Reflection Groups and Quiver Mutation: Diagrammatics

Patrick Wegener

Abstract

We extend Carter's notion of admissible diagrams and attach a Dynkin-like diagram to each reduced reflection factorization of an element in a finite Weyl group. We give a complete classification for the diagrams attached to reduced reflection factorizations. We show that such a diagram is cyclically orientable if and only if it is isomorphic to the underlying graph of a quiver which is mutation-equivalent to a Dynkin quiver. Furthermore, these diagrams encode a natural presentation of the Weyl group as reflection group, as shown by Cameron–Seidel–Tsaranov (1994) as well as Barot–Marsh (2015).

Mathematics Subject Classifications: 06B15, 05E10, 20F55, 05E18

1   Introduction

Any Weyl group is generated by reflections. In particular, each element of a Weyl group can be written as a product of reflections. We call this a *reflection factorization*. In 1972, Carter [10] used a particular class of reflection factorizations of an element in a Weyl group to define so-called admissible diagrams. Using these diagrams, he was able to give a complete classification of the conjugacy classes in the Weyl groups. We extend this method and define a diagram attached to each reduced reflection factorization of every element in a Weyl group. The vertices of the diagram correspond to the reflections appearing in the reflection factorization. Two reflections s and t are connected by w_{st} edges, where w_{st} is the order of st minus two. We call the resulting diagram a *Carter diagram* (the precise definition will be given in Definition 5).

In Section 2 we give a complete classification of Carter diagrams. A Carter diagram is said to be *of type* X_n if the reflection group generated by the reflections in the corresponding reduced reflection factorization is of type X_n. For the infinite families A_n, B_n and D_n we explicitly give a method to construct all these diagrams. Here we will

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Leibniz Universität Hannover, Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Fakultät für Mathematik und Physik, Welfengarten 1, D-30167 Hannover, Germany (`patrick.wegener@math.uni-hannover.de`).

benefit from work of Kluitmann [17], who implicitly constructed the Carter diagrams of type A_n. For the exceptional types we obtain these diagrams by computational methods. We should mention that in the simply-laced types, that is types A_n, D_n, E_6, E_7, E_8, these diagrams were already described by Cameron–Seidel–Tsaranov [9], while the diagrams for the remaining types appear in the work of Felikson [11].

In Section 3 we link these diagrams to those graphs arising as underlying unoriented graphs in the mutation classes of Dynkin quivers.

Theorem 1. *Let Q be a quiver which is mutation-equivalent to an orientation of a Dynkin diagram. Then the underlying undirected graph $\overline{Q}$ is a Carter diagram of the same Dynkin type.*

Moreover, let Γ be a Carter diagram of Dynkin type. Then there exists a quiver Q which is mutation-equivalent to an orientation of the corresponding Dynkin diagram such that Γ is isomorphic to $\overline{Q}$ if and only if Γ is cyclically orientable.

We shortly recall the concept of quiver mutation and the definition of a cyclically orientable graph in Section 3.

Given a crystallographic root system Φ and a Carter diagram Γ of the same Dynkin type as Φ, by work of Cameron–Seidel–Tsaranov [9] as well as Barot–Marsh [1] the diagram Γ encodes a natural presentation of the corresponding Weyl group $W = W_\Phi$. More precisely, for vertices i and j of Γ we define $m_{ij} := w_{ij} + 2$, where w_{ij} is the *weight* of i and j, that is, the number of edges connecting i and j in Γ. (Note that we possibly have $w_{ij} = 0$, that is, there is no edge between i and j.) Let $W(\Gamma)$ be the group with generators t_i, i a vertex of Γ, subject to the following relations:

(R1) $t_i^2 = 1$ for all vertices i of Γ;

(R2) $(t_i t_j)^{m_{ij}} = 1$ for all vertices $i \neq j$ of Γ;

(R3) for any chordless cycle $i_0 \xrightarrow{w_{i_0 i_1}} i_1 \xrightarrow{w_{i_1 i_2}} \ldots \xrightarrow{w_{i_{d-2} i_{d-1}}} i_{d-1} \xrightarrow{w_{i_{d-1} i_0}} i_0$, where either all weights are 1 or $w_{i_{d-1} i_0} = 2$, we have $(t_{i_0} t_{i_1} \cdots t_{i_{d-2}} t_{i_{d-1}} t_{i_{d-2}} \cdots t_{i_1})^2 = 1$.

Theorem 2 (Cameron–Seidel–Tsaranov, Barot–Marsh)**.** *Let Φ be a crystallographic root system and let Γ be a Carter diagram of the same Dynkin type as Φ. Then $W(\Gamma)$ is isomorphic to the Weyl group W_Φ.*

In Section 4 we also discuss to what extent this result can be generalized to the non-crystallographic cases.

Notation. For $n \in \mathbb{N} = \{1, 2, \ldots\}$ we set $[n] := \{1, \ldots, n\}$ and $[\pm n] := \{\pm k \mid k \in [n]\}$. If G is a group and $g, h \in G$, we put $g^h := hgh^{-1}$ for conjugation. As usual a graph Γ is a pair (Γ_0, Γ_1), where Γ_0 is a finite set (the set of vertices) and $\Gamma_1 \subseteq \Gamma_0 \times \Gamma_0$ (set of edges). A subgraph Ψ is called *full subgraph* (or *induced subgraph*) if $\Psi_0 \subseteq \Gamma_0$ and $\Psi_1 = \Gamma_1 \cap (\Psi_0 \times \Psi_0)$.

2 Reflection Groups and Carter Diagrams

2.1 Reflection Groups and Root systems

We collect some facts about reflection groups and root systems as can be found in [14].

Let V be a (real) euclidean vector space with positive definite symmetric bilinear form $(-\mid-)$. A *reflection* is a linear map $V \to V$ which sends some $\alpha \in V \setminus \{0\}$ to its negative and fixes the hyperplane $\alpha^\perp$ orthogonal to α. We denote such a reflection by s_α. It is given by

$$s_\alpha : V \to V, \ v \mapsto v - \frac{2(\alpha \mid v)}{(\alpha \mid \alpha)}\alpha.$$

A *finite reflection group* is a finite subgroup of the orthogonal group $O(V)$ which is generated by reflections. Such a group acts (by reflections) on the ambient vector space V.

Let $W \leqslant O(V)$ be a finite reflection group. Each reflection $s_\alpha \in W$ determines a reflecting hyperplane $\alpha^\perp$ and a line $\mathbb{R}\alpha$ orthogonal to $\alpha^\perp$. In the collection of these lines $\mathbb{R}\alpha$ induced by all reflections $s_\alpha \in W$, it is possible to select a collection of vectors which is stable under the action of W. This leads to the following definition:

Definition 3. A finite set Φ of nonzero vectors in V is called a *root system* if

(RS1) $\operatorname{span}_{\mathbb{R}}(\Phi) = V$;

(RS2) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \Phi$;

(RS3) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

The elements of Φ are called *roots* and we define the *rank* of Φ as the dimension of V.

Each finite reflection group can be realized as the group $\langle s_\alpha \mid \alpha \in \Phi \rangle$ for some root system Φ. We therefore write W_Φ for this group. Inside a root system Φ we always find a *positive system* Φ^+ which contains a unique *simple system* $\Delta \subseteq \Phi^+$; see [14, Ch.1.3]. Let $\Phi \neq \varnothing$ be a root system. Then Φ is *reducible* if $\Phi = \Phi_1 \cup \Phi_2$, where Φ_1 and Φ_2 are nonempty root systems such that $(\alpha \mid \beta) = 0$ whenever $\alpha \in \Phi_1$ and $\beta \in \Phi_2$. Otherwise Φ is called *irreducible*.

Given a simple system $\Delta \subseteq \Phi$, the group $W := W_\Phi$ is generated by the set $S := \{s_\alpha \mid \alpha \in \Delta\}$ subject to the relations

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \ (\alpha, \beta \in \Delta), \tag{1}$$

where $m(\alpha, \beta)$ is the order of $s_\alpha s_\beta$ in W; see [14, Ch.1.9]. In general, a group having such a presentation is called *Coxeter group*. The pair (W, S) is called *Coxeter system* and S is called the set of *simple reflections*. Moreover, note that all reflections in W_Φ are of the form s_α for some $\alpha \in \Phi$. We therefore call the set $T := \{s_\alpha \mid \alpha \in \Phi\}$ the *set of reflections*; see [14, Ch. 1.14].
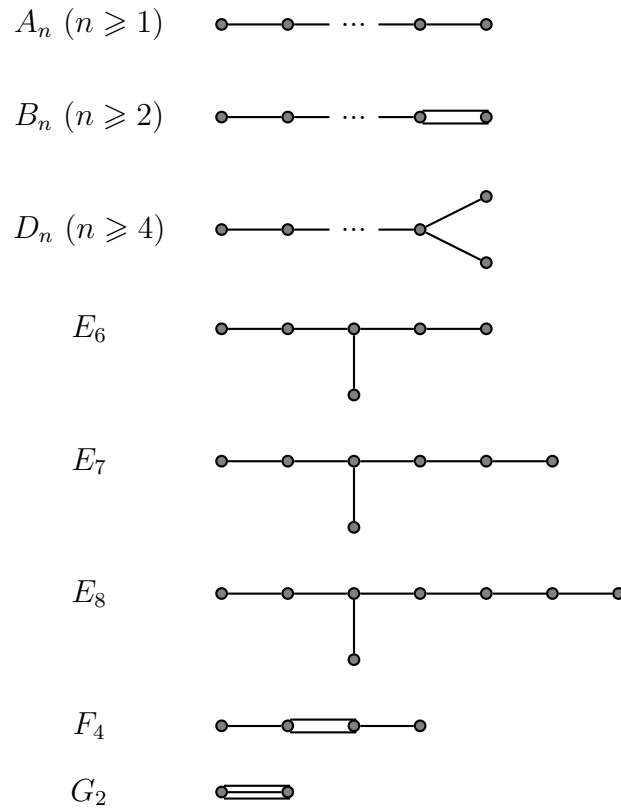
Figure 1: Dynkin diagrams.

The presentation of $W = W_\Phi$ given by (1) can be encoded in an undirected graph Γ with vertex set corresponding to Δ. Two vertices corresponding to $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$ are joined by $m(\alpha, \beta) - 2$ edges if $m(\alpha, \beta) \geqslant 3$.

The graph Γ is called *Coxeter graph*. If this graph is connected, the corresponding Coxeter system (W, S) is called *irreducible*. By abuse of notation, we sometimes just say that W is irreducible. The irreducible Coxeter systems (W, S) with W a finite reflection group, are classified by their Coxeter diagrams; see [14, Section 2.4].

A reflection group W_Φ is called *Weyl group* and the root system Φ is called *crystallographic* if

(RS4) $\frac{2(\alpha|\beta)}{(\beta|\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The irreducible Weyl groups are classified in terms of Dynkin diagrams, see Figure 1. When Φ is an irreducible crystallographic root system, there are at most two root lengths. Therefore roots are called *short* or *long*, depending on their respective lengths.

2.2 Reflection Length and Reflection Factorizations

Let $W = W_\Phi$ be a finite reflection group with root system Φ and set of reflections T. Each $w \in W$ is a product of reflections in T. We define

$$\ell_T(w) := \min\{k \in \mathbb{Z}_{\geqslant 0} \mid w = s_{\beta_1} \cdots s_{\beta_k}, \ \beta_i \in \Phi\}$$

and call $\ell_T(w)$ the *reflection length* of w. If $w = s_{\beta_1} \cdots s_{\beta_k}$ with $\beta_i \in \Phi$, we call $(s_{\beta_1}, \ldots, s_{\beta_k})$ a *reflection factorization* for w. If in addition $k = \ell_T(w)$, we call the factorization $(s_{\beta_1}, \ldots, s_{\beta_k})$ to be *reduced*. We denote by $\mathrm{Red}_T(w)$ the set of all reduced reflection factorizations for w. We have the following geometric criterion for Weyl groups to decide whether a reflection factorization is reduced.

Theorem 4 (Carter's Lemma, [10, Lemma 3]). *Let W_Φ be a Weyl group and $\beta_1, \ldots, \beta_k \in \Phi$. Then $\ell_T(s_{\beta_1} \cdots s_{\beta_k}) = k$ if and only if $\beta_1, \ldots, \beta_k \in \Phi$ are linearly independent.*

Since the set T of reflections is closed under conjugation, there is a natural way to obtain new reflection factorizations from a given (not necessarily reduced) reflection factorization. The *braid group* on n strands, denoted $\mathcal{B}_n$, is the group with generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| > 1,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \ldots, n-2.$$

It acts on the set T^n of n-tuples of reflections as

$$\sigma_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \ t_i t_{i+1} t_i, \quad t_i \quad , \ t_{i+2}, \ldots, t_n),$$
$$\sigma_i^{-1}(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \quad t_{i+1} \ , \ t_{i+1} t_i t_{i+1}, \ t_{i+2}, \ldots, t_n).$$

We call this action of $\mathcal{B}_n$ on T^n the *Hurwitz action* and an orbit of this action a *Hurwitz orbit*. It is easy to see that this action restricts to the reduced reflection factorizations of a given element $w \in W$, that is, it restricts to the set $\mathrm{Red}_T(w)$ for each $w \in W$.

2.3 Admissible diagrams and Carter diagrams

Let W_Φ be a Weyl group. It is a nontrivial result of Carter [10, Theorem C] that each $w \in W_\Phi$ can be written as $w = w_1 w_2$ with $w_1, w_2 \in W_\Phi$ involutions. Furthermore we can write

$$w_1 = s_{\beta_1} \cdots s_{\beta_k}, \ w_2 = s_{\beta_{k+1}} \cdots s_{\beta_{k+h}}$$

such that $\{\beta_1 \ldots, \beta_k\}$ and $\{\beta_{k+1} \ldots, \beta_{k+h}\}$ are sets of mutually orthogonal roots and $\ell_T(w) = k + h$; see [10, Lemma 5]. Corresponding to such a factorization, Carter defines a graph Γ:

(A) The vertices of Γ correspond to the roots $\beta_1 \ldots, \beta_{k+h}$.

(B) Two distinct roots β_i, β_j are joined by $\dfrac{2(\beta_i \mid \beta_j)}{(\beta_i \mid \beta_i)} \cdot \dfrac{2(\beta_j \mid \beta_i)}{(\beta_j \mid \beta_j)}$ edges.

Furthermore, the graph Γ is called an *admissible diagram* (for w) if each subgraph of Γ which is a cycle contains an even number of vertices. Admissible diagrams were classified by Carter; see [10, Section 5]. We extend Carter's definition as follows to all sets of linearly independent roots, that is, to all reduced reflection factorizations.

Definition 5. Let W_Φ be a Weyl group with crystallographic root system Φ. To each set of linearly independent roots $\{\beta_1, \ldots, \beta_m\} \subseteq \Phi$ we associate a diagram Γ given by the conditions (A) and (B) above. We call Γ a *Carter diagram*. The *type* of Γ is defined to be the (Dynkin-)type of the smallest root subsystem $\Phi' \subseteq \Phi$ which contains $\{\beta_1, \ldots, \beta_m\}$.

Remark 6. (a) For a set of roots $R \subseteq \Phi$ the smallest root subsystem $\Phi' \subseteq \Phi$ containing R is given by $W_R(R)$, where $W_R := \langle s_\alpha \mid \alpha \in R \rangle$. If Φ' is irreducible, then the Carter diagram associated to R is connected.

  (b) A Carter diagram does not change if we replace a root by its corresponding negative root.

Example 7. Each Dynkin diagram is a Carter diagram. In fact, a Carter diagram without cycles is a Dynkin diagram. This is proved in [10, Lemma 8] for admissible diagrams, but the proof given there also works for arbitrary Carter diagrams.

Example 8. We consider a root system Φ of type D_4. This can be realized as

$$\Phi = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant 4\},$$

where $\{e_1, e_2, e_3, e_4\}$ is the canonical base of $\mathbb{R}^4$. The Carter diagram Γ_1 attached to the set of roots $R_1 := \{e_1 - e_2, e_1 + e_2, e_2 - e_3, e_4 - e_1\} \subseteq \Phi$ is given by the left diagram in Figure 2.
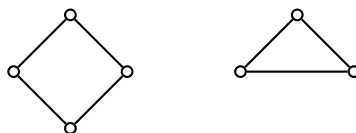

Figure 2: Carter diagrams in a root system of type D_4.

The smallest root subsystem of Φ containing R_1 is the root system Φ itself. Therefore Γ_1 is a Carter diagram of type D_4. On the other hand, the Carter diagram Γ_2 attached to the set of roots $R_2 := \{e_1 - e_3, e_2 - e_3, e_3 - e_4\} \subseteq \Phi$ is given by the right diagram in Figure 2. The smallest root subsystem of Φ containing R_2 is $\{e_i - e_j \mid i \neq j, \ 1 \leqslant i, j \leqslant 4\} \subseteq \Phi$. This system is of type A_3 and so is Γ_2.

Remark 9. In the definition of a Carter diagram we demand the set of roots $R := \{\beta_1, \ldots, \beta_m\}$ to be linearly independent. By Carter's Lemma 4 this is equivalent to $s_{\beta_1} \cdots s_{\beta_m}$ being reduced. In particular, to each $w \in W_\Phi$ and each reduced reflection factorization $(s_{\beta_1}, \ldots, s_{\beta_m})$ of w, we have an associated Carter diagram Γ induced by the set $\{\beta_1, \ldots, \beta_m\}$.

Therefore we can describe Carter diagrams entirely by reduced reflection factorizations. If $(s_{\beta_1}, \ldots, s_{\beta_m}) \in \mathrm{Red}_T(w)$, then the vertices of the diagram Γ correspond to the reflections s_{β_i} $(1 \leqslant i \leqslant m)$. By Carter's Lemma 4 there is an edge between s_{β_i} and s_{β_j} $(i \neq j)$ if and only if s_{β_i} and s_{β_j} do not commute. The number of edges between s_{β_i} and s_{β_j} is given by $m_{ij} - 2$, where m_{ij} is the order of $s_{\beta_i} s_{\beta_j}$. In this case we also call Γ a *Carter diagram associated to w*, or more precisely a *Carter diagram associated to $(s_{\beta_1}, \ldots, s_{\beta_m})$*. The linear independence of R also implies that the root subsystem $\Phi' := W_R(R)$ is of rank m and by [4, Theorem 1.1] we have $W_{\Phi'} = \langle s_{\beta_1}, \ldots, s_{\beta_m} \rangle$.

A direct calculation shows that Carter diagrams are invariant under conjugation in the following sense.

Lemma 10. *Let W_Φ be a Weyl group with crystallographic root system Φ, $w \in W_\Phi$ and $(s_{\beta_1}, \ldots, s_{\beta_m}) \in \mathrm{Red}_T(w)$. Then $(s_{\beta_1}, \ldots, s_{\beta_m})$ (resp. $\{\beta_1, \ldots, \beta_m\}$) and $(s_{\beta_1}^x, \ldots, s_{\beta_m}^x) \in \mathrm{Red}_T(w^x)$ (resp. $\{x(\beta_1), \ldots, x(\beta_m)\}$) yield the same Carter diagram for all $x \in W_\Phi$.*

Definition 11. Let $W = W_\Phi$ be a finite reflection group with root system Φ of rank n. An element $w \in W$ is called *quasi-Coxeter element* if there exists a reduced reflection factorization $(s_{\beta_1}, \ldots, s_{\beta_n}) \in \mathrm{Red}_T(w)$ such that $\langle s_{\beta_1}, \ldots, s_{\beta_n} \rangle = W$.

A very important example of a quasi-Coxeter element is the following: An element $c \in W$ is called *Coxeter element* if there exists a simple system $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ such that $c = s_{\alpha_1} \cdots s_{\alpha_n}$.

Remark 12. (a) Given a crystallographic root system Φ, the Carter diagrams of the same type as Φ are precisely the Carter diagrams attached to reduced reflection factorizations of quasi-Coxeter elements in W_Φ (see also Remark 9).

(b) Baumeister and the author give in [4] another characterization of quasi-Coxeter elements in terms of bases of the (co-)root lattice.

Example 13. Let Φ be a root system of type A_n and $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ a simple system. Then $c = s_{\alpha_1} \cdots s_{\alpha_n}$ is a Coxeter element. The Carter diagram associated to $\{\alpha_1, \ldots, \alpha_n\}$ resp. to $(s_{\alpha_1}, \ldots, s_{\alpha_n}) \in \mathrm{Red}_T(c)$ is the Dynkin diagram of type A_n. Applying the Hurwitz action yields

$$(\sigma_1 \cdots \sigma_{n-1})(\sigma_2 \cdots \sigma_{n-1}) \cdots \sigma_{n-1}(s_{\alpha_1}, \ldots, s_{\alpha_n}) = (s_{\alpha_n}^{s_{\alpha_1} \cdots s_{\alpha_{n-1}}}, \ldots, s_{\alpha_2}^{s_{\alpha_1}}, s_{\alpha_1})$$
$$= (s_{\alpha_1 + \cdots + \alpha_n}, \ldots, s_{\alpha_1 + \alpha_2}, s_{\alpha_1}) \in \mathrm{Red}_T(c).$$

The Carter diagram associated to the set $\{\alpha_1 + \cdots + \alpha_n, \ldots, \alpha_1 + \alpha_2, \alpha_1\}$ resp. to the reduced reflection factorization $(s_{\alpha_n}^{s_{\alpha_1} \cdots s_{\alpha_{n-1}}}, \ldots, s_{\alpha_2}^{s_{\alpha_1}}, s_{\alpha_1}) \in \mathrm{Red}_T(c)$ is the complete graph on n vertices.

The following lemma allows us to investigate Carter diagrams via quasi-Coxeter elements in irreducible Weyl groups.

Lemma 14. *Each Carter diagram is the disjoint union of connected Carter diagrams associated to reduced reflection factorizations of quasi-Coxeter elements in irreducible Weyl groups.*

Proof. Let $W = W_\Phi$ be a Weyl group with crystallographic root system Φ. Consider a set of linearly independent roots $R := \{\beta_1, \ldots, \beta_m\} \subseteq \Phi$ and put $\Phi' := W_R(R)$. By [4, Theorem 1.1] $s_{\beta_1} \cdots s_{\beta_m}$ is a quasi-Coxeter element in $W_{\Phi'}$. We can decompose Φ' as $\Phi' = \Phi_1 \cup \cdots \cup \Phi_k$ such that W_{Φ_i} is irreducible and such that the Φ_i are pairwise orthogonal. Again by [4, Theorem 1.1], we have $\Phi_i = W_{R_i}(R_i)$ with $R_i := \Phi_i \cap R$. Therefore each connected component of the Carter diagram attached to $\{\beta_1, \ldots, \beta_m\}$ is given by a Carter diagram associated to the reduced reflection factorization of a quasi-Coxeter element in W_{Φ_i} for some $i \in [k]$. $\qquad\square$

2.4 Construction of Carter diagrams

The aim of this section is to give a procedure of constructing all possible Carter diagrams of types A_n, B_n and D_n. For the exceptional types we will provide a complete list based on computations.

2.4.1 Carter diagrams of type A

Let $n \geqslant 1$ be an integer. For the type A_n we will use a result of Kluitmann [17]. It is well known that the Weyl group W of type A_n can be identified with the symmetric group $\mathrm{Sym}([n+1])$. In this setting the set of reflections can be identified with the set of transpositions. As in [17] we define for $m \geqslant n$ and $w \in W$ the following (possibly empty) sets

$$\Xi^{n,m} := \{(t_1, \ldots, t_m) \mid t_i \in \mathrm{Sym}([n+1]) \text{ a transposition}, \ \langle t_1, \ldots, t_m \rangle = \mathrm{Sym}([n+1])\}$$
$$\Xi_w^{n,m} := \{(t_1, \ldots, t_m) \in \Xi^{n,m} \mid w = t_1 \cdots t_m\}.$$

Remark 15. In type A_n each quasi-Coxeter element is a Coxeter element and Coxeter elements correspond to $(n+1)$-cycles in $\mathrm{Sym}([n+1])$ [3, Remark 6.6]. If c is a $(n+1)$-cycle in $\mathrm{Sym}([n+1])$, then $\Xi_c^{n,n} = \mathrm{Red}_T(c)$, where T is the set of transpositions in $\mathrm{Sym}([n+1])$. Moreover, we have

$$\Xi^{n,n} = \bigcup_{(n+1)-\text{cycle } c} \mathrm{Red}_T(c).$$

For each element $(t_1, \ldots, t_m) \in \Xi^{n,m}$, Kluitmann defines a graph as follows:

- The vertices correspond to the set $\{t_1, \ldots, t_m\}$.

- Two vertices corresponding to t_i and t_j are connected by an edge if $t_i t_j \neq t_j t_i$.

We call such a diagram a *Kluitmann diagram*. Denote by $\mathcal{A}^{n,m}$ (resp. by $\mathcal{A}_w^{n,m}$ for $w \in \mathrm{Sym}([n+1])$) the set of Kluimann diagrams given by the elements of $\Xi^{n,m}$ (resp. the elements of $\Xi_w^{n,m}$). As a direct consequence of our definitions of Carter diagrams and Kluitmann diagrams, we obtain:

Proposition 16. *The set of Carter diagrams of type A_n is given by the set $\mathcal{A}^{n,n}$.*

Kluitmann provides a procedure to construct all diagrams in $\mathcal{A}^{n,m}$.

Theorem 17 (Kluitmann, [17, Theorem 1])**.** *The following construction yields all elements in $\mathcal{A}^{n,m}$:*

(a) *Choose $m' \in \mathbb{N}$ with $n \leqslant m' \leqslant m$. Take any connected graph Γ on m' vertices with Γ the union of subgraphs $\Gamma_1, \ldots, \Gamma_k$ such that:*

    (i) *$\Gamma_1, \ldots, \Gamma_k$ are complete graphs.*

    (ii) *Γ_i and Γ_j $(i \neq j)$ intersect in at most one vertex.*

    (iii) *Every vertex of Γ belongs to at most two of the subgraphs Γ_i.*

    (iv) *$\sum_{i=1}^{k}(|\Gamma_i| - 1) = (m' - 1) + (m' - n)$.*

(b) *Attach additional vertices $v_1, v_2, \ldots, v_{m-m'}$ to Γ by the following procedure:*

    Suppose that $\Psi := \Gamma \cup \{v_1, \ldots, v_\ell\}$, with $v_1, \ldots, v_\ell$ already attached to Γ. Choose a vertex v of Ψ, and define v_{k+1} to be its "duplication", that is v and v_{k+1} are connected with exactly the same vertices; there is no edge between v and v_{k+1}.

Corollary 18 (Carter diagrams of type $\boldsymbol{A_n}$)**.** *The Carter diagrams of type A_n are given by the set $\mathcal{A}^{n,n}$. More precisely, a Carter diagram of type A_n is a connected graph Γ on n vertices which is the union of subgraphs $\Gamma_1, \ldots, \Gamma_k$ such that:*

(i) *$\Gamma_1, \ldots, \Gamma_k$ are complete graphs.*

(ii) *Γ_i and Γ_j $(i \neq j)$ intersect in at most one vertex.*

(iii) *Every vertex of Γ belongs to at most two of the subgraphs Γ_i.*

(iv) *$\sum_{i=1}^{k}(|\Gamma_i| - 1) = n - 1$.*

Remark 19. Note that a complete graph cannot be written as the union of proper subgraphs such that the properties (i)-(iv) of Corollary 18 hold. As a consequence, we obtain that the decomposition $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_k$ given by Corollary 18 is unique up to permutation of the factors.

Example 20. By Corollary 18, the graph
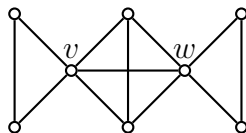
is a Carter diagram of type A_8. To see this, let Γ_1 and Γ_2 be the complete graphs on three vertices intersecting the complete graph Γ_3 on four vertices in the center of the picture in the vertices v and w, respectively.

2.4.2 Carter diagrams of type B

Let W_Φ be a Weyl group with Φ of type B_n. It is well known (see for instance [5, Chapter 8.1]) that W_Φ can be realized by signed permutations, that is, as the group

$$S_n^B := \{\pi \in \mathrm{Sym}([\pm n]) \mid \pi(-i) = -\pi(i)\},$$

where the group operation is given by composition. The set of reflections can be identified with the set

$$T_n^B := \{(i,j)(-i,-j) \mid 1 \leqslant i < |j| \leqslant n\} \cup \{(i,-i) \mid i \in [n]\}. \tag{2}$$

The group homomorphism

$$\theta : \mathrm{Sym}([n]) \to \mathrm{Aut}(\mathbb{Z}_2^n), \ \pi \mapsto [\mathbb{Z}_2^n \to \mathbb{Z}_2^n, (d_1, \ldots, d_n) \mapsto (d_{\pi(1)}, \ldots, d_{\pi(n)})]$$

yields an isomorphism $S_n^B \cong \mathbb{Z}_2^n \rtimes_\theta \mathrm{Sym}([n])$. For our purposes and for later use we state this isomorphism explicitly on the generating set T_n^B of S_n^B given in (2). Let $i, j \in \mathbb{N}$ with $i < j$:

$$\begin{aligned}
\varphi_B : S_n^B \to \mathbb{Z}_2^n \rtimes_\theta \mathrm{Sym}([n]), \ (i,j)(-i,-j) &\mapsto (0, (i,j)) \tag{3}\\
(i,-j)(-i,j) &\mapsto (e_i + e_j, (i,j))\\
(i,-i) &\mapsto (e_i, \mathrm{id})
\end{aligned}$$

We will now describe how to obtain all Carter diagrams of type B_n by describing all Carter diagrams of reduced reflection factorizations of quasi-Coxeter elements in S_n^B (see also Remark 9). Note that in type B_n each quasi-Coxeter element is a Coxeter element [3, Lemma 6.4]. Therefore let $(t_1, \ldots, t_n)$ be a reduced reflection factorization of a Coxeter element. By Lemma 10 and by the proof of [3, Lemma 6.4] we can assume that

$$\{t_1, \ldots, t_n\} = \{(1,-1), (2,i_2)(-2,-i_2), \ldots, (n,i_n)(-n,-i_n)\} =: R,$$

where $i_j \in \{1, \ldots, j-1\}$ for each $j \in \{2, \ldots, n\}$. Put

$$R' := \{(2,i_2)(-2,-i_2), \ldots, (n,i_n)(-n,-i_n)\} \subseteq R.$$

Proposition 21. *The Carter diagram on vertex set R' is (isomorphic to) a Carter diagram of type A_{n-1}.*

Proof. Two elements $(j,i_j)(-j,-i_j), (k,i_k)(-k,-i_k) \in R'$ commute in S_n^B if and only if (j,i_j) and (k,i_k) commute in $\mathrm{Sym}([n])$. The set $\{(2,i_2), \ldots, (n,i_n)\}$ generates $\mathrm{Sym}([n])$ and therefore yields a Carter diagram of type A_{n-1}. $\qquad\square$

Denote by Γ' the Carter diagram induced by R' and let $\{k_1, \ldots, k_\ell\}$ be the subset of $\{2, \ldots, n\}$ such that $i_{k_j} = 1$ for all $j \in [\ell]$. By property (i) in Corollary 18, the graph Γ' decomposes as the union of complete graphs.

Proposition 22. *The full subgraph Γ'' of Γ' on vertex set*

$$R'' := \{(k_1, 1)(-k_1, -1), \ldots, (k_\ell, 1)(-k_\ell, -1)\} \subseteq R'$$

is one of the complete graphs in the decomposition of Γ'.

Proof. It is easy to see that Γ'' is a complete graph. Therefore the assertion is true if $R' = R''$. Thus assume that $R'' \subsetneq R'$. To show that Γ'' respects the properties (i)-(iv) of Corollary 18 it is enough to show that Γ'' is connected by exactly one edge with the rest of the graph Γ'. Since Γ' is connected there exists at least one of those edges. Now assume that there exists $(i, i_j)(-i, -i_j) \in R' \setminus R''$ and $p, q \in [\ell]$, $p \neq q$ such that $(i, i_j)(-i, -i_j)$ is connected to both $(k_p, 1)(-k_p, -1) \in R''$ and $(k_q, 1)(-k_q, -1) \in R''$ by an edge. This implies that $\{i, i_j\} = \{k_p, k_q\}$ and therefore $(k_p, 1)(-k_p, -1)^{(k_q, 1)(-k_q, -1)} = (i, i_j)(-i, -i_j)$, contradicting the fact that $\langle R \rangle = S_n^B$. $\square$

Let Γ be the Carter diagram on vertex set R. By Proposition 21 the induced subgraph Γ' on vertex set R' is a Carter diagram of type A_{n-1}. This graph is described by the construction in Section 2.4.1. By Proposition 22 the induced subgraph Γ'' of Γ' on vertex set R'' is a complete graph in the decomposition of Γ'. The element $(1, -1) \in R$ commutes with all elements in $R' \setminus R''$ and does not commute with any element in R''. The full subgraph of Γ on vertex set $R'' \cup \{(1, -1)\}$ is a complete graph on $\ell + 1$ vertices. Denote by $\Gamma^{(1)}$ the graph obtained from Γ by suppressing the weights of the edges (that is, all weights are equal to 1). Our previous arguments show that $\Gamma^{(1)}$ is a Carter diagram of type A_n. The element $(1, -1) \in R$ corresponds to a reflection s_α in W_Φ with $\alpha \in \Phi$ a short root. All elements in $R \setminus \{(1, -1)\}$ correspond to reflections s_β with $\beta \in \Phi$ a long root. Therefore all edges in Γ adjacent with the vertex corresponding to $(1, -1)$ have the weight 2.

We summarize our construction:

Theorem 23 (Carter diagrams of type $\boldsymbol{B_n}$). *All Carter diagrams of type B_n are obtained by the following construction:*

- *Take any possible Carter diagram Γ of type A_n.*

- *Take any vertex v of Γ such that $\Gamma \setminus \{v\}$ is still connected.*

- *All edges adjacent with v have weight 2.*

We call v the distinguished vertex of Γ.

Example 24. Consider the type A_n Carter diagram from Example 20.
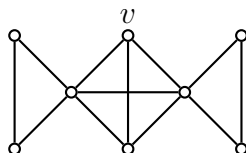
If we remove the vertex v, then the diagram is still connected. We choose v to be the distinguished vertex. The resulting type B_n Carter diagram is the following one.
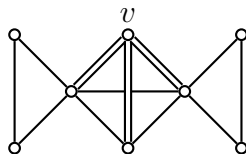
2.4.3 Carter diagrams of type D

Let W_Φ be a Weyl group of type D_n. It is well known (see for instance [5, Chapter 8.2]) that W_Φ can be realized as the group of even signed permutations. We denote this group by S_n^D. For our purposes it will be convenient to see S_n^D as the subgroup of S_n^B generated by the set

$$T_n^D := \{(i,j)(-i,-j) \mid 1 \leqslant i < |j| \leqslant n\}. \tag{4}$$

The set T_n^D can also be identified with the set of reflections for S_n^D. We denote by φ_D the restriction of the map φ_B given in (3) to the subgroup $S_n^D \leqslant S_n^B$. Further we put $\varphi := \mathrm{pr}_2 \circ \varphi_D$, where

$$\mathrm{pr}_2 : \mathbb{Z}_2^n \rtimes_\theta \mathrm{Sym}([n]) \to \mathrm{Sym}([n]), (d, \pi) \mapsto \pi,$$

and

$$\Xi_{D_n} := \{(t_1, \ldots, t_n) \mid t_i \in T_n^D, \ \langle t_1, \ldots, t_n \rangle = S_n^D\}.$$

Note that for $i, j \in \mathbb{N}$ with $i < j$ we have $\varphi((i,j)(-i,-j)) = \varphi((i,-j)(-i,j)) = (i,j)$. Therefore and since φ is surjective, φ induces a map

$$\overline{\varphi} : \Xi_{D_n} \to \Xi^{n-1,n}, \ (t_1, \ldots, t_n) \mapsto (\varphi(t_1), \ldots, \varphi(t_n)).$$

This map is well defined. Since the tuple $(t_1, \ldots, t_n)$ generates the group S_n^D and since φ is surjective, the tuple $(\varphi(t_1), \ldots, \varphi(t_n))$ generates the group $\mathrm{Sym}([n])$. Therefore $\varphi(t_1) \cdots \varphi(t_n)$ is quasi-Coxeter, hence an n-cycle (see [3, Lemma 6.3]).

The aim of this section is to show the following.

Theorem 25 (Carter diagrams of type $\boldsymbol{D_n}$). *All Carter diagrams of type D_n are given by the set $\mathcal{A}^{n-1,n}$. In particular, Theorem 17 provides a procedure to construct all Carter diagrams of type D_n.*

The strategy of the proof is the following: Each Carter diagram of type D_n is the Carter diagram of a reduced reflection factorization in Ξ_{D_n}. We first show that Carter diagrams are preserved under the map $\overline{\varphi}$. We then complete the proof of Theorem 25 by showing that $\overline{\varphi}$ is surjective. We state these observations separately.

Proposition 26. *Carter diagrams are preserved under the map $\overline{\varphi}$, that is, if $(t_1, \ldots, t_n) \in \Xi_{D_n}$ then the Carter diagram associated to $(t_1, \ldots, t_n)$ is the same as the Kluitmann diagram associated to $(\varphi(t_1), \ldots, \varphi(t_n)) \in \Xi^{n-1,n}$.*

Proposition 27. *The map $\overline{\varphi}$ is surjective.*

Definition 28. For a reflection $t = (i,j)(-i,-j) \in T_n^B$ (resp. $t = (i,-i) \in T_n^B$) we define the *support* of t as

$$\operatorname{supp}(t) := \{|i|, |j|\} \ (\text{resp.} \ \operatorname{supp}(t) := \{|i|\}.).$$

Proof of Proposition 26. Let $(t_1, \ldots, t_n) \in \Xi_{D_n}$ and $i, j \in [n]$ with $i < j$. We have to show that there is an edge between t_i and t_j in the Carter diagram associated to $(t_1, \ldots, t_n)$ if and only if there is an edge between $\varphi(t_i)$ and $\varphi(t_j)$ in the Kluitmann diagram associated to $(\varphi(t_1), \ldots, \varphi(t_n))$.

Let us first assume that there is an edge between t_i and t_j. By Carter's Lemma 4 this is equivalent to $t_i t_j \neq t_j t_i$. We conclude

$$
\begin{aligned}
t_i t_j \neq t_j t_i \text{ and } t_i \neq t_j \ &\Leftrightarrow \ |\operatorname{supp}(t_i) \cap \operatorname{supp}(t_j)| = 1 \\
&\Leftrightarrow \ |\operatorname{supp}(\varphi(t_i)) \cap \operatorname{supp}(\varphi(t_j))| = 1 \\
&\Leftrightarrow \ \varphi(t_i)\varphi(t_j) \neq \varphi(t_j)\varphi(t_i) \text{ and } \varphi(t_i) \neq \varphi(t_j),
\end{aligned}
$$

that is, $\varphi(t_i)$ and $\varphi(t_j)$ are connected by an edge in the Kluitmann diagram.

Now consider the case that there is no edge between t_i and t_j. This is equivalent to $|\operatorname{supp}(t_i) \cap \operatorname{supp}(t_j)| \in \{0, 2\}$. By the definition of φ we have

$$\operatorname{supp}(t_i) \cap \operatorname{supp}(t_j) = \operatorname{supp}(\varphi(t_i)) \cap \varphi(\operatorname{supp}(t_j)).$$

Hence there is no edge between t_i and t_j if and only if there is no edge between $\varphi(t_i)$ and $\varphi(t_j)$. $\qquad\square$

As a preparation for the proof of Proposition 27 we show:

Lemma 29. *The map $\overline{\varphi}$ is equivariant with respect to the Hurwitz action.*

Proof. Let $\underline{t} := (t_1, \ldots, t_i, t_{i+1}, \ldots, t_n) \in \Xi_{D_n}$ and $i \in [n-1]$. It is enough to show that

$$\sigma_i(\overline{\varphi}(\underline{t})) = \overline{\varphi}(\sigma_i(\underline{t})).$$

Let $t_i = (k_1, \ell_1)(-k_1, -\ell_1)$ and $t_{i+1} = (k_2, \ell_2)(-k_2, -\ell_2)$ with $1 \leqslant k_j < |\ell_j| \leqslant n$ for $j \in \{1, 2\}$.

Let us first assume that $t_i t_{i+1} = t_{i+1} t_i$. Then $\sigma_i(\underline{t}) = (\ldots, t_{i+1}, t_i, \ldots)$, thus $\overline{\varphi}(\sigma_i(\underline{t})) = (\ldots, (k_2, |\ell_2|), (k_1, |\ell_1|), \ldots)$. On the other hand we have

$$\sigma_i(\overline{\varphi}(\underline{t})) = \sigma_i(\ldots, (k_1, |\ell_1|), (k_2, |\ell_2|), \ldots) = (\ldots, (k_2, |\ell_2|)^{(k_1, |\ell_1|)}, (k_1, |\ell_1|), \ldots).$$

But $(k_2, |\ell_2|)^{(k_1, |\ell_1|)} = (k_2, |\ell_2|)$, since $|\operatorname{supp}(t_i) \cap \operatorname{supp}(t_{i+1})| \in \{0, 2\}$ (see also the proof of Proposition 26). Hence $\sigma_i(\overline{\varphi}(\underline{t})) = \overline{\varphi}(\sigma_i(\underline{t}))$.

Now assume that t_i and t_{i+1} do not commute. This implies $|\operatorname{supp}(t_i) \cap \operatorname{supp}(t_{i+1})| = 1$. Let us assume that $\ell_1 = k_2$ (the other cases can be treated analogously). Then

$$
\begin{aligned}
\overline{\varphi}(\sigma_i(\underline{t})) &= \overline{\varphi}(\ldots, (k_2, \ell_2)(-k_2, -\ell_2)^{(k_1, \ell_1)(-k_1, -\ell_1)}, (k_1, \ell_1)(-k_1, -\ell_1), \ldots) \\
&= \overline{\varphi}(\ldots, (k_2, \ell_2)(-k_2, -\ell_2)^{(k_1, k_2)(-k_1, -k_2)}, (k_1, k_2)(-k_1, -k_2), \ldots) \\
&= \overline{\varphi}(\ldots, (k_1, \ell_2)(-k_1, -\ell_2), (k_1, k_2)(-k_1, -k_2), \ldots) \\
&= (\ldots, (k_1, |\ell_2|), (k_1, k_2), \ldots)
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_i(\overline{\varphi}(\underline{t})) &= \sigma_i(\ldots, (k_1, |\ell_1|), (k_2, |\ell_2|), \ldots) \\
&= \sigma_i(\ldots, (k_1, k_2), (k_2, |\ell_2|), \ldots) \\
&= (\ldots, (k_2, |\ell_2|)^{(k_1, k_2)}, (k_1, k_2), \ldots) \\
&= (\ldots, (k_1, |\ell_2|), (k_1, k_2), \ldots).
\end{aligned}
$$
$\qquad\square$

Proof of Proposition 27. Let $(\tau_1, \ldots, \tau_n) \in \Xi^{n-1, n}$ be arbitrary. By Lemma 29 and by [18, Corollary 1.4], we can apply the Hurwitz action and assume that

$$
\tau_{n-1} = \tau_n \text{ and } \langle \tau_1, \ldots, \tau_{n-1} \rangle = \operatorname{Sym}([n]). \tag{5}
$$

Let $\tau_i = (k_i, \ell_i)$ with $k_i, \ell_i \in [n]$ and $k_i < \ell_i$ for all $i \in [n-1]$. Put

$$
\begin{aligned}
\overline{\tau}_i &:= (k_i, \ell_i)(-k_i, -\ell_i) \text{ for } 1 \leqslant i \leqslant n-1, \\
\overline{\tau}_n &:= (k_{n-1}, -\ell_{n-1})(-k_{n-1}, \ell_{n-1}).
\end{aligned}
$$

Since $\varphi(\overline{\tau}_i) = \tau_i$ for all $i \in [n]$, it remains to show that $(\overline{\tau}_1, \ldots, \overline{\tau}_{n-1}, \overline{\tau}_n) \in \Xi_{D_n}$, that is $W := \langle \overline{\tau}_1, \ldots, \overline{\tau}_{n-1}, \overline{\tau}_n \rangle = S_n^D$. Equivalentely, we have to show that $\varphi_D(W) \cong \mathbb{Z}_2^{n-1} \rtimes_\theta \operatorname{Sym}([n])$, where we identify $\mathbb{Z}_2^{n-1}$ with the subgroup $\{(d_1, \ldots d_n) \in \mathbb{Z}_2^n \mid d_1 + \ldots + d_n \text{ is even}\} \leqslant \mathbb{Z}_2^n$. We have

$$
\begin{aligned}
\varphi_D(\overline{\tau}_i) &= (0, \tau_i) \text{ for } 1 \leqslant i \leqslant n-1, \\
\varphi_D(\overline{\tau}_n) &= (e_{k_{n-1}} + e_{\ell_{n-1}}, \tau_{n-1}).
\end{aligned}
$$

Choose $\pi \in \operatorname{Sym}([n])$ with $\pi(k_{n-1}) = 1$ and $\pi(\ell_{n-1}) = 2$. By (5) there exist $i_1, \ldots, i_m \in [n-1]$ with $\pi = \tau_{i_1} \cdots \tau_{i_m}$.

$$
\begin{aligned}
((0, \tau_{i_1}), \ldots, (0, \tau_{i_m})) \cdot (e_{k_{n-1}} + e_{\ell_{n-1}}, \tau_{n-1}) &= (0, \pi) \cdot (e_{k_{n-1}} + e_{\ell_{n-1}}, \tau_{n-1}) \\
&= (e_{\pi(k_{n-1})} + e_{\pi(\ell_{n-1})}, \pi \tau_{n-1}) \\
&= (e_1 + e_2, \pi \tau_{n-1}) \in \varphi_D(W).
\end{aligned}
$$

By (5) we have $(0, (\pi\tau_{n-1})^{-1}(1,2)) \in \varphi_D(W)$. Therefore we obtain

$$(e_1 + e_2, \pi\tau_{n-1}) \cdot (0, (\pi\tau_{n-1})^{-1}(1,2)) = ((e_1 + e_2) + \theta(\pi\tau_{n-1})(0), (\pi\tau_{n-1})(\pi\tau_{n-1})^{-1}(1,2))$$
$$= (e_1 + e_2, (1,2)) \in \varphi_D(W).$$

By (5) we have $(0, (i, i+1)) \in \varphi_D(W)$ for all $i \in [n-1]$. These elements together with the element $(e_1 + e_2, (1,2)) \in \varphi_D(W)$ are a generating set of $\mathbb{Z}_2^{n-1} \rtimes_\theta \mathrm{Sym}([n])$. $\qquad\square$

2.4.4 Carter diagrams of exceptional types

We list all non-admissible Carter diagrams of the exceptional types E_6 and F_4 in Figure 3 and Figure 6, respectively. The admissible ones can be found in [10]. The only Carter diagram of type G_2 is the corresponding Dynkin diagram. For type E_7 there are up to isomorphism 233 Carter diagrams, some examples are shown in Figure 4. For type E_8 there are up to isomorphism 1242 Carter diagrams, one example is shown in Figure 5. The complete lists for the types E_7 and E_8 can be found on the author's webpage:

> https://www.iazd.uni-hannover.de/fileadmin/iazd/CarterDiagramsE7
> https://www.iazd.uni-hannover.de/fileadmin/iazd/CarterDiagramsE8
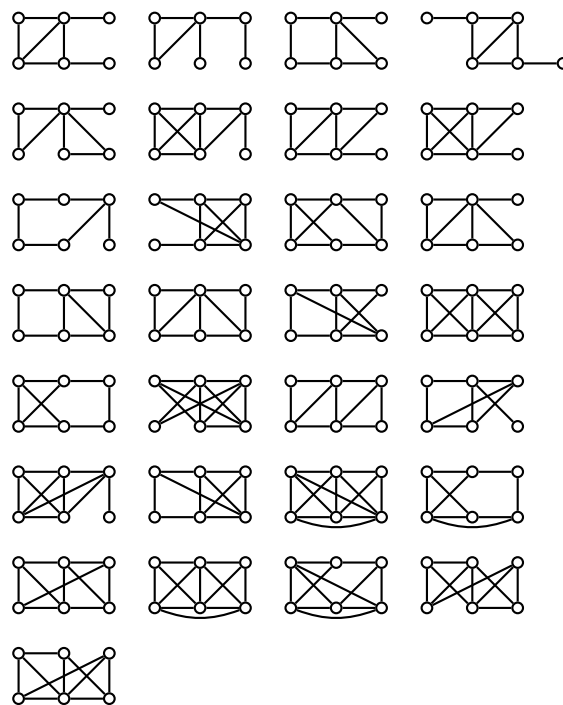
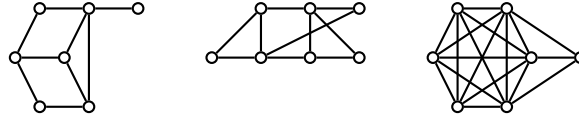Figure 3: Non-admissible Carter diagrams of type E_6.
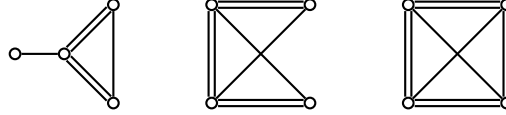
Figure 4: Examples of Carter diagrams of type E_7.

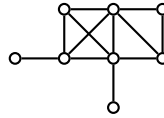Figure 6: Non-admissible Carter diagrams of type F_4.

Figure 5: A Carter diagram of type E_8.

Remark 30. Let us explain how we carried out the computations to obtain all Carter diagrams (up to isomorphism) of an exceptional type $X_n \in \{E_6, E_7, E_8, F_4\}$. By Remark 9 we just have to consider Carter diagrams associated to reduced reflection factorizations of quasi Coxeter elements in a Coxeter group of type X_n. By Lemma 10, we just have to consider one fixed quasi-Coxeter element for each conjugacy class. Given a quasi-Coxeter element w, all reduced reflection factorizations of w are precisely given by the Hurwitz orbit of one given reduced reflection factorization [3, Theorem 1.1]. The programs to carry out these computations in GAP [12] can be found at: https://www.math.uni-bielefeld.de/~baumeist/Dual-Coxeter/dual-Coxeter.html Given a reduced reflection factorization $(t_1, \ldots, t_n)$, the computation of the Carter diagram associated to this factorization is easily done by computing the order of $t_i t_j$ for all $1 \leqslant i < j \leqslant n$. Finally we used Sage [20] to obtain the complete list of Carter diagrams up to isomorphism. Sage provides the command G.is_isomorphic(H) to check whether two graphs G and H are isomorphic.

Remark 31. The Carter diagrams of exceptional types can also be found in [8].

3 Quiver Mutation

The aim of this section is to establish a connection between Carter diagrams and the mutation classes of Dynkin quivers. Namely we prove Theorem 1 in this section. Quiver mutation appears as an important concept in the theory of cluster algebras. We will shortly review the necessary definitions. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a directed graph on vertex set Q_0, edges given by the set Q_1 and maps $s, t : Q_1 \to Q_0$ such that $s(\alpha) = i$ and $t(\alpha) = j$ whenever $\alpha \in Q_1$ is an arrow from $i \in Q_0$ to $j \in Q_0$. We will assume throughout that quivers do not have loops or 2-cycles.

Definition 32. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. The *mutation at vertex* $k \in Q_0$ is the quiver $\mu_k(Q) = (Q_0^*, Q_1^*, s^*, t^*)$ obtained as follows:

- all arrows incident with k are reversed (that is, $s^*(\alpha) = t(\alpha)$ and $t^*(\alpha) = s(\alpha)$ for all $\alpha \in Q_1$ incident with k);

- whenever $i, j \in Q_0$ are such that there are $m > 0$ arrows from i to k (in Q) and $n > 0$ arrows from k to j (in Q), first add mn arrows from i to j. Then remove a maximal number of 2-cycles.

Definition 33. Let Γ be an undirected graph.

(a) A *cycle* in Γ is a subgraph which is isomorphic to the graph on vertex set $[n]$ whose edges are $(1, 2), \ldots, (n - 1, n), (n, 1)$.

(b) A full subgraph of Γ which is a cycle is called *chordless cycle*.

(c) The graph Γ is called *cyclically orientable* if it admits an orientation in which every chordless cycle of Γ is cyclically oriented.

Example 34. None of the graphs in Figure 4 is cyclically orientable. For instance, this can be easily seen by using a criterion provided by Gurvich [13].

We divide the proof of Theorem 1 into three parts. Since the types A_n and B_n are closely related, we prove them together. Then we will prove Theorem 1 for the type D_n. Lastly we treat the exceptional types.

Definition 35. Let Γ be a graph (possibly with directed or with weighted edges). Let v be a vertex of Γ. We define the *valency* of v, denoted by val(v), to be the number of vertices of Γ which are connected by a (directed or weighted) edge with v.

For example, in both graphs shown in Example 24 we have val$(v) = 3$.

Proposition 36. *Theorem 1 is true for type A_n (resp. type B_n).*

More precisley, let Q be a quiver which is mutation-equivalent to an orientation of the Dynkin diagram of type A_n (resp. B_n). Then the underlying undirected graph $\overline{Q}$ is a Carter diagram of type A_n (resp. B_n).

Moreover, let Γ be a Carter diagram of type A_n (resp. B_n). Then there exists a quiver Q which is mutation-equivalent to an orientation of the Dynkin diagram of type A_n (resp. B_n) such that Γ is isomorphic to $\overline{Q}$ if and only if Γ is cyclically orientable.

Proof. We begin with the A_n-case. By [7] the muation class of quivers of type A_n is given by the connected quivers on n vertices such that the following properties hold:

(I) All non-trivial cycles are oriented and of length 3.

(II) A vertex has valency at most 4.

(III) If a vertex has valency 4, then two of its adjacent arrows belong to one 3-cycle, the other two belong to another 3-cycle.

(IV) If a vertex has valency 3, then two of its adjacent arrows belong to a 3-cycle, the third arrow does not belong to another cycle.

Let Q be a quiver on n vertices fulfilling the properties (I)-(IV) above. By Corollary 18 we have to show that $\overline{Q}$ belongs to $\mathcal{A}^{n,n}$, that is $\overline{Q}$ fulfills the properties (i)-(iv) of Corollary 18. The claim is obvious for $n = 1$. So let $n > 1$. By (I) we conclude that we can write Γ as the union

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k \ (k \in \mathbb{N}),$$

where each Γ_i is a complete graph on two or three vertices, $\Gamma_i \nsubseteq \Gamma_j$ for $i \neq j$ and k is minimal (that is, for instance, we exclude the possibility that the union of three complete graphs on two vertices is a complete graph on three vertices).

Let $i \neq j$ be such that $\Gamma_i \cap \Gamma_j \neq \varnothing$ and let v be a vertex in $\Gamma_i \cap \Gamma_j$. By the previous arguments and by (II) we have $2 \leqslant \operatorname{val}(v) \leqslant 4$. By distinguishing all possible cases for the value of $\operatorname{val}(v)$, we can show that v is the only vertex in $\Gamma_i \cap \Gamma_j$, hence (ii) holds. We exhibit the case $\operatorname{val}(v) = 4$. In this case, condition (III) implies that Γ_i and Γ_j are both 3-cycles intersecting just in the vertex v.

Next we want to show (iii). Let v be a vertex of Γ. If $\operatorname{val}(v) \leqslant 2$, then (iii) holds obviously for v. If $\operatorname{val}(v) = 3$ (resp. $\operatorname{val}(v) = 4$) then condition (IV) (resp. (III)) implies that v belongs to exactly two of the subgraphs Γ_i. Thus (iii) holds.

Finally we have to show (iv). Since (iii) holds, we know that $\cap_{i \in I} \Gamma_i = \varnothing$ for $I \subseteq [k]$ with $|I| \geqslant 3$. By the inclusion-exclusion principle we conclude that

$$n = |\Gamma| = \sum_{i=1}^{k} |\Gamma_i| - \sum_{\substack{\{i,j\} \subseteq [k] \\ i \neq j}} |\Gamma_i \cap \Gamma_j|. \tag{6}$$

If $k = 1$, we are done. Therefore let $k > 1$. Since Γ is connected, the graph Γ_1 has non-trivial intersection with one of the graphs Γ_i with $i > 1$. After possible renumbering we can assume that $|\Gamma_1 \cap \Gamma_2| = 1$ by (ii). Again, if $k = 2$, we are done. Therefore let $k > 2$. Since Γ is connected, one of the graphs Γ_i with $i > 2$ has non-trivial intersection with Γ_1 or Γ_2. Again, after possible renumbering, we can assume that $i = 3$ and $|\Gamma_2 \cap \Gamma_3| = 1$ by (ii). Using the conditions (I)-(IV) and the minimality of k, it is straightforward to see that $\Gamma_1 \cap \Gamma_3 = \varnothing$. Proceeding in this manner we obtain that $|\Gamma_i \cap \Gamma_{i+1}| = 1$ for $i \in [k-1]$, while $\Gamma_i \cap \Gamma_j = \varnothing$ for $j \neq i, i+1$. We leave the details to the reader. Hence

$$\sum_{\substack{\{i,j\} \subseteq [k] \\ i \neq j}} |\Gamma_i \cap \Gamma_j| = k - 1$$

and we conclude by (6) that (iv) holds.

Now let Q be a quiver which is mutation-equivalent to an orientation of the Dynkin diagram of type B_n. By [19, Proposition 3.2] and what we have shown above for the A_n-case, $\overline{Q}$ is given by one of the graphs in Figure 7, where Γ_1 and Γ_2 are cyclically orientable Carter diagrams of type A_{k_1} and A_{k_2} for some $k_1, k_2 \geqslant 1$.
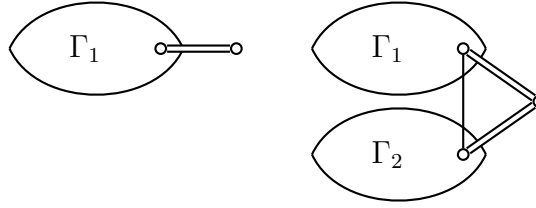
Figure 7: Quiver-mutation class B_n.

It follows immediately from our description of type B_n Carter diagrams that $\overline{Q}$ is a Carter diagram of type B_n.

To show the other assertion first note, that if Q is a quiver which is mutation-equivalent to an orientation of the Dynkin diagram of type A_n (resp. B_n), then in particular Q admits a cyclic orientation. By what we have shown above, $\overline{Q}$ is a cyclically orientable Carter diagram.

For the other direction let us begin with a Carter diagram Γ of type A_n, that is Γ lies in $\mathcal{A}^{n,n}$ by Theorem 17, and assume that Γ is cyclically orientable. We choose a cyclic orientation of Γ and show that the conditions (I)-(IV) hold.

The complete graph on four vertices is not cyclically orientable. In particular, the complete graph on $\ell \geqslant 4$ vertices is not cyclically orientable. Therefore condition (i) from Theorem 17 yields that Γ is the union of complete graphs on two or three vertices. Hence (I) holds. Conditions (ii) and (iii) imply (II), while conditions (ii)-(iv) imply (III) and (IV). We leave the details again to the reader.

Now let Γ be a Carter diagram of type B_n which is cyclically orientable. By our description of the type B_n Carter diagrams, there is a unique vertex v such that all edges adjacent with v have weight 2 and such that $\Gamma \setminus \{v\}$ is connected. Let us denote by Γ_0 the diagram obtained from Γ by replacing edges of weight two with edges of weight one. By our description of the type B_n Carter diagrams, the graph Γ_0 is a cyclically orientable Carter diagram of type A_n. In particular, Γ is cyclically orientable. $\qquad\square$

Before we show that Theorem 1 holds for type D_n, let us recall the description of the quiver mutation-class of type D_n given by Vatne.

Lemma 37 ([21, Theorem 3.1]). *For $n \geqslant 4$, a quiver Q is mutation equivalent to an orientation of the Dynkin diagram of type D_n if and only if Q is one of the types (D1)-(D4) shown in Figure 8*
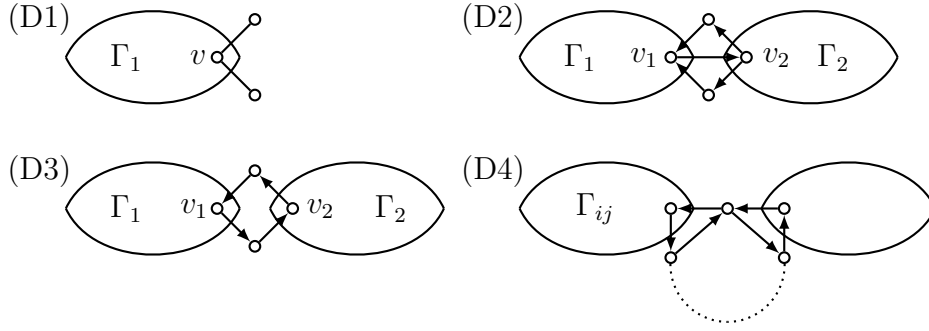
Figure 8: Quiver-mutation class D_n.

Let us make this description more precise (see [21, Chapter 2]). Therefore denote by $\mathcal{M}_{A_n}$ the type A_n mutation-class described by the properties (I)-(IV) in the proof of Proposition 36. Call a vertex v of a quiver Q a *connecting vertex* if v has valency at most 2 and, moreover, if v has valency 2, then v is a vertex in a 3-cycle in Q.

(D1): Γ_1 is in $\mathcal{M}_{A_{n-2}}$ and v is a connecting vertex for Γ_1

(D2): Γ_1 (resp. Γ_2) is in $\mathcal{M}_{A_{n_1}}$ (resp. $\mathcal{M}_{A_{n_2}}$) for some $n_1 \in \mathbb{N}$ (resp. $n_2 \in \mathbb{N}$) and v_1 (resp. v_2) is a connecting vertex for Γ_1 (resp. Γ_2).

(D3): See (D2).

(D4): The quiver Q described by this type has a full subquiver which is a directed k-cycle ($k \geqslant 3$), called *central cycle*. For each arrow $\alpha : i \to j$ in Q, there may (and may not) be a vertex c_{ij} which is not in the central cycle, such that there is an oriented 3-cycle $i \overset{\alpha}{\to} j \to v_{ij} \to i$. This 3-cycle is a full subquiver. It is called *spike*. There are no more arrows starting or ending in vertices on the central cycle. To each spike $i \overset{\alpha}{\to} j \to v_{ij} \to i$ there is a quiver Γ_{ij} from $\mathcal{M}_{A_{n_{ij}}}$ attached, for some $n_{ij} \in \mathbb{N}$. The vertex v_{ij} is a connecting vertex for Γ_{ij}.

Note that in types (D2)-(D4), the subquivers Γ_i resp. Γ_{ij} might be in $\mathcal{M}_{A_1}$.

Proposition 38. *Theorem 1 is true for type D_n.*

Proof. Let us first recall that by Theorem 25, all Carter diagrams of type D_n are given by the set $\mathcal{A}^{n-1,n}$. By Theorem 17 the shape of a diagram Γ in $\mathcal{A}^{n-1,n}$ depends on the choice of an integer m', which is either equal to $n-1$ or to n. If $m' = n-1$, we call Γ to be of type (D.I). In this type the diagram Γ is given by a Carter diagram of type A_{n-1} to which we attach a "duplicated" vertex. If $m' = n$, we call Γ to be of type (D.II).

Let us start with a quiver Q which is mutation-equivalent to an orientation of the Dynkin diagram of type D_n. We have to show that the underlying undirected graph $\overline{Q}$ is a Carter diagram of type D_n. We do this by showing that all four possible types of underlying undirected graphs in the mutation-class given by Lemma 37 can be realized by a Carter diagram of type D_n.

Let Q be of type (D1). Then $\overline{Q}$ is given by the following picture.
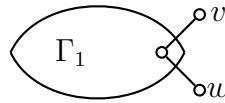
By Proposition 36, the graph Γ_1 is a Carter diagram of type A_{n-2}. In particular, the full subgraph $\Gamma_1 \cup \{v\}$ is a Carter diagram of type A_{n-1}. Duplication of the vertex v yields the above graph, which is therefore of type (D.I).

Let Q be of type (D2) or (D3). Then $\overline{Q}$ is given by the following picture (for type (D2) the vertices v_1 and v_2 are not connected, while for type (D3) they are).
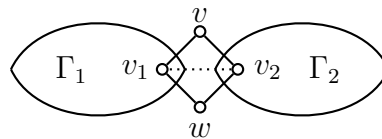
By Proposition 36, the full subgraph $\Gamma_1 \cup \Gamma_2 \cup \{v\}$ is a Carter diagram of type A_{n-1}. Duplication of the vertex v yields the above graph, which is therefore again of type (D.I).

Let Q be of type (D4). We want to show that $\overline{Q}$ is of type (D.II). By [17, Proposition 5 (iii) a)] any connected graph on n vertices that is the union of complete graphs which are arranged in the form of a circle with some side branches, is a graph of type (D.II). In particular, if we assume that all of these complete graphs are complete graphs on two or three vertices, eventually we will end up with the graph $\overline{Q}$.

Finally, let Γ be a Carter diagram of type D_n which is cyclically orientable. We have to show that Γ is isomorphic to $\overline{Q}$ for some quiver Q which is mutation-equivalent to an orientation of the Dynkin diagram of type D_n.

Let us first assume that Γ is of type (D.I), that is $\Gamma = \Gamma' \cup \{w\}$, where Γ' is a Carter diagram of type A_{n-1} and w is the "duplication" of a vertex $v \in \Gamma'$. In particular, Γ' is cyclically orientable and therefore $\mathrm{val}(v) \leqslant 4$. If $\mathrm{val}(v) = 1$, then it is easy to see that Γ is isomorphic to $\overline{Q}$ for some quiver Q of type (D1). Similarly, if $\mathrm{val}(v) = 2$, then Γ is isomorphic to $\overline{Q}$ for some quiver Q of type (D2) or (D3). Let $\mathrm{val}(v) = 3$. All three vertices v_1, v_2, v_3 adjacent to v have to be vertices of Γ'. Since Γ' is cyclically orientable of type A_{n-1}, property (IV) from the proof of Proposition 36 holds. Let us assume that we have the following situation in Γ':
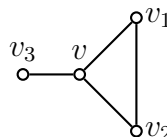
Duplication of v yields the following full subgraph of Γ:
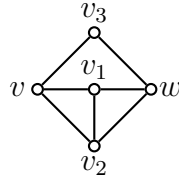
But this full subgraph is not cyclically orientable, hence Γ cannot be cyclically orientable. Therefore the case that v has valency 3 does not occur. Similarly, we can show that the case that v has valency 4 does not occur.

Let us assume that Γ is of type (D.II). By [17, Proposition 5 (iii) a)] the graph Γ is the union of complete graphs which are arranged in the form of a circle with some side branches. Each of these complete graphs has to be a complete graph on 2 or 3 vertices, since complete graphs on $\ell \geqslant 4$ vertices are not cyclically orientable and Γ is cyclically orientable. Therefore Γ is isomorphic to $\overline{Q}$ for some quiver Q of type (D4). $\qquad\square$

Proposition 39. *Theorem 1 is true for the exceptional types E_6, E_7, E_8, F_4 and G_2.*

Proof. This is done by inspection of our lists of Carter diagrams of exceptional type X_n and the mutation classes of Dynkin quivers of type X_n. The latter ones can be computed using Keller's quiver mutation applet [16]. $\qquad\square$

We have given a complete classification of Carter diagrams of Dynkin types. Thereafter we have seen that the cyclically orientable Carter diagrams are precisely the underlying undirected graphs of quivers which appear in the seeds of finite type cluster algebras. This leads to the following question:

Question 40. Let Φ be a crystallographic root system. Given a set of linearly independent roots $\{\beta_1, \ldots, \beta_m\} \subseteq \Phi$ and let Γ be its associated Carter diagram. Is there a criterion to determine whether Γ is cyclically orientable purely in terms of the root system?

4 Presentations of Reflection Groups

In [1] it is shown that each quiver which is mutation-equivalent to an orientation of a Dynkin diagram encodes a natural presentation of the corresponding finite Coxeter group (like the Dynkin diagram does; see Section 2.1).

We have seen in the previous chapters that the cyclically orientable Carter diagrams exactly provide the underlying graphs of quivers which are mutation-equivalent to a Dynkin diagram. In fact, all Carter diagrams, even those which are not cyclically orientable or simply-laced (that is, of type A_n, D_n, E_6, E_7 or E_8), provide a natural presentation (as given by Barot–Marsh [1]) of the corresponding finite Coxeter group.

Let us begin with the observation that Theorem 2 holds for admissible Carter diagrams.

Proposition 41. *Let Φ be a crystallographic root system and let Γ be an admissible Carter diagram of the same Dynkin type as Φ. Then $W(\Gamma)$ is isomorphic to the Weyl group W_Φ.*

Proof. For the cyclically orientable Carter diagrams, the assertion is a consequence of Theorem 1 and [1, Theorem A]. Inspection of Carter's list of admissible diagrams in [10] yields that all of them are cyclically orientable except for $E_7(a_4), E_8(a_7)$ and $E_8(a_8)$. For these three cases it has been checked using GAP [12] that the assertion holds. □

Before we start to explain the idea and to carry out the proof of Theorem 2, let us remark the following fact about the relations (R3) given in Section 1.

Proposition 42 ([1, Lemma 4.1, Proposition 4.6])**.** *For any chordless cycle C in a Carter diagram Γ, all relations of type (R3) attached to C are equivalent to one fixed relation of type (R3) attached to C (in the presence of the relations (R1) and (R2)). In particular, relation (R3) does neither depend on the choice of the vertex i_0 in the cycle C nor on the direction of the cycle.*

The Hurwitz action will plays an important role for Theorem 2. We illustrate this by two examples.

Example 43. Consider a Coxeter system (W, S) of type D_6 and let $(t_1, \ldots, t_6)$ be a tuple of reflections such that the corresponding Carter diagram is given by the left diagram in Figure 9. Note that this is in fact a Carter diagram of type D_6 by Theorem 25. We apply the Hurwitz move

$$(t_1, \ldots, t_6) \overset{\sigma_3}{\sim} (t_1, t_2, t_3 t_4 t_3, t_3, t_5, t_6) =: (r_1, \ldots, r_6).$$

The resulting diagram is given by the diagram on the right side in Figure 9.
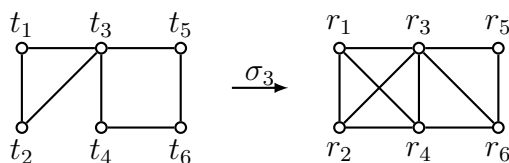

Figure 9: Hurwitz move applied to a Carter diagram.

Note that the diagram on the left side is cyclically orientable, while the diagram on the right side is not. Therefore the arguments of Barot–Marsh just yield a presentation attached to the diagram on the left.

Let $\langle t_1, \ldots, t_6 \mid R \rangle$ (resp. $\langle r_1, \ldots, r_6 \mid R' \rangle$) be the presentation attached to the diagram on the left side (resp. to the diagram on the right side) as described in Section 1 and consider the map

$$\varphi : \langle t_1, \ldots, t_6 \mid R \rangle \to \langle r_1, \ldots, r_6 \mid R' \rangle$$

$$t_j \mapsto \begin{cases} r_j & \text{if } j \neq 3, 4 \\ r_4 & \text{if } j = 3 \\ r_4 r_3 r_4 & \text{if } j = 4. \end{cases}$$

The relations R are preserved under the map φ. We exhibit some examples:

- We have the relation $(t_2 t_4)^2 = 1$ in R, but also

$$(\varphi(t_2)\varphi(t_4))^2 = (r_1 r_4 r_3 r_4)^2 = 1,$$

  since we have the chordless cycle $r_1 - r_4 - r_3 - r_1$.

- We have the relation $(t_4 t_3 t_5 t_6 t_5 t_3)^2 = 1$ in R, which is induced by the cycle of length 4. But also

$$\begin{aligned}
(\varphi(t_4)\varphi(t_3)\varphi(t_5)\varphi(t_6)\varphi(t_5)\varphi(t_3))^2 &= (r_4 r_3 r_4 r_4 r_5 r_6 r_5 r_4)^2 \\
&= (r_4 r_3 r_5 r_6 r_5 r_4)^2 \\
&= r_4 (r_3 r_5 r_6 r_5)^2 r_4 = r_4^2 = 1,
\end{aligned}$$

  where we used in the last line the relation in R' given by the cycle $r_3 - r_5 - r_6 - r_3$.

By [15, Ch. 4, Proposition 3] we obtain that the map φ extends to a surjective group homomorphism. Likewise we see that

$$\psi : \langle r_1, \ldots, r_6 \mid R' \rangle \to \langle t_1, \ldots, t_6 \mid R \rangle$$

$$r_j \mapsto \begin{cases} t_j & \text{if } j \neq 3, 4 \\ t_3 t_4 t_3 & \text{if } j = 3 \\ t_3 & \text{if } j = 4 \end{cases}$$

extends to a surjective group homomorphism. Since $\psi(\varphi(t_j)) = t_j$ and $\varphi(\psi(r_j)) = r_j$ for all $j \in \{1, \ldots, 6\}$, we obtain

$$\langle t_1, \ldots, t_6 \mid R \rangle \cong \langle r_1, \ldots, r_6 \mid R' \rangle.$$
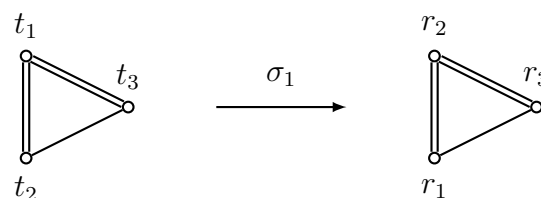
Example 44. We consider a root system Φ of type B_3 and use the description of W_Φ as group of signed permutations S_3^B given in Section 2.4.2. Consider

$$(t_1, t_2, t_3) := ((1, -1), (1, 2)(-1, -2), (1, 3)(-1, -3)).$$

This is a reduced reflection factorization of a Coxeter element. We have

$$\begin{aligned}
\sigma_1((1, -1), (1, 2)(-1, -2), (1, 3)(-1, -3)) &= ((1, -2)(-1, 2), (1, -1), (1, 3)(-1, -3)) \\
&=: (r_1, r_2, r_3).
\end{aligned}$$

The effect of this Hurwitz move on the Carter diagram is as follows.

That is, we obtain the same diagram. This comes from the fact that the distinguished vertex $t_1 = (1, -1) = r_2$ has not changed under the Hurwitz move σ_1.

On the other hand, we have

$$\sigma_1^{-1}((1, -1), (1, 2)(-1, -2), (1, 3)(-1, -3)) = ((1, 2)(-1, -2), (2, -2), (1, 3)(-1, -3))$$
$$=: (r_1', r_2', r_3').$$

The effect of this Hurwitz move on the Carter diagram is different since we change the distinguished vertex from $(1, -1)$ to $(2, -2)$.
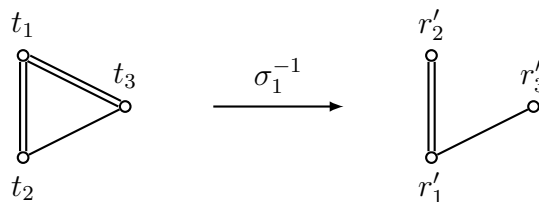
Before we begin with the proof of Theorem 2, let us emphasize again that the simply-laced case is already covered by the work of Cameron, Seidel and Tsaranov [9]. In fact, they also use the Hurwitz action, but call it *local switching*. The case that the Carter diagram is cyclically orientable is covered by the work of Barot and Marsh [1]. For our proof we will only have to fill the missing gaps, that is, Carter diagrams which are not simply-laced as well as not cyclically orientable. But since our arguments for type B_n are based on the simply laced case, we will give a complete proof of Theorem 2.

Let W_Φ be a Weyl group, $w \in W_\Phi$ and $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$ with Carter diagram Γ. For every braid $\sigma \in \mathcal{B}_m$, the Hurwitz action yields a new reduced reflection factorization $\sigma(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$ with Carter diagram Γ'. We put $\sigma(\Gamma) := \Gamma'$

Direct calculation yields the following.

Remark 45. Let Φ be a simply-laced root system (that is, of type A_n, D_n, E_6, E_7 or E_8). Let $w \in W_\Phi$ and $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$ with associated Carter diagram Γ. We can describe the impact of an elementary Hurwitz move on Γ, that is, how to obtain $\sigma_i(\Gamma)$ from Γ as follows:

Let $\sigma_i(t_1, \ldots, t_m) = (r_1, \ldots, r_m)$. Hence we have $t_j = r_j$ for $j \in [m] \setminus \{i, i+1\}$, $r_i = t_i t_{i+1} t_i$ and $r_{i+1} = t_i$. If $(t_i t_{i+1})^2 = 1$, then Γ and $\sigma_i(\Gamma)$ are identical. Therefore let us assume that $(t_i t_{i+1})^2 \neq 1$. The diagram $\sigma_i(\Gamma)$ is obtained as follows:

- The vertices of $\sigma_i(\Gamma)$ correspond to $r_1, \ldots, r_m$.

- For $j, k \in [m] \setminus \{i, i+1\}$, there is an edge between r_j and r_k if and only if there is an edge between t_j and t_k.

- For $j \in [m] \setminus \{i, i+1\}$, there is an edge between r_j and r_{i+1} if and only if there is an edge between t_j and t_i.

- There is an edge between r_i and r_{i+1}.

- For $j \in [m] \setminus \{i, i+1\}$, there is an edge between r_j and r_i if and only if either t_j and t_i or t_j and t_{i+1} (but not both) are connected by an edge in Γ.

Remark 46. Let Φ be a root system of type B_n. Let $w \in W_\Phi$ and $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$ with associated Carter diagram Γ. As in Remark 45 we describe how to obtain $\sigma_i(\Gamma)$ from Γ:

Let $\sigma_i(t_1, \ldots, t_m) = (r_1, \ldots, r_m)$. Hence we have $t_j = r_j$ for $j \in [m] \setminus \{i, i+1\}$, $r_i = t_i t_{i+1} t_i$ and $r_{i+1} = t_i$. If $(t_i t_{i+1})^2 = 1$, then Γ and $\sigma_i(\Gamma)$ are identical. Therefore let us assume that $(t_i t_{i+1})^2 \neq 1$. The diagram $\sigma_i(\Gamma)$ is obtained as follows:

1. If t_i does not correspond to the distinguished vertex of Γ, then the diagram $\sigma_i(\Gamma)$ is obtained from Γ as decribed in Remark 45. We just have to take care of the labeling. If t_{i+1} is the distinguished vertex in Γ, then r_i is the distinguished vertex in $\sigma_i(\Gamma)$. If t_j is the distinguished vertex in Γ for some $j \in [n] \setminus \{i, i+1\}$, then r_j is the distinguished vertex in $\sigma_i(\Gamma)$.

2. If t_i is the distinguished vertex, then we obtain $\sigma_i(\Gamma)$ as follows:

    - The vertices of $\sigma_i(\Gamma)$ correspond to $r_1, \ldots, r_m$.
    - For $j, k \in [m] \setminus \{i, i+1\}$, there is an edge between r_j and r_k if and only if there is an edge between t_j and t_k.
    - For $j \in [m] \setminus \{i, i+1\}$, there is an edge between r_j and r_{i+1} if and only if there is an edge between t_j and t_i.
    - There is an edge between r_i and r_{i+1}.
    - For $j \in [m] \setminus \{i, i+1\}$, there is an edge between r_j and r_i if and only if there is an edge between t_j and t_{i+1}: Since t_i is the distinguished vertex, we have

    $$\mathrm{supp}(t_{i+1}) = \mathrm{supp}(t_i t_{i+1} t_i) = \mathrm{supp}(r_i),$$

    and since $(t_1, \ldots, t_m)$ is reduced, we have $t_j \neq t_{i+1}$ for all $j \neq i+1$.
    - The distinguished vertex of $\sigma_i(\Gamma)$ is r_{i+1}.

In particular, we see that Γ and $\sigma_i(\Gamma)$ are isomorphic.

As a direct consequence of this remark we obtain:

Proposition 47. *If Φ is of type B_n and t_i is the distinguished vertex of Γ, then $W(\Gamma) \cong W(\sigma_i(\Gamma))$.*

Lemma 48. *Let Γ be a Carter diagram of simply-laced type or type B_n with vertex set corresponding to $t_1, \ldots, t_m$ and $i \in [m-1]$. If $(t_i t_{i+1})^2 \neq 1$ (and, in the B_n-case, if t_i is not the distinguished vertex of Γ), then the map*

$$\varphi : W(\Gamma) \to W(\sigma_i(\Gamma)), \ t_j \mapsto \begin{cases} r_j & \text{if } j \neq i, i+1 \\ r_{i+1} & \text{if } j = i \\ r_{i+1} r_i r_{i+1} & \text{if } j = i+1 \end{cases}$$

extends to a (surjective) group homomorphism.

Proof. We argue with the "substitution test" [15, Ch. 4, Proposition 3]. We therefore investigate all possible constellations in Γ and how these might change under the action of σ_i. We show that the relations in $W(\Gamma)$ are preserved under the map φ. We denote by $W(\sigma_i(\Gamma))$ the group with generators $r_1, \ldots r_m$ subject to the relations induced by the Carter diagram $\sigma_i(\Gamma)$.

By assumption, the vertices corresponding to t_i and t_{i+1} will be connected by an edge in Γ. Depending on whether the weight is 2 or not, we have $(t_i t_{i+1})^3 = 1$ or $(t_i t_{i+1})^4 = 1$. By Remark 45 and Remark 46 the relation $(\varphi(t_i)\varphi(t_{i+1}))^3 = 1$ or $(\varphi(t_i)\varphi(t_{i+1}))^4 = 1$ will always hold. Also by these remarks it is enough to just consider those relations which actually involve t_i and t_{i+1}. Therefore one would have to consider all possible constellations of t_i and t_{i+1} in Γ. To give an idea for the proof of the simply-laced case, we carry out two possible cases.

1) The vertex t_i is part of a cycle, while t_{i+1} is not part of that cycle.
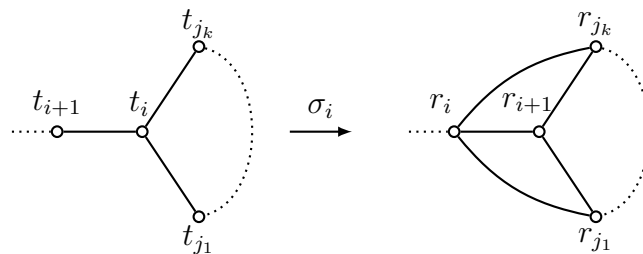
- $(t_{i+1} t_{j_1})^2 = 1 \to (\varphi(t_{i+1})\varphi(t_{j_1}))^2 = (r_{i+1} r_i r_{i+1} r_{j_1})^2 = 1$, since r_i, r_{i+1} and r_{j_1} are the vertices of a 3-cycle. Analogously we can argue for $(t_{i+1} t_{j_k})^2 = 1$.

- $(t_i t_{j_1})^3 = 1 \to (\varphi(t_i)\varphi(t_{j_1}))^3 = (r_{i+1} r_{j_1})^3 = 1$. Analogously we can argue for $(t_i t_{j_k})^3 = 1$.

- $(t_i t_{j_1} \cdots t_{j_{k-1}} t_{j_k} t_{j_{k-1}} \cdots t_{j_1})^2 = 1$. Applying φ yields
$$(r_{i+1} r_{j_1} \cdots r_{j_{k-1}} r_{j_k} r_{j_{k-1}} \cdots r_{j_1})^2 = 1,$$
which is exactly the realtion of type (R3) for the "big" cycle in $\sigma_i(\Gamma)$.

2) t_i and t_{i+1} are both vertices of a full subgraph which is an ℓ-cycle ($\ell \geqslant 4$).
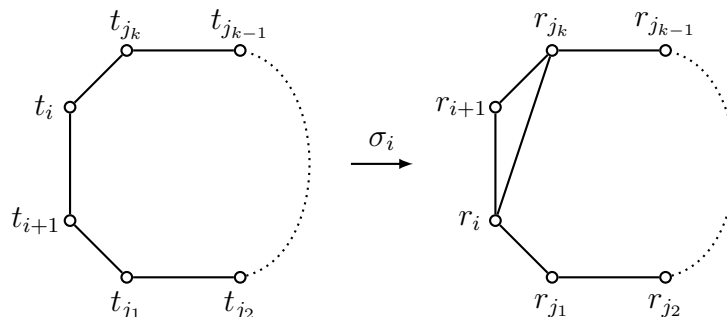
- $(t_i t_{j_k})^3 = 1 \rightarrow (\varphi(t_i)\varphi(t_{j_k}))^3 = (r_{i+1} r_{j_k})^3 = 1$, since r_{i+1} and r_{j_k} are connected by an edge.

- $(t_i t_{j_1})^2 = 1 \rightarrow (\varphi(t_i)\varphi(t_{j_1}))^2 = (r_{i+1} r_{j_1})^2 = 1$, since r_{i+1} and r_{j_1} are not connected by an edge.

- $(t_{i+1} t_{j_k})^2 = 1 \rightarrow (\varphi(t_{i+1})\varphi(t_{j_k}))^2 = (r_{i+1} r_i r_{i+1} r_{j_k})^2 = 1$, since the vertices are part of a 3-cycle.

- $(t_{i+1} t_{j_1})^3 = 1 \rightarrow (\varphi(t_{i+1})\varphi(t_{j_1}))^3 = (r_{i+1} r_i r_{i+1} r_{j_1})^3 = r_{i+1}(r_i r_{j_1})^3 r_{i+1} = 1$.

- $(t_i t_{i+1} t_{j_1} \cdots t_{j_{k-1}} t_{j_k} t_{j_{k-1}} \cdots t_{j_1} t_{i+1})^2 = 1$. Applying φ yields

$$
\begin{aligned}
&(r_{i+1} r_{i+1} r_i r_{i+1} r_{j_1} \cdots r_{j_{k-1}} r_{j_k} r_{j_{k-1}} \cdots r_{j_1} r_{i+1} r_i r_{i+1})^2\\
=&(r_i r_{i+1} r_{j_1} \cdots r_{j_{k-1}} r_{j_k} r_{j_{k-1}} \cdots r_{j_1} r_i r_{i+1} r_i)^2\\
=&r_i r_{i+1}(r_{j_1} \cdots r_{j_{k-1}} r_{j_k} r_{j_{k-1}} \cdots r_{j_1} r_i)^2 r_{i+1} r_i\\
=&r_i r_{i+1} r_{i+1} r_i = 1.
\end{aligned}
$$

Here we make use of the fact that $(r_{j_1} \cdots r_{j_{k-1}} r_{j_k} r_{j_{k-1}} \cdots r_{j_1} r_i)^2 = 1$ is equivalent to $(r_i r_{j_1} \cdots r_{j_{k-1}} r_{j_k} r_{j_{k-1}} \cdots r_{j_1})^2 = 1$, which is precisely the relation induced by the "big" cycle.

In the following we will consider all possible constellations in type B_n.

1) The vertices t_i and t_{i+1} are part of a full subgraph which is a line. We excluded the case that t_i is the distinguished vertex. If t_{i+1} is the distinguished vertex, then it has to be of valency one, because otherwise $\Gamma \setminus \{t_i\}$ would be disconnected.
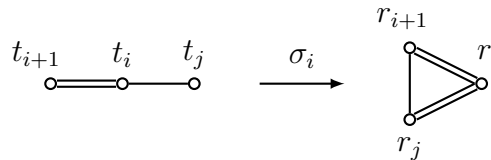
- $(t_i t_j)^3 = 1 \rightarrow (\varphi(t_i)\varphi(t_j))^3 = (r_{i+1} r_j)^3 = 1$,

- $(t_{i+1} t_j)^2 = 1 \rightarrow (\varphi(t_{i+1})\varphi(t_j))^2 = (r_{i+1} r_i r_{i+1} r_j)^2$. But $(r_{i+1} r_i r_{i+1} r_j)^2 = 1$ is equivalent to $(r_j r_{i+1} r_i r_{i+1})^2 = 1$ and this relation holds because of the cycle in $\sigma_i(\Gamma)$.

By the description of Carter diagrams of type B_n in Section 2.4.2, we know that if Γ contains a chordless cycle, then it has to be a 3-cycle.

2) The vertices t_i and t_{i+1} are part of a full subgraph which is a 3-cycle. The vertex t_{i+1} is the distinguished vertex.
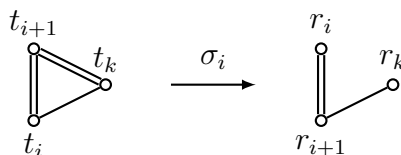
- $(t_i t_k)^3 = 1 \to (\varphi(t_i)\varphi(t_k))^3 = (r_{i+1} r_k)^3 = 1.$

- $(t_{i+1} t_k)^4 = 1 \to$ applying the map φ yields:

$$
\begin{aligned}
(\varphi(t_{i+1})\varphi(t_k))^4 &= (r_{i+1} r_i r_{i+1} r_k)^4 \\
&= (r_{i+1} r_i \underbrace{r_{i+1} r_k})(r_{i+1} r_i \underbrace{r_{i+1} r_k})(r_{i+1} r_i \underbrace{r_{i+1} r_k})(r_{i+1} r_i r_{i+1} r_k)) \\
&\qquad\quad {}_{=r_k r_{i+1} r_k} \qquad\quad {}_{=r_k r_{i+1} r_k} \qquad\quad {}_{=r_k r_{i+1} r_k} \\
&= r_{i+1} r_i r_k r_{i+1} \underbrace{r_k r_i r_k}\, r_{i+1} \underbrace{r_k r_i r_k}\, r_{i+1} r_k r_i r_{i+1} r_k \\
&\qquad\qquad\qquad {}_{=r_i} \qquad\qquad {}_{=r_i} \\
&= r_{i+1} r_i r_k r_{i+1} r_i r_{i+1} r_i r_{i+1} r_k r_i r_{i+1} r_k \\
&= r_{i+1} r_k \underbrace{r_i r_{i+1} r_i r_{i+1} r_i r_{i+1} r_i}\, r_k r_{i+1} r_k \\
&\qquad\qquad\qquad {}_{r_{i+1}} \\
&= (r_{i+1} r_k)^3 = 1.
\end{aligned}
$$

- $(t_i t_k t_{i+1} t_k)^2 = 1 \to$ applying the map φ yields:

$$
\begin{aligned}
(\varphi(t_i)\varphi(t_k)\varphi(t_{i+1})\varphi(t_k))^2 &= (r_{i+1} r_k r_{i+1} r_i r_{i+1} r_k)^2 \\
&= (r_{i+1} r_k r_{i+1} r_i \underbrace{r_{i+1} r_k})(r_{i+1} r_k r_{i+1}\, r_i r_{i+1} r_k) \\
&\qquad\qquad\qquad\qquad {}_{=r_k} \\
&= r_{i+1} r_k r_{i+1} \underbrace{r_i r_k r_i}\, r_{i+1} r_k \\
&\qquad\qquad\quad {}_{=r_k} \\
&= (r_{i+1} r_k)^3 = 1.
\end{aligned}
$$

3) The vertex t_{i+1} is part of a full subgraph which is a 3-cycle and which contains the distinguished vertex t_j, while t_i is not a vertex of that 3-cycle.
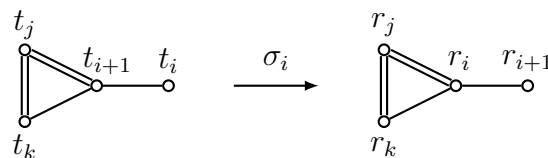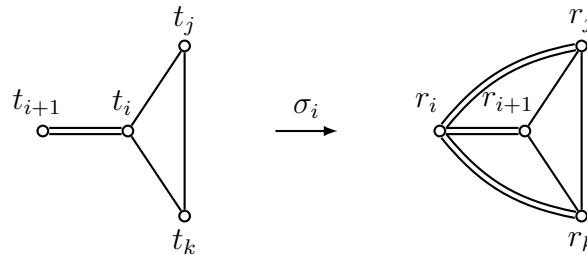

- $(t_i t_j)^2 = 1 \to (\varphi(t_i)\varphi(t_j))^2 = (r_{i+1} r_j)^2 = 1,$

- $(t_{i+1} t_j)^4 = 1 \to (\varphi(t_{i+1})\varphi(t_j))^4 = (r_i r_j)^4 = 1.$

- $(t_{i+1} t_k t_j t_k)^2 \to (\varphi(t_{i+1})\varphi(t_k)\varphi(t_j)\varphi(t_k))^2 = (r_{i+1} r_i r_{i+1} r_k r_j r_k)^2 = r_{i+1}(r_i r_k r_j r_k)^2 r_{i+1}$
  $= 1,$ where $(r_i r_k r_j r_k)^2 = 1$ holds because of the cycle in $\sigma_i(\Gamma)$.

4) The vertex t_i is part of a full subgraph which is a 3-cycle and which contains the distinguished vertex t_j, while t_{i+1} is not a vertex of that 3-cycle.
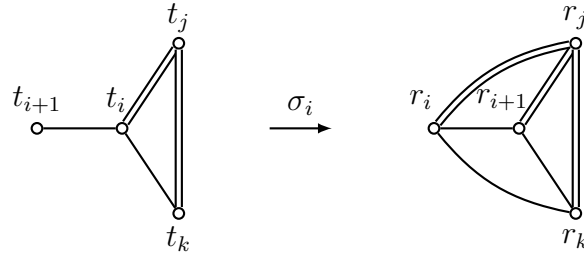
- $(t_i t_j)^4 = 1 \rightarrow (\varphi(t_i)\varphi(t_j))^4 = (r_{i+1}r_j)^4 = 1$. Analogously we can argue for $(t_i t_k)^4 = 1$.

- $(t_j t_k)^4 = 1 \rightarrow (\varphi(t_j)\varphi(t_k))^4 = (r_j r_k)^4 = 1$.

- $(t_i t_k t_j t_k)^2 \rightarrow (\varphi(t_i)\varphi(t_k)\varphi(t_j)\varphi(t_k))^2 = (r_{i+1}r_k r_j r_k)^2 = 1$, where $(r_{i+1}r_k r_j r_k)^2 = 1$ by one of the cycles in $\sigma_i(\Gamma)$.

- $(t_{i+1}t_j)^2 = 1 \rightarrow (\varphi(t_{i+1})\varphi(t_j))^2 = (r_{i+1}r_i r_{i+1}r_j)^2$, but the equation $(r_{i+1}r_i r_{i+1}r_j)^2 = 1$ is equivalent to $(r_i r_{i+1}r_j r_{i+1})^2 = 1$, which holds by one of the cycles in $\sigma_i(\Gamma)$. Analogously we can argue for $(t_{i+1}t_k)^2 = 1$.

5) The vertex t_i is part of a full subgraph which is a 3-cycle, while t_{i+1} is not a vertex of that 3-cycle but t_{i+1} is the distinguished vertex.

- $(t_i t_j)^3 = 1 \rightarrow (\varphi(t_i)\varphi(t_j))^3 = (r_{i+1}r_j)^3 = 1$. Analogously we can argue for $(t_i t_k)^3 = 1$.

- $(t_j t_k)^3 = 1 \rightarrow (\varphi(t_j)\varphi(t_k))^3 = (r_j r_k)^3 = 1$.

- $(t_i t_k t_j t_k)^2 \rightarrow (\varphi(t_i)\varphi(t_k)\varphi(t_j)\varphi(t_k))^2 = (r_{i+1}r_k r_j r_k)^2 = 1$, where $(r_{i+1}r_k r_j r_k)^2 = 1$ by one of the cycles in $\sigma_i(\Gamma)$.

- $(t_{i+1}t_j)^2 = 1 \rightarrow (\varphi(t_{i+1})\varphi(t_j))^2 = (r_{i+1}r_i r_{i+1}r_j)^2$, but the equation $(r_{i+1}r_i r_{i+1}r_j)^2 = 1$ is equivalent to $(r_i r_{i+1}r_j r_{i+1})^2 = 1$, which holds by one of the cycles in $\sigma_i(\Gamma)$. Analogously we can argue for $(t_{i+1}t_k)^2 = 1$. $\qquad\square$

We have $\sigma_i^{-1}(t_1, \ldots, t_m) = (r_1, \ldots, r_m)$, where $t_j = r_j$ for $j \in [m] \setminus \{i, i+1\}$, $r_i = t_{i+1}$ and $r_{i+1} = t_{i+1}t_i t_{i+1}$. Similar to Remarks 45 and 46 we can describe how to obtain the Carter diagram $\sigma_i^{-1}(\Gamma)$ on vertex set $\{r_1, \ldots, r_m\}$ from Γ. The same arguments as in the proof of Lemma 48 lead to the following result.

Lemma 49. *Let Γ be a Carter diagram of simply-laced type or type B_n with vertex set corresponding to $t_1, \ldots, t_m$ and $i \in [m-1]$. If $(t_i t_{i+1})^2 \neq 1$ (and, in the B_n-case, if t_{i+1} is not the distinguished vertex of Γ), then the map*

$$
\psi : W(\Gamma) \to W(\sigma_i^{-1}(\Gamma)), \ \ t_j \mapsto \begin{cases} r_j & \text{if } j \neq i, i+1 \\ r_i r_{i+1} r_i & \text{if } j = i \\ r_i & \text{if } j = i+1 \end{cases}
$$

extends to a (surjective) group homomorphism.

The maps φ from Lemma 48 and ψ from Lemma 49 are mutually inverse group homomorphisms. Therefore we obtain the following.

Proposition 50. *The groups $W(\Gamma)$ and $W(\sigma_i(\Gamma))$ (resp. $W(\sigma_i^{-1}(\Gamma))$) are isomorphic.*

Proof of Theorem 2. Let us begin with the case that the root system Φ is simply-laced or of type B_n. Let Γ' be a Carter diagram of the same Dynkin type as Φ. By Lemma 14, the Carter diagram Γ' can be realized by a reduced reflection factorization $(r_1, \ldots, r_m) \in \mathrm{Red}_T(w)$ for some quasi-Coxeter element $w \in W_\Phi$. By [3, Theorem 1.1 and Remark 8.3] we find $(t_1, \ldots, t_m)$ and $\sigma \in \mathcal{B}_m$ such that $\sigma(t_1, \ldots, t_m) = (r_1, \ldots, r_m)$ and the Carter diagram Γ corresponding to $(t_1, \ldots, t_m)$ is admissible. In particular we have $\sigma(\Gamma) = \Gamma'$. By Proposition 41 we have $W(\Gamma) \cong W_\Phi$ and repeated use of Proposition 50 yields $W(\Gamma) \cong W(\Gamma')$.

If Φ is of type F_4, the assertion can be checked directly for the non-admissible Carter diagrams in Figure 6. In type G_2 the Dynkin diagram is the only Carter diagram. $\square$

Remark 51. We used the Hurwitz action to prove Theorem 2, but we did not define a Hurwitz action on Carter diagrams. In fact, it might happen that there are elements $x, y \in W$ and reduced reflection factorizations $\underline{t} \in \mathrm{Red}_T(x)$, $\underline{r} \in \mathrm{Red}_T(y)$ such that $\underline{t}$ and $\underline{r}$ give rise to the same Carter diagram, but the set of Carter diagrams associated to the Hurwitz orbit of $\underline{t}$ and the set of Carter diagrams associated to the Hurwitz orbit of $\underline{r}$ are **not** equal (see the next example). The reason for this is that the Hurwitz orbit of a reflection factorizaton $\underline{t} = (t_1, \ldots, t_m)$ depends on the actual tuple, while the associated Carter diagram only depends on the set $\{t_1, \ldots, t_m\}$.

Example 52. Consider a Coxeter system $(W, \{s_1, s_2, s_3, s_4\})$ of type D_4, where s_2 does not commute with any of the other simple reflections. Let T be its set of reflections and $c = s_1 s_2 s_3 s_4$ be a Coxeter element. Inside the Hurwitz orbit of the corresponding factorization we find

$$
(s_1, s_2, s_3, s_4) \sim (s_2, s_2 s_1 s_2, s_3, s_4).
$$

The Carter diagram associated to $(s_2, s_2 s_1 s_2, s_3, s_4)$ is shown on the right side of the following picture, while the Carter diagram of (s_1, s_2, s_3, s_4) is the corresponding Dynkin diagram.
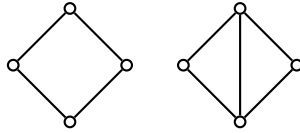
The element $w := (s_2 s_3 s_2) s_1 (s_1 s_2 s_1)(s_1 s_2 s_4 s_2 s_1) \in W$ is a quasi-Coxeter element, but not a Coxeter element. Inside the Hurwitz orbit we find

$$(s_2 s_3 s_2, s_1, s_1 s_2 s_1, s_1 s_2 s_4 s_2 s_1) \sim (s_2 s_3 s_2 s_1 s_2 s_3 s_2, s_2 s_3 s_2, s_1 s_2 s_1, s_1 s_2 s_4 s_2 s_1).$$

The Carter diagram associated to the left factorization is the one on the right side of the above picture, while the Carter diagram associated to the right factorization is the one on the left side. These two are the only Carter diagrams which appear in the Hurwitz orbit of $(s_2 s_3 s_2, s_1, s_1 s_2 s_1, s_1 s_2 s_4 s_2 s_1)$. In particular, the sets $\mathrm{Red}_T(c)$ and $\mathrm{Red}_T(w)$ have both diagrams shown above as associated Carter diagrams, while the Dynkin diagram of type D_4 just appears as a Carter diagram associated to a reduced reflection factorization in $\mathrm{Red}_T(c)$.

4.1 The non-crystallographic reflection groups

Let (W, S) be an arbitrary Coxeter system of rank n with set of reflections T. We generalize the definiton of a Carter diagram as follows.

Definition 53. Let $w \in W$ and $(t_1, \ldots, t_n) \in \mathrm{Red}_T(w)$. We define the *Carter diagram* Γ corresponding to this reduced reflection factorization to be the graph on n vertices corresponding to $t_1, \ldots, t_n$. Two vertices corresponding to t_i and t_j $(i \neq j)$ are joined by $w_{ij} := m_{ij} - 2$ edges, where m_{ij} is the order of $t_i t_j$.

Note that this definition is consistent with Definition 5 if $W = W_\Phi$ is a Weyl group (by Remark 9).

We call Γ to be of type $I_2(m)$ (resp. H_3 resp. H_4) if $\langle t_1, \ldots, t_n \rangle$ is a Coxeter group of type $I_2(m)$ (resp. H_3 resp. H_4).

The only Carter diagram of type $I_2(m)$ is the corresponding Coxeter diagram. For the types H_3 and H_4 we list all Carter diagrams in Figures 10 and 11.
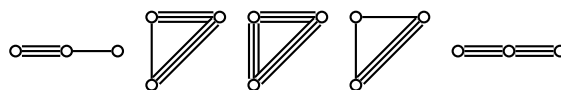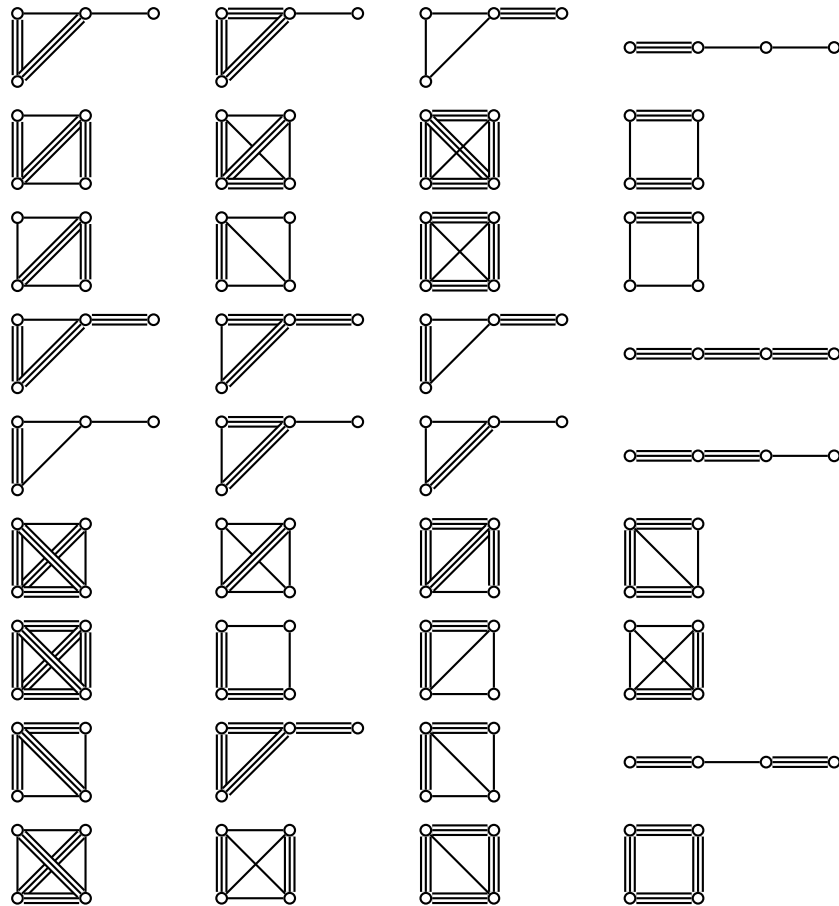

Figure 10: Carter diagrams of type H_3.

Figure 11: Carter diagrams of type H_4.

For these class of Carter diagrams, in general we do not obtain a presentation as for the crystallographic types. First we have to restrict ourselves to those Carter diagrams which are induced by reduced reflection factorizations of Coxeter elements. Those diagrams are given by the three leftmost diagrams in Figure 10 and by the diagrams in the first two lines in Figure 11. Then we have to replace the relation (R3) of Section 1 by the following two relations

(R3') for any chordless cycle

$$ i_0 \ \xrightarrow{w_{i_0 i_1}} \ i_1 \ \xrightarrow{w_{i_1 i_2}} \ \dots \ \xrightarrow{w_{i_{d-2} i_{d-1}}} \ i_{d-1} \ \xrightarrow{w_{i_{d-1} i_0}} \ i_0, $$

where either all weights are 1 or $w_{i_{d-1} i_0} = 3$, but not all of the other weights are 3, we have

$$ (t_{i_0} t_{i_1} \cdots t_{i_{d-2}} t_{i_{d-1}} t_{i_{d-2}} \cdots t_{i_1})^2 = 1. $$

(R3") for any chordless 3-cycle

$$ i_0 \equiv\!\!\equiv i_1 \equiv\!\!\equiv i_2 \equiv\!\!\equiv i_0 $$

we have the relations

$$(t_{i_0} t_{i_1} t_{i_2} t_{i_1})^3 = 1$$
$$(t_{i_1} t_{i_0} t_{i_1} t_{i_2} t_{i_1} t_{i_2})^2 = 1.$$

Using GAP [12] we arrive at the following statement.

Proposition 54. *Let (W, S) be a Coxeter system of type H_3, H_4 or $I_2(m)$, Γ a Carter diagram of the same type with vertex set given by the reduced reflection factorization of a Coxeter element in W. Then the group $W(\Gamma)$ with generators t_i, i a vertex of Γ, subject to the relations (R1), (R2), (R3') and (R3") is isomorphic to W.*

Remark 55. The statement of the preceding proposition is wrong if we replace the Coxeter element by a quasi-Coxeter element. That is, the diagrams in Figures 10 and 11 which are not induced by reduced reflection factorizations of a Coxeter element, do not yield a presentation of W given by the relations (R1), (R2), (R3') and (R3").

Acknowledgements

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