Mixed Graphs Determined by their Generalized Hermitian Adjacency Spectrum Based on Eisenstein Integers

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Abstract

A *mixed graph* is a graph obtained from a simple undirected graph by orientating a subset of edges. In 2020, Mohar introduced a new kind of Hermitian adjacency matrix (called Eisenstein adjacency matrix) of a mixed graph using a primitive sixth root of unity, which has some advantages over the one proposed by Guo and Mohar in 2017, and independently by Liu and Li in 2015 (called Gaussian adjacency matrix). We consider the problem of generalized spectral characterizations of mixed graphs based on the Eisenstein adjacency matrix. A simple sufficient condition is given for a self-converse mixed graph to be determined by its generalized Eisenstein spectrum based on the ring of Eisenstein integers. Numerical experiments are also presented which show that the generalized Eisenstein spectrum is superior to the generalized Gaussian spectrum in distinguishing mixed graphs.

Mathematics Subject Classifications: 05C50

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = [n] = \{1, 2, \ldots, n\}$ and edge set E. The *adjacency matrix* of graph G is an n by n matrix $A = (a_{uv})$, where $a_{uv} = 1$ if the vertices u and v of G are adjacent, and $a_{uv} = 0$ otherwise. The *characteristic polynomial* of G is defined as the characteristic polynomial of its adjacency matrix, $\chi(G) = \chi(G; x) = \det(xI_n - A(G))$, where I_n is the identity matrix of order n. The *spectrum* of G, denoted by $\mathrm{Spec}(G)$, consists of all the eigenvalues (including the multiplicities) of the adjacency

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matrix A of G. Two graphs G and H are called *cospectral* if they share the same spectrum, i.e., $\mathrm{Spec}(G) = \mathrm{Spec}(H)$. A graph G is said to be *determined by the spectrum* (DS for short) if any graph cospectral with G is isomorphic to G.

Characterizing which kinds of graphs are DS is a fundamental and challenging problem in the theory of graph spectra. A famous conjecture of Haemers [4] states that almost all graphs are DS. However, despite many efforts, little is known about this conjecture so far. Indeed, all the known DS graphs have very special properties. The fractions of both known DS graphs and non-DS graphs tend to zero, as the order n of the graphs goes to infinity. For the background and some known results about this problem, we refer the reader to van Dan and Haemers [4, 5].

In recent years, Wang and Xu [17, 18] and Wang [19, 20] considered a variant of the above problem in the context of the generalized spectrum. Here, the *generalized spectrum* of G is the ordered pair $(\mathrm{Spec}(G), \mathrm{Spec}(\bar{G}))$, where $\bar{G}$ is the complement of G. Similarly, a graph G is said to be *determined by its generalized spectrum* (DGS for short) if, whenever H is a graph that has the same generalized spectrum as G (i.e., $\mathrm{Spec}(G) = \mathrm{Spec}(H)$ and $\mathrm{Spec}(\bar{G}) = \mathrm{Spec}(\bar{H})$), then H is isomorphic to G.

Let $W(G) := [\mathbf{1}, A\mathbf{1}, \ldots, A^{n-1}\mathbf{1}]$ be the *walk-matrix* of G ($\mathbf{1}$ is the all-one vector). In Wang [19, 20], the author obtained a simple sufficient condition for a graph to be DGS.

Theorem 1 ([19, 20]). *If $2^{-\lfloor n/2 \rfloor} \det(W(G))$ (which is always an integer) is odd and square-free, then G is DGS.*

The problem of spectral determination for simple graphs can naturally extend to mixed graphs. A *mixed graph* $\Sigma = (V, E)$ can be regarded as a directed graph consisting of a finite vertex set $V = [n] = \{1, 2, \ldots, n\}$ together with an arc set $E \subseteq \{(u, v) : u, v \in V,\ u \neq v\}$. We say there is a directed edge (or an arc) from u to v if $(u, v) \in E$ and $(v, u) \notin E$, denoted by $u \to v$ (or $v \leftarrow u$), and say there is an undirected edge (or a digon) between u and v if the oppositely directed arcs $(u, v) \in E$ and $(v, u) \in E$, denoted by $u \sim v$.

The *complement* of a mixed graph $\Sigma = (V, E)$ is the mixed graph defined as $\bar{\Sigma} = (V, \bar{E})$, where $\bar{E} = \{(u, v) : u, v \in V,\ u \neq v,\ (u, v) \notin E\}$. The *converse* of a mixed graph Σ, denoted by Σ^{T}, is the mixed graph obtained from Σ by reversing every directed edge in Σ. A mixed graph Σ is said to be *self-converse* if Σ^{T} and Σ are isomorphic. Let $\Phi : V(\Sigma) \to V(\Sigma^{\mathrm{T}})$ be the isomorphism mapping from Σ to Σ^{T}. Note that Φ can also be regarded as a mapping $\Phi : V(\Sigma) \to V(\Sigma)$ such that $(u, v) \in E(\Sigma) \iff (\Phi(v), \Phi(u)) \in E(\Sigma)$, so we call Φ an *anti-automorphism* of Σ. The *number of fixed points* of Φ is defined as

$$\nu = \nu(\Phi) := \#\{v \in V(\Sigma) :\ \Phi(v) = v\},$$

which plays an important role in this paper, as we shall see later.

Here is an example of a self-converse graph and its anti-automorphism.

Example 2. Let Σ be a self-converse graph on 6 vertices, as shown in Figure 1.

Note that the mapping $\Phi : V(\Sigma) \to V(\Sigma)$ with $\Phi(1) = 3,\ \Phi(2) = 4,\ \Phi(3) = 1,\ \Phi(4) = 2,\ \Phi(5) = 5$ is an anti-automorphism. It is easy to see that $\nu(\Phi) = 1$.

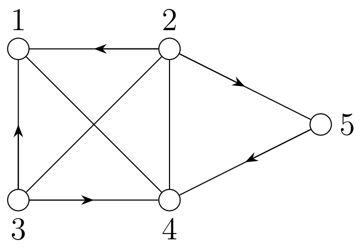

Figure 1: A graph Σ being self-converse

We denote the set of all mixed graphs with vertex set $V = [n] = \{1, 2, \ldots, n\}$ by $\mathcal{G}_n$, and the subset of all self-converse graphs in $\mathcal{G}_n$ is denoted by $\mathcal{G}_n^{sc}$.

Guo and Mohar [6], and Liu and Li [10] independently introduced the notion of the *Hermitian adjacency matrix* for a mixed graph Σ as a natural generalization of the adjacency matrix for an ordinary graph. For $\Sigma \in \mathcal{G}_n$, $A^{(1)}(\Sigma) = A^{(1)} = (a_{uv}^{(1)})$ is an n by n matrix, where $a_{uv}^{(1)}$ is defined as

$$a_{uv}^{(1)} = \begin{cases} 1, & \text{if } u \sim v; \\ \mathrm{i}, & \text{if } u \to v; \\ -\mathrm{i}, & \text{if } u \leftarrow v; \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. Some basic properties of this matrix are provided in [6, 10].

Very recently, Mohar [1] introduced a new kind of Hermitian matrices for mixed graphs, which will be the main focus of this paper.

Definition 3 ([1])**.** Let $\Sigma \in \mathcal{G}_n$. The second kind of Hermitian adjacency matrix of Σ is defined as $A^{(2)}(\Sigma) = A^{(2)} = (a_{uv}^{(2)})$, and

$$a_{uv}^{(2)} = \begin{cases} 1, & \text{if } u \sim v; \\ \omega, & \text{if } u \to v; \\ \overline{\omega}, & \text{if } u \leftarrow v; \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where $\omega = (1 + \mathrm{i}\sqrt{3})/2$ is the sixth root of unity.

For clarity, we refer to $A^{(1)}$ as the Gaussian adjacency matrix (G-adjacency matrix for short), since all the entries of $A^{(1)}$ are Gaussian integers. And refer to $A^{(2)}$ as the *Eisenstein adjacency matrix* (E-adjacency matrix for short), since all the entries of $A^{(2)}$ are Eisenstein integers.

The *Gaussian spectrum* (resp. *Eisenstein spectrum*), abbreviated as G-spectrum (resp. E-spectrum), consists of all the eigenvalues (including the multiplicities) of the G-adjacency matrix $A^{(1)}(\Sigma)$ (resp. E-adjacency matrix $A^{(2)}(\Sigma)$) of Σ.

Notice that for any mixed graph Σ, both kinds of adjacency matrices $A^{(1)}(\Sigma)$ and $A^{(2)}(\Sigma)$ are Hermitian matrices, i.e., $A^{(1)}(\Sigma) = (A^{(1)}(\Sigma))^*$, $A^{(2)}(\Sigma) = (A^{(2)}(\Sigma))^*$ (we use M^* denote the conjugate transpose of M), and hence all of their eigenvalues are real and both matrices are diagonalizable. Also note that $A^{(1)}(\Sigma^{\mathrm{T}}) = (A^{(1)}(\Sigma))^{\mathrm{T}}$ and $A^{(2)}(\Sigma^{\mathrm{T}}) = (A^{(2)}(\Sigma))^{\mathrm{T}}$, so Σ and Σ^{T} have the same G-spectrum, as well as the same E-spectrum. This explains why we focus on the self-converse graphs when dealing with the spectral determination problem of mixed graphs.

Recently, Wissing and van Dam [22] considered the Hemitian spectral determination of mixed graphs. A mixed graph is called *strongly determined by its Gaussian spectrum* (resp. *Eisenstein spectrum*), if it is isomorphic to each mixed graph to which it is cospectral. Wissing and van Dam constructed the first infinite family of connected mixed graphs that are strongly determined by their Gaussian spectrum in [22].

However, it seems that the mixed graphs that are strongly determined by their Gaussian spectrum or Eisenstein spectrum are extremely rare. For example, there are 15,224 self-converse mixed graphs on 6 vertices, among which there are 6 (resp. 2,560) graphs strongly determined by their G-spectrum (resp. E-spectrum); see Table 1 in Section 4. Thus, it appears that a single kind of spectrum is not powerful enough to distinguish mixed graphs. This observation leads us to consider the generalized spectrum.

Let J be the $n \times n$ all-one matrix and I be the identity matrix. Two mixed graphs Σ and Δ are said to be *cospectral w.r.t. the generalized G-spectrum*, if $\mathrm{Spec}(A^{(1)}(\Sigma)) = \mathrm{Spec}(A^{(1)}(\Delta))$ and $\mathrm{Spec}(J - I - A^{(1)}(\Sigma)) = \mathrm{Spec}(J - I - A^{(1)}(\Delta))$. Similarly, two mixed graphs Σ and Δ are said to be *cospectral w.r.t. the generalized E-spectrum if* $\mathrm{Spec}(A^{(2)}(\Sigma)) = \mathrm{Spec}(A^{(2)}(\Delta))$ and $\mathrm{Spec}(J - I - A^{(2)}(\Sigma)) = \mathrm{Spec}(J - I - A^{(2)}(\Delta))$.

Remark 4. Note that the E-adjacency matrix of complement $\bar{\Sigma}$ of a mixed graph Σ is precisely $A^{(2)}(\bar{\Sigma}) = J - I - A^{(2)}(\Sigma)$. However, $J - I - A^{(1)}(\Sigma)$ is not the G-adjacency matrix of the complement $\bar{\Sigma}$ of Σ. In this sense, the generalization of DGS problem over the ring of Eisenstein integers seems more natural.

We give the following definition as a natural generalization of the DGS problem for ordinary graphs.

Definition 5. A mixed graph $\Sigma \in \mathcal{G}_n$ is said to be strongly determined by its generalized Gaussian spectrum (SDGGS for short) (resp. strongly determined by its generalized Eisenstein spectrum (SDGES for short)) if, whenever $\Delta \in \mathcal{G}_n$ has the same generalized Gaussian spectrum (resp. generalized Eisenstein spectrum) as Σ, then Δ must be isomorphic to Σ.

Note that any mixed graph Σ has the same generalized G-spectrum (resp. the generalized E-spectrum) as its converse Σ^{T}. Hence, every SDGGS mixed graph (resp. every SDGES mixed graph) must be self-converse. Moreover, it is also interesting to consider the spectral determination problem within the range of all the self-converse mixed graphs $\mathcal{G}_n^{sc}$.

Definition 6. Let $\Sigma \in \mathcal{G}_n^{sc}$ be a self-converse mixed graph. Then Σ is said to be restrictively determined by its generalized Gaussian spectrum (RDGGS for short) (resp.

restrictively determined by its generalized Eisenstein spectrum (RDGES for short)) if, whenever $\Delta \in \mathcal{G}_n^{sc}$ has the same generalized Gaussian spectrum (resp. generalized Eisenstein spectrum) as Σ, then Δ must be isomorphic to Σ.

Wang et al. [21] extend Theorem 1 to the G-adjacency matrix for the mixed graphs. Before introducing their results, we need some definitions. We define the walk matrix $W^{(1)}(\Sigma) = [\mathbf{1}, A^{(1)}\mathbf{1}, \ldots, (A^{(1)})^{n-1}\mathbf{1}]$ (abbreviated as G-walk matrix), and the Eisenstein walk matrix $W^{(2)}(\Sigma) = [\mathbf{1}, A^{(2)}\mathbf{1}, \ldots, (A^{(2)})^{n-1}\mathbf{1}]$ (abbreviated as E-walk matrix). Wang et al. [21] propose the following conjecture as the generalization of Theorem 1.

Conjecture 7 ([21])**.** Let $\Sigma \in \mathcal{G}_n^{sc}$. If $2^{-\lfloor n/2 \rfloor} \det(W^{(1)}(\Sigma))$ is square-free and $(1+\mathrm{i}) \nmid 2^{-\lfloor n/2 \rfloor} \det(W^{(1)}(\Sigma))$, then Σ is RDGGS. Moreover, if $2^{-\lfloor n/2 \rfloor} \det(W^{(1)}(\Sigma))$ has no prime factor p such that $p \equiv 1 \pmod 4$, then Σ is SDGGS.

As a partial answer to Conjecture 7, Wang et al. [21] obtain the following theorem.

Theorem 8 ([21])**.** *For* $\Sigma, \Delta \in \mathcal{G}_n^{sc}$*, suppose* $2^{-\lfloor n/2 \rfloor} \det(W^{(1)}(\Sigma))$ *is square-free and* $(1+\mathrm{i}) \nmid 2^{-\lfloor n/2 \rfloor} \det(W^{(1)}(\Sigma))$*, and* Q *is a unitary matrix over Gaussian rational field* $\mathbb{Q}(\mathrm{i})$ *such that* $Q^* A^{(1)}(\Sigma) Q = A^{(1)}(\Delta)$ *and* $Q\mathbf{1} = \mathbf{1}$*. Then* $(1+\mathrm{i})Q$ *is a Gaussian integral matrix. Moreover, if* $2^{-\lfloor n/2 \rfloor} \det(W^{(1)}(\Sigma))$ *has no prime integer factor* $p \equiv 1 \pmod 4$*, then the assumption that* Δ *is self-converse can be deleted.*

Note that every entry of the G-adjacency matrix $A^{(1)}(\Sigma)$ of a mixed graph Σ is in the Gaussian integer ring $\mathbb{Z}[\mathrm{i}]$, and thus the proof of Theorem 8 needs some algebraic property of Gaussian integer ring $\mathbb{Z}[\mathrm{i}]$ and prime factorization over $\mathbb{Z}[\mathrm{i}]$. In Wang [19], the author shows that $2^{\lfloor n/2 \rfloor}$ always divides $\det(W(G))$ for any ordinary graph G. Wang et al. [21] extend this fact to the G-walk matrix of mixed graphs and prove that $2^{\lfloor n/2 \rfloor} \mid \det(W^{(1)}(\Sigma))$ always holds (or we may write $(1+\mathrm{i})^{2\lfloor n/2 \rfloor} \mid \det(W^{(1)}(G))$ since $2 = -\mathrm{i}(1+\mathrm{i})^2$ over $\mathbb{Z}[\mathrm{i}]$). However, this property does not hold for E-walk matrix. This makes the story quite different for G-adjacency matrix and E-adjacency matrix.

This paper is a continuation along this line of research. We are mainly concerned with the DGS problem of E-adjacency matrix of mixed graphs. The main result of the paper is the following theorem.

Theorem 9. (Main theorem) *Let* $\Sigma \in \mathcal{G}_n^{sc}$ *with E-walk matrix* $W^{(2)}(\Sigma)$ *and the determinant* $\det(W^{(2)}(\Sigma)) \neq 0$*. Let* ν *be the number of the fixed points of the anti-automorphism of* Σ*. If* $(\omega(1+\omega))^{-\frac{n-\nu}{2}} \det(W^{(2)}(\Sigma))$ *(which is always a rational integer) is square-free, then* Σ *is RDGES. Moreover, if* $(\omega(1+\omega))^{-\frac{n-\nu}{2}} \det(W^{(2)}(\Sigma))$ *has no prime factor* p *such that* $p \equiv 1 \pmod 3$*, then* Σ *is SDGES.*

The uniqueness of anti-automorphism Φ of $\Sigma \in \mathcal{G}_n^{sc}$ with $\det(W^{(2)}(\Sigma)) \neq 0$ will be proved in Lemma 19, and hence ν is well-defined. The proof of Theorem 9 follows essentially the general framework developed for showing a graph to be DGS; see e.g., [17, 19, 20]. The new ingredient is the discovery of a divisibility relation over the ring of Eisenstein integers, in terms of the number ν of fixed points of the anti-automorphism of Σ (i.e.,

$(1+\omega)^{\frac{n-\nu}{2}}$ always divides $\det(W^{(2)}(\Sigma)))$. Moreover, some numerical experiments are further presented to illustrate the effectiveness of Theorem 9, which show that the generalized E-spectrum is superior to the generalized G-spectrum in distinguishing mixed graphs to a certain extent; see Table 1 in Section 4.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries of Eisenstein integer ring $\mathbb{Z}[\omega]$ and matrices over $\mathbb{Z}[\omega]$. In Section 3, we present the proof of our main theorem. Some numerical results are given in Section 4.

2 Preliminaries

For convenience of the reader, we shall give some preliminaries that will be needed later in the paper. For simplicity, in Sections 2 and 3, we shall write A for $A^{(2)}$ and W for $W^{(2)}$ if no confusions arise.

2.1 Eisenstein integers

Since every entry of the E-adjacency matrix of a mixed graph is an Eisenstein integer, we shall briefly review some property of the ring $\mathbb{Z}[\omega]$ of Eisenstein integers, which is crucial to our proof. Let $z = a + b\omega \in \mathbb{Z}[\omega]$, where $a, b \in \mathbb{Z}$ and $\omega = \frac{1+\mathrm{i}\sqrt{3}}{2}$. The *norm* of z is $N(z) = \bar{z}z = a^2 + b^2 + ab$. The set of units of $\mathbb{Z}[\omega]$ is $U = \{1, -1, \omega, -\omega, \overline{\omega}, -\overline{\omega}\}$. Two Eisenstein integers z, z' are said to be *associates*, if $z = uz'$ for some unit u. An Eisenstein integer z is said to be *prime*, if $z \notin U$ and is only divisible by units and associates of z. $\mathbb{Z}[\omega]$ is a principal ideal domain (PID) and hence a unique factorization domain (UFD), i.e., the prime factorization of elements in the classical sense holds over $\mathbb{Z}[\omega]$. For clarity, when we talk about the greatest common divisor or the least common multiple of Eisenstein integers $z_1, z_2, \ldots, z_m$, we always assume $\arg(\gcd(z_1, z_2, \ldots, z_m))$ and $\arg(\mathrm{lcm}(z_1, z_2, \ldots, z_m)) \in [0, \frac{\pi}{3})$. For any Eisenstein integral vector v and any Eisenstein integral matrix M, $v^* = \overline{v}^{\mathrm{T}}$ and $M^* = \overline{M}^{\mathrm{T}}$ denote the conjugate transpose of v and M, respectively. For more background information on Eisenstein integers, see [9].

All the Eisenstein primes can be divided into the following three classes.
(i) $p = 1 + \omega$;
(ii) p is the ordinary (rational) prime and $p \equiv 2 \pmod 3$, e.g., $p = 2, 5, 11$;
(iii) $p = a + b\omega$, and $N(p) = a^2 + ab + b^2 \equiv 1 \pmod 3$ is a prime over $\mathbb{Z}$, e.g., $p = 2 + \omega, 3 + \omega, 3 + 2\omega$.

We call an Eisenstein integer $z \in \mathbb{Z}[\omega]$ *square-free* (resp. *norm-free*) if $p^2 \nmid z$ (resp. $N(p) \nmid z$) for any Eisenstein prime p.

Since $\mathbb{Z}[\omega]$ is a PID, the ideal (p) generated by the prime Eisenstein integer p is a maximal ideal, and hence $\mathbb{Z}[\omega]/(p)$ is a field. For Eisenstein integers z_1, z_2 and $z \neq 0$, we write $z_1 \equiv z_2 \pmod z$ if z divides $z_1 - z_2$. For two m by n Eisenstein integral matrices $M = (M_{ij})$ and $N = (N_{ij})$, we write $M \equiv N \pmod z$ if $M_{ij} \equiv N_{ij} \pmod z$ for $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$. The following properties of the field $\mathbb{Z}[\omega]/(1+\omega)$ are useful in our proof.

Lemma 10. (i) $\overline{\omega} \equiv \omega \equiv -1 \pmod{1+\omega}$;
(ii) *For* $z_1 \in \mathbb{Z}$, $(1+\omega) \mid z_1$ *if and only if* $3 \mid z_1$;
(iii) $\forall z_2 \in \mathbb{Z}[\omega]$ *and* $z_2 \not\equiv 0 \pmod{1+\omega}$, $N(z_2) \equiv 1 \pmod{1+\omega}$.

Proof. (i) $\overline{\omega} - \overline{\omega}(1+\omega) = \omega - (1+\omega) = -1$.
(ii) Since $3 = \overline{\omega}(1+\omega)^2$, and 3 is prime over $\mathbb{Z}$, the proof is complete.
(iii) $\forall z_2 = a + b\omega \in \mathbb{Z}[\omega]$, with $a, b \in \mathbb{Z}$ and $z_2 \not\equiv 0 \pmod{1+\omega}$. Then, $a - b \equiv z_2 \not\equiv 0 \pmod{1+\omega}$. By (ii), one obtains $3 \nmid a-b$. Then, $N(z_2) = a^2+ab+b^2 \equiv (a-b)^2+3ab \equiv 1+0 = 1 \pmod{1+\omega}$. □

With the notions above, our main theorem can be restated as follows.

Theorem 11. *Let* $\Sigma \in \mathcal{G}_n^{sc}$ *with E-walk matrix* $W(\Sigma)$ *and* $\det(W(\Sigma)) \neq 0$. *Let* ν *be the the number of fixed points of the anti-automorphism of* Σ. *If* $(1+\omega)^{-\frac{n-\nu}{2}} \det(W(\Sigma))$ *is square-free (resp. norm-free) as an Eisenstein integer, then* Σ *is RDGES (resp. SDGES).*

2.2 The Smith normal form of a matrix over a PID

The Smith Normal Form (SNF for short) is a useful tool, which plays an important role in our argument. Let R be a principal ideal domain (PID). A matrix U over R of order n is called *unimodular* if $\det(U)$ is a unit in R. The following theorem is well-known.

Theorem 12 ([8])**.** *For an* $n \times n$ *matrix* M *over* R, *there exist unimodular matrices* V_1 *and* V_2 *such that* $M = V_1 S V_2$, *where* $S = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ *is a diagonal matrix with* $d_i \mid d_{i+1}$ *for* $i = 1, 2, \ldots, n-1$ *over* R.

The above diagonal matrix S is called the *Smith normal form* of M, and d_i is called the i-th *invariant factor* of M. The SNF is unique up to associates. For convenience, we may assume that either $d_i = 0$ or $\arg(d_i) \in [0, \frac{\pi}{3})$ for $S = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ which is SNF of an Eisenstein integral matrix M.

The following lemma plays a key role in the proof of Theorem 9.

Lemma 13 ([19])**.** *Let* $S = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ *be the SNF of an Eisenstein integral matrix* M. *Then* $Mx \equiv 0 \pmod{p^2}$ *has a solution* $x \not\equiv 0 \pmod{p}$ *iff* $p^2 \mid d_n$.

2.3 The strategy for showing a graph to be DGS

Let $\mathbb{Q}(\omega) = \{a + b\omega : a, b \in \mathbb{Q}\}$ be the Eisenstein rational field obtained by adding ω to $\mathbb{Q}$. Let $U_n(\mathbb{Q}(\omega))$ be the set of all the unitary matrices of order n over $\mathbb{Q}(\omega)$.

The following lemma is the starting point for showing a mixed graph to be RDGES (resp. SDGES), which is a natural generalization of the result for the ordinary adjacency matrix of simple graph proposed in [7, 17]. The proof can be extended to E-adjacency matrix without much changes, and hence is omitted here.

Lemma 14 (cf. [7, 17])**.** *Let* Σ *and* Δ *be two mixed graphs. Then* Σ *and* Δ *are generalized cospectral w.r.t. the E-adjacency matrix if and only if there exists a unitary matrix* Q *such that* $Q^* A(\Sigma) Q = A(\Delta)$ *and* $Q\mathbf{1} = \mathbf{1}$. *Moreover, if* $\det(W(\Sigma)) \neq 0$, *then* $Q = (W(\Delta)(W(\Sigma))^{-1})^* \in U_n(\mathbb{Q}(\omega))$ *and hence is unique.*

Define

$$\mathcal{U}(\Sigma) := \{Q \in U_n(\mathbb{Q}(\omega)) : Q^* A(\Sigma) Q = A(\Delta) \text{ and } Q\mathbf{1} = \mathbf{1}, \text{ where } \Delta \in \mathcal{G}_n\}$$

and

$$\mathcal{U}^{sc}(\Sigma) := \{Q \in U_n(\mathbb{Q}(\omega)) : Q^* A(\Sigma) Q = A(\Delta) \text{ and } Q\mathbf{1} = \mathbf{1}, \text{ where } \Delta \in \mathcal{G}_n^{sc}\}.$$

Lemma 15 (cf. [7, 17])**.** *A mixed graph $\Sigma \in \mathcal{G}_n^{sc}$ is SDGES (resp. RDGES) if and only if $\mathcal{U}(\Sigma)$ (resp. $\mathcal{U}^{sc}(\Sigma)$) contains only permutation matrices.*

In order to show every Q in $\mathcal{U}(\Sigma)$ (resp. $\mathcal{U}^{sc}(\Sigma)$) is a permutation matrix, the following notion is proved to be useful, which is a generalization of that in [17].

Definition 16. Let Q be a unitary matrix over the Eisenstein rational field $\mathbb{Q}(\omega)$. The level of Q, denoted by ℓ or $\ell(Q)$, is an Eisenstein integer with $\arg(\ell) \in [0, \frac{\pi}{3})$ such that $\ell Q \in M_n(\mathbb{Z}[\omega])$ and $N(\ell)$ is minimal.

The following lemma shows that a unitary matrix Q over Eisenstein rational field $\mathbb{Q}(\omega)$ with $Q\mathbf{1} = \mathbf{1}$ is a permutation matrix if and only if the level $\ell(Q)$ of Q is one.

Lemma 17. *Let $Q \in U_n(\mathbb{Q}(\omega))$ with $Q\mathbf{1} = \mathbf{1}$. Then Q is a permutation matrix if and only if $\ell(Q) = 1$.*

Proof. The necessity part is clear. We only prove the sufficiency part. Suppose that $\ell(Q) = 1$, we shall show that Q must be a permutation matrix. Let $q = (q_1, q_2, \ldots, q_n)^*$ be any column of Q. Then $q_i \in \mathbb{Z}[\omega]$, and

$$1 = q^* q = \sum_{i=1}^{n} N(q_i).$$

Note that $N(z) \geqslant 1$ for any $0 \neq z \in \mathbb{Z}(\omega)$. Thus, there is exactly one entry, say q_i, of q satisfying $N(q_i) = 1$, i.e., q_i is a unit and all other entries of q are 0. Moreover, note that $Q\mathbf{1} = \mathbf{1}$, one obtains $q_i = 1$. By the arbitrariness of q, Q is a permutation matrix. $\square$

Thus, in order to show that a mixed graph $\Sigma \in \mathcal{G}_n^{sc}$ is SDGES (resp. RDGES), it suffices to show that any Eisenstein prime p does not divide the level $\ell(Q)$, for any $Q \in \mathcal{U}(\Sigma)$ (resp. $Q \in \mathcal{U}^{sc}(\Sigma)$).

The following lemma says that the level ℓ is always a divisor of the n-th invariant factor of the E-walk matrix $W(\Sigma)$.

Lemma 18. *Let $\Sigma \in \mathcal{G}_n^{sc}$ with E-walk matrix $\det(W(\Sigma)) \neq 0$. Let $Q \in \mathcal{U}(\Sigma)$ with level ℓ. Let $S = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the SNF of W^*. Then $\ell \mid d_n$.*

Proof. Let $W^*(\Sigma) = USV = U\mathrm{diag}(d_1, d_2, \ldots, d_n)V$. By Lemma 14, for $Q \in \mathcal{U}(\Sigma)$ and $Q^* A(\Sigma) Q = A(\Delta)$, then

$$d_n Q = d_n (W(\Delta)(W(\Sigma))^{-1})^* = U\mathrm{diag}(d_n/d_1, d_n/d_2, \ldots, d_n/d_n) V W(\Delta)^*.$$

Since U, V are unimodular, $d_n Q \in M_n(\mathbb{Z}[\omega])$. Hence $\ell \mid d_n$. $\square$

3 Proof of Theorem 9

In this section, we present the proof of Theorem 9. By Lemma 17, we only need to show that the level ℓ of unitary matrix in $\mathcal{U}(\Sigma)$ (resp. $\mathcal{U}^{sc}(\Sigma)$) is 1 under our assumptions. We shall show that any Eisenstein prime p which is not an associate of $1+\omega$ is not a prime factor of ℓ, and hence $\ell = 1$ or $1+\omega$ based on the SNF of the E-walk matrix $W^*(\Sigma)$. Then, we prove that $\ell \neq 1+\omega$, and hence $\ell = 1$ and Q is a permutation matrix.

Note that $\mathbb{Z}[\omega]/(p)$ is a field. We shall write $\mathrm{rank}_p(M)$ for the rank of an Eisenstein integral matrix M over the field $\mathbb{Z}[\omega]/(p)$ in what follows.

3.1 The SNF of $\boldsymbol{W^*(\Sigma)}$

In this subsection, we determine the SNF of $W(\Sigma)$. Before doing so, we need the following lemma which shows a particular structure possessed by a self-converse mixed graph with a non-vanishing determinant of E-walk matrix.

Lemma 19. *Suppose that Σ is a self-converse mixed graph with E-walk matrix W. Let ν be the number of the fixed points of the anti-automorphism Φ of Σ. If $\det(W(\Sigma)) \neq 0$, then Φ is unique and is a swap over $\frac{n-\nu}{2}$ pairs of vertices.*

Proof. Let Φ be an anti-automorphism of Σ and P be the corresponding permutation matrix. By the definition, $P^{\mathrm{T}}AP = A(\Sigma^{\mathrm{T}}) = A^{\mathrm{T}} = \overline{A}$. Since $P^{\mathrm{T}}\mathbf{1} = \mathbf{1}$, one obtains $\overline{A}^k\mathbf{1} = P^{\mathrm{T}}A^k\mathbf{1}$, and then

$$\begin{aligned} \overline{W} &= [\overline{\mathbf{1}}, \overline{A\mathbf{1}}, \ldots, \overline{A^{n-1}\mathbf{1}}] \\ &= [\mathbf{1}, \overline{A}\mathbf{1}, \ldots, \overline{A}^{n-1}\mathbf{1}] \\ &= [\mathbf{1}, P^{\mathrm{T}}A\mathbf{1}, \ldots, P^{\mathrm{T}}A^{n-1}\mathbf{1}] \\ &= P^{\mathrm{T}}W. \end{aligned} \tag{3}$$

Taking determinants on both sides, one obtains

$$\overline{\det(W)} = \det(\overline{W}) = \det(P^{\mathrm{T}})\det(W) = \pm\det(W). \tag{4}$$

Thus, $\det(W)$ is real or pure imaginary.

Note that $P^{\mathrm{T}}AP = \overline{A}$. It follows that $P^{\mathrm{T}}\overline{A}P = \overline{P^{\mathrm{T}}AP} = A$. Thus, $PAP^{\mathrm{T}} = \overline{A}$. Note that $\det(W) \neq 0$. By Lemma 14, one obtains that P is unique and $P = P^{\mathrm{T}}$ and $P^2 = I$. Hence Φ is unique and Φ is an involution, i.e., $\Phi^2 = Id$. Then for every vertex $u \in V(\Sigma)$ that is not a fixed point of Φ, there exists another vertex $v \in V(\Sigma)$ such that $\Phi(u) = v$ and $\Phi(v) = u$. There are exactly $\frac{n-\nu}{2}$ such pairs of vertices, and Φ is a swap over the $\frac{n-\nu}{2}$ pairs of vertices. □

According to the Lemma 19, let P be the corresponding permutation matrix of Φ, $\det(P) = 1$ if $\frac{n-\nu}{2}$ is even, and then $\det(P) = -1$ if $\frac{n-\nu}{2}$ is odd. By equation (4), $\det(W)$ is real if $\frac{n-\nu}{2}$ is even, and is pure imaginary if $\frac{n-\nu}{2}$ is odd. And we can get the following lemma for the structure of the nullspace of the E-walk matrix of self-converse mixed graph over $\mathbb{Z}[\omega]/(1+\omega)$.

Lemma 20. *Let $\Sigma \in \mathcal{G}_n^{sc}$ with E-adjacency matrix $A(\Sigma)$ and $\det(W(\Sigma)) \neq 0$. Let ν be the number of fixed points of the anti-automorphism of Σ. Then $\mathrm{rank}_{1+\omega}(W(\Sigma)) \leqslant \frac{n+\nu}{2}$.*

Proof. By Lemma 19, we can relabel the vertices of Σ such that the anti-automorphism Φ of Σ just swaps the vertices i and $i + \frac{n-\nu}{2}$ for $i \in \{1, 2, \dots, \frac{n-\nu}{2}\}$ and keeps the last ν vertices fixed. Let P be the permutation matrix corresponding to Φ. Then,

$$P = \begin{bmatrix} O_{\frac{n-\nu}{2}} & I_{\frac{n-\nu}{2}} & \\ I_{\frac{n-\nu}{2}} & O_{\frac{n-\nu}{2}} & \\ & & I_\nu \end{bmatrix} \tag{5}$$

By Lemma 10, $\overline{\omega} \equiv \omega \equiv -1 \pmod{1+\omega}$. Thus, $A \equiv \overline{A} \equiv A \pmod{1+\omega}$. Then, $PAP \equiv \overline{A} \pmod{1+\omega}$, and hence $(P-I)A^k\mathbf{1} \equiv 0 \pmod{1+\omega}$. Thus,

$$(P-I)W \equiv 0 \pmod{1+\omega}. \tag{6}$$

Note that

$$P - I = \begin{bmatrix} -I_{\frac{n-\nu}{2}} & I_{\frac{n-\nu}{2}} & \\ I_{\frac{n-\nu}{2}} & -I_{\frac{n-\nu}{2}} & \\ & & O_\nu \end{bmatrix}, \tag{7}$$

so $\mathrm{rank}_{1+\omega}(P-I) = \frac{n-\nu}{2}$. Therefore, $\mathrm{rank}_{1+\omega}(W) \leqslant n - \frac{n-\nu}{2} = \frac{n+\nu}{2}$. ◻

Now, we can get the structure of the SNF of $W(\Sigma)$.

Lemma 21. *Let $\Sigma \in \mathcal{G}_n^{sc}$ with adjacency matrix $A(\Sigma)$ and $\det(W(\Sigma)) \neq 0$. Let ν be the number of the fixed points of the anti-automorphism of Σ. If $(1+\omega)^{-\frac{n-\nu}{2}}\det(W)$ is square-free (resp. norm-free), then $W(\Sigma)$ and $W(\Sigma)^*$ have the same SNF given as follows:*

$$S = \mathrm{diag}(\underbrace{1, 1, \dots, 1}_{\frac{n+\nu}{2}}, \underbrace{1+\omega, 1+\omega, \dots, 1+\omega, (1+\omega)b}_{\frac{n-\nu}{2}}),$$

where b is a square-free (resp. norm-free) Eisenstein integer with $(1+\omega) \nmid b$.

Proof. Note that $\det(W)$ is real if $\frac{n-\nu}{2}$ is even, and is pure imaginary if $\frac{n-\nu}{2}$ is odd. It follows that $(\omega(1+\omega))^{-\frac{n-\nu}{2}}\det(W)$ is real and square-free (resp. norm-free). Let $b = |(\omega(1+\omega))^{-\frac{n-\nu}{2}}\det(W)|$. For a prime p which is not an associate of $1+\omega$ and $p \mid b$, then $\overline{p} \mid b$. Thus, either up is real for some unit $u \in U$ or $N(p) = \mathrm{lcm}(p, \overline{p})$ and $N(p) \mid b$. Since $u(1+\omega)$ cannot be real for any unit $u \in U$ and b is real and square-free (resp. norm-free), the exponent of $1+\omega$ is not 1 in the Eisenstein prime factorization of b and hence $(1+\omega) \nmid b$.

Let $b = p_1p_2\cdots p_s$, where p_i's are distinct primes which is not an associate of $1+\omega$ for each i. Then, $\det(W) = u(1+\omega)^{\frac{n-\nu}{2}}p_1p_2\cdots p_s$, where $u \in U$ is a unit. Thus the SNF of W can be written as

$$S = \mathrm{diag}(1, 1, \dots, 1, u_1(1+\omega)^{l_1}, u_2(1+\omega)^{l_2}, \dots, u_t(1+\omega)^{l_t}b),$$

where $u_i \in U$ for $i \in \{1, 2, \ldots, \nu\}$. It follows from Lemma 20 that $\mathrm{rank}_{1+\omega}(W) \leqslant \frac{n+\nu}{2}$, i.e., $n - t \leqslant \frac{n+\nu}{2}$. Thus, $t \geqslant n - \frac{n+\nu}{2} = \frac{n-\nu}{2}$. Moreover, since $\det(W) = u'\det(S)$ for some unit $u' \in U$, $l_1 + l_2 + \cdots + l_t = \frac{n-\nu}{2}$. It follows that $l_1 = l_2 = \cdots = l_t = 1$ and $t = \frac{n-\nu}{2}$.

Since $W \equiv \overline{W} \pmod{1+\omega}$, then $\mathrm{rank}_{1+\omega}(W) = \mathrm{rank}_{1+\omega}(\overline{W}) = \mathrm{rank}_{1+\omega}(W^*)$. Also, by Lemma 19, $\det(W^*) = \pm\det(W)$, and hence $(\omega(1+\omega))^{-\frac{n-\nu}{2}}\det(W)$ is real and square-free (resp. norm-free). By a similar argument, W^* has the same SNF as W. $\square$

3.2 Excluding the prime factors p not an associate of $1 + \omega$

The main result of this subsection is the following theorem.

Theorem 22. *Let $\Sigma \in \mathcal{G}_n^{sc}$ with $\det(W(\Sigma)) \neq 0$, and $Q \in \mathcal{U}^{sc}(\Sigma)$ (resp. $Q \in \mathcal{U}(\Sigma)$) with level ℓ. Let ν be the number of the fixed points of the anti-automorphism, and p be an Eisenstein prime which is not an associate of $1 + \omega$. If $(1 + \omega)^{-\frac{n-\nu}{2}}\det(W(\Sigma))$ is square-free (resp. norm-free), then $p \nmid \ell$.*

To prove Theorem 22, we need several lemmas below. For simplicity, we write A for $A(\Sigma)$ and W for $W(\Sigma)$ in what follows.

Lemma 23. *Let $\Sigma \in \mathcal{G}_n^{sc}$ with $\det(W(\Sigma)) \neq 0$. Let $Q \in \mathcal{U}(\Sigma)$ with level $\ell \neq 1$. Suppose that p is an Eisenstein prime which is not an associate of $1 + \omega$, and p is a divisor of $\gcd(\ell, \overline{\ell})$, and $\mathrm{rank}_p(W(\Sigma)) = n - 1$. Then there exists an Eisenstein integral vector $v \not\equiv 0 \pmod{p}$, $v \not\equiv 0 \pmod{\overline{p}}$ and an integer $\lambda_0 \in \mathbb{Z}$ such that*

$$v^* A(\Sigma)^k v \equiv 0 \pmod{\mathrm{lcm}(p, \overline{p})^2}, \text{ for any } k \geqslant 0, \tag{8}$$

$$W(\Sigma)^* v \equiv 0 \pmod{\mathrm{lcm}(p, \overline{p})}, \tag{9}$$

$$A(\Sigma)v \equiv \lambda_0 v \pmod{\mathrm{lcm}(p, \overline{p})}. \tag{10}$$

Proof. Suppose that $Q^*AQ = A(\Delta)$ for some mixed graph Δ. Let $\tilde{Q} = \ell Q$. Then $\tilde{Q}$ is an Eisenstein integral matrix. We distinguish the following two cases.

Case 1. $N(p)$ is a prime over $\mathbb{Z}$.

In this case, $\mathrm{lcm}(p, \overline{p}) = N(p)$. Note that $p \mid \gcd(\ell, \overline{\ell})$. It follows that $\overline{p} \mid \gcd(\ell, \overline{\ell})$. Hence, $N(p) \mid \gcd(\ell, \overline{\ell})$. Let v_1 and v_2 be two columns of $\tilde{Q}$ such that $v_1 \not\equiv 0 \pmod{p}$ and $v_2 \not\equiv 0 \pmod{\overline{p}}$. Let $v = a\overline{p}v_1 + bpv_2 = \tilde{Q}q$, where $a, b \in \mathbb{Z}[\omega]$, $a \not\equiv 0 \pmod{p}$ and $b \not\equiv 0 \pmod{\overline{p}}$ and $q \in \mathbb{Z}[\omega]^n$. Then $v \not\equiv 0 \pmod{p}$ and $v \not\equiv 0 \pmod{\overline{p}}$. It follows from $Q^*AQ = A(\Delta)$ that $Q^*A^kQ = A^k(\Delta)$ for any $k \geqslant 0$. Note that $N(p) \mid \gcd(\ell, \overline{\ell})$. It follows that

$$v^*A^kv \equiv q^*\tilde{Q}^*A^k\tilde{Q}q = \ell\overline{\ell}q^*A^k(\Delta)q \equiv 0 \pmod{N(p)^2}.$$

Since $W^*Q = W^*(\Delta)$ and $W^*AQ = W^*(\Delta)A(\Delta)$, then $W^*v \equiv W^*\tilde{Q}q = \ell W^*(\Delta)q \equiv 0 \pmod{N(p)}$ and $W^*Av \equiv W^*A\tilde{Q}q = \ell W^*(\Delta)A(\Delta)q \equiv 0 \pmod{N(p)}$. Since

$\operatorname{rank}_p(W^*) = \operatorname{rank}_{\overline{p}}(W^*) = \operatorname{rank}_{\overline{p}}(W) = n-1$, one obtains $Av \equiv \lambda_1 v \pmod{p}$ for some Eisenstein integer λ_1. Similarly, $Av \equiv \lambda_2 v \pmod{\overline{p}}$ for some Eisenstein integer λ_2. Let γ_1, γ_2 be two Eisenstein integers such that $\gamma_1\overline{p} \equiv 1 \pmod{p}$, $\gamma_2 p \equiv 1 \pmod{\overline{p}}$, and $\lambda_0 = \lambda_1\gamma_1\overline{p} + \lambda_2\gamma_2 p$. Then $Av \equiv \lambda_0 v \pmod{N(p)}$.

Case 2. up is a positive prime over $\mathbb{Z}$ for some $u \in U$.

In this case, $\operatorname{lcm}(p,\overline{p}) = up$. Let v be a column of $\tilde{Q}$ such that $v \not\equiv 0 \pmod{p}$. Since $Q^*A^kQ = A^k(\Delta)$ for any $k \geqslant 0$, then $\tilde{Q}^*A^k\tilde{Q} = \ell\overline{\ell}A^k(\Delta) \equiv 0 \pmod{p^2}$ and hence $v^*A^kv \equiv 0 \pmod{p^2}$. Since $W^*Q = W^*(\Delta)$ and $W^*AQ = W^*(\Delta)A(\Delta)$, then $W^*\tilde{Q} = \ell W^*(\Delta) \equiv 0 \pmod{p}$ and $W^*A \equiv W^*A\tilde{Q} = \ell W^*(\Delta)A(\Delta) \equiv 0 \pmod{p}$. Hence, $W^*v \equiv 0 \pmod{p}$ and $W^*Av \equiv 0 \pmod{p}$. Since $\operatorname{rank}_p(W^*) = \operatorname{rank}_{\overline{p}}(W^{\mathrm{T}}) = \operatorname{rank}_{\overline{p}}(W) = n-1$, one obtains $Av \equiv \lambda_0 v \pmod{p}$ for some Eisenstein integer $\lambda_0 \in \mathbb{Z}[\omega]$.

Next, we will show that $\operatorname{Im}(\lambda_0) \equiv 0 \pmod{\operatorname{lcm}(p,\overline{p})}$. Let P be the permutation matrix corresponding to the anti-automorphism Φ of Σ. Since $Av \equiv \lambda_0 v \pmod{\operatorname{lcm}(p,\overline{p})}$, then $\overline{A}\overline{v} \equiv \overline{\lambda_0}\overline{v} \pmod{\operatorname{lcm}(p,\overline{p})}$. Thus, $PAP\overline{v} \equiv \overline{\lambda_0}\overline{v} \pmod{\operatorname{lcm}(p,\overline{p})}$, so $A(P\overline{v}) \equiv \overline{\lambda_0}P\overline{v} \pmod{\operatorname{lcm}(p,\overline{p})}$.

Assume $\operatorname{Im}(\lambda_0) \not\equiv 0 \pmod{p}$. Then, v and $P\overline{v}$ are linearly independent over $\mathbb{Z}[\omega]/(p)$, because they are eigenvectors corresponding to the different eigenvalues. Note that $\mathbf{1}^{\mathrm{T}}A^kP\overline{v} \equiv \overline{\lambda_0}^k\mathbf{1}^{\mathrm{T}}P\overline{v} = \overline{\lambda_0}^k\mathbf{1}^{\mathrm{T}}\overline{v} \equiv 0 \pmod{p}$ for any $k \geqslant 0$. Therefore, $W^*P\overline{v} \equiv 0 \pmod{p}$, which contradicts with the fact that $\operatorname{rank}_p(W^*) = n-1$.

Using a similar argument over the field $\mathbb{Z}[\omega]/(\overline{p})$, one obtains $\operatorname{Im}(\lambda_0) \equiv 0 \pmod{\overline{p}}$, and hence $\operatorname{Im}(\lambda_0) \equiv 0 \pmod{\operatorname{lcm}(p,\overline{p})}$. Since $\mathbb{Z}[\omega] \cap \mathbb{R} = \mathbb{Z}$, we can choose $\lambda_0 \in \mathbb{Z}$. ☐

The following lemma shows that $A(\Sigma)-\lambda_0 I$ and v in Lemma 23 have a further property over $\mathbb{Z}[\omega]/(p)$.

Lemma 24. *Using the notations in Lemma 23, then* $\operatorname{rank}_p[A(\Sigma)-\lambda_0 I, v] = n-1$, *and there exists an Eisenstein integral vector* $w \not\equiv 0 \pmod{p}$ *such that* $\mathbf{1}^{\mathrm{T}}w \not\equiv 0 \pmod{p}$ *and* $(A(\Sigma)-\lambda_0 I)w \equiv sv \pmod{p}$, *where* $s = 0$ *or* 1.

Proof. Since $v \not\equiv 0 \pmod{\overline{p}}$, and

$$v^*[A-\lambda_0 I, v] \equiv 0 \pmod{p}, \tag{11}$$

then $\operatorname{rank}_p[A-\lambda_0 I, v] \leqslant n-1$.

Note that $W \sim [\mathbf{1}, (A-\lambda_0 I)\mathbf{1}, (A-\lambda_0 I)A\mathbf{1}, \ldots, (A-\lambda_0 I)A^{n-2}\mathbf{1}] = [\mathbf{1}, (A-\lambda_0 I)X]$ can be obtained from W by a simple elementary column transformation, where X is an Eisenstein integral matrix. Thus $\operatorname{rank}_p(A-\lambda_0 I) \geqslant \operatorname{rank}_p(W)-1 = n-2$. We consider the two following cases.

Case 1. If $\operatorname{rank}_p(A-\lambda_0 I) = n-1$, then $\operatorname{rank}_p[A-\lambda_0 I, v] = n-1$. Then, $v \equiv (A-\lambda_0 I)w \pmod{p}$ for some $w \in \mathbb{Z}[\omega]^n$. Thus,

$$\mathbf{1}^{\mathrm{T}}A^kw \equiv \mathbf{1}^{\mathrm{T}}A^{k-1}v + \lambda_0\mathbf{1}^{\mathrm{T}}A^{k-1}w \equiv \lambda_0 A^{k-1}\mathbf{1}^*w \equiv \lambda_0^k\mathbf{1}^{\mathrm{T}}w \pmod{p},$$

for any $k \geqslant 1$. It is obvious that v and w are linearly independent. Note that $W^*w \equiv \mathbf{1}^{\mathrm{T}}w(1, \lambda_0, \ldots, \lambda_0^{n-1})^*$. Since $\operatorname{rank}_p(W^*) = n-1$, then $\mathbf{1}^{\mathrm{T}}w \not\equiv 0 \pmod{p}$.

Case 2. If $\operatorname{rank}_p(A - \lambda_0 I) = n-2$, assume $\operatorname{rank}_p[A - \lambda_0 I, v] < n-1$. Then, $v \equiv (A - \lambda_0 I)w' \pmod{p}$ for some $w' \in \mathbb{Z}^n[\omega]$. Then,

$$\mathbf{1}^{\mathrm{T}}A^k w' \equiv \mathbf{1}^{\mathrm{T}}A^{k-1}v + \lambda_0 \mathbf{1}^{\mathrm{T}}A^{k-1}w' \equiv \lambda_0^k \mathbf{1}^* w' \pmod{p},$$

for any $k \geqslant 1$. Moreover, since $\operatorname{rank}_p(A - \lambda_0 I) = n-2$, there exists a vector y which is linearly independent with v over $\mathbb{Z}[\omega]/(p)$ such that $(A - \lambda_0 I)y \equiv 0 \pmod{p}$, and $\mathbf{1}^{\mathrm{T}}y \not\equiv 0 \pmod{p}$. It is easy to see that v, w' and y are linearly independent. Let $\alpha = (\mathbf{1}^{\mathrm{T}}y)w' - (\mathbf{1}^{\mathrm{T}}w')y$. Then $\alpha \not\equiv 0 \pmod{p}$, $\mathbf{1}^{\mathrm{T}}\alpha \equiv 0 \pmod{p}$, and $\mathbf{1}^{\mathrm{T}}A^k\alpha \equiv \lambda_0^k \mathbf{1}^{\mathrm{T}}y\mathbf{1}^{\mathrm{T}}w' - \lambda_0^k \mathbf{1}^{\mathrm{T}}w'\mathbf{1}^{\mathrm{T}}y \equiv 0 \pmod{p}$. Therefore, $W^*\alpha \equiv 0 \pmod{p}$, which contradicts the fact that $\operatorname{rank}_p(W^*) = n-1$.

Therefore, $\operatorname{rank}_p[A - \lambda_0 I, v] = n-1$. Since $\operatorname{rank}_p(A - \lambda_0 I) = n-2$, let w be the solution of $(A - \lambda_0 I)x \equiv 0 \pmod{p}$ linearly independent with v. Note that $W^*w \equiv \mathbf{1}^{\mathrm{T}}w(1, \lambda_0, \ldots, \lambda_0^{n-1})^*$. Since $\operatorname{rank}_p(W^*) = n-1$, then $\mathbf{1}^{\mathrm{T}}w \not\equiv 0 \pmod{p}$. $\square$

The following lemma shows that under certain conditions, the linear system of congruence equations $W^*v \equiv 0 \pmod{p^2}$ always has a non-trivial solution.

Lemma 25. *Let $\Sigma \in \mathcal{G}_n^{sc}$ with $\det(W(\Sigma)) \neq 0$, and $Q \in \mathcal{U}(\Sigma)$ with level $\ell \neq 1$. Suppose that p is an Eisenstein prime which is not an associate of $1 + \omega$, and p is a divisor of $\gcd(\ell, \overline{\ell})$, and $\operatorname{rank}_p(W(\Sigma)) = \operatorname{rank}_{\overline{p}}(W(\Sigma)) = n-1$. Then there exists an Eisenstein integral vector $v \not\equiv 0 \pmod{p}$ such that*

$$W(\Sigma)^* v \equiv 0 \pmod{p^2}. \tag{12}$$

Proof. Let v and λ_0 be the vector and integer in Lemma 23. By Lemma 23, one obtains $v^*Av \equiv \lambda_0 v^*v \equiv 0 \pmod{p^2}$. Hence,

$$v^*(A - \lambda_0 I)v \equiv 0 \pmod{p^2}.$$

Since $(A - \lambda_0 I)v \equiv 0 \pmod{p}$, one obtains that

$$v^* \frac{(A - \lambda_0 I)v}{p} \equiv 0 \pmod{p}.$$

Since $v \not\equiv 0 \pmod{\overline{p}}$, $v^*[A - \lambda_0 I, v] \equiv 0 \pmod{p}$, and $\operatorname{rank}_p[A - \lambda_0 I, v] = n-1$, there exists an Eisenstein integral vector $\begin{bmatrix} y \\ m \end{bmatrix}$, where $y \in \mathbb{Z}[\omega]^n$ and $m \in \mathbb{Z}[\omega]$. Thus,

$$\frac{(A - \lambda_0 I)v}{p} \equiv [A - \lambda_0 I, v] \begin{bmatrix} y \\ m \end{bmatrix} = (A - \lambda_0 I)y + mv \pmod{p}.$$

It follows that $\frac{(A-\lambda_0 I)(v-py)}{p} \equiv mv \pmod{p}$. Thus,

$$\mathbf{1}^{\mathrm{T}}A^k \frac{(A - \lambda_0 I)(v - py)}{p} \equiv m\mathbf{1}^{\mathrm{T}}A^k v \equiv 0 \pmod{p},$$

for any $k \geqslant 0$, and hence

$$\mathbf{1}^{\mathrm{T}}A^{k+1}(v-py) \equiv \lambda_0\mathbf{1}^{\mathrm{T}}A^k(v-py) \equiv \lambda_0^{k+1}\mathbf{1}^{\mathrm{T}}(v-py) \pmod{p^2}.$$

Therefore,

$$W^*(v-py) \equiv \mathbf{1}^{\mathrm{T}}(v-py)(1,\lambda_0,\ldots,\lambda_0^{n-1})^* \pmod{p^2}.$$

By Lemma 24, there exists an Eisenstein integral vector $w \not\equiv 0 \pmod{p}$ such that $\mathbf{1}^{\mathrm{T}}w \not\equiv 0 \pmod{p}$ and $(A-\lambda_0 I)w \equiv sv \pmod{p}$, where $s=0$ or 1. Thus,

$$\mathbf{1}^{\mathrm{T}}A^{k+1}w \equiv \lambda_0\mathbf{1}^{\mathrm{T}}A^k w + s\mathbf{1}^{\mathrm{T}}A^k v \equiv \lambda_0\mathbf{1}^{\mathrm{T}}A^k w \equiv \lambda_0^{k+1}\mathbf{1}^{\mathrm{T}}w$$

for any $k \geqslant 0$, and hence

$$W^*w \equiv \mathbf{1}^{\mathrm{T}}w(1,\lambda_0,\ldots,\lambda_0^{n-1})^* \pmod{p}.$$

Let $g \in \mathbb{Z}[\omega]$ such that $g\mathbf{1}^{\mathrm{T}}w \equiv 1 \pmod{p}$. Then it follows from $\mathbf{1}^{\mathrm{T}}(v-py) \equiv 0 \pmod{p}$ that

$$W^*(v-py-g\mathbf{1}^{\mathrm{T}}(v-py)w) \equiv (\mathbf{1}^{\mathrm{T}}(v-py)-g\mathbf{1}^{\mathrm{T}}(v-py)\mathbf{1}^{\mathrm{T}}w)(1,\lambda_0,\ldots,\lambda_0^{n-1})^* \equiv 0 \pmod{p^2},$$

and $v-py-g\mathbf{1}^{\mathrm{T}}(v-py)w \equiv v \not\equiv 0 \pmod{p}$, as desired. □

This leads to the following lemma.

Lemma 26. *Let $\Sigma \in \mathcal{G}_n^{sc}$ with $\det(W(\Sigma)) \neq 0$, and $Q \in \mathcal{U}(\Sigma)$ with level ℓ. Then for any prime p which is not an associate of $1+\omega$. If $p^2 \nmid \det(W(\Sigma))$, then $p \nmid \gcd(\ell,\overline{\ell})$.*

Proof. By Lemma 18 and Lemma 21, the n-th invariant factor of W^* is $d_n = (1+\omega)b = (1+\omega)|(\omega(1+\omega))^{-\frac{n-\nu}{2}}\det(W)|$ and $\ell \mid d_n$. Thus, for any prime p which is not an associate of $1+\omega$, if $p \mid \ell$, then $p \mid b$, and hence, $\mathrm{rank}_p(W) = \mathrm{rank}_{\overline{p}}(W) = n-1$. Then, the lemma follows immediately from Lemma 13 and Lemma 25. The proof is complete. □

Now, we present the proof of Theorem 22.

Proof of Theorem 22. If $Q \in \mathcal{U}^{sc}(\Sigma)$ and $(1+\omega)^{-\frac{n-\nu}{2}}\det(W)$ is square-free, then $Q^*A(\Sigma)Q = A(\Delta)$, where $\Delta \in \mathcal{G}_n^{sc}$. It follows that $Q = (W^*(\Sigma))^{-1}W^*(\Delta)$. Let P be the permutation matrix corresponding to the unique anti-automorphism of Σ, and R be the converse permutation matrix of Δ. Then, $\overline{(W^*(\Sigma))^{-1}} = (W^*(\Sigma)P)^{-1} = P(W^*(\Sigma))^{-1}$ and $\overline{W^*(\Delta)} = W^*(\Delta)R$. Then,

$$\overline{Q} = \overline{(W^*(\Sigma))^{-1}W^*(\Delta)} = P(W^*(\Sigma))^{-1}W^*(\Delta)R = PQR.$$

Thus, $\overline{\ell}Q = \overline{\ell\overline{Q}} = P\overline{\ell Q}R$ is an Eisenstein integral matrix. Then, $\ell \mid \overline{\ell}$, and hence $\ell = \gcd(\ell,\overline{\ell})$. By Lemma 26, $p \nmid \ell$.

If $Q \in \mathcal{U}(\Sigma)$ and $(1+\omega)^{-\frac{n-\nu}{2}}\det(W)$ is norm-free, by Lemma 18, $\ell \mid (1+\omega)^{-\frac{n-\nu}{2}}\det(W)$. Note that ℓ is norm-free, so $\ell = \overline{\ell} = \gcd(\ell,\overline{\ell})$. By Lemma 26, $p \nmid \ell$.

The proof is complete. □

3.3 Excluding the case $\ell = 1 + \omega$

In this subsection, we exclude the case $\ell = 1 + \omega$ with the help of Lemma 10 and present the proof of Theorem 9.

Theorem 27. *Let $\Sigma \in \mathcal{G}_n^{sc}$ with $\det(W(\Sigma)) \neq 0$, and $Q \in \mathcal{U}^{sc}(\Sigma)$ (resp. $Q \in \mathcal{U}(\Sigma)$) with level ℓ. Let ν be the number of the fixed points of the anti-automorphism of Σ. If $(1+\omega)^{-\frac{n-\nu}{2}} \det(W(\Sigma))$ is square-free (resp. norm-free), then $\ell \neq 1 + \omega$.*

Proof. For contradiction, assume $Q \in U_n(\mathbb{Q}(\omega))$ with level $\ell = 1 + \omega$ and $Q^*AQ = A(\Delta)$ and $Q\mathbf{1} = \mathbf{1}$. Let $\tilde{Q} = (1+\omega)Q$. Then, there is a column v of $\tilde{Q}$ such that $v \not\equiv 0 \pmod{1+\omega}$. Let $S \subset \{1, 2, \ldots, n\}$ be an index set and $S := \{i : v_i \not\equiv 0 \pmod{1+\omega}\}$ and $a = \#S$. Since $W^*Q = W^*(\Delta)$, then $W^*\tilde{Q} = (1+\omega)W^*(\Delta)$ and hence $W^*v \equiv 0 \pmod{1+\omega}$. By Lemma 20 and (7), over $\mathbb{Z}[\omega]/(1+\omega)$, v can be expressed as a linear combination of $e_1 - e_{1+(n-\nu)/2}, e_2 - e_{2+(n-\nu)/2}, \ldots, e_{(n-\nu)/2} - e_{n-\nu}$, where e_i is the i-th column of the identity matrix I. Therefore, a is even.

Since $Q^*Q = I$, then $\tilde{Q}^*\tilde{Q} = 3I$ and hence $v^*v = 3$. By Lemma 10,

$$0 \equiv v^*v \equiv \sum_{i \in S} N(v_i) \equiv \sum_{i \in S} 1 = a \pmod{1+\omega}.$$

It follows that $3 \mid a$, and hence $a \geqslant 3$. Besides,

$$a \leqslant \#\{i : v_i \neq 0\} \leqslant \sum_{i \in \{i : v_i \neq 0\}} N(v_i) = v^*v = 3.$$

It follows that $a = 3$; a contradiction. Therefore, $\forall Q \in \mathcal{U}^{sc}(\Sigma)$ (resp. $\forall Q \in \mathcal{U}(\Sigma)$), $\ell(Q) \neq 1 + \omega$. The proof is complete. □

Finally, we are ready to present the proof of Theorem 9.

Proof of Theorem 9. Let $\Delta \in \mathcal{G}_n^{sc}$ (resp. $\Delta \in \mathcal{G}_n$) be any mixed graph generalized cospectral with Σ. Then there exists a $Q \in \mathcal{U}^{sc}(\Sigma)$ (resp. $Q \in \mathcal{U}(\Sigma)$) with level ℓ such that $Q^*A(\Sigma)Q = A(\Delta)$ and $Q\mathbf{1} = \mathbf{1}$. By Lemma 18, $\ell \mid d_n$. Moreover, by Lemma 21, $d_n = (1+\omega)b$, where b is square-free (resp. norm-free). Thus, $p \mid (1+\omega)b$. By Theorem 22, one obtains $\ell = 1$ or $1 + \omega$. By Theorem 27, one obtains $\ell = 1$ and Q is a permutation matrix and hence Δ is isomorphic to Σ. □

4 Some numerical results

In this section, we shall exhibit some numerical results as an illustration of Theorem 9.

First, we exhibit a graph $\Sigma \in \mathcal{G}_n^{sc}$ which is SDGES by using Theorem 9.

Example 28. Let Σ be a self-converse graph on 6 vertices, as shown in Figure 2.

Note that the number ν of the fixed points of the anti-automorphism of Σ is $\nu = 0$. And $(1+\omega)^{-\frac{6-0}{2}} \det(W^{(2)}(\Sigma)) = 1$ is norm-free. Hence, Σ is SDGES.

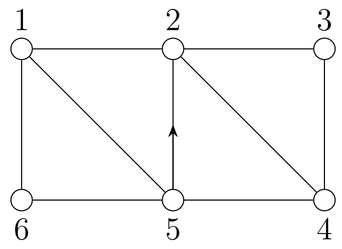

Figure 2: A graph Σ being SDGES

Next we exhibit a graph $\Sigma \in \mathcal{G}_n^{sc}$ which is RDGES by Theorem 9, but not SDGES, and we find a non-self-converse mixed graph having the same generalized spectrum as it.

Example 29. Let Σ be a self-converse mixed graph on 5 vertices, and Δ a non-self-converse mixed graph with 5 vertices, shown as in Figure 3.

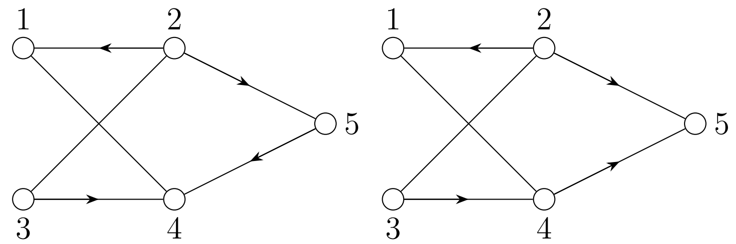

Figure 3: A graph Σ (on the left) being RDGES, but not SDGES, and its generalized cospectral mate Δ (on the right)

The number of the fixed points of the anti-automorphism of Σ is $\nu = 1$, and $(1+\omega)^{-\frac{5-1}{2}}\det(W^{(2)}(\Sigma)) = \omega(3+2\omega)(2+3\omega)$, which is square-free but not norm-free. Hence, Σ is RDGES but not SDGES. Note that Δ is not self-converse, so Σ and Δ are not isomorphic. The characteristic polynomials $\chi(A^{(2)}(\Sigma)) = x^5 - 6x^3 + 2x = \chi(A^{(2)}(\Delta))$ and $\chi(J - I - A^{(2)}(\Sigma)) = x^5 - 8x^3 - 5x^2 + 9x + 6 = \chi(J - I - A^{(2)}(\Delta))$, i.e., they are generalized cospectral w.r.t. the E-adjacency matrix.

Now, we show a pair of generalized cospectral self-converse graphs, and they do not satisfy the condition Theorem 9.

Example 30. Let Σ (on the left)and Δ (on the right) be a pair of self-converse graphs, as shown in Figure 4.

The numbers ν_Σ, ν_Δ of the fixed points of the anti-automorphism of Σ and Δ are $\nu_\Sigma = \nu_\Delta = 0$, and $(1+\omega)^{-\frac{6-0}{2}}\det(W^{(2)}(\Sigma)) = (1+\omega)^{-\frac{6-0}{2}}\det(W^{(2)}(\Delta)) = -\omega(1+2\omega)^2(2+\omega)^2$, which is neither square-free nor norm-free. Note that Σ has a vertex as terminal for 4 edges but Δ has no such vertex, so Σ and Δ are not isomorphic (actually, they are complementary graphs to each other). The characteristic polynomials $\chi(A^{(2)}(\Sigma)) = \chi(J-$

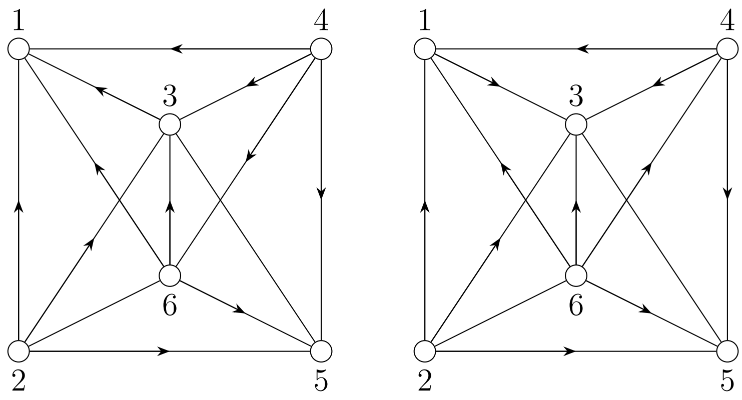

Figure 4: A pair of generalized cospectral self-converse mixed graphs

$I - A^{(2)}(\Sigma)) = x^6 - 13x^4 - 18x^3 + x^2 + 6x - 1 = \chi(A^{(2)}(\Delta)) = \chi(J - I - A^{(2)}(\Delta))$, i.e., they are generalized E-cospectral.

Moreover, we have enumerated all the mixed graphs with at most 6 vertices to find out how many graphs satisfy the conditions of Theorem 9; see Table 1.

Table 1: A comparison of E- and G- adjacency matrix

Order of graphs	2	3	4	5	6
# of self-converse mixed graphs	3	10	70	708	15224
# of mixed graphs strongly determined by its E-spectrum	1	3	9	83	1560
# of mixed graphs strongly determined by its G-spectrum	1	2	3	5	16
# of RDGES mixed graphs	3	10	70	708	15016
# of self-converse mixed graphs $\Sigma \in \mathcal{F}^{(2)}_{n,R}$	1	4	20	138	2794
# of RDGGS mixed graphs	3	10	64	603	12622
# of self-converse mixed graphs $\Sigma \in \mathcal{F}^{(1)}_{n,R}$	1	1	6	54	826
# of SDGES mixed graphs	3	10	68	666	14415
# of self-converse mixed graphs $\Sigma \in \mathcal{F}^{(2)}_{n,S}$	1	4	16	86	1300
# of SDGGS mixed graphs	3	10	61	530	11591
# of self-converse mixed graphs $\Sigma \in \mathcal{F}^{(1)}_{n,S}$	1	1	6	39	464

where

$$\mathcal{F}^{(1)}_{n,R} = \{\Sigma \in \mathcal{G}^{sc}_n : 2^{-\lfloor n/2 \rfloor} \det(W^{(1)}(\Sigma)) \text{ is odd and square-free}\};$$

$$\mathcal{F}^{(1)}_{n,S} = \{\Sigma \in \mathcal{G}^{sc}_n : 2^{-\lfloor n/2 \rfloor} \det(W^{(1)}(\Sigma)) \text{ is odd and norm-free}\};$$

$$\mathcal{F}^{(2)}_{n,R} = \{\Sigma \in \mathcal{G}^{sc}_n : (1+\omega)^{-(n-\nu)/2} \det(W^{(2)}(\Sigma)) \text{ is square-free}\};$$

$$\mathcal{F}^{(2)}_{n,S} = \{\Sigma \in \mathcal{G}^{sc}_n : (1+\omega)^{-(n-\nu)/2} \det(W^{(2)}(\Sigma)) \text{ is norm-free}\}.$$

It can be seen from Table 1 that the generalized E-spectrum is superior to the generalized G-spectrum in distinguishing mixed graphs to certain extent. For example, there are totally 15224 self-converse mixed graphs on 6 vertices, among which 15016 graphs are RDGES (resp. 12622 graphs are RDGGS), and there are 2794 RDGES graphs according to Theorem 9 (resp. 826 RDGGS graphs assuming Conjecture 7); there are 14415 graphs that are SDGES (resp. 11591 graphs are SDGGS), and there are 1300 SDGES graphs according to Theorem 9 (resp. 464 graphs SDGGS assuming Conjecture 7).

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