Turán Colourings in Off-Diagonal Ramsey Multiplicity

Joseph Hyde[a] Jae-baek Lee[b] Jonathan A. Noel[b]

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Abstract

The *Ramsey multiplicity constant* of a graph H is the limit as n tends to infinity of the minimum density of monochromatic labeled copies of H in a 2-edge colouring of K_n. Fox and Wigderson recently identified a large family of graphs whose Ramsey multiplicity constants are attained by sequences of "Turán colourings"; i.e. colourings in which one of the colour classes forms the edge set of a balanced complete multipartite graph. Each graph in their family comes from taking a connected non-3-colourable graph with a critical edge and adding many pendant edges. We extend their result to an off-diagonal variant of the Ramsey multiplicity constant which involves minimizing a weighted sum of red copies of one graph and blue copies of another.

Mathematics Subject Classifications: 05C35, 05C55

1 Introduction

The central question in the area of "Ramsey multiplicity" is: how should one colour the edges of the clique K_n, for large n, with red and blue to minimize the number of monochromatic labeled copies of a fixed graph H? As an "off-diagonal" generalization, one could instead minimize a "suitable linear combination" of the number of red copies of one graph H_1 and blue copies of another graph H_2; the coefficients of this linear combination will be specified in Section 2 but, for now, it suffices to think of them as arbitrary positive reals that may depend on n. Ramsey multiplicity problems have been extensively studied; see, for example, [6, 9, 10, 14, 17, 24, 29, 32, 33, 36].

One of the first strategies that comes to mind is to consider a uniformly random colouring. In the diagonal setting, i.e. when $H_1 = H_2 = H$ for some graph H, a random colouring has approximately $(1/2)^{e(H)-1}n^{v(H)}$ monochromatic copies of H with high probability, where $v(H) := |V(H)|$ and $e(H) := |E(H)|$. A graph H is said to be

[a]Department of Mathematics, King's College London, London, UK. (joseph.hyde@kcl.ac.uk).
[b]Department of Mathematics and Statistics, University of Victoria, Victoria, B.C., Canada. (dlwoqor0923@uvic.ca, noelj@uvic.ca).

common if every colouring has at least this many copies, up to a $(1+o(1))$ factor where the $o(1)$ term tends to 0 as $n \to \infty$. The notion of common graphs has its origins in the work of Goodman [20] and Erdős [11] in the 1950s and 60s and has been a popular area of research ever since [2, 3, 21, 23, 24, 27–29, 35].

After considering random colourings, perhaps the next most natural strategy is to "pack in" as many red edges as possible without creating a red copy of H_1. A well-known theorem of Simonovits [34] tells us that, if H_1 contains a *critical edge*, i.e. an edge e such that $\chi(H_1 - e) < \chi(H_1)$, then the maximum number of red edges in a red/blue colouring of $E(K_n)$ without a red copy of H_1 is attained by dividing $V(K_n)$ into $\chi(H_1)-1$ classes of cardinality $\lfloor n/(\chi(H_1)-1)\rfloor$ or $\lceil n/(\chi(H_1)-1)\rceil$ and colouring an edge red if it connects vertices in different classes or blue otherwise. Such a colouring, which is referred to as a *Turán colouring*,[1] has no red copy of H_1 and has at most

$$(1-o(1))\left(\frac{1}{\chi(H_1)-1}\right)^{v(H_2)-k(H_2)} n^{v(H_2)}$$

blue copies of H_2, where $k(H_2)$ denotes the number of connected components of H_2. We remark that Turán colourings feature prominently in the well-studied area of "Ramsey goodness" [4, 5, 7, 8, 13, 16, 19, 22, 30, 31]. In the diagonal setting, Fox and Wigderson [17] recently proved that this strategy is optimal for a fairly large family of graphs (see Theorem 1 below). Prior to their work, there were no examples of uncommon graphs for which the Ramsey multiplicity problem had been solved.

Our main result (Theorem 2 below) extends this theorem of Fox and Wigderson [17] to an off-diagonal setting. Stating our results precisely requires some technical definitions which we will formally provide in Section 2. For the time being, we informally say that (H_1, H_2) is a *bonbon pair* if, for large enough n, the only colourings minimizing the "suitable linear combination" of the number of red copies of H_1 and blue copies of H_2 alluded to in the first paragraph are the Turán colourings. Following [17], if H is a graph such that (H, H) is a bonbon pair, then H is said to be a *bonbon*.

For a graph F and $t \geqslant 0$, a *t-hairy F* is a graph with $v(F)+t$ vertices that is created by adding t copies of K_2 to F, one at a time, such that each added K_2 has exactly one endpoint in $V(F)$. If H is a t-hairy F for some t, then we simply say that H is a *hairy* F. We state the main result of [17] and our off-diagonal generalization of it.

Theorem 1 (Fox and Wigderson [17, Theorem 1.2])**.** *For any connected non-3-colourable graph F that contains a critical edge, there exists $t_0 = t_0(F)$ such that, for any $t \geqslant t_0$, every t-hairy F is a bonbon.*

Theorem 2. *Let $q \in (0,1]$ and let F_1 and F_2 be non-bipartite graphs, each of which contains a critical edge, such that $\chi(F_1)+\chi(F_2) \geqslant 7$. Then there exists $t_0 = t_0(F_1, F_2, q)$ such that if H_1 is a t_1-hairy F_1 and H_2 is a t_2-hairy F_2 with $t_1, t_2 \geqslant t_0$ and*

$$\min\{v(H_1), v(H_2)\} \geqslant q \cdot \max\{v(H_1), v(H_2)\},$$

[1] We also use the term *Turán colouring* to refer to a colouring in which there are $\chi(H_2)-1$ classes of almost equal size, edges between the classes are blue and edges within the classes are red.

then (H_1, H_2) *is a bonbon pair.*

One may wonder whether the presence of critical edges and the dependence of t_0 on the parameter q are essential in Theorem 2. The next proposition implies that both conditions are necessary. Let $\mathrm{crit}(F)$ denote the number of critical edges in a graph F. An explicit form of the function g in the following proposition will be provided in Section 5 (see Theorem 37).

Proposition 3. *There exists a function* $g : \mathbb{R}^4 \to \mathbb{R}$ *with the property that, if* H_1 *and* H_2 *are non-empty graphs such that*

$$e(H_1) > \mathrm{crit}(H_2) \cdot g(\chi(H_1), \chi(H_2), v(H_2), k(H_2)),$$

then (H_1, H_2) *is not a bonbon pair.*

The other conditions in Theorem 2—namely, that F_1 and F_2 are non-bipartite and $\chi(F_1) + \chi(F_2) \geqslant 7$—are also necessary. First, if F_2 is bipartite and H_2 is a t_2-hairy F_2 for some $t_2 \geqslant 1$, then H_2 does not contain a critical edge and so Proposition 3 implies that there cannot exist a graph H_1 with non-empty edge set such that (H_1, H_2) is a bonbon pair. Also, Fox and Wigderson [17, p. 4] observed that the conclusion of Theorem 1 is false for every 3-chromatic graph F. Therefore, the conclusion of Theorem 2 is false for $F_1 = F_2 = F$ whenever F is a 3-chromatic graph.

In Section 2, we provide a formal definition of bonbon pairs and an off-diagonal variant of the Ramsey multiplicity constant. The proof of Theorem 2, which is inspired by the proof of Theorem 1 in [17], is provided in Sections 3 and 4. First, in Section 3, we show that an optimal colouring has the "approximate" structure of a Turán colouring; i.e. the vertices can be partitioned into a small number of classes such that edges within the classes are nearly monochromatic. Then, in Section 4, we refine the structure of the colouring until it precisely matches that of a Turán colouring. In Section 5, we discuss various constructions of colourings which we use to prove a strong form of Proposition 3 (Theorem 37). We conclude the paper in Section 6 by proposing several open problems.

2 Formal Definitions

Given graphs H and G, a *homomorphism* from H to G is a function $f : V(H) \to V(G)$ such that adjacent pairs of vertices in H are mapped to adjacent pairs of vertices in G and the *homomorphism density* $t(H, G)$ is the probability that a random function from $V(H)$ to $V(G)$ is a homomorphism. That is, $t(H, G)$ is the number of homomorphisms from H to G divided by $v(G)^{v(H)}$. For graphs H and G with $v(H) \leqslant v(G)$, the *injective homomorphism density* of H in G, denoted $t_{\mathrm{inj}}(H, G)$, is the probability that a random injective function from $V(H)$ to $V(G)$ is a homomorphism. If H is a fixed graph and $v(G)$ is large, then there are only $O(v(G)^{v(H)-1})$ non-injective functions from $V(H)$ to $V(G)$, and so

$$t(H, G) = t_{\mathrm{inj}}(H, G) + o(1) \tag{1}$$

where the $o(1)$ term approaches zero as $v(G) \to \infty$. The following is essentially a rephrasing of [17, Definition 1.1], except that we generalize it slightly to include disconnected graphs.

Definition 4 (Fox and Wigderson [17])**.** A non-empty graph H is said to be a *bonbon* if there exists $n_0 = n_0(H)$ such that, if $n \geqslant n_0$ and G is an n-vertex graph such that

$$t_{\text{inj}}(H, G) + t_{\text{inj}}(H, \overline{G})$$

is minimized over all n-vertex graphs, then either G or $\overline{G}$ is a Turán graph with $\chi(H) - 1$ parts.

Let us now extend this definition to an off-diagonal setting.

Definition 5. A pair (H_1, H_2) of non-empty graphs is a *bonbon pair* if there exists $n_0 = n_0(H_1, H_2)$ such that, if $n \geqslant n_0$ and G is an n-vertex graph such that

$$(\chi(H_2) - 1)^{v(H_1)-k(H_1)} \cdot t_{\text{inj}}(H_1, G) + (\chi(H_1) - 1)^{v(H_2)-k(H_2)} \cdot t_{\text{inj}}(H_2, \overline{G})$$

is minimized over all n-vertex graphs, then either G is a Turán graph with $\chi(H_1) - 1$ parts or $\overline{G}$ is a Turán graph with $\chi(H_2) - 1$ parts.

Note that a graph H is a bonbon if and only if (H, H) is a bonbon pair. The coefficients on $t_{\text{inj}}(H_1, G)$ and $t_{\text{inj}}(H_2, \overline{G})$ in the above definition are chosen so that the two different Turán colourings achieve the same value, asymptotically. That is, if G is a Turán graph with $\chi(H_1) - 1$ parts or $\overline{G}$ is a Turán graph with $\chi(H_2) - 1$ parts, we have

$$(\chi(H_2) - 1)^{v(H_1)-k(H_1)} \cdot t_{\text{inj}}(H_1, G) + (\chi(H_1) - 1)^{v(H_2)-k(H_2)} \cdot t_{\text{inj}}(H_2, \overline{G}) = 1 - o(1).$$

Definition 5 was inspired by several recent papers focusing on off-diagonal generalizations of basic questions in Ramsey multiplicity. E.g., Parczyk, Pokutta, Spiegel and Szabó [33] proved asymptotic bounds on linear combinations of $t_{\text{inj}}(K_s, G)$ and $t_{\text{inj}}(K_t, \overline{G})$ for small s and t and Behague, Morrison and Noel [2,3] extended the notion of common graphs to an off-diagonal setting. Moss and Noel [32] recently introduced an off-diagonal notion of Ramsey multiplicity for general pairs of graphs. In our proof of Theorem 2, we will need the following notion and lemma from [32].

Definition 6 (Moss and Noel [32])**.** For non-empty graphs H_1 and H_2 and $\lambda \in [0, 2]$, define

$$c_\lambda(H_1, H_2) := \lim_{n\to\infty} \left[\min_{G:v(G)=n} \left(\lambda \cdot t(H_1, G) + (2 - \lambda) \cdot t(H_2, \overline{G}) \right) \right].$$

Lemma 7 (Moss and Noel [32, Lemma 2.11])**.** *For any non-empty graphs H_1 and H_2 and $\lambda \in (0, 2)$ we have $c_\lambda(H_1, H_2) > 0$.*

Remark 8. The key definition in [32] is the *balanced Ramsey multiplicity constant* of (H_1, H_2), defined by $c(H_1, H_2) := \sup_{\lambda\in[0,2]} c_\lambda(H_1, H_2)$. For and $\lambda \in [0, 2]$ and any graphs H_1 and H_2, we have

$$c_\lambda(H_1, H_2) \leqslant \frac{2}{(\chi(H_2) - 1)^{v(H_1)-k(H_1)} + (\chi(H_1) - 1)^{v(H_2)-k(H_2)}}.$$

Indeed, this can be shown by taking one of the two Turán colourings from Definition 5, where the specific choice depends on whether $\lambda \leqslant \frac{2(\chi(H_2)-1)^{v(H_1)-k(H_1)}}{(\chi(H_2)-1)^{v(H_1)-k(H_1)}+(\chi(H_1)-1)^{v(H_2)-k(H_2)}}$ or not. Combining this with the definition of a bonbon pair, we get that, if (H_1, H_2) is a bonbon pair, then

$$c(H_1, H_2) = \frac{2}{(\chi(H_2) - 1)^{v(H_1)-k(H_1)} + (\chi(H_1) - 1)^{v(H_2)-k(H_2)}}.$$

3 Proof of Theorem 2: Rough Structure

The focus of this section is on obtaining an approximate version of Theorem 2 (Lemma 11 below) which will be refined in the next section to complete the proof of the theorem.

Remark 9. Fox and Wigderson [17] cleverly avoided using the Graph Removal Lemma in their proof of Theorem 1. Doing so added a few steps to their argument, but resulted in much better bounds on t_0. To keep our paper to a reasonable length, and to differentiate it from [17], we have chosen to present a shorter argument which uses the Removal Lemma at the expense of having poorer control over t_0. We remark that better bounds on our t_0 could be obtained by following the proof of [17, Theorem 1.2] more closely.

Throughout the next two sections, we let $q \in (0, 1]$ and let F_1 and F_2 be non-bipartite graphs, each of which contains a critical edge, such that $\chi(F_1) + \chi(F_2) \geqslant 7$. Define $f := \max\{v(F_1), v(F_2)\}$ and $\chi := \max\{\chi(F_1), \chi(F_2)\}$. We let t_0 be an integer chosen large with respect to F_1, F_2 and q, which will be specified later. Actually, t_0 is defined in terms of a throng of other parameters $\theta, \varepsilon, \delta, \beta, \xi, \gamma$ and τ, where each parameter depends on F_1, F_2 and q and the parameters that come before it in the list. The relationships between $F_1, F_2, q, \theta, \varepsilon, \delta, \beta, \xi, \gamma, \tau$ and t_0 will be revealed "as needed" throughout this section and the next, and will be summarized in the final proof of Theorem 2 at the end of Section 4.

Let $t_1, t_2 \geqslant t_0$ and let H_1 be a t_1-hairy F_1 and H_2 be a t_2-hairy F_2 satisfying

$$\min\{v(H_1), v(H_2)\} \geqslant q \cdot \max\{v(H_1), v(H_2)\}. \tag{2}$$

Note that $\chi(H_i) = \chi(F_i)$ and $k(H_i) = k(F_i)$ for $i \in \{1, 2\}$. For the sake of brevity, let

$$\rho_1 := (\chi(H_2) - 1)^{v(H_1)-k(H_1)}$$
$$\rho_2 := (\chi(H_1) - 1)^{v(H_2)-k(H_2)}.$$

Note that, by definition,

$$\rho_1 \cdot \left(\frac{1}{\chi(F_2) - 1}\right)^{v(H_1)-k(F_1)} = \rho_2 \left(\frac{1}{\chi(F_1) - 1}\right)^{v(H_2)-k(F_2)} = 1. \tag{3}$$

We may assume that H_1 and H_2 have no singleton components, since any such components do not affect injective homomorphism densities into large enough graphs, nor do they affect ρ_1 or ρ_2 (since adding a singleton component to a graph H increases both of $v(H)$ and $k(H)$ by one and does not affect $\chi(H)$).

Let n_0 be a large integer which may depend on H_1, H_2 and all of the parameters discussed so far, and assume that $n \geqslant n_0$. For any graph G, define

$$m(H_1, H_2; G) := \rho_1 \cdot t_{\mathrm{inj}}(H_1, G) + \rho_2 \cdot t_{\mathrm{inj}}(H_2, \overline{G}).$$

Here, the letter m stands for "monochromatic." Let G_1 be a graph on n vertices chosen so that $\min_{G:v(G)=n} m(H_1, H_2; G) = m(H_1, H_2; G_1)$ and let $G_2 := \overline{G_1}$. Our goal in the proof of Theorem 2 is to show that either G_1 is a Turán graph with $\chi(F_1) - 1$ parts or G_2 is a Turán graph with $\chi(F_2) - 1$ parts. Since $m(H_1, H_2; G_1)$ is at most the value of $m(H_1, H_2; G)$ when G is an n-vertex Turán graph with $\chi(F_1) - 1$ parts, we have that

$$m(H_1, H_2; G_1) \leqslant (1 - o(1))\rho_2 \left(\frac{1}{\chi(F_1) - 1}\right)^{v(H_2)-k(F_2)} = 1 - o(1) \tag{4}$$

where the last equality follows from (3). Note that $m(H_2, H_1; G_2) = m(H_1, H_2; G_1)$ and so it is also at most $1 - o(1)$.

It is useful to classify vertices based on their degrees in G_1 and G_2. Let $V := V(G_1) = V(G_2)$. For a graph G with vertex set V and a vertex $v \in V$, the *degree* of v in G is the number of edges of G that are connected to v, denoted by $d_G(v)$. For $i \in \{1, 2\}$, we let $d_i(v) := d_{G_i}(v)$. When interpreting the next definition, recall that ξ is one of the many parameters that appears throughout this section and the next and will be specified in the final proof of Theorem 2.

Definition 10. For $i \in \{1, 2\}$, define

$$V_i := \left\{ v \in V : d_i(v) \geqslant \left(1 - \frac{1 + 2\xi}{\chi(F_i) - 1}\right)(n - 1) \right\}.$$

Also, let $V_0 := V \setminus (V_1 \cup V_2)$ and $V_3 = V_1 \cap V_2$.

We may assume the following, without loss of generality.

Assumption 1. $|V_1| \leqslant |V_2|$.

The focus of the rest of this section is on proving the following lemma which determines the "rough structure" of G_1 and G_2. For any two subsets $S, T \subseteq V$ and a graph G with vertex set V, define $e_G(S, T)$ to be the number of ordered pairs $(u, v) \in S \times T$ such that $uv \in E(G)$ and let $e_G(S) := \frac{1}{2} e_G(S, S)$. For any $S, T \subseteq V$ and $i \in \{1, 2\}$, we let $e_i(S, T) := e_{G_i}(S, T)$ and $e_i(S) := e_{G_i}(S)$.

Lemma 11. *There exists a partition $A_1, A_2, \ldots, A_{\chi(F_2)-1}$ of V such that*

$$\sum_{i=1}^{\chi(F_2)-1} e_2(A_i) \leqslant \varepsilon n^2.$$

It is worth noting that the proof of Lemma 11 does not require F_1 and F_2 to have critical edges, nor does it require the inequality (2); these conditions come into play when seeking the exact structure of an optimal colouring in the next section.

A high-level overview of the proof of Lemma 11 is as follows. We first show that $V_3 = \emptyset$ and that V_0 and V_1 are both quite small; specifically $|V_0| \leqslant \xi n$ and $|V_1| \leqslant 25\xi n$. Therefore, most of the vertices reside in V_2, and thus have a large degree in G_2. If the density of F_2 in G_2 is sufficiently far from zero, then there must be several copies of F_2 whose vertices are contained in V_2, and each of these copies can be "extended" to a copy of H_2 in G_2 in many ways due to the high G_2-degree of vertices in V_2. This would lead to a large density of H_2 in G_2, which would violate (4). The Graph Removal Lemma then implies that G_2 can be made F_2-free by deleting a small proportion of its edges. After deleting these edges, we obtain a graph with close to $\left(1 - \frac{1}{\chi(F_2)-1}\right)\binom{n}{2}$ edges which is F_2-free. The classical Erdős–Simonovits Stability Theorem then states that such a graph must be "close" to a complete $(\chi(F_2) - 1)$-partite graph, which gives us Lemma 11. The rest of the section is devoted to fleshing out the details of these arguments.

3.1 Analyzing Degrees

We show that V_1 and V_2 have empty intersection. The following assumption is useful for proving this, and will be used again later as well:

$$0 < \xi < \frac{1}{39}. \tag{5}$$

Lemma 12. $V_3 = V_1 \cap V_2 = \emptyset$.

Proof. Suppose not and let $v \in V_1 \cap V_2$. Then, since $\chi(F_1), \chi(F_2) \geqslant 3$ and $\chi(F_1) + \chi(F_2) \geqslant 7$, we have

$$\begin{aligned} n - 1 &= d_1(v) + d_2(v) \\ &\geqslant \left(1 - \frac{1+2\xi}{\chi(F_1)-1}\right)(n-1) + \left(1 - \frac{1+2\xi}{\chi(F_2)-1}\right)(n-1) \\ &\geqslant \left(1 - \frac{1+2\xi}{2}\right)(n-1) + \left(1 - \frac{1+2\xi}{3}\right)(n-1) = \left(\frac{7}{6} - \frac{5\xi}{3}\right)(n-1). \end{aligned}$$

This implies that $\xi \geqslant 1/10$; however, this contradicts (5). $\square$

We obtain a bound on the degrees of vertices in V_i for $i \in \{1, 2\}$ via a similar argument.

Observation 13. *Let $\{i, j\} = \{1, 2\}$. If $v \in V_i$, then $d_i(v) > \left(\frac{5}{4} \cdot \frac{1+\xi}{\chi(F_j)-1}\right) n$.*

Proof. If not, then, since $v \in V_i$ and n is large,

$$\left(1 - \frac{1+3\xi}{\chi(F_i)-1}\right) n \leqslant \left(1 - \frac{1+2\xi}{\chi(F_i)-1}\right)(n-1) \leqslant d_i(v) \leqslant \left(\frac{5}{4} \cdot \frac{1+\xi}{\chi(F_j)-1}\right) n.$$

Recall that $\chi(F_1), \chi(F_2) \geqslant 3$ and $\chi(F_1) + \chi(F_2) \geqslant 7$. In the case that $\chi(F_i) = 3$, then we must have $\chi(F_j) \geqslant 4$ and so the above inequality implies that

$$1 - \frac{1+3\xi}{2} \leqslant \frac{5}{4} \cdot \frac{1+\xi}{3}.$$

However, this can only hold if $\xi \geqslant 1/23$ which contradicts (5). On the other hand, if $\chi(F_1) \geqslant 4$, then we instead get

$$1 - \frac{1+3\xi}{3} \leqslant \frac{5}{4} \cdot \frac{1+\xi}{2}.$$

This can only hold if $\xi \geqslant 1/39$ which again contradicts (5). $\square$

Next, we prove that both V_0 and V_1 are small. Given a graph G on vertex set V and a set $S \subseteq V$, let $G[S]$ be the subgraph of G *induced* by S; i.e. the graph with vertex set S and edge set $\{uv \in E(G) : u, v \in S\}$. The next lemma says that, for $i \in \{1, 2\}$, there cannot be a fairly sizeable set S such that $t(F_i, G_i[S])$ is bounded away from zero and $d_i(v)$ is relatively large for every $v \in S$. To prove this, we assume that t_0 is chosen large enough so that the following holds:

$$(1+\xi)^{t_0} > 3/\tau. \tag{6}$$

Lemma 14. *Let $\{i, j\} = \{1, 2\}$. If S is a non-empty subset of V such that*

$$d_i(v) \geqslant \left(\frac{1+\xi}{\chi(F_j) - 1}\right) n$$

for all $v \in S$, then $t(F_i, G_i[S]) \leqslant \tau \cdot (n/|S|)^{v(F_i)}$.

Proof. Suppose to the contrary that the hypotheses hold but

$$t(F_i, G_i[S]) > \tau \cdot (n/|S|)^{v(F_i)}.$$

The probability that a uniformly random function φ from $V(H_i)$ to V is a homomorphism from H_i to G_i is at least the probability that the restriction of φ to $V(F_i)$ is a homomorphism from F_i to $G_i[S]$ multiplied by the probability that, for every $w \in V(H_i) \setminus V(F_i)$, if v is the unique neighbour of w in H_i, then $\varphi(w)$ is adjacent to $\varphi(v)$ in G_i. Thus,

$$\begin{aligned}
t(H_i, G_i) &\geqslant (|S|/n)^{v(F_i)}\, t(F_i, G_i[S]) \left(\frac{1+\xi}{\chi(F_j)-1}\right)^{v(H_i)-v(F_i)} \\
&> \tau \left(\frac{1+\xi}{\chi(F_j)-1}\right)^{v(H_i)-v(F_i)} \\
&= \tau(1+\xi)^{t_i} \left(\frac{1}{\chi(F_j)-1}\right)^{v(H_i)-v(F_i)} \\
&> 3 \left(\frac{1}{\chi(F_j)-1}\right)^{v(H_i)-k(F_i)}
\end{aligned}$$

where the last inequality follows from (6) and the facts that $t_1, t_2 \geqslant t_0$ and $v(F_i) \geqslant k(F_i)$. So, by (1) and the fact that n is large, we have

$$t_{\text{inj}}(H_i, G_i) = t(H_i, G_i) - o(1) > 2 \left(\frac{1}{\chi(F_j) - 1} \right)^{v(H_i) - k(F_i)}.$$

Consequently, by (3),

$$m(H_1, H_2; G_1) = \rho_1 \cdot t_{\text{inj}}(H_1, G_1) + \rho_2 \cdot t_{\text{inj}}(H_2, G_2) > 2$$

which contradicts (4) and thus completes the proof. □

Next, we prove that V_0 is quite small. For this, we assume the following bound on τ:

$$0 < \tau < \frac{c_1(F_1, F_2) \cdot \xi^f}{4}. \tag{7}$$

Note that $c_1(F_1, F_2) > 0$ by Lemma 7 and so it is possible to choose τ to satisfy this condition. The next lemma is analogous to [17, Claim 3.3].

Lemma 15. $|V_0| < \xi n$.

Proof. Suppose that $|V_0| \geqslant \xi n$. Our goal is to obtain a contradiction via an application of Lemma 14 with $S = V_0$. By definition of V_0, for each $i \in \{1, 2\}$, every $v \in V_0$ satisfies

$$d_i(v) \leqslant \left(1 - \frac{1 + 2\xi}{\chi(F_i) - 1} \right) (n - 1).$$

Since $d_1(v) + d_2(v) = n - 1$, this tells us that

$$d_1(v) \geqslant \left(\frac{1 + 2\xi}{\chi(F_2) - 1} \right) (n - 1) > \left(\frac{1 + \xi}{\chi(F_2) - 1} \right) n$$

and

$$d_2(v) \geqslant \left(\frac{1 + 2\xi}{\chi(F_1) - 1} \right) (n - 1) > \left(\frac{1 + \xi}{\chi(F_1) - 1} \right) n$$

for every vertex $v \in V_0$ and large enough n. Therefore, by Lemma 14, for each $i \in \{1, 2\}$, we must have

$$t(F_i, G_i[V_0]) \leqslant \tau (n/|V_0|)^{v(F_i)} \leqslant \tau / \xi^{v(F_i)}.$$

According to (7), this last expression is less than $c_1(F_1, F_2)/4$. Thus, for $i \in \{1, 2\}$,

$$t(F_i, G_i[V_0]) < c_1(F_1, F_2)/4.$$

On the other hand, by definition of $c_1(F_1, F_2)$, we have

$$t(F_1, G_1[V_0]) + t(F_2, G_2[V_0]) \geqslant c_1(F_1, F_2) - o(1)$$

where the $o(1)$ term tends to 0 as $\xi n \to \infty$. Since $c_1(F_1, F_2) > 0$ by Lemma 7 this implies that, for large n,

$$t(F_1, G_1[V_0]) + t(F_2, G_2[V_0]) \geqslant c_1(F_1, F_2)/2.$$

Therefore, we can let $i \in \{1, 2\}$ such that

$$t(F_i, G_i[V_0]) \geqslant c_1(F_1, F_2)/4.$$

Combining the upper and lower bound on $t(F_i, G_i[V_0])$ that we have proven leads to a contradiction, thereby completing the proof. □

Next, we show that V_1 is small which, when combined with the fact that $|V_0| < \xi n$, implies that the vast majority of the vertices are in V_2. To do this, we use the following form of the Erdős–Simonovits Supersaturation Theorem.

Theorem 16 (Erdős–Simonovits Supersaturation Theorem [12])**.** *For every non-empty graph F and $\xi > 0$ there exists $\gamma = \gamma(F, \xi) > 0$ such that if G is a graph with $t(K_2, G) \geqslant 1 - \frac{1-\xi}{\chi(F)-1}$, then $t(F, G) \geqslant \gamma$.*

Using Theorem 16, we define γ by

$$\gamma := \min\{\gamma(F_1, \xi), \gamma(F_2, \xi)\}. \tag{8}$$

We also assume that τ is chosen so that

$$0 < \tau < \gamma \cdot (25\xi)^f. \tag{9}$$

The next lemma is analogous to [17, Claim 3.4].

Lemma 17. *$|V_1| < 25\xi n$.*

Proof. Let us begin by establishing the following claim.

Claim 18. *For each $i \in \{1, 2\}$, if $|V_i| \geqslant 25\xi n$, then*

$$e_i(V_i) < \left(1 - \frac{1-\xi}{\chi(F_i) - 1}\right) \frac{|V_i|^2}{2}.$$

Proof of Claim 18. Suppose not. Then there exists $i \in \{1, 2\}$ such that

$$t(K_2, G_i[V_i]) = \frac{2e_i(V_i)}{|V_i|^2} \geqslant 1 - \frac{1-\xi}{\chi(F_i) - 1}.$$

Consequently, Theorem 16 implies that $t(F_i, G_i[V_i]) \geqslant \gamma$. Using the hypothesis $|V_i| \geqslant 25\xi n$,

$$\gamma \geqslant \gamma \cdot \left(\frac{25\xi n}{|V_i|}\right)^{v(F_i)} \geqslant \gamma \cdot (25\xi)^{v(F_i)} \left(\frac{n}{|V_i|}\right)^{v(F_i)}$$

which, by (9), is greater than $\tau \cdot (n/|V_i|)^{v(F_i)}$. By Observation 13, we have $d_i(v) \geqslant \left(\frac{1+\xi}{\chi(F_j)-1}\right) n$ for all $v \in V_i$, where $j \in \{1, 2\} \setminus \{i\}$. So, the set $S = V_i$ contradicts Lemma 14. Therefore, the claim holds. □

We now use Claim 18 to complete the proof of the lemma. If $|V_1| < 25\xi n$, then we are done; so, we assume $|V_1| \geqslant 25\xi n$. By Assumption 1, $|V_2| \geqslant 25\xi n$ as well. In particular, both V_1 and V_2 are non-empty and satisfy the hypothesis, and therefore the conclusion, of Claim 18. For $i \in \{1, 2\}$, define

$$\eta_i := \frac{e_i(V_1, V_2)}{|V_1||V_2|}$$

and note that $\eta_1 + \eta_2 = 1$ because V_1 and V_2 are disjoint (by Lemma 12) and G_2 is the complement of G_1. By definition of V_1, we have

$$\sum_{v \in V_1} d_1(v) \geqslant |V_1| \left(1 - \frac{1 + 2\xi}{\chi(F_1) - 1}\right)(n - 1) \geqslant |V_1| \left(1 - \frac{1 + 3\xi}{\chi(F_1) - 1}\right) n.$$

On the other hand,

$$\begin{aligned} \sum_{v \in V_1} d_1(v) &= 2e_1(V_1) + e_1(V_1, V \setminus V_1) \\ &= 2e_1(V_1) + e_1(V_1, V_2) + e_1(V_1, V_0) \\ &\leqslant 2e_1(V_1) + \eta_1 |V_1||V_2| + |V_1||V_0|. \end{aligned}$$

Claim 18 tells us that the above expression is less than

$$\left(1 - \frac{1 - \xi}{\chi(F_1) - 1}\right) |V_1|^2 + \eta_1 |V_1||V_2| + |V_1||V_0|.$$

Combining the lower and upper bounds on $\sum_{v \in V_1} d_1(v)$ obtained above and cancelling a factor of $|V_1|$, we get

$$\left(1 - \frac{1 + 3\xi}{\chi(F_1) - 1}\right) n \leqslant \left(1 - \frac{1 - \xi}{\chi(F_1) - 1}\right) |V_1| + \eta_1 |V_2| + |V_0|.$$

By Lemma 12, we have $|V_0| + |V_1| + |V_2| = n$ and so this inequality becomes

$$\begin{aligned} \left(1 - \frac{1}{\chi(F_1) - 1} - \frac{3\xi}{\chi(F_1) - 1}\right) n &\leqslant n - \left(\frac{1 - \xi}{\chi(F_1) - 1}\right) |V_1| + (\eta_1 - 1)|V_2| \\ &= n - \frac{|V_1|}{\chi(F_1) - 1} + \frac{\xi |V_1|}{\chi(F_1) - 1} + (\eta_1 - 1)|V_2|. \end{aligned}$$

Adding and subtracting $\frac{|V_2| + |V_0|}{\chi(F_1) - 1}$ in this final expression and using $|V_0| + |V_1| + |V_2| = n$ again yields

$$\begin{aligned} n - \frac{|V_1|}{\chi(F_1) - 1} + \frac{\xi |V_1|}{\chi(F_1) - 1} + (\eta_1 - 1)|V_2| + \frac{|V_2| + |V_0|}{\chi(F_1) - 1} - \frac{|V_2| + |V_0|}{\chi(F_1) - 1} \\ = n - \frac{n}{\chi(F_1) - 1} + \frac{\xi |V_1|}{\chi(F_1) - 1} + (\eta_1 - 1)|V_2| + \frac{|V_2| + |V_0|}{\chi(F_1) - 1}. \end{aligned}$$

Since $|V_1| \leqslant n$ trivially and $|V_0| < \xi n$ by Lemma 15, we get that this last expression is strictly less than

$$n - \frac{n}{\chi(F_1) - 1} + \left(\eta_1 - 1 + \frac{1}{\chi(F_1) - 1}\right)|V_2| + \frac{2\xi n}{\chi(F_1) - 1}.$$

To recap, the inequality that we have just derived is

$$\left(1 - \frac{1}{\chi(F_1) - 1} - \frac{3\xi}{\chi(F_1) - 1}\right)n < n - \frac{n}{\chi(F_1) - 1} + \left(\eta_1 - 1 + \frac{1}{\chi(F_1) - 1}\right)|V_2| + \frac{2\xi n}{\chi(F_1) - 1}.$$

By rearranging, we get

$$\left(1 - \eta_1 - \frac{1}{\chi(F_1) - 1}\right)|V_2| < \frac{5\xi n}{\chi(F_1) - 1}. \tag{10}$$

Applying the same argument, but with the roles of (F_1, V_1, η_1) and (F_2, V_2, η_2) reversed, we get that

$$\left(1 - \eta_2 - \frac{1}{\chi(F_2) - 1}\right)|V_1| < \frac{5\xi n}{\chi(F_2) - 1}. \tag{11}$$

We now divide the proof into cases depending on the values of η_1 and η_2.

Case 1. $\eta_i \leqslant \frac{2}{5}$ for some $i \in \{1, 2\}$.

Let $j \in \{1, 2\} \setminus \{i\}$. Since $\chi(F_i) \geqslant 3$, we get the following by applying (10) or (11):

$$\frac{5\xi n}{2} \geqslant \frac{5\xi n}{\chi(F_i) - 1} > \left(1 - \eta_i - \frac{1}{\chi(F_i) - 1}\right)|V_j| \geqslant \left(1 - \frac{2}{5} - \frac{1}{2}\right)|V_j| = \frac{|V_j|}{10}$$

and so $|V_j| < 25\xi n$. Since $|V_1| \leqslant |V_2|$ by Assumption 1, this implies that $|V_1| < 25\xi n$.

Case 2. $\eta_1, \eta_2 > \frac{2}{5}$.

Since $\chi(F_1) + \chi(F_2) \geqslant 7$, we can let $j \in \{1, 2\}$ so that $\chi(F_j) \geqslant 4$ and let $i \in \{1, 2\} \setminus \{j\}$. Since $\eta_i > \frac{2}{5}$ and $\eta_i + \eta_j = 1$, we have $\eta_j < \frac{3}{5}$. Now, by applying (10) or (11),

$$\frac{5\xi n}{3} \geqslant \frac{5\xi n}{\chi(F_j) - 1} > \left(1 - \eta_j - \frac{1}{\chi(F_j) - 1}\right)|V_i| > \left(1 - \frac{3}{5} - \frac{1}{3}\right)|V_i| = \frac{|V_i|}{15}$$

and so $|V_i| < 25\xi n$. Since $|V_1| \leqslant |V_2|$ by Assumption 1, this completes the proof. □

3.2 Obtaining the Partition

Now that we know that most vertices are in V_2 and, thus, have high degree in G_2, the next step is to show that G_2 can be made F_2-free by deleting a small proportion of its edges. For this, we apply the well-known Graph Removal Lemma of Alon, Duke, Lefmann, Rödl and Yuster [1] and Füredi [18]. As discussed in Remark 9, this is one place in which our argument deviates from that of [17].

Theorem 19 (Graph Removal Lemma [1,18])**.** *For every graph F and any given $\delta > 0$, there is a $\beta = \beta(F, \delta) > 0$ such that if G is a graph with $t(F, G) \leqslant \beta$, then there is a spanning subgraph G' of G with $t(F, G') = 0$ and $t(K_2, G') \geqslant t(K_2, G) - \delta$.*

Using Theorem 19, define

$$\beta := \min \{\beta(F_1, \delta/2), \beta(F_2, \delta/2)\}. \tag{12}$$

We also assume that ξ and τ satisfy the following:

$$0 < \xi < \frac{\beta}{52 \cdot f}, \tag{13}$$

$$0 < \tau < \beta/2. \tag{14}$$

The following lemma allows us to apply Theorem 19.

Lemma 20. $t(F_2, G_2) < \beta$.

Proof. Suppose, to the contrary, that $t(F_2, G_2) \geqslant \beta$. The number of homomorphisms from F_2 to G_2 which map a vertex to $V_0 \cup V_1$ is at most $v(F_2) \cdot n^{v(F_2)-1} \cdot |V_0 \cup V_1|$ which, by Lemmas 15 and 17, is no more than $26v(F_2)\xi n^{v(F_2)}$. Therefore,

$$t(F_2, G_2[V_2]) \geqslant \frac{t(F_2, G_2)n^{v(F_2)} - 26v(F_2)\xi n^{v(F_2)}}{|V_2|^{v(F_2)}} \geqslant (n/|V_2|)^{v(F_2)}(\beta - 26v(F_2)\xi).$$

By (13) and (14), this is greater than $\tau \cdot (n/|V_2|)^{v(F_2)}$. Recall that, by Observation 13, we have $d_2(v) \geqslant \left(\frac{1+\xi}{\chi(F_1)-1}\right) n$ for all $v \in V_2$. So, the set $S = V_2$ contradicts Lemma 14, and the proof is complete. □

The last step in verifying Lemma 11 involves utilizing the Erdős–Simonovits Stability Theorem [34] in the following form.

Theorem 21 (Erdős–Simonovits Stability Theorem [34])**.** *For every non-empty graph F and $\varepsilon > 0$, there exists $\delta = \delta(F, \varepsilon) > 0$ such that if G is a graph with $t(F, G) = 0$ and $t(K_2, G) \geqslant 1 - \frac{1}{\chi(F)-1} - \delta$, then there exists a partition $A_1, \ldots, A_{\chi(F)-1}$ of $V(G)$ such that $\sum_{i=1}^{\chi(F)-1} e(A_i) \leqslant \varepsilon n^2$.*

Using Theorem 21, we define

$$\delta := \min \{\delta(F_1, \varepsilon/2), \delta(F_2, \varepsilon/2), 2\varepsilon\}. \tag{15}$$

We also assume that ξ satisfies

$$0 < \xi < \frac{\delta}{52}. \tag{16}$$

The following lemma establishes a lower bound on $t(K_2, G_2)$ which will facilitate our application of Theorem 21.

Lemma 22. $t(K_2, G_2) \geqslant 1 - \frac{1}{\chi(F_2)-1} - \frac{\delta}{2}$.

Proof. By Lemmas 15 and 17, we have $|V_2| = n - |V_0 \cup V_1| \geqslant (1-26\xi)n$. Therefore,

$$\begin{aligned}
t(K_2, G_2) &= \frac{2e(G_2)}{n^2} \\
&= \frac{1}{n^2}\sum_{v\in V} d_2(v) \\
&\geqslant \frac{1}{n^2}\sum_{v\in V_2} d_2(v) \\
&\geqslant \frac{|V_2|}{n^2}\left(1 - \frac{1+2\xi}{\chi(F_2)-1}\right)(n-1) \\
&\geqslant \frac{|V_2|}{n^2}\left(1 - \frac{1+3\xi}{\chi(F_2)-1}\right)n \\
&\geqslant (1-26\xi)\left(1 - \frac{1+3\xi}{\chi(F_2)-1}\right) \\
&\geqslant 1 - \frac{1}{\chi(F_2)-1} - 26\xi.
\end{aligned}$$

By (16), this is at least $1 - \frac{1}{\chi(F_2)-1} - \frac{\delta}{2}$ and so the proof is complete. $\square$

Finally, we present the proof of Lemma 11, thereby accomplishing our primary objective of this section.

Proof of Lemma 11. By Lemma 20, Theorem 19 and (12), there exists a spanning subgraph G_2' of G_2 such that $t(F_2, G_2') = 0$ and $t(K_2, G_2') \geqslant t(K_2, G_2) - \delta/2$. So, Lemma 22 implies that $t(K_2, G_2') \geqslant 1 - \frac{1}{\chi(F_2)-1} - \delta$. By Theorem 21 and (15), there is a partition $A_1, A_2, \ldots, A_{\chi(F_2)-1}$ of V such that $\sum_{i=1}^{\chi(F_2)-1} e_{G_2'}(A_i) \leqslant (\varepsilon/2)n^2$. Since $t(K_2, G) = 2e(G)/v(G)^2$ for any graph G, the inequality $t(K_2, G_2') \geqslant t(K_2, G_2) - \delta/2$ is equivalent to $e(G_2') \geqslant e(G_2) - (\delta/4)n^2$. Therefore,

$$\sum_{i=1}^{\chi(F_2)-1} e_2(A_i) \leqslant \sum_{i=1}^{\chi(F_2)-1} e_{G_2'}(A_i) + (\delta/4)n^2 \leqslant (\varepsilon/2 + \delta/4)n^2 \leqslant \varepsilon n^2$$

where the last inequality follows from (15). $\square$

4 Proof of Theorem 2: Exact Structure

The aim of this section is to complete the proof of Theorem 2. The way that this breaks down is as follows. We start by obtaining control over the number of copies of H_1 in G_1 and H_2 in G_2 that contain any given vertex $v \in V$. In particular, we show that any two vertices in V "contribute" roughly the same amount to $m(H_1, H_2; G_1)$. Thus, if one vertex contributes "too much," then all vertices do, which leads to a violation of (4).

After this, we refine the rough structure afforded to us by Lemma 11 until we get that G_2 is simply a Turán graph with $\chi(F_2)-1$ parts. The first step in this process is to show that the parts have nearly the same size and almost all edges between pairs of parts are in G_2. We then show that the G_2-neighbourhood of every vertex $v \notin V_1$ roughly "respects" the partition. Next, we prove that the G_2-degree of any vertex is within a small window around $\left(1-\frac{1}{\chi(F_2)-1}\right)n$, which implies that $V_1=\emptyset$. After that, we can use the critical edge in F_2 to show that all edges within A_i must be in G_1 for all $1 \leqslant i \leqslant \chi(F_2)-1$. From this point, the theorem is easily deduced via a convexity argument.

4.1 Every Vertex Contributes the Same

For a graph H, a graph G on vertex set V and $v \in V$, define $t_{\mathrm{inj}}(H,G)(v)$ to be the probability that a random function from $V(H)$ to V is an injective homomorphism from H to G whose image contains v. The main idea of the next lemma is that, if u and w are vertices such that an appropriate weighted sum of $t_{\mathrm{inj}}(H_1,G_1)(u)$ and $t_{\mathrm{inj}}(H_2,G_2)(u)$ is significantly smaller than the analogous sum for w, then one can get a better colouring by "deleting" w and "cloning" u. This is a standard idea in extremal combinatorics going back at least as far as Zykov's proof of Turán's Theorem [38]. This lemma is analogous to [17, Lemma 2.1].

Lemma 23. *There exists a constant $C=C(H_1,H_2)>0$ such that, for any $u,w \in V$,*

$$\rho_1 \cdot t_{\mathrm{inj}}(H_1,G_1)(u)+\rho_2 \cdot t_{\mathrm{inj}}(H_2,G_2)(u) \geqslant \rho_1 \cdot t_{\mathrm{inj}}(H_1,G_1)(w)+\rho_2 \cdot t_{\mathrm{inj}}(H_2,G_2)(w)-\frac{C}{n^2}.$$

Proof. Suppose, to the contrary, that the inequality does not hold for some $u,w \in V$. If we remove all edges incident to the vertex w from G_1, then we lose all of the injective homomorphisms from H_1 to G_1 which map at least one vertex to w. Likewise, if we delete all edges incident to w from G_2, then we lose all of the injective homomorphisms from H_2 to G_2 which map at least one vertex to w. (Note that, here, we are subtly using the assumption that the graphs H_1 and H_2 have no singleton components.)

After deleting all such edges from G_1 and G_2, suppose that we add to G_1 all edges of the form wv such that $uv \in E(G_1)$ and $v \neq w$ to form a new graph G_1'. Similarly, add to G_2 all edges of the form wv such that v is a vertex with $uv \notin E(G_1)$ and $v \neq w$ to get a graph G_2'. Note that G_2' is the complement of G_1'. In adding these edges, we gain one injective homomorphism from H_1 to G_1' per injective homomorphism from H_1 to G_1 that includes u and not w. Similarly, we gain one injective homomorphsim from H_2 to G_2' per injective homomorphism from H_2 to G_2 that includes u and not w. Additionally, for each $i \in \{1,2\}$, we may also gain $O(n^{v(H_i)-2})$ injective homomorphisms which map to both u and w. Thus, for each $i \in \{1,2\}$,

$$t_{\mathrm{inj}}(H_i,G_i') \leqslant t_{\mathrm{inj}}(H_i,G_i)-t_{\mathrm{inj}}(H_i,G_i)(w)+t_{\mathrm{inj}}(H_i,G_i)(u)+O(1/n^2)$$

where the constant factor on the $O(1/n^2)$ term is bounded by a function of H_i. Thus, assuming that the inequality in the lemma is not true, we have that $m(H_1,H_2;G_1')$ is at most $m(H_1, H_2; G_1)$ plus a $O(1/n^2)$ term, where the constant factor depends on H_1 and H_2, minus C/n^2. So, if C is chosen large enough with respect to H_1 and H_2, we get that G_1' contradicts our choice of G_1. Thus, the lemma holds. □

Analogous to the definition of $t_{\text{inj}}(H, G)(v)$, let $t(H, G)(v)$ be the probability that a uniformly random function from $V(H)$ to V is a homomorphism from H to G whose image contains v. The following lemma restricts $t(H_i, G_i)(v)$ for every vertex v.

Lemma 24. *Suppose that $\{i, j\} = \{1, 2\}$. For every $v \in V$,*

$$t(H_i, G_i)(v) \leqslant \frac{3 \max\{v(H_1), v(H_2)\}}{n} \left(\frac{1}{\chi(F_j) - 1}\right)^{v(H_i) - k(F_i)}.$$

Proof. Suppose, to the contrary, that there exists $v \in V$ such that

$$t(H_i, G_i)(v) > \frac{3 \max\{v(H_1), v(H_2)\}}{n} \left(\frac{1}{\chi(F_j) - 1}\right)^{v(H_i) - k(F_i)}. \tag{17}$$

By Lemma 23, (3), (17) and the fact that $t_{\text{inj}}(H_i, G_i)(v) = t(H_i, G_i)(v) + O(1/n^2)$, we get that, for large n, every $u \in V$ satisfies

$$\rho_1 \cdot t_{\text{inj}}(H_1, G_1)(u) + \rho_2 \cdot t_{\text{inj}}(H_2, G_2)(u) > \frac{2 \max\{v(H_1), v(H_2)\}}{n}.$$

Summing this inequality over all $u \in V$ yields

$$\begin{aligned}
2 \max\{v(H_1), v(H_2)\} &< \sum_{u \in V} \left(\rho_1 \cdot t_{\text{inj}}(H_1, G_1)(u) + \rho_2 \cdot t_{\text{inj}}(H_2, G_2)(u)\right) \\
&= \rho_1 \cdot v(H_1) \cdot t_{\text{inj}}(H_1, G_1) + \rho_2 \cdot v(H_2) \cdot t_{\text{inj}}(H_2, G_2) \\
&\leqslant \max\{v(H_1), v(H_2)\} \left(\rho_1 \cdot t_{\text{inj}}(H_1, G_1) + \rho_2 \cdot t_{\text{inj}}(H_2, G_2)\right).
\end{aligned}$$

This contradicts (4), and thus the proof is complete. □

4.2 Refining the Partition

We assume, throughout the remainder of this section, that $A_1, \ldots, A_{\chi(F_2)-1}$ is a partition of V as in Lemma 11. Let us show that the sets $A_1, \ldots, A_{\chi(F_2)-1}$ have approximately the same size and that G_1 contains almost no edges between different parts. In order to prove this, we make the following assumption on ε. Recall that $\chi = \max\{\chi(F_1), \chi(F_2)\}$.

$$0 < \varepsilon < \frac{1}{12\chi^4}. \tag{18}$$

The next lemma is analogous to [17, Claim 3.8].

Lemma 25. *For $1 \leqslant i \neq j \leqslant \chi(F_2) - 1$,* *(a)* $\left||A_i| - \frac{n}{\chi(F_2)-1}\right| \leqslant \sqrt{3\varepsilon} \cdot n$ *and*

(b) $e_2(A_i, A_j) \geqslant (1 - 2\chi(F_2)^2\varepsilon)|A_i||A_j|$.

Proof. First observe that, since $\sum_{i=1}^{\chi(F_2)-1} |A_i| = n$,

$$\begin{aligned}\sum_{i=1}^{\chi(F_2)-1} \left(\frac{|A_i|}{n} - \frac{1}{\chi(F_2)-1}\right)^2 &= \sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|^2}{n^2} - 2\sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|}{n(\chi(F_2)-1)} + \sum_{i=1}^{\chi(F_2)-1} \left(\frac{1}{\chi(F_2)-1}\right)^2 \\ &= \sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|^2}{n^2} - \frac{1}{\chi(F_2)-1}\end{aligned}$$

and, also,

$$1 = \left(\sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|}{n}\right)^2 = \sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|^2}{n^2} + 2\left(\sum_{1\leqslant i<j\leqslant \chi(F_2)-1} \frac{|A_i||A_j|}{n^2}\right).$$

Solving for $\sum_{i=1}^{\chi(F_2)-1} \frac{|A_i|^2}{n^2}$ in one of these two equations and substituting into the other yields

$$\sum_{i=1}^{\chi(F_2)-1} \left(\frac{|A_i|}{n} - \frac{1}{\chi(F_2)-1}\right)^2 + \frac{1}{\chi(F_2)-1} = 1 - 2\left(\sum_{1\leqslant i<j\leqslant \chi(F_2)-1} \frac{|A_i||A_j|}{n^2}\right)$$

which is equivalent to

$$1 - \frac{1}{\chi(F_2)-1} = \sum_{i=1}^{\chi(F_2)-1} \left(\frac{|A_i|}{n} - \frac{1}{\chi(F_2)-1}\right)^2 + 2\left(\sum_{1\leqslant i<j\leqslant \chi(F_2)-1} \frac{|A_i||A_j|}{n^2}\right). \tag{19}$$

Also, by (15) and Lemma 22, we have $t(K_2, G_2) \geqslant 1 - \frac{1}{\chi(F_2)-1} - \varepsilon$. So,

$$\begin{aligned}1 - \frac{1}{\chi(F_2)-1} - \varepsilon &\leqslant t(K_2, G_2) \\ &= \frac{2e(G_2)}{n^2} \\ &= \sum_{i=1}^{\chi(F_2)-1} \frac{2e_2(A_i)}{n^2} + \sum_{1\leqslant i<j\leqslant \chi(F_2)-1} \frac{2e_2(A_i, A_j)}{n^2} \\ &\leqslant 2\varepsilon + 2\left(\sum_{1\leqslant i<j\leqslant \chi(F_2)-1} \frac{e_2(A_i, A_j)}{n^2}\right)\end{aligned}$$

where the last inequality is by Lemma 11. Substituting the expression for $1 - \frac{1}{\chi(F_2)-1}$ in (19) into this inequality and rearranging yields

$$\sum_{i=1}^{\chi(F_2)-1} \left(\frac{|A_i|}{n} - \frac{1}{\chi(F_2)-1} \right)^2 + 2 \sum_{1 \leqslant i<j \leqslant \chi(F_2)-1} \left(\frac{|A_i||A_j|}{n^2} - \frac{e_2(A_i, A_j)}{n^2} \right) \leqslant 3\varepsilon.$$

Since all summands on the left side are non-negative, we get

$$\left(\frac{|A_i|}{n} - \frac{1}{\chi(F_2)-1} \right)^2 \leqslant 3\varepsilon$$

for all i, which proves (a). Similarly, for each $i \neq j$, the above inequality implies that

$$2 \left(\frac{|A_i||A_j|}{n^2} - \frac{e_2(A_i, A_j)}{n^2} \right) \leqslant 3\varepsilon$$

and so

$$e_2(A_i, A_j) \geqslant |A_i||A_j| - (3\varepsilon/2)n^2 = \left(1 - \frac{3\varepsilon n^2}{2|A_i||A_j|}\right)|A_i||A_j|.$$

By (a), the right side is at least

$$\left(1 - \frac{3\varepsilon}{2\left(\frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon}\right)^2}\right)|A_i||A_j|$$

and by (18), this is at least $(1 - 2\chi(F_2)^2\varepsilon)|A_i||A_j|$ (with room to spare). Therefore, (b) holds. $\square$

Next, we show that the G_2-neighbourhood of every vertex that is not in V_1 roughly "respects" the partition $A_1, \ldots, A_{\chi(F_2)-1}$ (see Lemma 27 below). We assume that ε satisfies the following condition:

$$0 < \varepsilon < \frac{\theta^2}{4 \cdot \chi^2 f^2}. \tag{20}$$

Also, we assume that t_0 is chosen large enough so that, for all $t \geqslant t_0$, we have

$$\left(\frac{5}{4}\right)^{t/2} > \frac{6(t+f)}{q\left(\frac{\theta}{2\chi}\right)^f}. \tag{21}$$

Definition 26. Say that a vertex $v \in V$ is *bad* if, for all $1 \leqslant i \leqslant \chi(F_2) - 1$, the number of G_2-neighbours of v in $A_i \cap V_2$ is at least $\theta|A_i|$. Let B be the set of all bad vertices.

Lemma 27. *$B \subseteq V_1$.*

Proof. Suppose, to the contrary, that there exists a bad vertex $v \notin V_1$. For each vertex x of F_2, let $p(x)$ be the number of pendant edges incident to x which were added during the construction of H_2 from F_2. Let $v_0 z$ be a critical edge of F_2 where we assume, without loss of generality, that $p(v_0) \leqslant p(z)$. Then, in particular, at most half of the pendant edges added in the construction of H_2 are incident to v_0; i.e. $p(v_0) \leqslant \frac{v(H_2)-v(F_2)}{2}$. Let $F_2' := F_2 \setminus \{v_0\}$ and note that, since $v_0 z$ is a critical edge, $\chi(F_2') < \chi(F_2)$. Fix a proper colouring $\psi : V(F_2') \to \{1, \dots, \chi(F_2)-1\}$ of F_2'. Our aim is to prove a lower bound on $t(H_2, G_2)(v)$ which is large enough to contradict Lemma 24.

Let S be the G_2-neighbourhood of v. The probability that a random function φ from $V(H_2)$ to V is a homomorphism from H_2 to G_2 such that $\varphi(v_0) = v$ is at least the probability that the restriction of φ to $V(F_2')$ is a homomorphism from F_2' to $G_2[S \cap V_2]$, times $1/n$ (the probability that $\varphi(v_0) = v$), times the probability that every vertex of $V(H_2) \setminus V(F_2)$ is mapped to a G_2-neighbour of the image of its unique neighbour in H_2. Taking into account that $v \notin V_1$, we have

$$d_2(v) = n - 1 - d_1(v) \geqslant \left(\frac{1+2\xi}{\chi(F_1)-1}\right)(n-1) > \left(\frac{1}{\chi(F_1)-1}\right)n.$$

Thus, since $d_2(w) \geqslant \left(\frac{5}{4} \cdot \frac{1}{\chi(F_1)-1}\right)n$ for all $w \in V_2$ by Observation 13, we get that $t(H_2, G_2)(v)$ is greater than

$$\left(\frac{|S \cap V_2|}{n}\right)^{v(F_2')} t(F_2', G[S \cap V_2]) \frac{1}{n} \left(\frac{1}{\chi(F_1)-1}\right)^{p(v_0)} \left(\frac{5}{4} \cdot \frac{1}{\chi(F_1)-1}\right)^{v(H_2)-v(F_2)-p(v_0)}. \tag{22}$$

Next, we bound $(|S \cap V_2|/n)^{v(F_2')}\, t(F_2', G[S \cap V_2])$ from below. First, since v is bad, we have that $|S \cap A_i \cap V_2| \geqslant \theta|A_i|$ for all $1 \leqslant i \leqslant \chi(F_2)-1$. So, if we map $V(F_2')$ randomly to V, then the probability that every vertex w of F_2' is mapped to $S \cap A_{\psi(w)} \cap V_2$ is at least $\prod_{w \in V(F_2')}(\theta|A_{\psi(w)}|/n)$ which, by Lemma 25 (a), is at least $\theta^{v(F_2')}\left(\frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon}\right)^{v(F_2')}$. By Lemma 25 (b), the number of non-edges in G_2 from $S \cap A_i \cap V_2$ to $S \cap A_j \cap V_2$ for $i \neq j$ is at most $2\chi(F_2)^2\varepsilon|A_i||A_j|$ which, since v is bad, is at most

$$\frac{2\chi(F_2)^2\varepsilon|S \cap A_i \cap V_2||S \cap A_j \cap V_2|}{\theta^2}.$$

Thus, for any fixed edge wy of F_2', the conditional probability that $\varphi(w)$ is not adjacent to $\varphi(y)$ given that $\varphi(w) \in S \cap A_{\psi(w)} \cap V_2$ and $\varphi(y) \in S \cap A_{\psi(y)} \cap V_2$ is at most $\frac{2\chi(F_2)^2\varepsilon}{\theta^2}$. By taking a union bound over all edges of F_2', we get that the probability that every vertex w of F_2' is mapped to $S \cap A_{\psi(w)} \cap V_2$ and no edge of F_2' is mapped to a non-edge of G_2 is at least

$$\theta^{v(F_2')}\left(\frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon}\right)^{v(F_2')}\left(1 - \frac{2e(F_2')\chi(F_2)^2\varepsilon}{\theta^2}\right).$$

By (18), the product of the first two factors is at least $\left(\frac{\theta}{\chi(F_2)}\right)^{v(F_2)}$ and, by (20), the third factor is at least 1/2. So, the expression in (22) is at least

$$\frac{1}{2}\left(\frac{\theta}{\chi(F_2)}\right)^{v(F_2)}\frac{1}{n}\left(\frac{1}{\chi(F_1)-1}\right)^{p(v_0)}\left(\frac{5}{4}\cdot\frac{1}{\chi(F_1)-1}\right)^{v(H_2)-v(F_2)-p(v_0)}$$
$$=\frac{1}{2}\left(\frac{\theta}{\chi(F_2)}\right)^{v(F_2)}\frac{1}{n}\left(\frac{4}{5}\right)^{p(v_0)}\left(\frac{5}{4}\cdot\frac{1}{\chi(F_1)-1}\right)^{v(H_2)-v(F_2)}.$$

Now, since $p(v_0)\leqslant\frac{v(H_2)-v(F_2)}{2}$, we get that this is at least

$$\frac{1}{2}\left(\frac{\theta}{\chi(F_2)}\right)^{v(F_2)}\frac{1}{n}\left(\frac{4}{5}\right)^{\frac{v(H_2)-v(F_2)}{2}}\left(\frac{5}{4}\cdot\frac{1}{\chi(F_1)-1}\right)^{v(H_2)-v(F_2)}$$
$$=\frac{1}{2}\left(\frac{\theta}{\chi(F_2)}\right)^{v(F_2)}\frac{1}{n}\left(\frac{5}{4}\right)^{\frac{v(H_2)-v(F_2)}{2}}\left(\frac{1}{\chi(F_1)-1}\right)^{v(H_2)-v(F_2)}$$
$$=\frac{1}{2}\left(\frac{\theta}{\chi(F_2)}\right)^{v(F_2)}\frac{1}{n}\left(\frac{5}{4}\right)^{t_2/2}\left(\frac{1}{\chi(F_1)-1}\right)^{v(H_2)-v(F_2)}.$$

By (21) and the fact that $t_2\geqslant t_0$, this is at least

$$\frac{1}{2}\left(\frac{\theta}{\chi(F_2)}\right)^{v(F_2)}\frac{1}{n}\left(\frac{6v(H_2)}{q\left(\frac{\theta}{\chi(F_2)}\right)^{v(F_2)}}\right)\left(\frac{1}{\chi(F_1)-1}\right)^{v(H_2)-v(F_2)}$$
$$>\frac{3v(H_2)}{q\cdot n}\left(\frac{1}{\chi(F_1)-1}\right)^{v(H_2)-k(F_2)}$$
$$\geqslant\frac{3\max\{v(H_1),v(H_2)\}}{n}\left(\frac{1}{\chi(F_1)-1}\right)^{v(H_2)-k(F_2)}$$

where the penultimate step uses $v(F_2)>k(F_2)$ and the last step uses (2). This contradicts Lemma 24 and completes the proof. □

Using the above lemma, it follows relatively easily that $d_2(v)$ cannot be too large for any vertex $v\in V$. To verify this, we use the following assumption:

$$0<\xi<\frac{\theta}{26\chi}. \tag{23}$$

Lemma 28. *For every $v\in V$,*

$$d_2(v)\leqslant\left(1-\frac{1-3\theta}{\chi(F_2)-1}\right)(n-1).$$

Proof. If $v \in V_1$, then, by Observation 13,

$$\begin{aligned} d_2(v) &= n - 1 - d_1(v) \\ &\leqslant n - \left(\frac{5}{4} \cdot \frac{1+\xi}{\chi(F_2)-1}\right) n \\ &< \left(1 - \frac{1}{\chi(F_2)-1}\right) n \\ &< \left(1 - \frac{1-3\theta}{\chi(F_2)-1}\right)(n-1). \end{aligned}$$

On the other hand, if $v \notin V_1$, then, by Lemma 27, we have that $v \notin B$. So, there exists i such that v has fewer than $\theta|A_i|$ neighbours in $A_i \cap V_2$. Since $|V \setminus V_2| \leqslant 26\xi n$ by Lemmas 12, 15 and 17, we have

$$d_2(v) \leqslant \sum_{j \neq i} |A_j| + \theta \cdot |A_i| + |A_i \setminus V_2| \leqslant n - (1-\theta)|A_i| + 26\xi n.$$

By Lemma 25 (a) this is at most

$$n - (1-\theta)\left(\frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon}\right) n + 26\xi n \leqslant \left(1 - \frac{1-\theta}{\chi(F_2)-1} + \sqrt{3\varepsilon} + 26\xi\right) n.$$

Note that (20) implies that $\varepsilon < \frac{\theta^2}{3(\chi(F_2)-1)^2}$. Using this bound, together with (23), tells us that the above expression is at most $\left(1 - \frac{1-3\theta}{\chi(F_2)-1}\right)(n-1)$ as desired. $\square$

Next, we show that $d_2(v)$ is reasonably large for every vertex $v \in V$. This will then be used to show that $V_1 = \emptyset$. To prove it, we assume the following:

$$0 < \varepsilon < \frac{1}{4f^2\chi^4}. \tag{24}$$

Also, we assume that t_0 is chosen large enough that, for all $t \geqslant t_0$,

$$e^{\theta \cdot t} \geqslant \frac{6\chi^{2f}(t+f)}{q}. \tag{25}$$

Lemma 29. *For every $v \in V$,*

$$d_2(v) \geqslant \left(1 - \frac{1+15\theta f}{\chi(F_2)-1}\right)(n-1).$$

Proof. Suppose that the lemma is false. Then there exists $v \in V$ such that

$$\begin{aligned} d_1(v) = n - 1 - d_2(v) &> n - 1 - \left(1 - \frac{1+15\theta f}{\chi(F_2)-1}\right)(n-1) \\ &\geqslant \left(\frac{1+14\theta f}{\chi(F_2)-1}\right) n. \end{aligned}$$

Our goal is to show that v is contained in a large number of copies of H_1 in G_1 which will contradict Lemma 24. For each $w \in V(F_1)$, let $p(w)$ be the number of pendant edges incident to w added in the construction of H_1 from F_1 and let $v_0 \in V(F_1)$ so that $p(v_0)$ is maximum. Then, by the Pigeonhole Principle,

$$p(v_0) \geqslant \frac{v(H_1) - v(F_1)}{v(F_1)}. \tag{26}$$

Let $F_1' := F_1 \setminus \{v_0\}$.

By the lower bound on $d_1(v)$ proven above, there must exist some $1 \leqslant i \leqslant \chi(F_2) - 1$ such that v has at least

$$\frac{1}{\chi(F_2)-1} \cdot \left(\frac{1+14\theta f}{\chi(F_2)-1}\right) n \geqslant \frac{n}{(\chi(F_2)-1)^2}$$

G_1-neighbours in A_i. Let S be the set of G_1-neighbours of v in A_i. Recall that, by Lemma 11, the number of non-edges of G_1 in S is at most

$$\varepsilon n^2 \leqslant \varepsilon n^2 \left(\frac{|S|}{n/(\chi(F_2)-1)^2}\right)^2 = \varepsilon(\chi(F_2)-1)^4 |S|^2.$$

Therefore, for large enough n,

$$t(K_2, G_1[S]) = \frac{2e_1(S)}{|S|^2} \geqslant \frac{2\binom{|S|}{2} - 2\varepsilon(\chi(F_2)-1)^4|S|^2}{|S|^2} \geqslant 1 - 2\varepsilon\chi(F_2)^4.$$

Thus, if $V(F_1')$ is mapped to S randomly, then the probability that any individual edge of F_1' is mapped to a non-edge of G_1 is at most $2\varepsilon\chi(F_2)^4$. So, by a union bound and (24), we have that

$$t(F_1', G_1[S]) \geqslant 1 - 2\varepsilon e(F_1')\chi(F_2)^4 > 1/2. \tag{27}$$

Now, if φ is a random function from $V(H_1)$ to V, then the probability that φ is a homomorphism mapping v_0 to v is at least the probability that the restriction of φ to $V(F_1')$ is a homomorphism from F_1' to $G[S]$, times $1/n$ (the probability that v_0 maps to v) times the probability that every vertex of $V(H_1) \setminus V(F_1)$ is mapped to a G_1-neighbour of the image of its unique neighbour in H_1. So, by Lemma 28, $t(H_1, G_1)(v)$ is at least

$$(|S|/n)^{v(F_1')}\, t(F_1', G_1[S]) \cdot \frac{1}{n} \left(\frac{1+14\theta f}{\chi(F_2)-1}\right)^{p(v_0)} \left(\frac{1-3\theta}{\chi(F_2)-1}\right)^{v(H_1)-v(F_1)-p(v_0)}.$$

Using the fact that $|S| \geqslant n/(\chi(F_2)-1)^2 > n/\chi(F_2)^2$ and (27), we get that this is at least

$$\frac{1}{2\chi(F_2)^{2v(F_1)}} \frac{1}{n} \left(\frac{1}{\chi(F_2)-1}\right)^{v(H_1)-v(F_1)} (1+14\theta f)^{p(v_0)} (1-3\theta)^{v(H_1)-v(F_1)-p(v_0)}.$$

Using the inequalities $1 + r \geqslant e^{r/2}$ and $1 - r \geqslant e^{-2r}$, which are valid for all $r \in [0, 1/2]$, we can bound the product of the last two factors as follows:

$$(1+14\theta f)^{p(v_0)} (1-3\theta)^{v(H_1)-v(F_1)-p(v_0)}$$

$$\geqslant \exp(7\theta f p(v_0) - 6\theta(v(H_1) - v(F_1) - p(v_0))).$$

By (26), this is at least

$$\exp(7\theta(v(H_1) - v(F_1)) - 6\theta(v(H_1) - v(F_1) - p(v_0))) \geqslant e^{\theta t_1}.$$

So, by (25) and the fact that $t_1 \geqslant t_0$, we have that

$$\begin{aligned} t(H_1, G_1)(v) &> \frac{1}{n}\left(\frac{1}{\chi(F_2)-1}\right)^{v(H_1)-v(F_1)}\left(\frac{3\,(t_1+f)}{q}\right) \\ &> \frac{3v(H_1)}{q\cdot n}\left(\frac{1}{\chi(F_2)-1}\right)^{v(H_1)-k(F_1)} \\ &\geqslant \frac{3\max\{v(H_1), v(H_2)\}}{n}\left(\frac{1}{\chi(F_2)-1}\right)^{v(H_1)-k(F_1)} \end{aligned}$$

where the penultimate step used $v(F_1) > k(F_1)$ and the last step applied (2). This contradicts Lemma 24 and completes the proof. □

As a consequence of the previous lemma, we will show next that $V_1 = \emptyset$. This also implies $B = \emptyset$ by virtue of Lemma 27. For this, we assume

$$0 < \theta < \frac{1}{60f}. \tag{28}$$

Lemma 30. *We have $V_1 = \emptyset$. Consequently, $B = \emptyset$.*

Proof. Assuming $v \in V_1$, Lemma 29 implies that

$$d_1(v) = n - 1 - d_2(v) \leqslant \left(\frac{1 + 15\theta f}{\chi(F_2) - 1}\right)(n-1)$$

which, by (28), is less than

$$\left(\frac{5}{4}\cdot\frac{1}{\chi(F_2)-1}\right)n.$$

This contradicts Observation 13, and so V_1 must be empty. Lemma 27 then implies that B is also empty. □

From here forward, we impose an additional assumption that $\sum_{i=1}^{\chi(F_2)-1} e_2(A_i)$ is minimum among all partitions $A_1, \ldots, A_{\chi(F_2)-1}$ of V. This allows us to prove the next lemma, which is analogous to [17, Claim 3.11]. We assume that ξ satisfies

$$0 < \xi < \theta\left(\frac{1}{f-1} - \sqrt{3\varepsilon}\right). \tag{29}$$

Note that the expression on the right side of the rightmost inequality above is positive by (24), and so it is possible to choose ξ in this way. We use the assumption on the choice of partition to show that, for each i, every vertex in A_i has few G_2-neighbours in $A_i \cap V_2$.

Lemma 31. *For $1 \leqslant i \leqslant \chi(F_2) - 1$, every $v \in A_i$ is adjacent in G_2 to fewer than $\theta|A_i|$ vertices of $A_i \cap V_2$.*

Proof. Let $v \in A_i$. Then v is not bad by Lemma 30, so there must exist an index i' such that the number of G_2-neighbours of v in $A_{i'} \cap V_2$ is at most $\theta|A_{i'}|$. If $i' = i$, then we are done; so, we assume that $i' \neq i$. Since $V = V_0 \sqcup V_2$ by Lemmas 12 and 30, the number of G_2-neighbours of v in $A_{i'}$ overall is at most $\theta|A_{i'}| + |A_{i'} \cap V_0|$ which, by Lemma 15, is at most

$$\theta|A_{i'}| + \xi n < 2\theta|A_{i'}|$$

where the last step applies Lemma 25 (a) and (29). Since $i \neq i'$, the vertex v must have at most $2\theta|A_{i'}|$ neighbours in A_i as well; otherwise, moving v from A_i to $A_{i'}$ would decrease $\sum_{i=1}^{\chi(F_2)-1} e_2(A_i)$, contradicting our choice of partition. Thus,

$$d_2(v) \leqslant \sum_{j \notin \{i,i'\}} |A_j| + 4\theta|A_{i'}|.$$

By Lemma 25 (a) and (28), this is at most

$$(\chi(F_2) - 3 + 4\theta)\left(\frac{1}{\chi(F_2)-1} + \sqrt{3\varepsilon}\right) n \leqslant \left(1 - \frac{2}{\chi(F_2)-1} + 4\theta + \chi(F_2)\sqrt{3\varepsilon}\right) n.$$

Using (20), we can bound this above by

$$\begin{aligned}\left(1 - \frac{2}{\chi(F_2)-1} + 5\theta\right) n &< \left(1 - \frac{2}{\chi(F_2)-1} + \frac{5\theta\chi(F_2)}{\chi(F_2)-1}\right)(n-1)\\ &= \left(1 - \frac{1 + (1 - 5\theta\chi(F_2))}{\chi(F_2)-1}\right)(n-1).\end{aligned}$$

By (28), we have $\theta \leqslant \frac{1}{15f + 5\chi(F_2)}$. Plugging this into the above expression yields an upper bound of

$$\left(1 - \frac{1 + 15\theta f}{\chi(F_2)-1}\right)(n-1)$$

contradicting Lemma 29 and completing the proof of the claim. □

Next, let us show that every vertex $v \in A_i$ has many neighbours in A_j for $j \neq i$.

Lemma 32. *For $1 \leqslant i \neq j \leqslant \chi(F_2) - 1$, every $v \in A_i$ is adjacent in G_2 to at least $(1 - 33\theta f)\,|A_j|$ vertices of A_j.*

Proof. Let $v \in A_i$. Suppose that v has fewer than $(1 - 33\theta f)|A_j|$ G_2-neighbours in A_j. Let S be the G_2-neighbourhood of v. Then

$$d_2(v) = \sum_{\ell=1}^{\chi(F_2)-1} |A_\ell \cap S| \leqslant |S \cap A_i \cap V_2| + |S \cap A_j| + |V_0| + \sum_{\ell \notin \{i,j\}} |A_\ell \cap S|$$

which, by Lemmas 15 and 31, is at most

$$\theta|A_i| + (1-33\theta f)|A_j| + \xi n + n - |A_i| - |A_j| = \left(1 + \xi - \frac{(1-\theta)|A_i| + 33\theta f|A_j|}{n}\right) n.$$

Using the lower bound on $|A_i|$ and $|A_j|$ in Lemma 25 (a) yields an upper bound of

$$\begin{aligned}\left(1 + \xi - (1-\theta+33\theta f)\left(\frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon}\right)\right) n \\ \leqslant \left(1 - \frac{1}{\chi(F_2)-1} + \xi + \sqrt{3\varepsilon} - 32\theta f\left(\frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon}\right)\right) n.\end{aligned}$$

By (18), (20) and (29), this is less than

$$\left(1 - \frac{1+15\theta f}{\chi(F_2)-1}\right)(n-1)$$

which contradicts Lemma 29 and completes the proof. □

Next, we prove that, in fact, there are no edges within $G_2[A_i]$ for any $1 \leqslant i \leqslant \chi(F_2)-1$. The presence of critical edges in F_1 and F_2 is crucial in this step. After this, the proof of Theorem 2 will follow relatively easily. The following lemma is analogous to [17, Claim 3.12]. Assume that t_0 is large enough that the following holds for all $t \geqslant t_0$:

$$\left(\frac{10}{9}\right)^{qt} > 4\chi\left(\frac{4\chi}{3}\right)^{f}(t+f)^2. \tag{30}$$

Also, choose θ small enough so that

$$0 < \theta < \frac{1}{66f^3} \tag{31}$$

and

$$(1 - 20\cdot\theta f) \geqslant (5/6)\cdot(1+15\theta f)^{1/q}. \tag{32}$$

Note that such a θ exists because the limit as $\theta \to 0$ of the left side is 1 and the limit of the right size is 5/6.

Lemma 33. $e_2(A_i) = 0$ *for* $1 \leqslant i \leqslant \chi(F_2)-1$.

Proof. Suppose that the lemma is not true; without loss of generality, the set A_1 contains an edge of G_2. Let u_0 and v_0 be the endpoints of such an edge. Let G_2' be the graph obtained from G_2 by deleting the edge u_0v_0 and let $G_1' = \overline{G_2'}$. We estimate the number of copies of H_1 in G_1 that are "gained" and the number of copies of H_2 in G_2 that are "lost" when replacing (G_1, G_2) by (G_1', G_2') with a goal of contradicting the choice of G_1.

We begin by bounding from above the number of injective homomorphisms of H_1 to G_1' which are not homomorphisms from H_1 to G_1. Any such homomorphism can be described as follows. First, we pick an edge $e = wz$ of H_1 and map its endpoints to u_0v_0 (in one of two possible ways). Now, imagine that we list the vertices of H_1 so that w and z are listed first (in this order), followed by the other vertices of the component of H_1 containing w and z, and then the vertices of another (arbitrary) component, and so on, so that each vertex in the list is either the first vertex of its component or has a neighbour which comes before it in the list, which we refer to as its "parent." Then, in a homomorphism, each vertex in the list after w and z must be mapped to a G_1-neighbour of its parent (if it has one). Thus, since $k(H_1) = k(F_1)$ and each vertex has at most $\left(\frac{1+15\theta f}{\chi(F_2)-1}\right)(n-1)$ neighbours in G_1 by Lemma 29, the number of such mappings is at most

$$\begin{aligned}2e(H_1)n^{k(F_1)-1}&\left(\frac{1+15\theta f}{\chi(F_2)-1}\right)^{v(H_1)-2-(k(F_1)-1)} n^{v(H_1)-2-(k(F_1)-1)}\\ &= 2e(H_1)(\chi(F_2)-1)\left(1+15\theta f\right)^{v(H_1)-k(F_1)-1}\left(\frac{1}{\chi(F_2)-1}\right)^{v(H_1)-k(F_1)} n^{v(H_1)-2}.\end{aligned}$$

Thus, by (3),

$$\begin{aligned}\rho_1\left(t_{\mathrm{inj}}(H_1, G_1') - t_{\mathrm{inj}}(H_1, G_1)\right)&\\ &\leqslant 2e(H_1)(\chi(F_2)-1)\left(1+15\theta f\right)^{v(H_1)-k(F_1)-1} n^{-2} + O(n^{-3})\\ &\leqslant 2e(H_1)(\chi(F_2)-1)\left(1+15\theta f\right)^{v(H_1)} n^{-2} + O(n^{-3}).\end{aligned}$$

Next, let us bound from below the number of injective homomorphisms of H_2 to G_2' which are not homomorphisms from H_2 to G_2. Let $e_0 = w_0z_0$ be a critical edge of F_2, let $F_2' = F_2 \setminus \{e_0\}$ and let $\psi : V(F_2') \to \{1, \ldots, \chi(F_2)-1\}$ be a proper colouring of F_2' such that $\psi(w_0) = \psi(z_0) = 1$. Now, suppose that φ is a function that maps w_0 to u_0 and z_0 to v_0 and then maps every other vertex of F_2' to V randomly. The probability that every other vertex u of F_2' is mapped by φ to $A_{\psi(u)}$ is

$$\prod_{u\in V(F_2')\setminus\{w_0,z_0\}}\left(\frac{|A_{\psi(u)}|}{n}\right) \geqslant \left(\frac{1}{\chi(F_2)-1} - \sqrt{3\varepsilon}\right)^{v(F_2')-2} \geqslant \left(\frac{1}{\chi}\right)^f.$$

by Lemma 25 (a) and (18). Given this, by Lemma 32, the probability that every edge of F_2' maps to an edge of G_2 is, by a union bound, at least

$$1 - 33\theta e(F_2')f \geqslant 1/2$$

where the inequality is by (31). Finally, given these two events, if each vertex of $V(H_2) \setminus V(F_2)$ is mapped randomly to V, an application of Lemma 29 combined with the above inequalities tells us that the probability that the final function is a homomorphism is at least

$$\frac{1}{2}\left(\frac{1}{\chi}\right)^f\left(1 - \frac{1+16\theta f}{\chi(F_2)-1}\right)^{v(H_2)-v(F_2)} = \frac{1}{2}\left(\frac{1}{\chi}\right)^f\left(\frac{\chi(F_2)-2-16\theta f}{\chi(F_2)-1}\right)^{v(H_2)-v(F_2)}.$$

The last factor can be bounded as follows:

$$\begin{aligned}
&\left(\frac{\chi(F_2)-2-16\theta f}{\chi(F_2)-1}\right)^{v(H_2)-v(F_2)} \\
&\qquad = \frac{(\chi(F_1)-1)^{v(H_2)-v(F_2)}}{(\chi(F_1)-1)^{v(H_2)-v(F_2)}}\left(\frac{\chi(F_2)-2-16\theta f}{\chi(F_2)-1}\right)^{v(H_2)-v(F_2)} \\
&\qquad = \left(\frac{1}{\chi(F_1)-1}\right)^{v(H_2)-v(F_2)}\left(\frac{(\chi(F_1)-1)(\chi(F_2)-2-16\theta f)}{\chi(F_2)-1}\right)^{v(H_2)-v(F_2)} \\
&\qquad \geqslant \left(\frac{1}{\chi(F_1)-1}\right)^{v(H_2)-k(F_2)}\left(\frac{(\chi(F_1)-1)(\chi(F_2)-2-16\theta f)}{\chi(F_2)-1}\right)^{v(H_2)-v(F_2)}
\end{aligned}$$

where in the inequality we used that $v(F_2) \geqslant k(F_2)$. Now, since $\chi(F_1) \geqslant 3$ and $\chi(F_1) + \chi(F_2) \geqslant 7$, we have

$$\frac{(\chi(F_1)-1)(\chi(F_2)-2-16\theta f)}{\chi(F_2)-1} \geqslant \frac{4}{3}(1-20\cdot\theta f).$$

Putting this all together and applying (3), we get that

$$\begin{aligned}
\rho_2(t_{\text{inj}}(H_2,G_2)-t_{\text{inj}}(H_2,G_2')) &\geqslant \frac{1}{2}\left(\frac{1}{\chi}\right)^f\left(\frac{4}{3}(1-20\cdot\theta f)\right)^{v(H_2)-v(F_2)} n^{-2}-O(n^{-3}) \\
&\geqslant \frac{1}{2}\left(\frac{3}{4\chi}\right)^f\left(\frac{4}{3}(1-20\cdot\theta f)\right)^{v(H_2)} n^{-2}-O(n^{-3})
\end{aligned}$$

which, by (2), is at least

$$\frac{1}{2}\left(\frac{3}{4\chi}\right)^f\left(\frac{4}{3}(1-20\cdot\theta f)\right)^{qv(H_1)} n^{-2}-O(n^{-3}).$$

Now, by (32), this is at least

$$\begin{aligned}
&\frac{1}{2}\left(\frac{3}{4\chi}\right)^f\left(\frac{4}{3}\cdot\frac{5}{6}(1+15\theta f)^{1/q}\right)^{qv(H_1)} n^{-2}-O(n^{-3}) \\
&\qquad\qquad = \frac{1}{2}\left(\frac{3}{4\chi}\right)^f\left(\frac{10}{9}\right)^{qv(H_1)}(1+15\theta f)^{v(H_1)} n^{-2}-O(n^{-3}).
\end{aligned}$$

Combining the upper bound that we have proven on $\rho_1\left(t_{\text{inj}}(H_1,G_1')-t_{\text{inj}}(H_1,G_1)\right)$ and the lower bound on $\rho_2(t_{\text{inj}}(H_2,G_2)-t_{\text{inj}}(H_2,G_2'))$, we get that

$$\begin{aligned}
&n^2(m(H_1,H_2;G_1)-m(H_1,H_2;G_1')) \\
&\geqslant \frac{1}{2}\left(\frac{3}{4\chi}\right)^f\left(\frac{10}{9}\right)^{qv(H_1)}(1+15\theta f)^{v(H_1)}-2e(H_1)(\chi(F_2)-1)\left(1+15\theta f\right)^{v(H_1)}-O(n^{-1})
\end{aligned}$$

which is positive for large n by (30). This contradicts the definition of G_1 and completes the proof. $\square$

Finally, we present the proof of Theorem 2.

Proof of Theorem 2. Given F_1, F_2 and q satisfying the hypotheses of the theorem, we select our parameters in the following order, subject to the given conditions:

- choose θ to satisfy (28), (31) and (32),
- choose ε so that (18), (20) and (24) hold,
- choose δ as in (15),
- choose β as in (12),
- choose ξ so that (5), (13), (16), (23) and (29) all hold,
- choose γ as in (8),
- choose τ to satisfy (7), (9) and (14),
- choose t_0 large enough so that (6), (21), (25) and (30) all hold.

Let $t_1, t_2 \geqslant t_0$ and let H_1 be a t_1-hairy F_1 and H_2 be a t_2-hairy F_2. We may assume that H_1 and H_2 have no singleton components. Let n_0 be large with respect to H_1 and H_2 and the parameters chosen in the previous paragraph and let G_1 be a graph on n vertices minimizing $m(H_1, H_2; G_1)$ and $G_2 = \overline{G_1}$. Without loss of generality, $|V_1| \leqslant |V_2|$. As a result of our parameter choices, all of the statements in Sections 3 and 4 hold. In particular, Lemma 33 implies that there is a partition $A_1, \ldots, A_{\chi(F_2)-1}$ of $V = V(G_1)$ such that G_2 contains no edge with endpoints in A_i for $1 \leqslant i \leqslant \chi(F_2) - 1$, and Lemma 25 (a) guarantees that all of the sets of the partition have approximately the same size, $\frac{n}{\chi(F_2)-1}$.

We assert that G_1 has no edges between A_i and A_j for $i \neq j$. To prove this, suppose that such an edge exists in G_1. If we move this edge from G_1 to G_2, it would destroy at least one injective homomorphism from H_1 to G_1 (since $|A_i| > v(H_1)$ for large n and H_1 has at least $t_1 \geqslant 1$ vertices of degree one). At the same time, this would not create any injective homomorphism from H_2 to G_2, since G_2 is still $(\chi(F_2) - 1)$-partite after adding such an edge to it. This contradicts our choice of G_1. Therefore, G_2 is a complete $(\chi(F_2) - 1)$-partite graph. In particular, $t(H_2, G_2) = 0$ and G_1 is a disjoint union of $\chi(F_2) - 1$ cliques.

Finally, we show that the cardinalities of any two sets A_i and A_j differ by at most one. Each homomorphism from H_1 to G_1 gives rise to a partition of $V(H_1)$ into at most $\chi(F_2)-1$ classes such that each partition class is a union of components of H_1 and all vertices of each class are mapped to the same component of G_1. We think of these partitions as being "unlabelled" in the sense that they contain information about which components of H_1 are mapped to the same component of G_1 but not about which component of G_1 they are mapped to. Given such a partition $\mathcal{P} = \{P_1, \ldots, P_{\chi(F_2)-1}\}$ (where we allow some of the sets P_j to be empty), we show that the number of injective homomorphisms of H_1 to G_1 giving rise to the partition $\mathcal{P}$ is minimized when the cardinalities any two of the sets A_i and A_j differ by at most one. For each $1 \leqslant i \leqslant \chi(F_2) - 1$, let us count the number of choices for the mapping of vertices in P_i given the mapping of the vertices in $\bigcup_{j=1}^{i-1} P_j$. Let T_i be the set of indices t such that there does not exist $1 \leqslant j \leqslant i-1$ such that the vertices of P_j are mapped to A_t. Then the number of choices for the mapping of P_i given that of P_j for all $j < i$ is

$$\sum_{t \in T_i} \frac{|A_t|!}{(|A_t| - |P_i|)!}. \tag{33}$$

For an integer $c \geqslant 2$, define $f_c : \mathbb{R} \to \mathbb{R}$ by $f_c(z) = z(z-1)\dots(z-c+1)$. Then f_c has $c-1$ distinct (integer) roots in the interval $[0, c-1]$. The derivative $f_c'(z)$ is a polynomial of degree $c-1$ with $c-1$ real roots which interlace the roots of f_c; in particular, its roots are also contained in the interval $[0, c-1]$. By similar logic, the roots of the second derivative f_c'' are in $[0, c-1]$ as well. From this, we see that f_c'' is positive on $[c, \infty)$, and so f_c is strictly convex on this set. Thus, by Jensen's Inequality, for any i such that $|P_i| \geqslant 2$, the sum in (33) is uniquely minimized when the cardinalities of the sets A_t for $t \in T_i$ are as similar as possible. Thus, the number of injective homomorphisms from H_1 to G_1 is minimized by taking G_2 to be a $(\chi(F_2) - 1)$-partite Turán graph. $\square$

5 Beating the Turán Colourings

In this section, we show that if (H_1, H_2) is a bonbon pair, then $e(H_1)$ cannot be "excessively large;" see Theorem 37 below. This result will then be used to derive Proposition 3. We will use the following result of [15] which was previously known as Tomescu's Graph Colouring Conjecture [37].

Theorem 34 (Fox, He and Manners [15]). *For $m \neq 3$, every connected m-chromatic graph on n vertices has at most $m!(m-1)^{n-m}$ proper m-colourings.*

The case $m = 2$ of the above theorem is trivial, as every connected bipartite graph has precisely two proper 2-colourings. It is also necessary to exclude the case $m = 3$, as an odd cycle of length $k \geqslant 5$ has more than $3!2^{k-3}$ proper 3-colourings. Knox and Mohar [25,26] established the cases $m = 4$ and $m = 5$ before the full conjecture was proven by Fox, He and Manners [15]. Note that every hairy K_m on n vertices has exactly $m!(m-1)^{n-m}$ proper m-colourings and so Theorem 34 is tight. We will use the following corollary of Theorem 34. Given a graph H, say that a vertex colouring $f : V(H) \to [\chi(H) - 1]$ of H is *nearly proper* if there is a unique edge of H whose endpoints are monochromatic.

Corollary 35. *If H is a graph such that $\chi(H) \neq 4$, then the number of nearly proper colourings of H is at most*

$$\mathrm{crit}(H) \cdot (\chi(H) - 2)! \cdot (\chi(H) - 2)^{v(H) - \chi(H) - k(H) + 1} \cdot (\chi(H) - 1)^{k(H)}.$$

Proof. Given an edge e of H, let H/e be the graph obtained by *contracting* e; i.e. by identifying the two endpoints of e and removing any multi-edges that arise. Let z_e be the vertex formed by contracting the edge e. The number of nearly proper colourings of H is equal to the number of ways to select

- a critical edge e of H,
- a proper $(\chi(H)-1)$-colouring of the component of H/e containing z_e and
- a proper $(\chi(H)-1)$-colouring of the components of H/e that do not contain z_e.

The number of choices in the first step is clearly $\mathrm{crit}(H)$.

Assuming that a critical edge e has been chosen, let p denote the number of vertices in the component of H/e containing z_e. Note that the chromatic number of this component is exactly $\chi(H)-1$ which, since $\chi(H) \neq 4$, is not equal to three. So, by Theorem 34, the number of choices in the second step is at most $(\chi(H)-1)!(\chi(H)-2)^{p-(\chi(H)-1)}$.

In the last step, for each component that does not contain z_e, there are at most $\chi(H)-1$ choices for the colour of an arbitrary "root" vertex of this component and then at most $\chi(H)-2$ choices for each subsequent vertex. Since H/e has $v(H)-1$ vertices, the number of vertices in the components of H/e that do not contain z_e is $v(H)-1-p$. Thus, the number of choices in the last step is at most

$$(\chi(H)-1)^{k(H)-1}(\chi(H)-2)^{v(H)-1-p-(k(H)-1)}.$$

Putting this all together, we get that the number of nearly proper colourings of H is at most

$$\mathrm{crit}(H)\cdot(\chi(H)-1)!\cdot(\chi(H)-2)^{v(H)-\chi(H)-k(H)+1}\cdot(\chi(H)-1)^{k(H)-1}$$

as desired. □

We also need the following simple bound on the number of nearly proper colourings in the case that $\chi(H)=4$. The proof is analogous to that of the previous corollary, except that, instead of Theorem 34, we use the (trivial) fact that every connected 3-chromatic graph on n vertices has at most $3\cdot 2^{n-1}$ proper 3-colourings.

Lemma 36. *If H is a 4-chromatic graph, then the number of nearly proper colourings of H is at most*

$$\mathrm{crit}(H)\cdot 3^{k(H)}\cdot 2^{v(H)-k(H)-1}.$$

Next, we use Corollary 35 and Lemma 36 to prove the following result which restricts the number of edges in a graph contained in a bonbon pair. In fact, it applies to a slightly more general class of graphs. Say that (H_1,H_2) is *multiplicity good* if

$$(\chi(H_2)-1)^{v(H_1)-k(H_1)}t(H_1,G)+(\chi(H_1)-1)^{v(H_2)-k(H_2)}t(H_2,\overline{G})\geqslant 1-o(1)$$

for all graphs G. Clearly, every bonbon pair is multiplicity good. Say that a graph H is *multiplicity good* if the pair (H,H) is.

Theorem 37. *Let H_1 and H_2 be graphs such that, if $\chi(H_2)\neq 4$, then*

$$e(H_1) > \frac{\mathrm{crit}(H_2)\cdot(\chi(H_2)-2)!\cdot(\chi(H_2)-2)^{v(H_2)-\chi(H_2)-k(H_2)+1}\cdot(\chi(H_1)-1)^{v(H_2)-k(H_2)}}{(\chi(H_2)-1)^{v(H_2)-k(H_2)}}$$

and, otherwise,

$$e(H_1) > \frac{\mathrm{crit}(H_2)\cdot 2^{v(H_2)-k(H_2)-1}(\chi(H_1)-1)^{v(H_2)-k(H_2)}}{3^{v(H_2)-k(H_2)}}.$$

Then (H_1, H_2) is not multiplicity good.

Proof. Suppose that H_1 and H_2 are graphs satisfying the hypotheses of the theorem. Let $\varepsilon > 0$ be very small and, for each $n \geqslant \chi(H_2) - 1$, let $G_{n,\varepsilon}$ be a graph on n vertices obtained from the complement of the Turán graph with n vertices and $\chi(H_2) - 1$ parts by jettisoning each edge of this graph with probability ε independently of all other such edges. Define

$$f_1(\varepsilon) := \lim_{n\to\infty} t(H_1, G_{n,\varepsilon}), \qquad f_2(\varepsilon) := \lim_{n\to\infty} t(H_2, \overline{G_{n,\varepsilon}})$$

and note that both of these limits exist with probability 1. Moreover, with probability one,

$$(\chi(H_2)-1)^{v(H_1)-k(H_1)}\cdot f_1(\varepsilon) = (1-\varepsilon)^{e(H_1)} = 1 - e(H_1)\varepsilon + O(\varepsilon^2)$$

where the asymptotics here (and throughout the proof) are as $\varepsilon \to 0$.

Let K be the number of nearly proper colourings of H_2. Then, with probability one,

$$(\chi(H_1)-1)^{v(H_2)-k(H_2)}\cdot f_2(\varepsilon) = (\chi(H_1)-1)^{v(H_2)-k(H_2)}\cdot\varepsilon K\left(\frac{1}{\chi(H_2)-1}\right)^{v(H_2)} + O(\varepsilon^2). \quad (34)$$

At this point, we divide the proof into cases.

Case 1. $\chi(H_2) \neq 4$.

By Corollary 35, the linear term of (34) (with respect to ε) is at most

$$(\chi(H_1)-1)^{v(H_2)-k(H_2)}\cdot\varepsilon\cdot\mathrm{crit}(H_2)\cdot(\chi(H_2)-2)! \cdot(\chi(H_2)-2)^{v(H_2)-\chi(H_2)-k(H_2)+1}\left(\frac{1}{\chi(H_2)-1}\right)^{v(H_2)-k(H_2)}$$

which is equal to

$$\frac{\varepsilon\cdot\mathrm{crit}(H_2)\cdot(\chi(H_2)-2)!\cdot(\chi(H_2)-2)^{v(H_2)-\chi(H_2)-k(H_2)+1}\cdot(\chi(H_1)-1)^{v(H_2)-k(H_2)}}{(\chi(H_2)-1)^{v(H_2)-k(H_2)}}.$$

Therefore, the lower bound on $e(H_1)$ assumed at the beginning of the proof implies that the linear term with respect to ε in $(\chi(H_2)-1)^{v(H_1)-k(H_1)}\cdot f_1(\varepsilon)+(\chi(H_1)-1)^{v(H_2)-k(H_2)}\cdot f_2(\varepsilon)$ has a negative coefficient. So, for ε sufficiently small, we have that

$$(\chi(H_2)-1)^{v(H_1)-k(H_1)}t(H_1, G_{n,\varepsilon}) + (\chi(H_1)-1)^{v(H_2)-k(H_2)}t(H_2, \overline{G_{n,\varepsilon}}) = 1 - \Omega(\varepsilon)$$

as $n \to \infty$ which implies that (H_1, H_2) is not multiplicity good.

Case 2. $\chi(H_2) = 4$.

In this case, by Lemma 36, the linear term of (34) with respect to ε is at most

$$(\chi(H_1)-1)^{v(H_2)-k(H_2)}\varepsilon \cdot \mathrm{crit}(H_2)\cdot 2^{v(H_2)-k(H_2)-1}\left(\frac{1}{3}\right)^{v(H_2)-k(H_2)}$$
$$=\frac{\varepsilon\cdot \mathrm{crit}(H_2)\cdot 2^{v(H_2)-k(H_2)-1}(\chi(H_1)-1)^{v(H_2)-k(H_2)}}{3^{v(H_2)-k(H_2)}}.$$

Thus, analogous to the previous case, by taking ε sufficiently close to zero, we get a certificate that (H_1, H_2) is not multiplicity good. □

Proof of Proposition 3. The proposition follows immediately from Theorem 37. □

6 Conclusion

We conclude by stating some open problems. A result of Goodman [20] implies that $c_1(K_3) = 1/4$ and so K_3 is multiplicity good. However, for odd n, $\hom_{\mathrm{inj}}(K_3, G) + \hom_{\mathrm{inj}}(K_3, \overline{G})$ is minimized among all n-vertex graphs by every n-vertex graph G which is $((n-1)/2)$-regular; therefore, K_3 is multiplicity good but not a bonbon. We are currently unaware of any non-3-colourable graph which is multiplicity good but not a bonbon, which leads us to the following question.

Question 38. Is it true that every non-3-colourable multiplicity good graph is a bonbon?

It would also be interesting to explore off-diagonal variants of the above question, such as the following.

Question 39. Suppose that (H_1, H_2) is multiplicity good such that H_1 and H_2 are non-bipartite and $\chi(H_1) + \chi(H_2) \geqslant 7$. Does it follow that (H_1, H_2) is a bonbon pair?

Currently, all of the known examples of bonbons contain vertices of degree one. It is unclear whether a bonbon of minimum degree at least two can exist. The analogous question for non-3-colourable multiplicity good graphs is also intriguing (the case of chromatic number three is settled, since K_3 is multiplicity good).

Question 40. Does there exist a bonbon H such that $\delta(H) \geqslant 2$?

Question 41. Does there exist a non-3-colourable multiplicity good graph H such that $\delta(H) \geqslant 2$?

Let us conclude with an observation that was shared with us by an anonymous referee. We claim that, if H is a bonbon, then H must contain a bridge. Indeed, consider the colouring obtained from a Turán colouring where the red edges form a complete $(\chi(H)-1)$-partite graph and change one of the red edges to blue. If H does not have a bridge, then this colouring has the same number of monochromatic copies of H as the Turán colouring and so H is not a bonbon. In particular, this implies that the answer to Question 40 would be "no" if the condition $\delta(H) \geqslant 2$ was replaced with the stronger condition that H is 2-edge connected. However, this example does not seem to preclude the existence of 2-edge connected multiplicity good graphs.

Question 42. Does there exist a non-3-colourable multiplicity good graph H that is 2-edge connected?

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