leqslant s)$ the *corresponding parameters*.

A *distance partition around* x of a (directed) graph Γ with vertex set X is a partition $\{\Gamma_0(x) = \{x\}, \Gamma(x), \ldots, \Gamma_{\varepsilon(x)}(x)\}$ of X where $\varepsilon(x)$ is eccentricity of x. A *x-distance-faithful partition* $\{\mathcal{P}_0 = \{x\}, \mathcal{P}_1, \ldots, \mathcal{P}_s\}$ with $s \geqslant \varepsilon(x)$ is a refinement of the distance partition around x (here refinement means that some of $\Gamma_i(x)$ can be equal to a union of some $\mathcal{P}_h$'s).

The *intersection diagram* of an equitable partition Π of a graph Γ is a collection of circles indexed by the sets of Π with lines (or directed edges) between some of them. If there is no line (directed edge) between $\mathcal{P}_i$ and $\mathcal{P}_j$, then it means that there is no (directed) edge yz for any $y \in \mathcal{P}_i$ and $z \in \mathcal{P}_j$. If there is a line (directed edge) between $\mathcal{P}_i$ and $\mathcal{P}_j$, then a number on the line (from $\mathcal{P}_i$ to $\mathcal{P}_j$) near the circle $\mathcal{P}_i$ denotes the corresponding parameter $d^{\rightarrow}_{ij}$. A number above or below a circle $\mathcal{P}_i$ denotes the corresponding parameter $d^{\rightarrow}_{ii} (= d^{\leftarrow}_{ii})$. A similar explanation holds for the corresponding parameter $d^{\leftarrow}_{ij}$ (see Figures 1 and 2 for an example).

We say that the combinatorial structure of the intersection diagram is *the same* around every vertex if for every vertex x there exists an x-distance-faithful equitable partition with the same number of cells of same cardinality and (same) corresponding parameters do not depend on the choice of x.

2.2 Elementary algebraic graph theory

In this section, we recall some definitions and basic concepts from algebraic graph theory.

The *adjacency matrix* $A \in \mathrm{Mat}_X(\mathbb{C})$ of a directed graph Γ (with vertex set X) is indexed by the vertices from X, and is defined in the following way

$$(A)_{yz} = \text{the number of arcs from } y \text{ to } z \qquad (y, z \in X)$$

(note that $(A)_{yz} \geqslant 0$). Moreover, if we allow loops, the diagonal entries of A can be different from zero. Note that the yz-entry of the power A^ℓ ($\ell \in \mathbb{N}$) corresponds to the number of ℓ-walks from the vertex y to the vertex z in Γ.

Lemma 5 (see, for example, [39, Lemma 2.8]). *Let Γ denote a simple strongly connected digraph with vertex set X, diameter D and adjacency matrix A. The number of walks of length $\ell \in \mathbb{N}$ in Γ from x to y is equal to (x, y)-entry of the matrix A^ℓ.*

The distance-i matrix A_i of a digraph Γ with diameter D and vertex set X is defined by

$$(A_i)_{zy} = \begin{cases} 1 & \text{if } \partial(z, y) = i, \\ 0 & \text{otherwise.} \end{cases} \qquad (z, y \in X,\ 0 \leqslant i \leqslant D).$$

In particular, $A_0 = I$ and $A_1 = A$. A matrix $A \in \mathrm{Mat}_X(\mathbb{C})$ is said to be a *reducible* when there exists a permutation matrix P such that $P^\top AP = \begin{pmatrix} X & Y \\ \boldsymbol{O} & Z \end{pmatrix}$, where X and Z are both square, and $\boldsymbol{O}$ is a zero matrix. Otherwise, A is said to be *irreducible*. Recall that a directed graph Γ with adjacency matrix A is strongly connected if and only if A is an irreducible matrix (see, for example, [37, Section 8.3]).

A matrix $A \in \mathrm{Mat}_X(\mathbb{C})$ is called *normal* if it commutes with its adjoint, i.e. if $A\overline{A}^\top = \overline{A}^\top A$. Two matrices $A, B \in \mathrm{Mat}_X(\mathbb{C})$ are said to be *simultaneously diagonalizable* if there is a nonsingular $S \in \mathrm{Mat}_X(\mathbb{C})$ such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal.

Theorem 6 (see, for example, [29, Subsection 1.3]). *Let $\mathcal{M}$ denote a space of commutative normal matrices. Then, there exists a unitary matrix $U \in \mathrm{Mat}_X(\mathbb{C})$ which diagonalizes $\mathcal{M}$.*

Let Γ denote a regular graph with vertex set X and $\circ$ denote the elementwise–Hadamard product of matrices. Let us call two 01-matrices B, C *disjoint* if $B \circ C = 0$. For the moment, let $\mathcal{B}$ denote some algebra of $|X| \times |X|$ matrices. A basis $\{B_0, B_1, \ldots, B_d\}$ of $\mathcal{B}$ is called a *standard basis* of $\mathcal{B}$ if and only if the B_i's are mutually disjoint 01-matrices which satisfy the following properties: (i) the sum of some of these matrices gives I; (ii) the sum of all of these matrices gives the all-1 matrix J; (iii) for each $i \in \{0, \ldots, d\}$, the conjugate transpose of B_i belongs to $\{B_0, B_1, \ldots, B_d\}$; and (iv) the vector space spanned by $\{B_0, B_1, \ldots, B_d\}$ is closed under both ordinary and elementwise–Hadamard multiplication.

2.3 Commutative association scheme

Let X denote a finite set and $\mathrm{Mat}_X(\mathbb{C})$ the set of complex matrices with rows and columns indexed by X. Let $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ denote a set of cardinality $d + 1$ of nonempty subsets of $X \times X$. The elements of the set $\mathcal{R}$ are called *relations* (or *classes*) on X. For each integer i ($0 \leqslant i \leqslant d$), let $A_i \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of the graph (X, R_i) (directed, in general). The pair $\mathfrak{X} = (X, \mathcal{R})$ is a *commutative d-class association scheme* (or a *d-class scheme* for short) if

(AS1) $A_0 = I$, the identity matrix.

(AS2) $\sum_{i=0}^{d} A_i = J$, the all-ones matrix.

(AS3) ${A_i}^{\top} \in \{A_0, A_1, \ldots, A_d\}$ for $0 \leqslant i \leqslant d$.

(AS4) A_iA_j is a linear combination of $A_0, A_1, \ldots, A_d$ for $0 \leqslant i, j \leqslant d$ (i.e., for every i, j $(0 \leqslant i, j \leqslant d)$ there exist *intersection numbers* p^h_{ij}, $0 \leqslant h \leqslant d$, such that $A_iA_j = \sum_{h=0}^{d} p^h_{ij} A_h$).

(AS5) $A_iA_j = A_jA_i$ for every i, j $(0 \leqslant i, j \leqslant d)$ (i.e., for the intersection numbers p^h_{ij}, $0 \leqslant i, j, h \leqslant d$, from (AS4) we have that $p^h_{ij} = p^h_{ji}$).

By (AS1)–(AS5) the vector space $\mathcal{M} = \mathrm{span}\{A_0, A_1, \ldots, A_d\}$ is a commutative algebra; we call it the *Bose–Mesner algebra* of $\mathfrak{X}$. The set of $(0,1)$-matrices $\{A_0, A_1, \ldots, A_d\}$ is linearly independent by (AS2) and thus forms a basis of $\mathcal{M}$. We say that $\mathfrak{X}$ is *symmetric* if the A_i's are symmetric matrices.

For the moment, pick h $(0 \leqslant h \leqslant d)$ and let $x, y \in X$ denote two vertices such that $(A_h)_{xy} = 1$. By (AS2) and (AS4), $(A_iA_j)_{xy} = p^h_{ij}$ $(0 \leqslant i, j \leqslant d)$. On the other hand

$$\begin{aligned}(A_iA_j)_{xy} &= \sum_{z \in X} (A_i)_{xz}(A_j)_{zy} \\ &= |\{z \in X \mid (A_i)_{xz} = 1 \text{ and } (A_j)_{zy} = 1\}| \\ &= |\{z \in X \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}|,\end{aligned}$$

which yields $p^h_{ij} = |\{z \in X \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}|$. This suggests an equivalent *combinatorial* definition of a commutative association scheme (the following axioms are the combinatorial analogs of those given in (AS1)–(AS5)):

(AS1') $R_0 = \{(x, x) \mid x \in X\}$ (that is, R_0 is the *diagonal relation*).

(AS2') $\{R_i\}_{i=0}^{d}$ is a partition of the Cartesian product $X \times X$.

(AS3') Relation $R_j^{\top} = \{(y, x) \mid (x, y) \in R_j\}$ is in $\{R_i\}_{0 \leqslant i \leqslant d}$, for each $j \in \{0, \ldots, d\}$ (that is, $\{R_i\}_{i=0}^{d}$ is closed under taking the *transpose relation* $^{\top}$).

(AS4') For each triple i, j, h $(0 \leqslant i, j, h \leqslant d)$, and $(x, y) \in R_h$, a scalar

$$|\{z \in X \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}| \tag{1}$$

does not depend on the choice of the pair $(x, y) \in R_h$. The scalars obtained from line (1) we denote by p^h_{ij} and call the *intersection numbers* of $\mathfrak{X}$.

(AS5') For each triple i, j, h $(0 \leqslant i, j, h \leqslant d)$, $p^h_{ij} = p^h_{ji}$.

Note that association scheme is *symmetric* if $R_i = R_i^\top$, for each i $(0 \leqslant i \leqslant d)$. Immediately from the combinatorial definition, for example, we can get some properties on the intersection numbers that we will use later: in particular, pick j $(0 \leqslant j \leqslant d)$, let $(x, y) \in R_j$, and note that

$$\begin{aligned}\sum_{\ell=0}^{d} p^j_{k\ell} &= \sum_{\ell=0}^{d} |\{z \in X \mid (x,z) \in R_k \text{ and } (z,y) \in R_\ell\}| \\ &= |\{z \in X \mid (x,z) \in R_k\}| \\ &= |\{z \in X \mid (x,z) \in R_k \text{ and } (z,x) \in R_{k^*}\}| \quad (\text{where } R_k^\top = R_{k^*}) \\ &= p^0_{kk^*}.\end{aligned}$$

We abbreviate $n_k := p^0_{kk^*}$ $(0 \leqslant k \leqslant d)$. The number n_k is the so-called *valency* of the relation R_k. For any $w \in X$, the comments from above imply

$$(A_k \boldsymbol{j})_w = |\{z \in X \mid (w,z) \in R_k\}| = p^0_{kk^*} = n_k = \sum_{\ell=0}^{d} p^j_{k\ell} \qquad (0 \leqslant k \leqslant d). \tag{2}$$

Equation (2) also yields that all-1 vector $\boldsymbol{j}$ is an eigenvector of A_k $(0 \leqslant k \leqslant d)$ that corresponds to the eigenvalue n_k.

Lemma 7. *With respect to the above notations, the Bose–Mesner algebra $\mathcal{M}$ of a commutative d-class association scheme $\mathfrak{X}$ is a space of commutative normal matrices.*

Proof. From the definition of $\mathfrak{X}$, $A_i\overline{A_i}^\top = \overline{A_i}^\top A_i$ $(0 \leqslant i \leqslant d)$, and the result follows. $\square$

Note that the Bose–Mesner algebra $\mathcal{M}$ is semisimple, because it does not contain nilpotent elements. So let $\{E_0, E_1, \ldots, E_d\}$ denote the set of primitive idempotents of $\mathcal{M}$. For $\{E_0, E_1, \ldots, E_d\}$ the following hold: (ei) $E_iE_j = \delta_{ij}E_i$ $(0 \leqslant i, j \leqslant d)$; (eii) $\sum_{i=0}^{d} E_i = I$, the identity matrix; (eiii) there exists a complex scalar $p_h(i)$ $(0 \leqslant i, h \leqslant d)$ such that $A_hE_i = p_h(i)E_i$ (moreover, $p_h(i)$ is the eigenvalue of A_h on the eigenspace V_i); (eiv) $A_h \in \operatorname{span}\{E_0, E_1, \ldots, E_d\}$ $(0 \leqslant h \leqslant d)$; (ev) $\overline{E_i}^\top = E_i$ $(0 \leqslant i \leqslant d)$; (evi) the idempotents E_i are the orthogonal projectors of V onto the spaces $V_i := E_i\mathbb{C}^{|X|}$.

The change-of-basis matrices P and Q are defined by

$$A_i = \sum_{h=0}^{d} (P)_{hi}E_h, \qquad E_i = \frac{1}{|X|}\sum_{h=0}^{d} (Q)_{hi}A_h.$$

We shall refer to P and Q as the *first* and *second eigenmatrices* of the association scheme, respectively. Moreover, we set

$$P = \begin{pmatrix} p_0(0) & p_1(0) & p_2(0) & \cdots & p_d(0) \\ p_0(1) & p_1(1) & p_2(1) & \cdots & p_d(1) \\ p_0(2) & p_1(2) & p_2(2) & \cdots & p_d(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_0(d) & p_1(d) & p_2(d) & \cdots & p_d(d) \end{pmatrix} = \begin{pmatrix} \text{——} & (P)_{0*} & \text{——} \\ \text{——} & (P)_{1*} & \text{——} \\ \text{——} & (P)_{2*} & \text{——} \\ & \vdots & \\ \text{——} & (P)_{d*} & \text{——} \end{pmatrix}. \tag{3}$$

Lemma 8 ([3])**.** *With respect to the above notation, let $\mathfrak{X} = (X, \mathcal{R})$ denote a d-class association scheme, $V_i = E_i\mathbb{C}^{|X|}$ $(0 \leqslant i \leqslant d)$ and and let $\boldsymbol{j}$ denote the all-1 vector. The first and the second eigenmatrix have the following form*

$$P = \begin{array}{c} \\ V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_d \end{array} \begin{array}{c} \begin{array}{ccccc} R_0 & R_1 & R_2 & \cdots & R_d \end{array} \\ \begin{pmatrix} 1 & n_1 & n_2 & \cdots & n_d \\ 1 & p_1(1) & p_2(1) & \cdots & p_d(1) \\ 1 & p_1(2) & p_2(2) & \cdots & p_d(2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & p_1(d) & p_2(d) & \cdots & p_d(d) \end{pmatrix} \end{array}, \quad Q = \begin{array}{c} \\ R_0 \\ R_1 \\ R_2 \\ \vdots \\ R_d \end{array} \begin{array}{c} \begin{array}{ccccc} V_0 & V_1 & V_2 & \cdots & V_d \end{array} \\ \begin{pmatrix} 1 & m_1 & m_2 & \cdots & m_d \\ 1 & q_1(1) & q_2(1) & \cdots & q_d(1) \\ 1 & q_1(2) & q_2(2) & \cdots & q_d(2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & q_1(d) & q_2(d) & \cdots & q_d(d) \end{pmatrix} \end{array}$$

where n_i are positive integers and $m_i = \dim(V_i)$. Moreover, $A_i\boldsymbol{j} = n_i\boldsymbol{j}$ $(0 \leqslant i \leqslant d)$ and for any i $(0 \leqslant i \leqslant d)$, the scalars $n_i, p_i(1), \ldots, p_i(d)$ are the eigenvalues (not necessarily pairwise distinct) of A_i on $V_0, V_1, \ldots, V_d$, respectively.

The matrix P is also called the *character table* of an association scheme, and in fact can be viewed as a natural generalization of the character table of a finite group (see, for example, [2, 10, 32, 36, 50]). Lemma 9 follows immediately from the definitions of P and Q.

Lemma 9. *Let P and Q denote the first and second eigenmatrices of an association scheme $\mathfrak{X} = (X, \mathcal{R})$, respectively. Then, $PQ = QP = |X|I$.*

Corollary 10. *Let P denote the first eigenmatrix of an association scheme $\mathfrak{X} = (X, \mathcal{R})$. Then, $P\boldsymbol{j} = \begin{pmatrix} |X| & 0 & \cdots & 0 \end{pmatrix}^\top$, i.e., for every i $(i \neq 0, 1 \leqslant i \leqslant d)$ the sum of the entries of the V_i row in P is equal to 0.*

Proof. Immediate from Lemmas 8 and 9. □

An association scheme $(X, \mathcal{S})$ on the same vertex set X is called a *fusion* of $(X, \mathcal{R})$ if each $S_i \in \mathcal{S}$ is the union of some of the R_i. Note that $R_0 \in \mathcal{S}$. As an extreme case, we call $(X, \mathcal{R})$ *amorphous* (or *amorphic*) if every "merging" operation on $\{R_0, R_1, \ldots, R_d\}$ yields a fusion (see [15] for survey on this topic). For fusion schemes, readers are refereed to [1, Section 2.3.2] and [15].

2.4 On 2-class association schemes: strongly-regular graphs

In order to better understand some arguments and results in Section 3, it is convenient to recall what is widely known about strongly-regular graphs. In particular, we define a strongly-regular graph by using the language of association schemes, and deduce its particular combinatorial properties from this definition. Furthermore, we recall a well-known result about strongly-regular graphs, namely, that the adjacency matrix A of a connected strongly-regular graph generates its corresponding association scheme (see Proposition 14). We refer the reader to [4, 6, 9, 42] for further details on the general theory of strongly-regular graphs, and we also point out the more recent [5, 7, 23]. Our main source for what follows is [8].

Let $\Gamma = (X, R)$ denote a graph with vertex set X and edge set R. Define $\overline{R} = \{(x, y) \in X \times X \mid (x, y) \notin R\}$ and $R_0 = \{(x, x) : x \in X\}$. The graph Γ is said to be *strongly-regular* if exactly one of the following two properties holds:

(i) $(X, \{R_0, R\})$ is the trivial association scheme (i.e., Γ is a complete graph),

(ii) $(X, \{R_0, R, \overline{R}\})$ is a (symmetric) 2-class association scheme.

A *clique* C of an undirected graph Γ is an induced subgraph of Γ such that every two distinct vertices of C are adjacent (i.e., a clique of Γ is a complete subgraph of Γ). The number of vertices of C is called the *size* of the clique C.

Theorem 11 ([8, Theorem 3.11]). *A disconnected strongly-regular graph is a disjoint union of cliques of the same size. Conversely, if a graph is a disjoint union of $t > 1$ $(t \in \mathbb{N})$ cliques of the same size, then it is a disconnected strongly-regular graph.*

Proposition 12 ([7, page 2]). *Let Γ denote a strongly-regular graph. If Γ is a disjoint union of cliques, then -1 is an eigenvalue for Γ, and vice-versa.*

Throughout this section, we assume that $\Gamma = (X, R)$ is a connected strongly-regular graph for which the corresponding association scheme $\mathfrak{X} = (X, \{R_0, R, \overline{R}\})$ has 2 classes. Directly from our definition, it is not hard to obtain the following combinatorial properties: the graph Γ has valency p^0_{11}, and the number of common neighbors of two vertices x, y of Γ is p^1_{11} or p^2_{11}, depending on whether x and y are adjacent or not (see Proposition 14 below for more details). Moreover, Γ has diameter 2. By convention, the intersection numbers p^0_{11}, p^1_{11} and p^2_{11} of the scheme $\mathfrak{X}$ are denoted by k, λ and μ, respectively. Furthermore, we use k_2 to refer to p^0_{22}, i.e., the number of vertices that are not adjacent to a given one; thus, the number of vertices $|X|$ is equal to

$$w = 1 + k + k_2. \tag{4}$$

Counting in two different ways the edges between vertices which are adjacent and nonadjacent to a fixed $x \in X$, we get the well-known identity

$$k(k - \lambda - 1) = k_2\mu. \tag{5}$$

The *ith intersection matrix* L_i of a d-class association scheme is defined to be a $(d+1)$-matrix whose generic entry is $(L_i)_{j,k} = p^k_{ij}$, for $i, j, k \in \{0, \ldots, d\}$. Following the monumental thesis of Delsarte [17], after *diagonalizing* both sides of the equation in (AS4), we deduce that $PL_iP^{-1} = \operatorname{diag}(p_i(0), p_i(1), \ldots, p_i(d))$ (see [17, page 13]). Consequently, the matrices A_i and L_i have the same eigenvalues (but with different multiplicities), and it follows that the map $A_i \to L_i$ defines an isomorphism between the Bose–Mesner algebra of the scheme and the algebra generated by the L_i's. Thus, following [8, pages 76, 77], the eigenvalues of Γ are the zeros of the minimal polynomial of the matrix L_1 of the 2-class association scheme $\mathfrak{X}$. In particular, the valency k is an eigenvalue with multiplicity 1, and the other eigenvalues are

$$r, s = \frac{(\lambda - \mu) \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}{2}. \tag{6}$$

Furthermore, the multiplicities f, g of eigenvalues r and s can be computed by solving the following equations:

$$f + g = k + k_2 \qquad \text{and} \qquad k + fr + gs = \operatorname{trace}(\mathrm{A}_1) = 0 \tag{7}$$

(see, for example, [8, page 77]).

Remark 13. We show that $k = \mu$ if and only if $r = 0$. First assume that $k = \mu$. Equation (6) yields $r = \frac{\lambda-\mu+\sqrt{(\lambda-k)^2}}{2} = \frac{\lambda-\mu+|\lambda-k|}{2}$, and with it $r = 0$. Now assume that $r = 0$. From equation (6) we now have $(\lambda-\mu) + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)} = 0$ which implies $k = \mu$. The claim follows.

Note that from the first eigenmatrix P it follows that $\Gamma = (X, R)$ has eigenvalue $r = 0$ if and only if -1 is an eigenvalue for $\overline{\Gamma} = (X, \overline{R})$, i.e., $\overline{\Gamma}$ (which is also strongly-regular) is a disjoint union of cliques (by Proposition 12). Since $\overline{\Gamma}$ is disjoint union of cliques, Γ is a complete multipartite graph (by construction). Now, for the case $k = \mu$, using (4), (5), and (6), we find

$$w = 2k - \lambda, \qquad k_2 = k - \lambda - 1, \qquad r = 0, \qquad s = \lambda - k \; (< -1). \tag{8}$$

In above Remark 13, we considered the case $k = \mu$. For the rest of the current section assume $k \neq \mu$. By combining (6) and (7), we get

$$f, g = \frac{1}{2}\left((k + k_2) \pm \frac{(k+k_2)(\mu-\lambda) - 2k}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right) \tag{9}$$

These numbers must be positive integers. We distinguish two cases in (9), so yielding two classes of strongly-regular graphs.

Case 1: $(k + k_2)(\mu - \lambda) - 2k = 0$. Here, we find $k_2 = k$ (since $k + k_2 > k$ and $k + k_2$ divides $2k$). It follows that $\lambda = \mu - 1$ and $k = f = g$. Moreover, (5) yields $k = 2\mu$. Also, since $w = 1 + 2k = 1 + 4\mu$, we get

$$r, s = \frac{-1 \pm \sqrt{w}}{2}, \tag{10}$$

with $r > 0$ and $s < 0$. Strongly-regular graphs with these parameters are known as *conference graphs*. See [4, 31, 44] or [8, page 77] for more details.

Case 2: $(k + k_2)(\mu - \lambda) - 2k \neq 0$. Now, following explanations in [8, page 77], we deduce that r and s are integers, with $r > 0$ and $s < 0$ still holding, and use the eigenvalues k, r, s to write the parameters of Γ:

$$\lambda = k + r + s + rs, \qquad \mu = k + rs, \qquad k_2 = -\frac{k(r+1)(s+1)}{k+rs},$$

from which

$$w = \frac{(k-r)(k-s)}{k+rs}. \tag{11}$$

Therefore, with reference to (6), for a connected strongly-regular, if $k \neq \mu$ we can always assume that

$$r > 0 \quad \text{and} \quad s < 0 \quad (s \neq -1). \tag{12}$$

We conclude the subsection by providing a well-known result, so to prepare the reader for what is the spirit that runs through the upcoming sections.

Proposition 14. *Let Γ denote a connected strongly-regular graph. Then, the following hold.*

(i) *Γ is regular with valency k. Moreover, there exists a positive integer λ such that any two adjacent vertices have λ common neighbors. In addition, if Γ is not a complete graph, there exists a positive integer μ such that the number of common neighbors of any two nonadjacent vertices is equal to μ.*

(ii) *If $\mathcal{M}$ is the Bose–Mesner algebra of the association scheme of Γ and A is the adjacency matrix of Γ, then $\mathcal{M} = (\langle A \rangle, +, \cdot)$.*

Proof. Routine. □

2.5 Some algebraic properties of $\mathcal{M}$

In this subsection we prove some results that can be found implicitly (or explicitly) in the literature, and that we use latter in the paper. Without this subsection our paper is not readable as we want it to be.

Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme. In the next few claims we survey basic algebraic properties under which a matrix $A \in \mathcal{M}$ (not necessarily a 01-matrix) generates $\mathcal{M}$. Main results of this subsection which we use latter in the paper are Lemmas 15, 20 and Proposition 21. Note that if A is a 01-matrix then A is an adjacency matrix of some (directed) graph Γ; we study the combinatorial structure of such a graph in Sections 4 and 5.

Lemma 15. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme. If $A \in \mathcal{M}$ has $d+1$ distinct eigenvalues, then $\{A^0, A^1, \ldots, A^d\}$ is a linearly independent set. Moreover, $\mathcal{M} = (\langle A \rangle, +, \cdot)$.*

Proof. Let $\{E_0, E_1, \ldots, E_d\}$ denote a basis of primitive idempotents of $\mathcal{M}$. We show that $E_i \in \operatorname{span}\{A^0, A^1, \ldots, A^d\}$ $(0 \leqslant i \leqslant d)$. Since $A \in \mathcal{M}$, there exist scalars λ_i $(0 \leqslant i \leqslant d)$ such that $A = \sum_{i=0}^{d} \lambda_i E_i$. Note that $\{\lambda_0, \lambda_1, \ldots, \lambda_d\}$ is the set of the distinct eigenvalues of A. We can now get the following system

$$A^\ell = \lambda_0^\ell E_0 + \lambda_1^\ell E_1 + \cdots + \lambda_d^\ell E_d \qquad (0 \leqslant \ell \leqslant d),$$

which can be written as

$$\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^d \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_0 & \lambda_1 & \dots & \lambda_d \\ \lambda_0^2 & \lambda_1^2 & \dots & \lambda_d^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^d & \lambda_1^d & \dots & \lambda_d^d \end{bmatrix}}_{=B^\top} \begin{bmatrix} E_0 \\ E_1 \\ E_2 \\ \vdots \\ E_d \end{bmatrix}.$$

The matrix B from above is a Vandermonde matrix which is invertible (see, for example, [37, page 185]). The result follows. □

Corollary 16. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme. For an arbitrary $A \in \mathcal{M}$ the following hold.*

(i) *If A is a symmetric 01-matrix with $d+1$ distinct eigenvalues, then A is an adjacency matrix of a connected undirected graph Γ.*

(ii) *If A is a non-symmetric 01-matrix with $d+1$ distinct eigenvalues, then A is an adjacency matrix of a strongly connected directed graph Γ.*

Proof. From Lemma 15, $\{A^0, A^1, \dots, A^d\}$ is a basis of $\mathcal{M}$ (recall $\mathcal{M} = \operatorname{span}\{A_0, A_1, \dots, A_d\}$). Since $\sum_{i=0}^d A_i = J$, the all-1 matrix J belongs to $\mathcal{M}$, which yields $J \in \operatorname{span}\{A^0, A^1, \dots, A^d\}$. In other words, for any choice of vertices $y, z \in X$ there exists ℓ $(0 \leqslant \ell \leqslant d)$ such that $(A^\ell)_{yz} \neq 0$ (otherwise $J \notin \mathcal{M}$, a contradiction). Recall that the (y, z) entry of A^ℓ represents the number of walks of length ℓ between y and z. The result follows. □

Lemma 17. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme and let A denote an arbitrary matrix from $\mathcal{M}$. Then the following hold.*

(i) *The sum of the row entries of A is the same for every vertex.*

(ii) *The sum of the column entries of A is the same for every vertex.*

(iii) *The sum of the row entries of A is equal to the sum of the column entries of A for every vertex.*

Proof. Immediately from (2). □

Corollary 18. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme. For an arbitrary $A \in \mathcal{M}$ the following hold.*

(i) *If A is a non-symmetric 01-matrix, then there exists a polynomial $H(t) \in \mathbb{R}[t]$ such that $J = H(A)$ if and only if $\Gamma = \Gamma(A)$ is a regular and strongly connected directed graph.*

(ii) *If A is a symmetric 01-matrix, then there exists a polynomial $H(t) \in \mathbb{R}[t]$ such that $J = H(A)$ if and only if $\Gamma = \Gamma(A)$ is a regular and connected undirected graph.*

Proof. (i) By Lemma 17, for a given 01-matrix $A \in \mathcal{M}$ we have $A\boldsymbol{j} = A^{\top}\boldsymbol{j} = k\boldsymbol{j}$ for some k. The result now follows from [28, Theorem 1]. (ii) The claim follows from Lemma 17 and [27, Theorem 1]. □

Recall that A generates $\mathcal{M}$ if every element of $\mathcal{M}$ can be written as a polynomial in A.

Corollary 19. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme and let A denote a 01-matrix in $\mathcal{M}$. Then A generates $\mathcal{M}$ if and only if A has $d+1$ distinct eigenvalues.*

Proof. ($\Leftarrow$) If A has $d+1$ distinct eigenvalues the result follows from Lemma 15.

($\Rightarrow$) Assume that A generates $\mathcal{M}$. Then the matrix J is polynomial in A. By Corollary 18 the graph $\Gamma = \Gamma(A)$ is (strongly) connected (directed) graph. For the moment let $r+1$ denote the number of distinct eigenvalues of Γ. We know that $\{A^0, A^1, \ldots, A^r\}$ is linearly independent set (see, for example, [41, Proposition 5.04]). Since every element of the standard basis $\{A_0, A_1, \ldots, A_d\}$ of $\mathcal{M}$ can be written as polynomial in A, we can conclude $d \leqslant r$. On the other hand, since A belong to $\mathcal{M}$, $A^h \in \mathcal{M} = \operatorname{span}\{A_0, A_1, \ldots, A_d\}$ $(0 \leqslant h \leqslant r)$ which yields $r \leqslant d$. The result follows. □

Lemma 20. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme with adjacency matrices $\{A_i\}_{i=0}^d$. For any complex scalars α_i $(0 \leqslant i \leqslant d)$ the set of eigenvalues of a matrix $A = \sum_{i=0}^d \alpha_i A_i$ is*

$$\{(P)_{0*}\boldsymbol{\alpha}, (P)_{1*}\boldsymbol{\alpha}, \ldots, (P)_{d*}\boldsymbol{\alpha}\} \tag{13}$$

where $(P)_{i*}$ denotes the ith row of the first eigenmatrix P, and $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_d)^{\top}$. (Note that we do not know the cardinality of (13)*, i.e., we do not know whether the products $(P)_{i*}\boldsymbol{\alpha}$ $(0 \leqslant i \leqslant d)$ are pairwise distinct.)*

Proof. Let $\mathcal{B}_i$ $(0 \leqslant i \leqslant d)$ denote a basis of the space $V_i = E_i\mathbb{C}^{|X|}$ $(0 \leqslant i \leqslant d)$. With the notation of Lemma 8, for any $u \in \mathcal{B}_h$ $(0 \leqslant h \leqslant d)$ we have

$$Au = \sum_{i=0}^{d} \alpha_i A_i u = \sum_{i=0}^{d} \alpha_i p_i(h) u = \big((P)_{h*}\boldsymbol{\alpha}\big)u.$$

The result follows. □

Note that the vector $\boldsymbol{\alpha}$ from Lemma 20 is arbitrary. For our goal, the interesting case is when this vector is a 01-vector (with $\alpha_0 = 0$) (see Proposition 21). Our result from Proposition 21 is very similar to [30, Lemma 6.2]. Some results on symmetric association schemes generated by a relation can be found in [49].

Proposition 21. *Let $\mathfrak{X}$ denote a commutative d-class association scheme with adjacency matrices $\{A_i\}_{i=0}^d$, first eigenmatrix P, and let $(P)_{*i}$ denote the ith column of P for all $1 \leqslant i \leqslant d$. The column vector $\sum_{i\in\Phi}(P)_{*i}$ has $d+1$ distinct entries (for some set of indices $\Phi \subseteq \{1, 2, \ldots, d\}$) if and only if the matrix $A = \sum_{i\in\Phi} A_i$ generates the Bose–Mesner algebra $\mathcal{M}$ of $\mathfrak{X}$.*

Proof. ($\Rightarrow$) Assume that the column vector $\sum_{i\in\Phi}(P)_{*i}$ has $d+1$ distinct entries (for some set of indices $\Phi \subseteq \{1, 2, \ldots, d\}$). Then, by Lemma 20, $A = \sum_{i\in\Phi} A_i$ has $d+1$ distinct eigenvalues. The result now follows from Lemma 15.

($\Leftarrow$) Assume now that $A = \sum_{i\in\Phi} A_i$ generates $\mathcal{M}$. Then, by Corollary 19, A has $d+1$ distinct eigenvalues. It follows from Lemma 20 that the column vector $\sum_{i\in\Phi}(P)_{*i}$ has $d+1$ distinct entries. $\square$

3 On 3-class association schemes generated by a graph

In the current section, we answer the question of whether a commutative 3-class association scheme $(X, \{R_i\}_{i=0}^3)$ (not necessarily symmetric) is generated by a *graph in the scheme*, i.e., by a graph $G_\Phi = (X, \{R_i\}_{i\in\Phi})$ with some nonempty index set $\Phi \subseteq \{1, \ldots, d\}$.

Let $(X, \{R_i\}_{i=0}^d)$ denote a commutative d-class association scheme, and consider the graph in the scheme $G_\Phi = (X, \{R_i\}_{i\in\Phi})$, with a nonempty set of indices $\Phi \subseteq \{1, \ldots, d\}$. With reference to Lemma 8 and Lemma 20, for every $0 \leqslant \ell \leqslant d$, the number $\sum_{i\in\Phi} p_i(\ell)$ is the eigenvalue of such a graph associated with the ℓth (maximal common) eigenspace of the scheme, where $p_i(\ell)$ (the (ℓ, i)-entry of the first eigenmatrix P) is the eigenvalue of (X, R_i) for the ℓth eigenspace. To remain consistent with our notations, we index the columns (resp. rows) of P with the ordered set $(R_0, \ldots, R_d)$ (resp. $(V_0, \ldots, V_d)$) such that the ith column (resp. row) represents the ith relation (resp. eigenspace). For sake of simplicity, when $\Phi = \{i\}$, for some i $(1 \leqslant i \leqslant d)$, we denote the graph $G_{\{i\}} = (X, R_i)$ by G_i.

Going back to our goal of determining if a commutative 3-class scheme is generated by a graph, we need to distinguish two cases:

- symmetric 3-class schemes,
- non-symmetric 3-class schemes.

The result in Proposition 21, for $d = 3$, will play a fundamental role in both cases.

3.1 Symmetric 3-class association schemes

Let's start with the symmetric case.

In [14], Van Dam completely classified symmetric 3-class schemes.

Lemma 22 ([14, Section 7]). *Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^3)$ denote a symmetric 3-class scheme, and let $G_i = (X, R_i)$ $(1 \leqslant i \leqslant 3)$. Then, $\mathfrak{X}$ belongs to exactly one of the following categories:*

(a) 3-*class schemes which are amorphic, i.e., every graph* G_i $(1 \leqslant i \leqslant 3)$ *is strongly-regular;*

(b) 3-*class schemes with at least one graph* G_i *that is the disjoint union of* $N > 1$ $(N \in \mathbb{N})$ *connected strongly-regular graphs (which are not complete graphs) with the same parameters;*

(c) 3-*class schemes with at least one graph* G_i *having* 4 *distinct eigenvalues.*

In Lemma 23 we describe the shape of the first eigenmatrix of the first class of graphs from Lemma 22.

Lemma 23. *Let* $\mathfrak{X}$ *denote an amorphic symmetric* 3-*class scheme. Then, its first eigenmatrix* P *has the following form:*

$$P = \begin{array}{c} \begin{array}{cccc} R_0 & R_1 & R_2 & R_3 \end{array} \\ \begin{pmatrix} 1 & n_1 & n_2 & n_3 \\ 1 & p_1(1) & p_2(1) & p_3(1) \\ 1 & p_1(1) & p_2(2) & p_3(2) \\ 1 & p_1(3) & p_2(2) & p_3(1) \end{pmatrix} \end{array},$$

where n_i *is the* i*th valency of the scheme.*

Proof. Note that, as P is not singular, the rows of P are pairwise different. Furthermore, since the entries of the first column of Q are all one (see Lemma 8), the identity $PQ = |X|I$ (Lemma 9) implies that each row of P, except for the V_0-row, has sum zero (see Corollary 9). The result follows (see [14, page 76] for further details).

Notice that the result also follows from [15, Proposition 2] after a reordering of the columns. □

In Lemma 24 and Proposition 25 we we are dealing with the second class of graphs from Lemma 22.

Lemma 24. *Let* $\mathfrak{X} = (X, \{R_i\}_{i=0}^3)$ *denote a symmetric* 3-*class scheme. Assume that* $G_1 = (X, R_1)$ *is the disjoint union of* $N > 1$ $(N \in \mathbb{N})$ *connected, non-complete strongly-regular graphs, each one with valency* k*, eigenvalues* r *and* s $(r \neq k, s \neq k)$*, and* w *vertices. Then, the following hold.*

(i) *The first eigenmatrix* P *of* $\mathfrak{X}$ *has the form*

$$P = \begin{array}{c} \begin{array}{cccc} R_0 & R_1 & R_2 & R_3 \end{array} \\ \begin{pmatrix} 1 & k & -1-k+w & (-1+N)w \\ 1 & k & -1-k+w & -w \\ 1 & r & -1-r & 0 \\ 1 & s & -1-s & 0 \end{pmatrix} \end{array}$$

(ii) *None of the graphs* $G_i = (X, R_i)$ $(1 \leqslant i \leqslant 3)$ *generates the scheme* $\mathfrak{X}$

(iii) *The graph* $G_{\{1,2\}} = (X, \{R_1, R_2\})$ *does not generate the scheme* $\mathfrak{X}$.

Proof. (i) The shape of P is given in the first table of [14, page 88]. The proof is done by construction in [14, Section 3.1]. In the next few lines, we provide a detailed description of the first eigenmatrix P. The R_1-column of P represents (the eigenvalues of) the graph which is the disjoint union of N strongly-regular graphs with the same parameters and (distinct) eigenvalues k, r, s; the R_2-column represents the disjoint union of the complements of the N strongly-regular graphs (with eigenvalues $-1-k+w$, $-1-r$, $-1-s$); the R_3-column represents the complete N-partite graph (on the N sets of w vertices).

(ii) Since we are dealing with connected (non-complete) strongly-regular graphs, we may assume $r \geqslant 0$ and $s < 0$ such that $s \neq -1$ (by (12) and Remark 13). It is evident that none of the graphs $G_i = (X, R_i)$ $(1 \leqslant i \leqslant 3)$ generates the scheme since each one has at most 2 distinct eigenvalues (for example, from the first eigenmatrix P, the eigenvalues of $G_2 = (X, R_2)$ are $-1-k+w$, $-1-r$, $-1-s$; but the first and the third ones are equal when $r = 0$, as $s = k - w$ by (8)).

(iii) The graph $G_{\{1,2\}} = (X, \{R_1, R_2\})$ has 2 distinct eigenvalues, namely $w-1$ and -1, obtained by adding the R_1-column and the R_2-column of P. Thus, $G_{\{1,2\}} = (X, \{R_1, R_2\})$ cannot generate the scheme, so proving the claim. □

Note that, in light of our aim (to figure out if there is a generating graph in the scheme), part (ii) allows us to set aside all graphs that we do not need to consider (see next result).

Proposition 25. *Let* $\mathfrak{X} = (X, \{R_i\}_{i=0}^{3})$ *denote a symmetric* 3*-class scheme, and let* $G_i = (X, R_i)$ $(1 \leqslant i \leqslant 3)$. *Assume that* $\mathfrak{X}$ *has at least one graph* G_i *(we can set* $i = 1$*) which is the disjoint union of* $N > 1$ $(N \in \mathbb{N})$ *connected strongly-regular graphs (which are not complete graphs) with the same parameters: valency* k*, eigenvalues* r *and* s $(r \neq k, s \neq k)$*, and* w *vertices. If* $r > 0$ *then* $\mathfrak{X}$ *can be generated only by the following two graphs:* $G_{\{1,3\}}$ *and* $G_{\{2,3\}}$.

Proof. By Lemma 24, the first eigenmatrix P has the following form:

$$P = \begin{array}{c} \begin{array}{cccc} R_0 & R_1 & R_2 & R_3 \end{array} \\ \begin{pmatrix} 1 & k & -1-k+w & (-1+N)w \\ 1 & k & -1-k+w & -w \\ 1 & r & -1-r & 0 \\ 1 & s & -1-s & 0 \end{pmatrix} \end{array},$$

where k, r, s are the distinct eigenvalues of a connected, non-complete strongly-regular graph of valency k on w vertices, with $r \geqslant 0$ and $s < 0, \neq -1$ (see (12) and Remark 13); the parameter $N(> 1)$ is the number of the (connected, non-complete) strongly-regular graphs, with the same parameters, which appear in the definition of the scheme. In light of Lemma 24(ii), (iii), two graphs remain to be checked: $G_{\{i,3\}} = (X, \{R_i, R_3\})$, $i \in \{1, 2\}$.

Recall that $r > 0$. First, consider the graph $G_{\{1,3\}} = (X, \{R_1, R_3\})$, whose eigenvalues (arising from the sum of the R_1-column and the R_3-column in P) are $k+(-1+N)w$, $k-w$, r, s. We verify that they are distinct from each other in the next few lines. Note that $w > k$ (by (4)), $k > r$ (this is a well-known fact, see [17, Section 2.3] for example), and $r > 0 > s$ (by (12)). Thus, the first eigenvalue $k+(-1+N)w$ cannot be equal to any of the other ones, since it is positive and greater than each of the others ($(-1+N)w > 0$). The second $k - w$, being negative ($w > k$), could be equal to s. Then, to get a contradiction, assume $k - w = s$. Following Subsection 2.4, for a connected strongly-regular graph, only one of the two cases can occur:

(a) $w = 2k + 1,\ r, s = \frac{-1\pm\sqrt{w}}{2}$ (see Case 1 at page 14, and (10));

(b) $w = \frac{(k-r)(k-s)}{k+rs},\ r, s \in \mathbb{Z}$ (see Case 2 at page 14, and (11)).

Suppose (a) holds. Then, applying the appropriate substitutions in the equation $k - w = s$, we get $-k - 1 = \frac{-1-\sqrt{w}}{2}$, and since $w = 2k + 1$ this yields $2k + 1 = \sqrt{2k+1}$. Thus we obtain $k = 0$, which is impossible by definition ($k > 0$ is the valency of the graph). Now, assume (b) holds. If we replace w in the equation $k - w = s$ with the expression provided in (b), then we have $k(k + rs) - (k - r)(k - s) = s(k + rs)$ which yields $(rs + r)(k - s) = 0$. Thus we get three solutions: $k = s$ or $r = 0$ or $s = -1$. All of them would obviously yield a contradiction (indeed, we have $k > 0$, $r > 0$, $s < -1$ by (12)). Therefore, it follows that the graph $G_{\{1,3\}} = (X, \{R_1, R_3\})$ has 4 distinct eigenvalues, thus generating the scheme.

Now, consider the graph $G_{\{2,3\}} = (X, \{R_2, R_3\})$ (by assumption recall that $r > 0$). Eigenvalues of $\Gamma_{\{2,3\}}$ (arising from the sum of the R_2-column and the R_3-column in P) are $-1-k+Nw$, $-1-k$, $-1-r$, $-1-s$. Applying the same arguments as before (or, for example, putting equality between some of them to get a contradiction), it turns out that they are all distinct unless $k - Nw = s$ holds. Suppose $k - Nw = s$. Again, we need to distinguish the two cases (a) and (b). If (a) holds, then from $\frac{-1-\sqrt{2k+1}}{2} = k - N(2k + 1)$ we get $2N(2k + 1) = 2k + 1 + \sqrt{2k+1}$. Thus we have two solutions: $N = \frac{1+2k+\sqrt{1+2k}}{2(1+2k)}$ or $k = -\frac{1}{2}$. Both of them are impossible since $N > 1$ and $k > 0$. If (b) holds, then $s = k - N\frac{(k-r)((k-s))}{k+rs}$ yields $(s - k)(k + rs - N(k - r)) = 0$. From it two solutions arise: $k = s$ or $N = \frac{k+rs}{k-r}$. As $s < 0 < r$ (by (12)) and $k + rs < k - r$ (note that r, s are integers), both of them are not acceptable. Thus, also the graph $G_{\{2,3\}} = (X, \{R_2, R_3\})$ has 4 distinct eigenvalues, thus generating the scheme. The result follows. □

Theorem 26. *Let $\mathfrak{X}$ denote a symmetric 3-class scheme. Then, the scheme $\mathfrak{X}$ is generated by a (undirected) graph if and only if it is not amorphic.*

Proof. We take advantage of the Van Dam classification given in Lemma 22.

Case 1. Assume that $\mathfrak{X} = (X, \{R_i\}_{i=0}^3)$ is a symmetric 3-class scheme which is amorphic, i.e., every graph $G_i = (X, R_i)$ ($1 \leqslant i \leqslant 3$) is a strongly-regular graph (not necessarily connected). Such a scheme is never generated by a graph in the scheme: we prove that any time we take the union of 2 classes, we get a graph with at most 3 distinct eigenvalues. In order to understand this, it is enough to look at the first eigenmatrix P of the scheme $\mathfrak{X}$, whose form is provided by Lemma 23:

$$P = \begin{array}{c} \begin{matrix} R_0 & R_1 & R_2 & R_3 \end{matrix} \\ \begin{pmatrix} 1 & n_1 & n_2 & n_3 \\ 1 & p_1(1) & p_2(1) & p_3(1) \\ 1 & p_1(1) & p_2(2) & p_3(2) \\ 1 & p_1(3) & p_2(2) & p_3(1) \end{pmatrix} \end{array}.$$

None of the graphs $G_i = (X, R_i)$ $(1 \leqslant i \leqslant 3)$ generates the scheme since each one has at most 3 distinct eigenvalues. For example, the eigenvalues of $G_1 = (X, R_1)$ are n_1, $p_1(1)$, $p_1(3)$; note that 2 of them can be equal to each other but not all 3. Now, consider the graph $G_{\{1,2\}} = (X, \{R_1, R_2\})$, whose eigenvalues are obtained by adding the R_1-column and the R_2-column of P; precisely, they are n_1+n_2, $p_1(1)+p_2(1)$, $p_1(1)+p_2(2)$, $p_1(3)+p_2(2)$. Since the sum of the R_0-row of P, $1+n_1+n_2+n_3$, equals n (the number of vertices of the scheme) and the sum of any other row of P is zero (see Corollary 10), these eigenvalues are respectively equal to $n-n_3-1$, $-1-p_3(1)$, $-1-p_3(2)$, $-1-p_3(1)$. This yields that the graph $G_{\{1,2\}} = (X, \{R_1, R_2\})$ has at most 3 distinct eigenvalues, namely, $n-n_3-1$, $-1-p_3(1)$, $-1-p_3(2)$. Thus, the graph $G_{\{1,2\}} = (X, \{R_1, R_2\})$ cannot generate the scheme. The same conclusions arise if we choose the remaining graphs $G_{\{1,3\}} = (X, \{R_1, R_3\})$ and $G_{\{2,3\}} = (X, \{R_2, R_3\})$.

Case 2. Suppose that $\mathfrak{X}$ has a graph $G_i = (X, R_i)$ which is the disjoint union of connected strongly-regular graphs with the same parameters which are not complete graphs. We may assume $i = 1$. For $r > 0$, by Proposition 25 each of $G_{\{1,3\}}$ and $G_{\{2,3\}}$ generate the scheme. For $r = 0$, in the next few lines, we prove that only $G_{\{2,3\}} = (X, \{R_2, R_3\})$ has 4 distinct eigenvalues, thus generating the scheme.

Assume $r = 0$. Then, by (8), the first eigenmatrix P is as follows:

$$P = \begin{array}{c} \begin{matrix} R_0 & R_1 & R_2 & R_3 \end{matrix} \\ \begin{pmatrix} 1 & k & -1+k-\lambda & (-1+N)(2k-\lambda) \\ 1 & k & -1+k-\lambda & -(2k-\lambda) \\ 1 & 0 & -1 & 0 \\ 1 & -k+\lambda & -1+k-\lambda & 0 \end{pmatrix} \end{array}.$$

In this case, the graph $G_{\{1,3\}} = (X, \{R_1, R_3\})$ has 3 (distinct by (8)) eigenvalues, i.e., $k+(-1+N)(2k-\lambda)$, $-k+\lambda$, 0; so it cannot generate the scheme. Then, consider the graph $G_{\{2,3\}} = (X, \{R_2, R_3\})$, whose eigevalues are $-1-k+N(2k-\lambda)$, $-1-k$, -1, $-1+k-\lambda$. The last three are all different from each other as $k > 0$, $w = 2k-\lambda > 0$, and $-s = k-\lambda > 0$ (see (8)). Now, comparing the first eigenvalue with each of the others, we never get an equality since $N(2k-\lambda) = Nw > 0$, $k < Nw$, and $(-1+N)w > 0$, respectively (by $N > 1$, $N \in \mathbb{N}$, and (8)). This means that the graph $G_{\{2,3\}} = (X, \{R_2, R_3\})$ has 4 distinct eigenvalues, and so it generates the scheme.

CASE 3. Suppose that $\mathfrak{X}$ has a graph $G_i = (X, R_i)$ having 4 distinct eigenvalues. Then, G_i generates $\mathfrak{X}$. $\square$

3.2 Non-symmetric 3-class association schemes

Now, we consider the non-symmetric case. In [24], GOLDBACH found the general structure of the first eigenmatrix P of a non-symmetric 3-class scheme.

Lemma 27 ([24, Theorem 2.3])**.** *Let $\mathfrak{X}$ denote a non-symmetric 3-class scheme with n points, intersection numbers p_{ij}^k $(0 \leqslant k, i, j \leqslant 3)$, valencies $n_i = p_{ii^*}^0$ $(0 \leqslant i \leqslant 3)$, and multiplicities m_i $(0 \leqslant i \leqslant 3)$. Then, the first eigenmatrix P of the scheme $\mathfrak{X}$ has the following form:*

$$P = \begin{array}{c} \begin{array}{cccc} R_0 & R_1 & R_2 = R_1^\top & R_3 \end{array} \\ \begin{pmatrix} 1 & n_1 & n_1 & n_3 \\ 1 & p_1(1) & \overline{p_1(1)} & p_3(1) \\ 1 & \overline{p_1(1)} & p_1(1) & p_3(1) \\ 1 & p_1(3) & p_1(3) & p_3(3) \end{pmatrix} \end{array}, \tag{14}$$

where $p_1(1) = \frac{1}{2}(p_{11}^1 - p_{11}^2 + \mathrm{i}\sqrt{\frac{nn_1}{m_1}}) \in \mathbb{C} \setminus \mathbb{R}$ and $p_1(3) = \frac{n_1}{n_3} p_{33}^1 - p_{23}^1 \in \mathbb{Q}$.

Remark 28. *With reference to Lemma 27, let us make some considerations on the first eigenmatrix P. Since the sum of every row of P except the V_0-row is zero (see Corollary 10), it turns out that $p_3(1) = p_{11}^2 - p_{11}^1 - 1 \in \mathbb{Z}$ and $p_3(3) = -1 - 2p_1(3) \in \mathbb{Q}$. Furthermore, as $p_{11}^1 - p_{11}^2 + p_{13}^1 - p_{23}^1 = -1$ (see [24, Lemma 2.4]), we can write $p_3(1) = p_{13}^1 - p_{23}^1$. In the end, note that $n_1 \neq p_1(1)$ and $p_1(3) \neq p_1(1)$ since $n_1 \in \mathbb{N}$, $p_1(1) \in \mathbb{C} \setminus \mathbb{R}$, and $p_1(3) \in \mathbb{Q}$.*

To make the next arguments clearer, let us recall the following definitions. An association scheme $(X, \{R_i\}_{i=0}^d)$ is said to be *primitive* if every graph $G_i = (X, R_i)$ $(1 \leqslant i \leqslant d)$ is connected; otherwise, it is said to be *imprimitive*. An association scheme $(X, \overline{\mathcal{R}})$, in which $\overline{\mathcal{R}} = \{R \cup R^\top \mid R \in \mathcal{R}\}$, is said to be the *symmetric closure* of the scheme $(X, \mathcal{R})$.

Theorem 29. *Let $\mathfrak{X}$ be a non-symmetric 3-class scheme. Then, there always exists a directed graph which generates the scheme $\mathfrak{X}$.*

Proof. Since $\mathfrak{X} = (X, \{R_i\}_{i=0}^3)$ is a non-symmetric 3-class scheme, its first eigenmatrix P looks like the one in (14). Thus, we will use here the same notation as in Lemma 27 as well as the contents of Remark 28. Two cases arise: the scheme $\mathfrak{X}$ is primitive or imprimitive.

We first suppose $\mathfrak{X}$ is primitive, i.e., each of its relations but the diagonal one is connected. In our case, $(X, \{R_1 \cup R_2, R_3\})$ is the symmetric closure of $\mathfrak{X}$. By [24, Theorem 2.2], a 3-class scheme is primitive if and only if its symmetric closure is primitive. This implies that the (undirected) graph $(X, R_1 \cup R_2)$ is connected with eigenvalues $2n_1$, $p_1(1) + \overline{p_1(1)}$, and $2p_1(3)$ (see (14)). It is known that the number of connected components of a regular (undirected) graph is the multiplicity of its valency (see [7, page 1]). Since the valency of the graph $(X, R_1 \cup R_2)$ is $2n_1$, we have that $2n_1 \neq 2p_1(3)$, i.e., $n_1 \neq p_1(3)$. It follows that the entries in the R_1-column of P are distinct, that is, the graph $G_1 = (X, R_1)$ has 4 distinct eigenvalues, and so it generates the scheme. The same arguments hold for $G_2 = (X, R_2)$. Observe that $G_3 = (X, R_3)$ has at most 3 distinct eigenvalues, and so it cannot generate the scheme.

We explore now the case in which $\mathfrak{X}$ is imprimitive. According to [24, Theorem 4.1], this means that $p^1_{33}(p^1_{13} + p^1_{23}) = 0$. Since the intersection numbers p^k_{ij} are non-negative integers by definition, then either $p^1_{33} = 0$ or $p^1_{13} = p^1_{23} = 0$, which never occur together, otherwise $n_3 = p^1_{31} + p^1_{32} + p^1_{33}$ (see equation (2)) would be zero.

If $p^1_{33} = 0$, the first eigenmatrix P appears as follows:

$$\begin{array}{cc} & \begin{array}{cccc} R_0 & R_1 & R_2 = R_1^\top & R_3 \end{array} \\ P = & \begin{pmatrix} 1 & n_1 & n_1 & n_3 \\ 1 & p_1(1) & \overline{p_1(1)} & p_3(1) \\ 1 & \overline{p_1(1)} & p_1(1) & p_3(1) \\ 1 & -p^1_{23} & -p^1_{23} & -1 + 2p^1_{23} \end{pmatrix}, \end{array}$$

where $n_1 \neq -p^1_{23}$. Then, $G_i = (X, R_i)$, $i \in \{1, 2\}$, having 4 distinct eigenvalues, generates the scheme. Note that $G_3 = (X, R_3)$ is disconnected.

If $p^1_{13} = p^1_{23} = 0$, then $n_3 = p^1_{33}$ and $p_3(1) = 0$. The first eigenmatrix P is now the following:

$$\begin{array}{cc} & \begin{array}{cccc} R_0 & R_1 & R_2 = R_1^\top & R_3 \end{array} \\ P = & \begin{pmatrix} 1 & n_1 & n_1 & n_3 \\ 1 & p_1(1) & \overline{p_1(1)} & 0 \\ 1 & \overline{p_1(1)} & p_1(1) & 0 \\ 1 & n_1 & n_1 & -1 - 2n_1 \end{pmatrix}. \end{array}$$

None among the graphs $G_i = (X, R_i)$ $(1 \leqslant i \leqslant 3)$ can generate the scheme, as each of them has exactly 3 distinct eigenvalues. Let us then consider the graph $G_{\{1,3\}} = (X, \{R_1, R_3\})$, whose eigenvalues are obtained by adding the R_1-column and the R_3-column of P. Thus, this graph has 4 distinct eigenvalues, namely $n_1 + n_3$, $p_1(1)$, $\overline{p_1(1)}$, $-1-n_1$, thus generating the scheme. The same holds if we consider $G_{\{2,3\}} = (X, \{R_2, R_3\})$. Note that $G_{\{1,2\}} = (X, \{R_1, R_2\})$ has distinct eigenvalues $2n_1$ and -1; hence it cannot generate the scheme. □

3.3 Proof of Theorem 2

Let $\mathfrak{X}$ be a commutative 3-class association scheme. If $\mathfrak{X}$ is symmetric, then the result follows from Theorem 26. Otherwise, $\mathfrak{X}$ is non-symmetric and the result follows from Theorem 29.

4 The distance-faithful intersection diagram

Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X}$ and $A \in \mathcal{M}$ denote a 01-matrix which generates $\mathcal{M}$. In this section, we study combinatorial properties of $\Gamma = \Gamma(A)$. We prove that, whenever a 01-matrix $A \in \mathcal{M}$ represents a (strongly) connected (directed) graph, then for every vertex $x \in X$ there exists an x-distance-faithful intersection diagram of an equitable partition Π_x with $d+1$ cells. Moreover, the structure of the x-distance-faithful intersection diagram does not depend on x (see Theorem 3). We use this fact to describe combinatorial properties of a graph which generates a commutative 3-class association scheme (see Corollary 34).

Lemma 30. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$ with adjacency matrices $\{A_i\}_{i=0}^{d}$. For a given $x \in X$ we define the partition $\Pi_x = \{\mathcal{P}_0(x), \mathcal{P}_1(x), \dots, \mathcal{P}_d(x)\}$ of X in the following way*

$$\mathcal{P}_i(x) = \{z \mid (A_i)_{xz} = 1\} \qquad (0 \leqslant i \leqslant d).$$

Let A denote an arbitrary 01-matrix in $\mathcal{M}$, and consider the (directed) graph $\Gamma = \Gamma(A)$. If Γ is a (strongly) connected (directed) graph then in Γ all vertices in $\mathcal{P}_i(x)$ are at the same distance from x.

Proof. We first show that for any $z, w \in \mathcal{P}_i(x)$ the number of walks of length ℓ from x to z is the same as the number of walks of length ℓ from x to w (i.e., $(A^\ell)_{xz} = (A^\ell)_{xw}$ $(0 \leqslant \ell \leqslant d)$). Since $\{A_h\}_{h=0}^{d}$ is a basis of $\mathcal{M}$, there exist scalars α_{ij} $(0 \leqslant i, j \leqslant d)$ such that

$$A^\ell = \sum_{j=0}^{d} \alpha_{\ell j} A_j \qquad (0 \leqslant \ell \leqslant d).$$

For any $z, w \in \mathcal{P}_i(x)$, we have $(A_i)_{xz} = (A_i)_{xw} = 1$ and $(A_j)_{xz} = (A_j)_{xw} = 0$ if $j \neq i$. This yields $(A^\ell)_{xz} = \alpha_{\ell i} = (A^\ell)_{xw}$.

We now prove our claim by a contradiction. Assume that $z, w \in \mathcal{P}_i(x)$ and that $\partial(x, z) > \partial(x, w) = \ell$. Then, we have $(A^\ell)_{xw} \neq 0$ but $(A^\ell)_{xz} = 0$, a contradiction. $\square$

Lemma 31. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$ with the adjacency matrices $\{A_i\}_{i=0}^{d}$. Pick $x, y \in X$ and define the partitions $\Pi_x = \{\mathcal{P}_0(x), \mathcal{P}_1(x), \dots, \mathcal{P}_d(x)\}$ and $\Pi_y = \{\mathcal{P}_0(y), \mathcal{P}_1(y), \dots, \mathcal{P}_d(y)\}$ of X in the following way:*

$$\mathcal{P}_i(x) = \{z \mid (A_i)_{xz} = 1\}, \quad \mathcal{P}_i(y) = \{z \mid (A_i)_{yz} = 1\} \qquad (0 \leqslant i \leqslant d).$$

Let A denote an arbitrary 01-matrix in $\mathcal{M}$, and consider the (directed) graph $\Gamma = \Gamma(A)$. If Γ is a (strongly) connected (directed) graph then for any i, j $(0 \leqslant i, j \leqslant d)$ there exist scalars $D_{ij}^{\rightarrow}$ such that in Γ the following hold:

$$|\Gamma_1^{\rightarrow}(z) \cap \mathcal{P}_j(x)| = D_{ij}^{\rightarrow} \qquad \textit{for every } z \in \mathcal{P}_i(x) \tag{15}$$

and

$$|\Gamma_1^{\rightarrow}(w) \cap \mathcal{P}_j(y)| = D_{ij}^{\rightarrow} \qquad \textit{for every } w \in \mathcal{P}_i(y). \tag{16}$$

Proof. We give a proof for a directed graph. The proof for an undirected graph is similar.

Pick some i, j $(0 \leqslant i, j \leqslant d)$ and let k and ℓ denote the unique indices such that $A_k = A_j^\top = A_{j^*}$ and $A_\ell = A_i^\top = A_{i^*}$ (such indices exists since $A_i^\top, A_j^\top \in \{A_0, A_1, \ldots, A_d\}$). Note that $A_k^\top = A_j = A_{k^*}$ and $A_\ell^\top = A_i = A_{\ell^*}$. Since $AA_k \in \operatorname{span}\{A_0, A_1, \ldots, A_d\}$, there exist scalars α_k^h $(0 \leqslant h \leqslant d)$ such that

$$AA_k = \sum_{h=0}^{d} \alpha_k^h A_h. \tag{17}$$

Pick $x, y \in X$ and consider the partitions Π_x and Π_y. We show that for any $z \in \mathcal{P}_i(x)$ and $w \in \mathcal{P}_i(y)$, we have $|\Gamma_1^{\rightarrow}(z) \cap \mathcal{P}_j(x)| = \alpha_{j^*}^{i^*} (= \alpha_k^\ell)$ and $|\Gamma_1^{\rightarrow}(w) \cap \mathcal{P}_j(y)| = \alpha_{j^*}^{i^*} (= \alpha_k^\ell)$.

Note that for any matrix B, $(B)_{zx} = (B^\top)_{xz}$. From the left-hand side of (17), we have

$$\begin{aligned}(AA_k)_{zx} &= \sum_{u \in X} (A)_{zu} (A_k)_{ux} = \sum_{u \in X} (A)_{zu} (A_j)_{xu} \\ &= |\Gamma_1^{\rightarrow}(z) \cap \mathcal{P}_j(x)|\end{aligned}$$

and

$$\begin{aligned}(AA_k)_{wy} &= \sum_{u \in X} (A)_{wu} (A_k)_{uy} = \sum_{u \in X} (A)_{wu} (A_j)_{yu} \\ &= |\Gamma_1^{\rightarrow}(w) \cap \mathcal{P}_j(y)|.\end{aligned}$$

For the same choices of $z \in \mathcal{P}_i(x)$ and $w \in \mathcal{P}_i(y)$ as above, from the right-hand side of (17), we have

$$\begin{aligned}(AA_k)_{zx} &= (AA_k)^\top_{xz} = \left(\sum_{h=0}^{d} \alpha_k^h A_h \right)^\top_{xz} = \sum_{h=0}^{d} \alpha_k^h (A_h^\top)_{xz} \\ &= \alpha_k^\ell (A_i)_{xz} \qquad (\text{where } (A_\ell)^\top = A_i) \\ &= \alpha_k^\ell = \alpha_{j^*}^{i^*}\end{aligned}$$

and

$$\begin{aligned}(AA_k)_{wy} &= (AA_k)^\top_{yw} = \left(\sum_{h=0}^{d} \alpha_k^h A_h \right)^\top_{yw} = \sum_{h=0}^{d} \alpha_k^h (A_h^\top)_{yw} \\ &= \alpha_k^\ell (A_i)_{yw} \qquad (\text{where } (A_\ell)^\top = A_i) \\ &= \alpha_k^\ell = \alpha_{j^*}^{i^*}.\end{aligned}$$

With it, if we define $D_{ij}^{\rightarrow}$ as α_k^ℓ (the index i uniquely determines ℓ, and the index j uniquely determines k), we get that (15) and (16) hold. $\square$

4.1 Proof of Theorem 3

In this subsection we prove Theorem 3. The proof is in the same spirit as [22, Theorem 4.1].

Assume that A is a non-symmetric matrix. Using the same notations as in Lemma 30, for a given $x \in X$ we define the partition Π_x in the following way:

$$\Pi_x = \{\mathcal{P}_0(x), \mathcal{P}_1(x), \dots, \mathcal{P}_d(x)\}, \qquad \text{where} \qquad \mathcal{P}_i(x) = \{z \mid (A_i)_{xz} = 1\} \ (0 \leqslant i \leqslant d).$$

To prove the claim, we need to show that the following (a)–(d) hold.

(a) All vertices in $\mathcal{P}_i(x)$ are at the same distance from x.

(b) $|\mathcal{P}_i(x)| = |\mathcal{P}_i(u)|$ $(0 \leqslant i \leqslant d)$ for every $x, u \in X$.

(c) There exist numbers $D_{ij}^{\rightarrow}$, $D_{ij}^{\leftarrow}$ $(0 \leqslant i, j \leqslant d)$ such that, for every $x \in X$, Π_x is an equitable partition of Γ with corresponding parameters $D_{ij}^{\rightarrow}$, $D_{ij}^{\leftarrow}$ (which do not depend on x).

(d) With respect to (c), $D_{ij}^{\rightarrow} = \sum_{m\in\Phi} p_{mj^*}^{i^*}$ and $D_{ij}^{\leftarrow} = \sum_{m\in\Phi} p_{mj}^{i}$ $(0 \leqslant i, j \leqslant d)$, where Φ is some nonempty index set such that $A = \sum_{m\in\Phi} A_m$.

The claim (a) follows immediately from Lemma 30.

For the claim (b) first note that every matrix in $\mathcal{M}$ has constant row sums (see Lemma 17). Thus $|\mathcal{P}_i(x)| = \sum_{z\in X}(A_i)_{xz} = \sum_{w\in X}(A_i)_{uw} = |\mathcal{P}_i(u)|$ holds for every $x, u \in X$. (Furthermore, note that the cardinality of $\mathcal{P}_i(x)$ for every $x \in X$ is equal to $|R_i(x)| = |\{z \in X \mid (x,z) \in R_i\}|$ and that $|R_i(x)| = n_i$ where R_i is ith relation of the association scheme $\mathfrak{X}$ and n_i is valency of R_i (see Subsection 2.3)).

Next we prove claim (c). In Lemma 31 we showed that for any i, j $(0 \leqslant i, j \leqslant d)$ and $x, y \in X$ there exists scalars $D_{ij}^{\rightarrow}$ such that in Γ, $|\Gamma_1^{\rightarrow}(z) \cap \mathcal{P}_j(x)| = D_{ij}^{\rightarrow}$ holds for every $z \in \mathcal{P}_i(x)$; and that $|\Gamma_1^{\rightarrow}(w) \cap \mathcal{P}_j(y)| = D_{ij}^{\rightarrow}$ holds for every $w \in \mathcal{P}_i(y)$.

For $D_{ij}^{\leftarrow}$ we have something similar. Pick i, j $(0 \leqslant i, j \leqslant d)$ and $x, y \in X$. First, note that

$$A_j A = \sum_{h=0}^{d} \beta_j^h A_h. \tag{18}$$

for some scalars β_j^h $(0 \leqslant h \leqslant d)$. For any $z \in \mathcal{P}_i(x)$ and $w \in \mathcal{P}_i(y)$, from the left-hand side of (18), we have

$$\begin{aligned}(A_j A)_{xz} &= \sum_{u\in X}(A_j)_{xu}(A)_{uz} = \sum_{u\in\mathcal{P}_j(x)}(A)_{uz} \\ &= \sum_{u\in\mathcal{P}_j(x)} |\Gamma_1^{\rightarrow}(u) \cap \{z\}|\end{aligned}$$

and

$$(A_j A)_{yw} = \sum_{u\in X}(A_j)_{yu}(A)_{uw} = \sum_{u\in\mathcal{P}_j(y)}(A)_{uw}$$

$$= \sum_{u \in \mathcal{P}_j(y)} |\Gamma_1^{\rightarrow}(u) \cap \{w\}|.$$

For the same choices of $z \in \mathcal{P}_i(x)$ and $w \in \mathcal{P}_i(y)$, from the right-hand side of (18), we have

$$(A_j A)_{xz} = \left(\sum_{h=0}^{d} \beta_j^h A_h \right)_{xz} = \beta_j^i (A_i)_{xz} = \beta_j^i$$

and

$$(A_j A)_{yw} = \left(\sum_{h=0}^{d} \beta_j^h A_h \right)_{yw} = \beta_j^i (A_i)_{yw} = \beta_j^i.$$

With it, if we define $D_{ij}^{\leftarrow}$ as β_j^i, we get that

$$\sum_{u \in \mathcal{P}_j(x)} |\Gamma_1^{\rightarrow}(u) \cap \{z\}| = D_{ij}^{\leftarrow} \qquad \text{for every } z \in \mathcal{P}_i(x),$$

and

$$\sum_{u \in \mathcal{P}_j(y)} |\Gamma_1^{\rightarrow}(u) \cap \{w\}| = D_{ij}^{\leftarrow} \qquad \text{for every } w \in \mathcal{P}_i(y).$$

Thus, Π_x and Π_y are equitable partitions of Γ with the same corresponding parameters $D_{ij}^{\rightarrow}, D_{ij}^{\leftarrow}$ $(0 \leqslant i, j \leqslant d)$.

It is left to prove claim (d). Pick i, j $(0 \leqslant i, j \leqslant d)$ and let i^* and j^* denote indices such that $A_{j^*} = A_k = A_j^{\top}$ and $A_{i^*} = A_\ell = A_i^{\top}$. Since $A = \sum_{m \in \Phi} A_m$ (for some nonempty index set Φ),

$$AA_{j^*} = AA_k = \sum_{m \in \Phi} \left(\sum_{h=0}^{d} \underbrace{p_{mk}^h}_{=p_{mj^*}^h} A_h \right) = \sum_{h=0}^{d} \left(\sum_{m \in \Phi} p_{mj^*}^h \right) A_h. \tag{19}$$

Pick $z \in \mathcal{P}_i(x)$ and note that zx-entry of the right-hand side of (19) is $\sum_{m \in \Phi} p_{mj^*}^{i^*}$. From the proof of Lemma 31, for any $z \in \mathcal{P}_i(x)$, the left-hand side of (19) is $(AA_{j^*})_{zx} = |\Gamma_1^{\rightarrow}(z) \cap \mathcal{P}_j(x)|$. This implies that $D_{ij}^{\rightarrow} = |\Gamma_1^{\rightarrow}(z) \cap \mathcal{P}_j(x)| = \sum_{m \in \Phi} p_{mj^*}^{i^*}$, and the first part of the claim follows. The second part of the claim follows immediately from the equation $A_j A = \sum_{h=0}^{d} \left(\sum_{m \in \Phi} p_{jm}^h \right) A_h$ and the observations from the proof of the claim (c).

4.2 Some corollaries of Theorem 3

Theorem 3 gives us a useful combinatorial property for a (strongly) connected (directed) graph which 'lives' in a d-class association scheme. See Corollary 34 to understand what is happening in a 3-class association scheme.

Recall that a graph is *walk-regular* if the number of closed walks of length ℓ rooted at vertex x only depends on ℓ, for each $\ell \geqslant 0$ (i.e., the $(A^\ell)_{xx}$ entry for every $x \in X$ only depends on ℓ).

Corollary 32. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$. If a (strongly) connected (directed) graph Γ 'lives' in the association scheme $\mathfrak{X}$ (i.e., if the adjacency matrix A of Γ belongs to $\mathcal{M}$), then Γ is a walk-regular graph.*

Proof. Immediate from Theorem 3. □

In Corollary 33 we deal with a symmetric d-class association scheme.

Corollary 33. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a symmetric d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$, and $A \in \mathcal{M}$ denote a* 01*-matrix. If $\Gamma = \Gamma(A)$ generates $\mathfrak{X}$ then the following hold.*

(i) *For every vertex $x \in X$, there exists an x-distance-faithful intersection diagram (of an equitable partition Π_x) with $d+1$ cells.*

(ii) *The structure of the x-distance-faithful intersection diagram (of the equitable partition Π_x) from* (i) *does not depend on x.*

(iii) *Graph Γ does not have an x-distance-faithful intersection diagram whose number of cells is less than $d+1$ (i.e., $d+1$ is the smallest number of cells for which there exists an x-distance-faithful equitable partition).*

Proof. By assumption A generates $\mathcal{M}$, so by Corollary 19 A has $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. Note that Corollary 32 yields that Γ is a walk-regular graph. By Theorem 3, for every vertex $x \in X$, there exists an x-distance-faithful intersection diagram (of an equitable partition Π_x) with $d+1$ cells and the structure of the intersection diagram does not depend on x (so claims (i) and (ii) hold). For the moment let B denote the $(d+1) \times (d+1)$ quotient matrix of the x-distance-faithful intersection diagram. By [13, Proposition 4.1] every λ_i $(0 \leqslant i \leqslant d)$ is an eigenvalue of B. Now our proof is by a contradiction. Assume that there exists an x-distance-faithful intersection diagram with less than $d+1$ cells. Then quotient matrix C of such intersection diagram has less than $d+1$ distinct eigenvalues, and by [13, Proposition 4.1] every of $d+1$ distinct eigenvalues λ_i $(0 \leqslant i \leqslant d)$ of Γ are also eigenvalues of C, a contradiction. The claim (iii) follows. □

Corollary 34. *Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative* 3*-class association scheme $\mathfrak{X} = (X, \mathcal{R})$, $A \in \mathcal{M}$ denote a* 01*-matrix, and let $\Gamma = \Gamma(A)$ denote a (directed) graph of diameter D with adjacency matrix A. If Γ generates $\mathfrak{X}$ then $D \in \{2, 3\}$, Γ has the same x-distance-faithful intersection diagram around every vertex $x \in X$ and such a diagram has* 4 *cells. Moreover, the following hold.*

(i) *If $D = 3$, then the partition $\{\Gamma_i(x)\}_{0 \leqslant i \leqslant 3}$ is equitable, and the corresponding parameters do not depend on the choice of $x \in X$.*

(ii) *If $D = 2$, then exactly one of the following* (a), (b) *holds.*

(a) *Any two adjacent vertices have a constant number of common neighbors, and the number of common neighbors of any two nonadjacent vertices takes precisely two values. Moreover, for any* $x \in X$ *there exists an equitable partition* $\Pi_x = \{\{x\}, \Gamma_1(x), \mathcal{P}(x), \mathcal{P}'(x)\}$, *for which* $\Gamma_2(x) = \mathcal{P}(x) \cup \mathcal{P}'(x)$.

(b) *Any two nonadjacent vertices have a constant number of common neighbors, and the number of common neighbors of any two adjacent vertices takes precisely two values. Moreover, for any* $x \in X$ *there exists an equitable partition* $\Pi_x = \{\{x\}, \mathcal{P}(x), \mathcal{P}'(x), \Gamma_2(x)\}\}$, *for which* $\Gamma_1(x) = \mathcal{P}(x) \cup \mathcal{P}'(x)$.

Proof. Corollary 19 yields that Γ has 4 distinct eigenvalues, and by Corollary 16, Γ is a (strongly) connected (directed) graph.

We first show that $D \leqslant 3$. Since $\{A^0, A^1, \ldots, A^D\}$ is a linearly independent set (this is a well-known fact, see for example [41, Proposition 5.06]) and since $\{A^0, A^1, \ldots, A^d\}$ is a basis of $\mathcal{A}$, we have $D \leqslant d$, and consequently $D \leqslant 3$. Next we show that $D = 1$ is not possible. If $D = 1$ then every two different vertices are adjacent, which yields that Γ is a complete graph. Then, we have that $A = J - I$ is the adjacency matrix, which yields that A has less than 4 distinct eigenvalues, a contradiction. Case $D = 1$ is not possible.

By Theorem 3, the number of cells of a distance-faithful equitable partition is equal to 4.

Assume that $D = 3$. Pick $x \in X$. The only possibility to get a x-distance-faithful equitable partition with 4 cells is to take distance partition $\{\Gamma_i(x)\}_{0 \leqslant i \leqslant 3}$ of X. An example of a directed graph with $D = 3$ which generates 3-class association scheme is given in Figure 1.

Assume that $D = 2$. For the moment let $\{B_0 = I, B_1, B_2, B_3\}$ denote the standard basis of $\mathcal{M}$, and let A denote the adjacency matrix of a graph $\Gamma = \Gamma(A)$. Since $A \in \mathcal{M}$, the matrix A is equal to some linear combination of $\{B_0, B_1, B_2, B_3\}$. Moreover, since A and the B_i's are 01-matrices, in total six cases are possible $A \in \{B_1, B_2, B_3\}$ or $A \in \{B_1 + B_2, B_1 + B_3, B_2 + B_3\}$. (Case $A = B_1 + B_2 + B_3$ is not possible since then we would have a complete graph.) First three cases $A \in \{B_1, B_2, B_3\}$ give claim (a). Cases $A \in \{B_1 + B_2, B_1 + B_3, B_2 + B_3\}$ yield claim (b). Note that, if we do not have two different values (in both cases), then Γ is a strongly-regular graph, a contradiction (by assumption, Γ generates $\mathfrak{X}$).

The result follows. $\square$

5 Algebraic property of Γ when Γ generates a commutative association scheme

In this section we prove Theorem 4. For that purpose we need Proposition 35. In order to distinguish between the distance-i matrices of a graph Γ and the adjacency matrices of a scheme $\mathfrak{X}$, in this section, we use A_i's to the denote distance-i matrices of a graph, and B_i's to denote the adjacency matrices of an association scheme.

Proposition 35. *Let $\Gamma = \Gamma(A)$ denote a directed graph with vertex set X and adjacency matrix A. Assume that A generates the Bose–Mesner algebra $\mathcal{M}$ of a commutative d-class association scheme, and let $\{B_0, B_1, \ldots, B_d\}$ denote the standard basis of $\mathcal{M}$. Then, the following hold.*

(i) *For any i $(0 \leqslant i \leqslant d)$ and $y, z, u, v \in X$, if $(B_i)_{zy} = (B_i)_{uv} = 1$ then $\partial(z, y) = \partial(u, v)$.*

(ii) *Every distance-i matrix A_i of $\Gamma = \Gamma(A)$ belongs to $\mathcal{M}$, i.e., $A_i \in \mathcal{M}$ $(0 \leqslant i \leqslant D)$.*

Proof. Since A generate the Bose–Mesner algebra $\mathcal{M}$, and $J \in \mathcal{M}$, there exists a polynomial $p(t)$ such that $J = p(A)$. This implies that Γ is regular and strongly connected (see Corollary 18).

(i) For every $\ell \in \mathbb{N}$, there exists complex scalars $\alpha_i^{(\ell)}$ $(0 \leqslant i \leqslant d)$ such that $A^\ell = \sum_{i=0}^{d} \alpha_i^{(\ell)} B_i$. Recall that $\sum_{i=0}^{d} B_i = J$ and $B_i \circ B_j = \delta_{ij} B_i$ $(0 \leqslant i, j \leqslant d)$. This yields that for any $y, z, u, v \in X$ and i $(0 \leqslant i \leqslant d)$, if $(B_i)_{zy} \neq 0$ and $(B_i)_{uv} \neq 0$ then $(A^\ell)_{zy} = (A^\ell)_{uv} = \alpha_i^{(\ell)}$, i.e., the number of walks of length ℓ from z to y is equal to the the number of walks of length ℓ from u to v (see Lemma 5). Moreover, $(A^\ell)_{zy} = (A^\ell)_{uv}$ holds for any ℓ $(\ell \in \mathbb{N})$. To prove the claim, we use the proof by a contradiction. Assume that $\partial(z, y) > \partial(u, v) = m$. Then, $(A^m)_{uv} \neq 0$ and $(A^m)_{zy} = 0$, a contradiction. The result follows.

(ii) From the proof of (i) above it follows that, if $y, z \in X$ are two arbitrary vertices such that $\partial(z, y) = i$, then there exists B_j (for some $0 \leqslant j \leqslant d$) such that $(B_j)_{zy} = 1$. Recall also that $(A_i)_{zy} = 1$. In fact, for such a choice of j and any nonzero (u, v)-entry of B_j, we have $\partial(u, v) = i$. This yields

$$A_i = \sum_{j : A_i \circ B_j \neq \boldsymbol{O}} B_j \qquad (0 \leqslant i \leqslant D).$$

The result follows. □

5.1 Proof of Theorem 4

We show that (i)⇒(ii), (ii)⇒(iii), and (iii)⇒(i). Recall that

$$\Delta = \{(i, j) \mid i = \partial(x, y),\ j = \partial(y, x),\ x, y \in X\}, \tag{20}$$

and by our assumption $|\Delta| = d + 1$.

(i)⇒(ii). Assume that $\mathcal{A}$ is the Bose–Mesner algebra of a d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$, and let $\{B_i\}_{i=0}^{d}$ denote the adjacency matrices of $\mathfrak{X}$. Note that $\mathcal{A} = \operatorname{span}\{A^0, A^1, \ldots, A^d\} = \operatorname{span}\{B_0, B_1, \ldots, B_d\}$. Assume that Γ has diameter D, and let A_i's denote the distance-i matrices of Γ. For a given $x \in X$ we define a partition $\Pi_x = \{\mathcal{P}_0(x), \mathcal{P}_1(x), \ldots, \mathcal{P}_d(x)\}$ of X as in Theorem 3, i.e.,

$$\mathcal{P}_i(x) = \{z \mid (B_i)_{xz} = 1\} \qquad (0 \leqslant i \leqslant d).$$

From the proof of Theorem 3, the partition Π_x is equitable and the corresponding parameters do not depend of the choice of x. Note that the number of cells of the partition Π_x is equal to $d+1$ (i.e., $|\Pi_x| = |\Delta|$). Moreover, by Proposition 35, and since $B_i^\top \in \{B_0, B_1, \ldots, B_d\}$ $(0 \leqslant i \leqslant d)$, for any i $(0 \leqslant i \leqslant d)$ we can produce an element $(h,r) \in \Delta$. Namely, for a fixed i, if $(B_i)_{xy} = 1$ and $(B_i^\top)_{yx} = (B_{i^*})_{yx} = 1$, since $|\Pi_x| = |\Delta| = d+1$, there exists $(h,r) \in \Delta$ such that $h = \partial(x,y)$ and $r = \partial(y,x)$. On the other hand, for any element $(h,r) \in \Delta$ we can produce an element i $(0 \leqslant i \leqslant d)$. Namely, for a given $(h,r) \in \Delta$ we first need to find vertices $x, y \in X$ such that $h = \partial(x,y)$ and $r = \partial(y,x)$ (such vertices exist from the definition of Δ). Since $\sum_{i=0}^d B_i = J$, there exists a unique i such that $(B_i)_{xy} = 1$. For such an i we also have $(B_i^\top)_{yx} = 1$, i.e., $(B_{i^*})_{yx} = 1$. Note that i^* is also unique and will depend only on the choice of i (we can also say that it will only depend on $r = \partial(y,x)$). Note that the pair (B_i, B_{i^*}) uniquely corresponds to (h,r). Therefore, it is enough to have an index i $(0 \leqslant i \leqslant d)$ to derive an element (h,r) of the set Δ, and vice versa, and with that

$$\{R_i\}_{0 \leqslant i \leqslant d} = \{R_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \Delta},$$

where

$$\begin{aligned} R_i &= \{(x,y) \in X \times X \mid (B_i)_{xy} = 1\} \qquad (0 \leqslant i \leqslant d), \\ R_{\boldsymbol{i}} &= \{(x,y) \in X \times X \mid (\partial(x,y), \partial(y,x)) = \boldsymbol{i}\} \qquad (\boldsymbol{i} \in \Delta), \end{aligned}$$

i.e., for every i $(0 \leqslant i \leqslant d)$ there exists $\boldsymbol{i} \in \Delta$ such that $R_i = R_{\boldsymbol{i}}$, and vice versa.

Let $\mathbf{0} = (0,0)$. Then, (AS1') $R_{\mathbf{0}} = \{(x,x) \mid x \in X\}$; and (AS2') $\{R_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \Delta}$ is a partition of the Cartesian product $X \times X$. Furthermore (AS3') $R_{\boldsymbol{j}}^\top = \{(y,x) \mid (x,y) \in R_{\boldsymbol{j}}\}$ is in $\{R_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \Delta}$; as well as (AS4') for each triple $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{h}$ $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{h} \in \Delta)$, and $(x,y) \in R_{\boldsymbol{h}}$, the scalar

$$|\{z \in X \mid (x,z) \in R_{\boldsymbol{i}} \text{ and } (z,y) \in R_{\boldsymbol{j}}\}|$$

does not depend on the choice of the pair $(x,y) \in R_{\boldsymbol{h}}$. Namely,

$$(B_i B_j)_{xy} = |\{z \in X \mid (x,z) \in R_{\boldsymbol{i}} \text{ and } (z,y) \in R_{\boldsymbol{j}}\}|.$$

Since $\mathcal{A}$ is generated by A and is also the Bose–Mesner algebra of $(X, \{R_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \Delta})$ by the above lines, we have that $(X, \{R_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \Delta})$ is a commutative d-class association scheme. The result follows.

(ii)$\Rightarrow$(iii). Assume that $\mathfrak{X} = (X, \{R_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \Delta})$ is a commutative d-class association scheme, and let $\mathcal{M}$ denote the corresponding Bose–Mesner algebra. Since $|\Delta| = d+1$ we can relabel this set in some way and write, for example, $\Delta = \{\mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{d}\}$ where $\mathbf{0} = (0,0)$. Note that $\{R_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \Delta}$ are relations on X indexed by the set Δ. Define the set of adjacency matrices of $\mathfrak{X}$ in the following way

$$(B_{\boldsymbol{i}})_{zy} = \begin{cases} 1 & \text{if } (z,y) \in R_{\boldsymbol{i}}, \\ 0 & \text{if } (z,y) \notin R_{\boldsymbol{i}} \end{cases} \qquad (\boldsymbol{i} \in \Delta,\ z, y \in X).$$

The set $\{B_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\Delta}$ is a basis of the Bose–Mesner algebra $\mathcal{M}$ of $\mathfrak{X}$ indexed by the set Δ. For the moment let Φ_1 denote subset of Δ with first coordinate equal to 1, i.e., let

$$\Phi_1 = \{(1, \partial(y,x)) \mid \partial(x,y) = 1,\, x, y \in X\} \subseteq \Delta.$$

Note that $A \in \mathcal{M}$ because $A = \sum_{\boldsymbol{h}\in\Phi_1} B_{\boldsymbol{h}}$. This yields that A is a normal matrix. Similarly, if $\Phi_i = \{(i, \partial(y,x)) \mid \partial(x,y) = i,\, x, y \in X\}$, it is not hard to see that every distance-i matrix A_i ($0 \leqslant i \leqslant D$) belongs to $\mathcal{M}$, i.e., $A_i = \sum_{\boldsymbol{h}\in\Phi_i} B_{\boldsymbol{h}}$. For the rest of the proof we only need the fact that $A \in \mathcal{M}$.

The fact that $A \in \mathcal{M}$ yields that there exist complex scalars w_{ij} ($0 \leqslant i, j \leqslant d$) such that

$$\begin{aligned} I &= w_{00}B_{\boldsymbol{0}} + w_{01}B_{\boldsymbol{1}} + \cdots + w_{0d}B_{\boldsymbol{d}},\\ A &= w_{10}B_{\boldsymbol{0}} + w_{11}B_{\boldsymbol{1}} + \cdots + w_{1d}B_{\boldsymbol{d}},\\ A^2 &= w_{20}B_{\boldsymbol{0}} + w_{21}B_{\boldsymbol{1}} + \cdots + w_{2d}B_{\boldsymbol{d}},\\ &\vdots\\ A^d &= w_{d0}B_{\boldsymbol{0}} + w_{d1}B_{\boldsymbol{1}} + \cdots + w_{dd}B_{\boldsymbol{d}}, \end{aligned} \tag{21}$$

i.e.

$$\begin{bmatrix} I\\ A\\ A^2\\ \vdots\\ A^d \end{bmatrix} = \underbrace{\begin{bmatrix} w_{00} & w_{01} & \dots & w_{0d}\\ w_{10} & w_{11} & \dots & w_{1d}\\ w_{20} & w_{21} & \dots & w_{2d}\\ \vdots & \vdots & & \vdots\\ w_{d0} & w_{d1} & \dots & w_{dd} \end{bmatrix}}_{=B} \begin{bmatrix} B_{\boldsymbol{0}}\\ B_{\boldsymbol{1}}\\ B_{\boldsymbol{2}}\\ \vdots\\ B_{\boldsymbol{d}} \end{bmatrix}. \tag{22}$$

By (22), $\mathcal{A} \subseteq \mathcal{M}$. Since B from (22) is invertible (it is a change-of-basis matrix), we also have $\mathcal{M} \subseteq \mathcal{A}$.

In the end, since $\sum_{\boldsymbol{i}\in\Delta} B_{\boldsymbol{i}} = J$, note that (21) yields that the number of walks from x to y of every given length $\ell \geqslant 0$ only depends on the index from the set $\Delta = \{\boldsymbol{0}, \boldsymbol{1}, \ldots, \boldsymbol{d}\}$. On the other hand, by definition of Δ, every index depends only on the distances $\partial(x,y)$ and $\partial(y,x)$. The result follows.

(iii)$\Rightarrow$(i). Assume that A is a normal matrix, $|\Delta| = d+1$ and the number of walks from x to y of every given length $\ell \geqslant 0$ only depends on the distances $\partial(x,y)$ and $\partial(y,x)$ (and does not depend on the choice of the pair (x,y)). For any $y, z \in X$, define a column vector $\boldsymbol{w}(y,z) \in \mathbb{R}^{d+1}$ in the following way

$$\boldsymbol{w}(y,z) := \Big((A^0)_{yz}, (A^1)_{yz}, \ldots, (A^d)_{yz}\Big)^{\top}.$$

By our assumption, for any $\boldsymbol{h} \in \Delta$ and $x, y, u, v \in X$ such that $(\partial(x,y), \partial(y,x)) = (\partial(u,v), \partial(v,u)) = \boldsymbol{h}$, we have

$$\boldsymbol{w}(u,v) = \boldsymbol{w}(x,y).$$

Define the matrices $B_{\boldsymbol{i}}$ ($\boldsymbol{i} \in \Delta$) in the following way

$$(B_{\boldsymbol{i}})_{zy} = \begin{cases} 1 & \text{if } \ (\partial(z,y), \partial(y,z)) = \boldsymbol{i}, \\ 0 & \text{otherwise} \end{cases} \qquad (\boldsymbol{i} \in \Delta,\ y, z \in X).$$

Thus, if $(B_{\boldsymbol{i}})_{xy} = (B_{\boldsymbol{i}})_{uv} = 1$ then $\boldsymbol{w}(x,y) = \boldsymbol{w}(u,v)$. If $(B_{\boldsymbol{i}})_{uv} = 1$, we can write $\boldsymbol{w}(u,v) = (w_{0i}, w_{1i}, \ldots, w_{di})^\top$. Let $\boldsymbol{0} = (0,0)$. By the above comments we have that the system of linear equations (21) holds, i.e., we have

$$\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^d \end{bmatrix} = \underbrace{\begin{bmatrix} w_{00} & w_{01} & \ldots & w_{0d} \\ w_{10} & w_{11} & \ldots & w_{1d} \\ w_{20} & w_{21} & \ldots & w_{2d} \\ \vdots & \vdots & & \vdots \\ w_{d0} & w_{d1} & \ldots & w_{dd} \end{bmatrix}}_{=W} \begin{bmatrix} B_{\boldsymbol{0}} \\ B_{\boldsymbol{1}} \\ B_{\boldsymbol{2}} \\ \vdots \\ B_{\boldsymbol{d}} \end{bmatrix} \tag{23}$$

for some real scalars w_{ij} $(0 \leqslant i, j \leqslant d)$. Since A is a normal matrix with $d+1$ distinct eigenvalues, $\{A^0, A^1, \ldots, A^d\}$ is a linearly independent set. On the other hand, $\{B_{\boldsymbol{0}}, B_{\boldsymbol{1}}, \ldots, B_{\boldsymbol{d}}\}$ is also a linearly independent set by definition. For the moment let $\mathcal{B} = \operatorname{span}\{B_{\boldsymbol{0}}, B_{\boldsymbol{1}}, \ldots, B_{\boldsymbol{d}}\}$ denote an algebra with respect to the elementwise–Hadamard $\circ$-product. Note that (23) yields that $\mathcal{A} \subseteq \mathcal{B}$. On the other hand, since the matrix W from (23) is invertible (it is a change-of-basis matrix of the vector spaces $\mathcal{A}$ and $\mathcal{B}$, both of dimension $d+1$), we also have $\mathcal{B} \subseteq \mathcal{A}$. This yields that $\mathcal{B} = \mathcal{A}$. Now, we have (AS1) $B_0 = I$, the identity matrix; as well as (AS2) $\sum_{i=0}^{d} B_i = J$. Since $\mathcal{A} = \mathcal{B}$, every B_i can be written as a polynomial in A, i.e., there exists some polynomial $p_i(t) \in \mathbb{R}[t]$ of degree less or equal to d such that $B_i = p_i(A)$. This yields that (AS5) $B_i B_j = B_j B_i$ $(0 \leqslant i, j \leqslant d)$. The assumption that A is a normal matrix gives $A^\top \in \mathcal{A}$ (see, for example, [11, Theorem 1.1]), so we have (AS3) $B_i^\top \in \{B_0, \ldots, B_d\}$ (recall, every B_i can be written as a polynomial in A); and since $\mathcal{A} = \mathcal{B}$, (AS4) $B_i B_j$ is a linear combination of $B_0, B_1, \ldots, B_d$ for any i, j $(0 \leqslant i, j \leqslant d)$. The result follows.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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