Generalized Spectral Characterization of Signed Trees

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Abstract

Let T be a tree on n vertices with an irreducible characteristic polynomial $\phi(x)$ over $\mathbb{Q}$. Let $\Delta(T)$ be the discriminant of $\phi(x)$. It is proved that if $2^{-\lfloor \frac{n}{2} \rfloor}\sqrt{\Delta(T)}$ (which is always an integer) is odd and square free, then every signed tree with underlying graph T is determined by its generalized spectrum.

Mathematics Subject Classifications: 05C50

1 Introduction

It is well known that the spectra of graphs encode a lot of combinatorial information about the given graphs. A major unsolved question in spectral graph theory is: "What kinds of graphs are determined (up to isomorphism) by their spectrum (DS for short)?". The problem originates from chemistry and was raised in 1956 by Günthard and Primas [2], which relates Hückel's theory in chemistry to graph spectra. The above problem is also closely related to a famous problem of Kac [15]: "Can one hear the shape of a drum?" Fisher [14] modelled the drum by a graph, and the frequency of the sound was characterized by the eigenvalues of the graph. Hence, the two problems are essentially the same.

It was commonly believed that every graph is DS until the first counterexample (a pair of cospectral but non-isomorphic trees) was found by Collatz and Sinogowitz [3] in 1957. Another famous result on cospectral graphs was given by Schwenk [19], which states that almost every tree is not DS. For more constructions of cospectral graphs, see, e.g., [13, 16, 20]. However, it turns out that showing a given graph to be DS is generally very hard and challenging. Up to now, only a few graphs with very special structures are known to be DS. We refer the reader to [10, 11] for more background and known results.

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In recent years, Wang and Xu [22] and Wang [23, 24] considered a variant of the above problem. For a simple graph G, they defined the *generalized spectrum* of G as the spectrum of G together with that of its complement $\bar{G}$. A graph G is said to be *determined by its generalized spectrum* (DGS for short), if any graph having the same generalized spectrum as G is necessarily isomorphic to G.

Let G be a graph on n vertices with adjacency matrix $A = A(G)$. The *walk-matrix* of G is defined as

$$W(G) = [e, Ae, \dots, A^{n-1}e],$$

where e is the all-one vector. Wang [23, 24] proved the following theorem.

Theorem 1 ([23, 24]). *If $2^{-\lfloor \frac{n}{2} \rfloor} \det(W)$ is odd and square-free, then G is DGS.*

The problem of spectral determination of ordinary graphs naturally extends to signed graphs. A *signed graph* $\Gamma = (G, \sigma)$ is a graph obtained from a simple graph $G = (V, E)$ by assigning a sign $+1$ or -1 to every edge according to a mapping $\sigma : E \to \{+1, -1\}$. We call G the *underlying graph* of Γ and σ the *sign function (signature)* of Γ. Note that an ordinary graph can be regard as a signed graph, in which every edge has been assigned a positive sign $+1$. We call Γ a *signed bipartite graph*, if its underlying graph is bipartite.

The *adjacency matrix* of Γ is an $n \times n$ symmetric $(0, \pm 1)$-matrix defined as $A(\Gamma) = (a_{ij})$, where $a_{ij} = \sigma(\{i, j\})$ if $\{i, j\}$ is an edge in E, and $a_{ij} = 0$ otherwise. The *characteristic polynomial* of Γ is defined as the characteristic polynomial of $A(\Gamma)$, i.e., $\phi(\Gamma; x) = \det(xI - A(\Gamma))$, where I is the identity matrix. Let Γ and Γ' be two signed graphs with adjacency matrices $A(\Gamma)$ and $A(\Gamma')$, respectively. Γ and Γ' are called *generalized cospectral* if

$$\det(xI - A(\Gamma)) = \det(xI - A(\Gamma')) \text{ and } \det(xI - (J - I - A(\Gamma))) = \det(xI - (J - I - A(\Gamma'))),$$

where J is the all-one matrix and $J - I - A(\Gamma)$ formally denotes the 'complement' of Γ (it is indeed the complement of Γ if Γ is a simple graph, we remark however, the complement of a signed graph usually cannot be defined in a satisfactory way; see **Problem** 3.29 in [1]). A signed graph Γ is said to be *determined by the generalized spectrum* (DGS for short), if any signed graph that is generalized cospectral with Γ is isomorphic to Γ.

This paper is a continuation along this line of research for signed graphs in the flavour of Theorem 1. Let T be a tree and let $\Delta(T)$ denote its discriminant, see Section 4 for the definition. The main result of the paper is the following theorem.

Theorem 2. *Let T be a tree on n vertices and suppose that its characteristic polynomial is irreducible over $\mathbb{Q}$. If $2^{-\lfloor \frac{n}{2} \rfloor} \sqrt{\Delta(T)}$ (which is always an integer) is odd and square-free, then every signed tree with underlying graph T is DGS.*

As an immediately consequence of Theorem 2, we have

Corollary 3. *Let T and T' be two non-isomorphic trees which share the same irreducible characteristic polynomial. Suppose that $2^{-\lfloor \frac{n}{2} \rfloor} \sqrt{\Delta(T)}$ is odd and square free. Then no two signed trees with underlying graphs T and T' respectively are generalized cospectral.*

Theorem 2 shows that the signing of the tree is DGS while the underlying graph itself is prescribed, i.e., whenever the underlying tree T with n vertices satisfies a simple arithmetic condition, then all the 2^{n-1} signed trees (including T itself) whose underlying tree is T are DGS. Thus, the DGS property of all these signed trees only depends on the underlying graph T. This is somewhat unexpected, since given a pair of trees T and T', it seems time consuming even to check whether there exist two signed trees with underlying graphs T and T' respectively that are generalized cospectral; see Example 1.

Example 4. Let T and T' be two cospectral non-isomorphic trees on 14 vertices in Figure 1. Then they have the same irreducible characteristic polynomial

$$\phi(T) = \phi(T') = -1 + 16x^2 - 79x^4 + 157x^6 - 143x^8 + 63x^{10} - 13x^{12} + x^{14}.$$

It can be easily computed that $2^{-7}\sqrt{\Delta(T)} = 2^{-7}\sqrt{\Delta(T')} = 5 \times 11 \times 4754599$, which is odd and square-free. Thus, according to Theorem 2, every signed tree with underlying graph T (resp. T') is DGS. In particular, no two signed trees with underlying graphs T and T', respectively, are generalized cospectral.

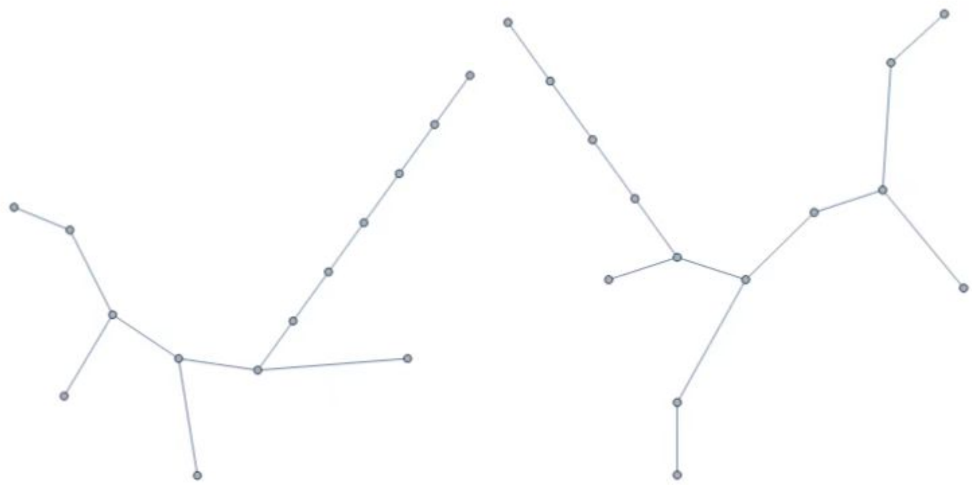

Figure 1: A pair of cospectral non-isomorphic trees on 14 vertices.

We mention that Theorem 2 is the best possible in the sense that it is no longer true if $2^{-\lfloor \frac{n}{2} \rfloor}\sqrt{\Delta(T)}$ has a multiple odd prime factor. Moreover, the irreducibility assumption of the characteristic polynomial of the tree is essential which cannot be removed; see Remarks 18 and 19 in Section 4.

The rest of the paper is organized as follows. In Section 2, we give some preliminary results that will be needed in the proof of Theorem 2. In Section 3, we give a structure theorem, which plays a key role in the paper. In Section 4, we present the proof of Theorem 2. Conclusions and future work are given in Section 5.

2 Preliminaries

For the convenience of the reader, we give some preliminary results that will be needed later in the paper. For more results in spectral graphs theory, we refer to [5, 9].

Let $\Gamma = (G, \sigma)$ be a signed graph with the underlying $G = (V, E)$. Let U be a subset of V such that $(U, V \setminus U)$ is a partition of V. A *switching* w.r.t. U (or $V \setminus U$) is an operation that changes all the signs of edges between U and $V \setminus U$, while keeps the others unchanged. Two signed graphs Γ and Γ' are *switching-equivalent* if Γ' can be obtained from Γ by a switching operation, or equivalently, there exists a diagonal matrix D with all diagonal entry ± 1 such that $DA(\Gamma)D = A(\Gamma')$. A signed graph is *balanced* if every cycle contains an even number of edges with sign -1. It is well-known that a signed graph is balanced if and only it is switching equivalent to an unsigned graph; see [6, 7].

A polynomial $f(x) \in \mathbb{Q}[x]$ is *irreducible* if it cannot be factored into two polynomials with rational coefficients of lower degree. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial with degree n and α be one of its root. Then $\mathbb{Q}(\alpha) = \{c_0 + c_1\alpha + \cdots + c_{n-1}\alpha^{n-1} : c_i \in \mathbb{Q},\ 0 \leqslant i \leqslant n-1\}$ is a *number field* which is isomorphic to $\mathbb{Q}[x]/(f(x))$ and is obtained by adding α to $\mathbb{Q}$; see e.g. [12].

An *orthogonal* matrix Q is a square matrix such that $Q^{\mathrm{T}}Q = I_n$. It is called *rational* if every entry of Q is a rational number, and *regular* if each row sum of Q is 1, i.e., $Qe = e$, where e is the all-one column vector. Denote by $\mathrm{RO}_n(\mathbb{Q})$ the set of all n by n regular orthogonal matrices with rational entries.

In 2006, Wang and Xu [22] initiated the study of the generalized spectral characterization of graphs. For two generalized cospectral graphs G and H, they obtained the following result (see also [8]), which plays a fundamental role in their method.

Theorem 5 ([8],[22])**.** *Let G be a graph. Then there exists a graph H such that G and H are generalized cospectral if and only if there exists a regular orthogonal matrix Q such that*

$$Q^{\mathrm{T}}A(G)Q = A(H). \tag{1}$$

Moreover, if $\det W(G) \neq 0$, then $Q \in \mathrm{RO}_n(\mathbb{Q})$ is unique and $Q = W(G)W^{-1}(H)$.

A graph G with $\det W(G) \neq 0$ is called *controllable* (see [17]). Denote by $\mathcal{G}_n$ the set of all controllable graphs on n vertices. For a graph $G \in \mathcal{G}_n$, define

$$\mathcal{Q}(G) := \{Q \in \mathrm{RO}_n(\mathbb{Q}) :\ Q^{\mathrm{T}}A(G)Q = A(H) \text{ for some graph } H\}.$$

Then according to Theorem 5, it is easy to obtain the following

Theorem 6 ([22])**.** *Let G be a controllable graph. Then G is DGS if and only if the set $\mathcal{Q}(G)$ contains only permutation matrices.*

The above theorems extend naturally to signed graphs. By Theorem 6, finding out the possible structure of all $Q \in \mathcal{Q}(G)$ is a key to determine whether a (signed) graph G is DGS.

Notations: We use e_n (or e if there is no confusion arises) to denote an n-dimensional column all-one vector, and J the all-one matrix. For a vector $\alpha = (a_1, a_2, \ldots, a_n)^{\mathrm{T}} \in \mathbb{R}^n$, we use $||\alpha||_2 = (a_1^2 + a_2^2 + \cdots + a_n^2)^{1/2}$ to denote the Euclidean norm of α.

3 A structure theorem for Q

The key observation of this paper is the following theorem which shows that for two generalized cospectral signed bipartite graphs with a common irreducible characteristic polynomial, the regular rational orthogonal matrix carried out the similarity of their adjacency matrices has a special structure.

Theorem 7. *Let Γ and $\tilde{\Gamma}$ be two generalized cospectral signed bipartite graphs with a common irreducible characteristic polynomial $\phi(x)$ over $\mathbb{Q}$. Suppose that the adjacency matrices of Γ and $\tilde{\Gamma}$ are given as follows, respectively:*

$$A = A(\Gamma) = \begin{bmatrix} O & M \\ M^{\mathrm{T}} & O \end{bmatrix}, \ \tilde{A} = A(\tilde{\Gamma}) = \begin{bmatrix} O & \tilde{M} \\ \tilde{M}^{\mathrm{T}} & O \end{bmatrix}.$$

Then there exists a regular orthogonal matrix Q such that $Q^{\mathrm{T}}AQ = \tilde{A}$, where

$$Q = \begin{bmatrix} Q_1 & O \\ O & Q_2 \end{bmatrix} \text{ or } Q = \begin{bmatrix} O & Q_1 \\ Q_2 & O \end{bmatrix}$$

with Q_1 and Q_2 being regular rational orthogonal matrices, respectively.

Corollary 8. *The matrix Q in Theorem 7 is the unique rational orthogonal matrix such that $Q^{\mathrm{T}}AQ = \tilde{A}$.*

Proof. The irreducibility assumption of the characteristic polynomial of A implies that Γ is controllable. Then the corollary follows immediately from Theorem 5. $\square$

To give the proof of Theorem 7, we need several lemmas below.

Lemma 9. *Let Γ and $\tilde{\Gamma}$ be two generalized cospectral signed graphs with adjacency matrices A and $\tilde{A}$, respectively. Suppose that λ is not an eigenvalue of A. Then $e^{\mathrm{T}}(\lambda I - A)^{-1}e = e^{\mathrm{T}}(\lambda I - \tilde{A})^{-1}e$.*

Proof. It can be easily computed that

$$\begin{aligned} & \det(\lambda I - (A + tJ)) \\ = \ & \det(\lambda I - A)\det(I - t(\lambda I - A)^{-1}ee^{\mathrm{T}}) \\ = \ & \det(\lambda I - A)(1 - te^{\mathrm{T}}(\lambda I - A)^{-1}e). \end{aligned}$$

Similarly, $\det(\lambda I - (\tilde{A} + tJ)) = \det(\lambda I - \tilde{A})(1 - te^{\mathrm{T}}(\lambda I - \tilde{A})^{-1}e)$. Thus, the lemma follows. $\square$

Lemma 10 ([18])**.** $(\lambda I - A)^{-1} = \sum_{i=1}^{n} \frac{\xi_i \xi_i^{\mathrm{T}}}{\lambda - \lambda_i}$, *where* ξ_i*'s are orthonormal eigenvectors of* A *associated with* λ_i*, for* $1 \leqslant i \leqslant n$.

Lemma 11 ([21])**.** *Let* $A = (a_{ij})$ *be a symmetric integral matrix with an irreducible characteristic polynomial* $\phi(x)$*. Let* $\lambda_1, \ldots, \lambda_n$ *be the distinct eigenvalues of* A*. Then there exist polynomials* $\phi_i(x) \in \mathbb{Q}[x]$ *with* $\deg \phi_i < n$ *such that the eigenvectors* ξ_i *of* A *associated with* λ_i *can be expressed as*

$$\xi_i = (\phi_1(\lambda_i), \phi_2(\lambda_i), \ldots, \phi_n(\lambda_i))^{\mathrm{T}}$$

for $1 \leqslant i \leqslant n$.

Proof. Let λ_1 be an eigenvalue of A with corresponding eigenvector ξ_1. Consider the linear system of equations $(\lambda_1 I - A)\xi_1 = 0$. By Gaussian elimination, there exist $x_i \in \mathbb{Q}(\lambda_1)$ such that $\xi_1 = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$. Note $\mathbb{Q}(\lambda_1)$ is a number field. There exist polynomials $\phi_i(x) \in \mathbb{Q}[x]$ with $\deg \phi_i < n$ such that $x_i = \phi_i(\lambda_1)$.

By the k-th equation of $(\lambda_1 I - A)\xi_1 = 0$, we have $\psi(\lambda_1) := \sum_{j=1}^{n} a_{k,j}\phi_j(\lambda_1) - \lambda_1 \phi_j(\lambda_1) = 0$, for $1 \leqslant k \leqslant n$. Note $\psi(x) \in \mathbb{Q}[x]$ and $\psi(\lambda_1) = 0$. By the irreducibility of $\phi(x)$, we have $\phi(x)$ divides $\psi(x)$. Thus, $\psi(\lambda_i) = 0$ for $1 \leqslant i \leqslant n$, and $\xi_i = (\phi_1(\lambda_i), \phi_2(\lambda_i), \ldots, \phi_n(\lambda_i))^{\mathrm{T}}$ is an eigenvector associated with λ_i. □

Next, we collect some simple facts about the relationships of eigenvalues/eigenvectors between the adjacency matrix A of a signed bipartite graph Γ and its bipartite-adjacency matrix M.

Lemma 12 ([5])**.** *Let* Γ *be a signed bipartite graph with an irreducible characteristic polynomial over* $\mathbb{Q}$*. Let the adjacency matrix of* Γ *be* $A = A(\Gamma) = \begin{bmatrix} O & M \\ M^{\mathrm{T}} & O \end{bmatrix}$*. Suppose that* $\begin{bmatrix} u \\ v \end{bmatrix}$ *is an eigenvector of* A *associated with an eigenvalue* λ*. Then*

1. λ^2 *is an eigenvalue of* MM^{T} *and* $M^{\mathrm{T}}M$ *with corresponding eigenvectors* u *and* v*, respectively;*

2. u *and* v *have the same length, i.e.,* $||u||_2 = ||v||_2$*;*

3. $\begin{bmatrix} u \\ -v \end{bmatrix}$ *is an eigenvector of* A *associated with eigenvalue* $-\lambda$*;*

4. *The characteristic polynomials of* MM^{T} *and* $M^{\mathrm{T}}M$ *are irreducible over* $\mathbb{Q}$.

Proof. We present a proof for completeness. Note that the characteristic polynomial $\phi(x)$ of A is irreducible, it follows that zero can never be an eigenvalue of A, and hence M must be a square matrix of order $m := n/2$.

Let $\lambda \neq 0$ be any eigenvalue of A with corresponding eigenvector $\begin{bmatrix} u \\ v \end{bmatrix}$. Then

$$A \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} Mv \\ M^{\mathrm{T}}u \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix} \Longleftrightarrow \begin{cases} Mv = \lambda u, \\ M^{\mathrm{T}}u = \lambda v. \end{cases} \tag{2}$$

Thus, we have $u \neq 0$ and $v \neq 0$, for otherwise we would have $u = v = 0$, since $\lambda \neq 0$. It follows that

$$MM^{\mathrm{T}}u = \lambda^2 u,\ M^{\mathrm{T}}Mv = \lambda^2 v.$$

It follows from $Mv = \lambda u$ that $u^{\mathrm{T}}Mv = \lambda u^{\mathrm{T}}u$. By $M^{\mathrm{T}}u = \lambda v$ we get $v^{\mathrm{T}}M^{\mathrm{T}}u = \lambda v^{\mathrm{T}}v$. Note $u^{\mathrm{T}}Mv = (u^{\mathrm{T}}Mv)^{\mathrm{T}} = v^{\mathrm{T}}M^{\mathrm{T}}u$. It follows that $\lambda u^{\mathrm{T}}u = \lambda v^{\mathrm{T}}v$, and hence $u^{\mathrm{T}}u = v^{\mathrm{T}}v$ since $\lambda \neq 0$.

Note that

$$A \begin{bmatrix} u \\ -v \end{bmatrix} = \begin{bmatrix} -Mv \\ M^{\mathrm{T}}u \end{bmatrix} = -\lambda \begin{bmatrix} u \\ -v \end{bmatrix}.$$

Hence, $\begin{bmatrix} u \\ -v \end{bmatrix}$ is an eigenvector of A associated with eigenvalue $-\lambda$. Since the characteristic polynomial of A is irreducible, the set of all the eigenvalues of A can be written as $\{\lambda_1, \lambda_2, \ldots, \lambda_m, -\lambda_1, -\lambda_2, \ldots, \lambda_m\}$.

Hence, the set of all the eigenvalues of MM^{T} (or $M^{\mathrm{T}}M$) can be written as $\{\lambda_1^2, \lambda_2^2, \ldots, \lambda_m^2\}$. Since $\phi(A;x) = (x^2-\lambda_1^2)\cdots(x^2-\lambda_m^2)$ is irreducible over $\mathbb{Q}$, $\phi(MM^{\mathrm{T}};x) = \phi(M^{\mathrm{T}}M;x) = (x-\lambda_1^2)\cdots(x-\lambda_m^2)$ is also irreducible over $\mathbb{Q}$. □

Proof of Theorem 7. Set $m := n/2$. By Lemma 12, let $\lambda_1, \lambda_2, \ldots, \lambda_m, -\lambda_1, -\lambda_2, \ldots, -\lambda_m$ be the eigenvalues of A and $\tilde{A}$ with corresponding normalized eigenvectors

$$\frac{1}{\sqrt{2}} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}} \begin{bmatrix} u_m \\ v_m \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} u_1 \\ -v_1 \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}} \begin{bmatrix} u_m \\ -v_m \end{bmatrix}, \tag{3}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{u}_1 \\ \tilde{v}_1 \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{u}_m \\ \tilde{v}_m \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{u}_1 \\ -\tilde{v}_1 \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{u}_m \\ -\tilde{v}_m \end{bmatrix}, \tag{4}$$

respectively, where $u_i, \tilde{u}_i, v_i, \tilde{v}_i \in \mathbb{R}^n$ are m-dimensional unit vectors.

By Lemma 9, we have $e^{\mathrm{T}}(xI - A)^{-1}e = e^{\mathrm{T}}(xI - \tilde{A})^{-1}e$. It follows from Lemma 10 that

$$\sum_{i=1}^{m} \frac{(\frac{1}{\sqrt{2}}e_{2m}^{\mathrm{T}} \begin{bmatrix} u_i \\ v_i \end{bmatrix})^2}{x - \lambda_i} + \sum_{i=1}^{m} \frac{(\frac{1}{\sqrt{2}}e_{2m}^{\mathrm{T}} \begin{bmatrix} u_i \\ -v_i \end{bmatrix})^2}{x + \lambda_i} = \sum_{i=1}^{m} \frac{(\frac{1}{\sqrt{2}}e_{2m}^{\mathrm{T}} \begin{bmatrix} \tilde{u}_i \\ \tilde{v}_i \end{bmatrix})^2}{x - \lambda_i} + \sum_{i=1}^{m} \frac{(\frac{1}{\sqrt{2}}e_{2m}^{\mathrm{T}} \begin{bmatrix} \tilde{u}_i \\ -\tilde{v}_i \end{bmatrix})^2}{x + \lambda_i}. \tag{5}$$

Hence, we have that for each $1 \leqslant i \leqslant m$,

$$\begin{cases} (e_m^{\mathrm{T}}u_i + e_m^{\mathrm{T}}v_i)^2 = & (e_m^{\mathrm{T}}\tilde{u}_i + e_m^{\mathrm{T}}\tilde{v}_i)^2, \\ (e_m^{\mathrm{T}}u_i - e_m^{\mathrm{T}}v_i)^2 = & (e_m^{\mathrm{T}}\tilde{u}_i - e_m^{\mathrm{T}}\tilde{v}_i)^2. \end{cases} \tag{6}$$

For a fixed i, we distinguish the following two cases:

Case 1. $e_m^{\mathrm{T}}u_i+e_m^{\mathrm{T}}v_i$ and $e_m^{\mathrm{T}}\tilde{u}_i+e_m^{\mathrm{T}}\tilde{v}_i$ have the same sign (or opposite sign), and $e_m^{\mathrm{T}}u_i-e_m^{\mathrm{T}}v_i$ and $e_m^{\mathrm{T}}\tilde{u}_i-e_m^{\mathrm{T}}\tilde{v}_i$ have the same sign (or opposite sign). It follows from (6) that

$$\begin{cases} e_m^{\mathrm{T}}u_i+e_m^{\mathrm{T}}v_i = & e_m^{\mathrm{T}}\tilde{u}_i+e_m^{\mathrm{T}}\tilde{v}_i, \\ e_m^{\mathrm{T}}u_i-e_m^{\mathrm{T}}v_i = & e_m^{\mathrm{T}}\tilde{u}_i-e_m^{\mathrm{T}}\tilde{v}_i, \end{cases} \text{or} \begin{cases} e_m^{\mathrm{T}}u_i+e_m^{\mathrm{T}}v_i = & -(e_m^{\mathrm{T}}\tilde{u}_i+e_m^{\mathrm{T}}\tilde{v}_i), \\ e_m^{\mathrm{T}}u_i-e_m^{\mathrm{T}}v_i = & -(e_m^{\mathrm{T}}\tilde{u}_i-e_m^{\mathrm{T}}\tilde{v}_i), \end{cases}$$

which implies that either i) $e_m^{\mathrm{T}}u_i = e_m^{\mathrm{T}}\tilde{u}_i$ and $e_m^{\mathrm{T}}v_i = e_m^{\mathrm{T}}\tilde{v}_i$; or ii) $e_m^{\mathrm{T}}u_i = -e_m^{\mathrm{T}}\tilde{u}_i$ and $e_m^{\mathrm{T}}v_i = -e_m^{\mathrm{T}}\tilde{v}_i$.

Case 2. $e_m^{\mathrm{T}}u_i+e_m^{\mathrm{T}}v_i$ and $e_m^{\mathrm{T}}\tilde{u}_i+e_m^{\mathrm{T}}\tilde{v}_i$ have the same sign (or opposite sign), and $e_m^{\mathrm{T}}u_i-e_m^{\mathrm{T}}v_i$ and $e_m^{\mathrm{T}}\tilde{u}_i-e_m^{\mathrm{T}}\tilde{v}_i$ have the opposite sign (or same sign). Then

$$\begin{cases} e_m^{\mathrm{T}}u_i+e_m^{\mathrm{T}}v_i = & e_m^{\mathrm{T}}\tilde{u}_i+e_m^{\mathrm{T}}\tilde{v}_i, \\ e_m^{\mathrm{T}}u_i-e_m^{\mathrm{T}}v_i = & -(e_m^{\mathrm{T}}\tilde{u}_i-e_m^{\mathrm{T}}\tilde{v}_i), \end{cases} \text{or} \begin{cases} e_m^{\mathrm{T}}u_i+e_m^{\mathrm{T}}v_i = & -(e_m^{\mathrm{T}}\tilde{u}_i+e_m^{\mathrm{T}}\tilde{v}_i), \\ e_m^{\mathrm{T}}u_i-e_m^{\mathrm{T}}v_i = & e_m^{\mathrm{T}}\tilde{u}_i-e_m^{\mathrm{T}}\tilde{v}_i, \end{cases}$$

which implies that either i) $e_m^{\mathrm{T}}u_i = e_m^{\mathrm{T}}\tilde{v}_i$ and $e_m^{\mathrm{T}}v_i = e_m^{\mathrm{T}}\tilde{u}_i$; or ii) $e_m^{\mathrm{T}}u_i = -e_m^{\mathrm{T}}\tilde{v}_i$ and $e_m^{\mathrm{T}}v_i = -e_m^{\mathrm{T}}\tilde{u}_i$.

Thus, for a fixed i, we may assume that either $e_m^{\mathrm{T}}u_i = \tau_i e_m^{\mathrm{T}}\tilde{u}_i$ and $e_m^{\mathrm{T}}v_i = \tau_i e_m^{\mathrm{T}}\tilde{v}_i$ or $e_m^{\mathrm{T}}u_i = \sigma_i e_m^{\mathrm{T}}\tilde{v}_i$ and $e_m^{\mathrm{T}}v_i = \sigma_i e_m^{\mathrm{T}}\tilde{u}_i$, where $\tau_i, \sigma_i \in \{1,-1\}$. Next, we show that uniformly, either $e_m^{\mathrm{T}}u_i = \tau_i e_m^{\mathrm{T}}\tilde{u}_i$ and $e_m^{\mathrm{T}}v_i = \tau_i e_m^{\mathrm{T}}\tilde{v}_i$ for all $1 \leqslant i \leqslant m$ or $e_m^{\mathrm{T}}u_i = \sigma_i e_m^{\mathrm{T}}\tilde{v}_i$ and $e_m^{\mathrm{T}}v_i = \sigma_i e_m^{\mathrm{T}}\tilde{u}_i$ for all $1 \leqslant i \leqslant m$. This is the key technical part of the proof, which highly depends on the irreducibility assumption of ϕ.

According to Lemma 11, the eigenvectors of MM^{T} associated with eigenvalues λ_i^2 can be expressed as $\xi_i = (\phi_1(\lambda_i), \phi_2(\lambda_i), \dots, \phi_m(\lambda_i))^{\mathrm{T}}$, where $\phi_j(x) \in \mathbb{Q}[x]$ with $\deg \phi_j < n$.

By Lemma 12, u_i is an eigenvector of MM^{T} associated with λ_i^2. Note u_i is a unit vector. It follows that u_i and $\xi_i/||\xi_i||_2$ differ by at most a sign, i.e., there exists a $\epsilon_i \in \{1,-1\}$ such that $u_i = \epsilon_i \frac{\xi_i}{||\xi_i||_2}$, and

$$\begin{aligned} v_i &= \frac{1}{\lambda_i} M^{\mathrm{T}} u_i \\ &= \frac{\epsilon_i}{\lambda_i} M^{\mathrm{T}} (\phi_1(\lambda_i), \phi_2(\lambda_i), \dots, \phi_m(\lambda_i))^{\mathrm{T}} / ||\xi_i||_2 \\ &= \epsilon_i (\varphi_1(\lambda_i), \varphi_2(\lambda_i), \dots, \varphi_m(\lambda_i))^{\mathrm{T}} / ||\xi_i||_2, \end{aligned}$$

for some $\varphi_j(x) \in \mathbb{Q}[x]$ with degree less than n, for $1 \leqslant j \leqslant m$. The last equality follows since the entries of the vector $\frac{1}{\lambda_i} M^{\mathrm{T}} (\phi_1(\lambda_i), \phi_2(\lambda_i), \dots, \phi_m(\lambda_i))^{\mathrm{T}}$ belong to $\mathbb{Q}(\lambda_i)$, which is a number field. Further note that $||u_i||_2 = ||v_i||_2 = 1$, we have $\varphi_1(\lambda_i)^2 + \varphi_2(\lambda_i)^2 + \cdots + \varphi_m(\lambda_i)^2 = ||\xi_i||^2$, for $1 \leqslant i \leqslant m$.

The above discussions apply similarly to the signed bipartite graph $\tilde{\Gamma}$ with adjacency matrix $\tilde{A}$. In this case, one obtains that $\tilde{u}_i = \tilde{\epsilon_i} \frac{\tilde{\xi_i}}{||\tilde{\xi_i}||_2}$ for $\tilde{\epsilon_i} \in \{1,-1\}$, where $\tilde{\xi_i} = (\tilde{\phi_1}(\lambda_i), \tilde{\phi_2}(\lambda_i), \dots, \tilde{\phi}_m(\lambda_i))^{\mathrm{T}}$, $\tilde{\phi_j}(x) \in \mathbb{Q}[x]$ with $\deg \tilde{\phi_j} < n$. Moreover, $\tilde{v}_i = \tilde{\epsilon}_i (\tilde{\varphi}_1(\lambda_i), \tilde{\varphi}_2(\lambda_i), \dots, \tilde{\varphi}_m(\lambda_i))^{\mathrm{T}} / ||\tilde{\xi_i}||_2$ with $\tilde{\varphi}_j(x) \in \mathbb{Q}[x]$ with degree less than n, and $\tilde{\varphi}_1(\lambda_i)^2 + \tilde{\varphi}_2(\lambda_i)^2 + \cdots + \tilde{\varphi}_m(\lambda_i)^2 = ||\tilde{\xi_i}||^2$.

Claim 13. *If $e_m^{\mathrm{T}}u_1 = \tau_1 e_m^{\mathrm{T}}\tilde{u}_1$ and $e_m^{\mathrm{T}}v_1 = \tau_1 e_m^{\mathrm{T}}\tilde{v}_1$ with $\tau_1 \in \{1, -1\}$, then $e_m^{\mathrm{T}}u_i = \tau_i e_m^{\mathrm{T}}\tilde{u}_i$ and $e_m^{\mathrm{T}}v_i = \tau_i e_m^{\mathrm{T}}\tilde{v}_i$ for some $\tau_i \in \{1, -1\}$, for all $2 \leqslant i \leqslant m$.*

Proof. Actually, it follows from $e_m^{\mathrm{T}}u_1 = \tau_1 e_m^{\mathrm{T}}\tilde{u}_1$ that

$$\epsilon_1 \frac{\sum_{j=1}^m \phi_j(\lambda_1)}{\sqrt{\sum_{j=1}^m \phi_j^2(\lambda_1)}} = \tau_1 \tilde{\epsilon_1} \frac{\sum_{j=1}^m \tilde{\phi}_j(\lambda_1)}{\sqrt{\sum_{j=1}^m \tilde{\phi}_j^2(\lambda_1)}}. \tag{7}$$

Taking squares on both sides of (7), it follows that

$$\Phi(\lambda_1) := (\sum_{j=1}^m \phi_j(\lambda_1))^2 \sum_{j=1}^m \tilde{\phi}_j^2(\lambda_1) - (\sum_{j=1}^m \tilde{\phi}_j(\lambda_1))^2 \sum_{j=1}^m \phi_j^2(\lambda_1) = 0.$$

Note that $\phi(x)$ is irreducible and $\Phi(x) \in \mathbb{Q}[x]$. It follows that $\phi(x) \mid \Phi(x)$. Hence $\Phi(\lambda_i) = 0$ and $e_m^{\mathrm{T}}u_i = \tau_i e_m^{\mathrm{T}}\tilde{u}_i$ for some $\tau_i \in \{1, -1\}$, for $2 \leqslant i \leqslant m$. Similarly, we have $e_m^{\mathrm{T}}v_i = \tilde{\tau}_i e_m^{\mathrm{T}}\tilde{v}_i$ for some $\tilde{\tau}_i \in \{1, -1\}$, for $2 \leqslant i \leqslant m$. Next, we show that τ_i and $\tilde{\tau}_i$ coincide, i.e., $\tau_i = \tilde{\tau}_i = \pm 1$, for all $2 \leqslant i \leqslant m$.

In fact, it follows from $e_m^{\mathrm{T}}v_1 = \tau_1 e_m^{\mathrm{T}}\tilde{v}_1$ that

$$\epsilon_1 \frac{\sum_{j=1}^m \varphi_j(\lambda_1)}{\sqrt{\sum_{j=1}^m \phi_j^2(\lambda_1)}} = \tau_1 \tilde{\epsilon_1} \frac{\sum_{j=1}^m \tilde{\varphi}_j(\lambda_1)}{\sqrt{\sum_{j=1}^m \tilde{\phi}_j^2(\lambda_1)}}. \tag{8}$$

It is easy to see that all the numerators in Eqs. (7) and (8) are non-zero. For example, if $\sum_{j=1}^m \phi_j(\lambda_1) = 0$, then $\sum_{j=1}^m \phi_j(\lambda_i) = 0$ for $1 \leqslant i \leqslant m$ by the irreducibility of ϕ. That is, $e_m^{\mathrm{T}}\xi_i = 0$ for $1 \leqslant i \leqslant m$, which is ridiculous since ξ_i $(1 \leqslant i \leqslant m)$ are eigenvectors of MM^{T} constituting a basis of $\mathbb{R}^m$.

By dividing Eq. (8) by Eq. (7), it follows that

$$\frac{\sum_{j=1}^m \phi_j(\lambda_1)}{\sum_{j=1}^m \varphi_j(\lambda_1)} = \frac{\sum_{j=1}^m \tilde{\phi}_j(\lambda_1)}{\sum_{j=1}^m \tilde{\varphi}_j(\lambda_1)}, \tag{9}$$

or equivalently, $\Psi(\lambda_1) := \sum_{j=1}^m \phi_j(\lambda_1) \sum_{j=1}^m \tilde{\varphi}_j(\lambda_1) - \sum_{j=1}^m \varphi_j(\lambda_1) \sum_{j=1}^m \tilde{\phi}_j(\lambda_1) = 0$. By the irreducibility of $\phi(x)$, we obtain that $\phi(x) \mid \Psi(x)$, and hence $\Psi(\lambda_i) = 0$ for $2 \leqslant i \leqslant m$. So Eq. (9) still holds if we replace λ_1 with any λ_i, i.e.,

$$\frac{\sum_{j=1}^m \phi_j(\lambda_i)}{\sum_{j=1}^m \varphi_j(\lambda_i)} = \frac{\sum_{j=1}^m \tilde{\phi}_j(\lambda_i)}{\sum_{j=1}^m \tilde{\varphi}_j(\lambda_i)}, \text{ for } 2 \leqslant i \leqslant m. \tag{10}$$

By the previous discussions, we get that

$$\epsilon_i \frac{\sum_{j=1}^m \phi_j(\lambda_i)}{\sqrt{\sum_{j=1}^m \phi_j^2(\lambda_i)}} = \tau_i \tilde{\epsilon}_i \frac{\sum_{j=1}^m \tilde{\phi}_j(\lambda_i)}{\sqrt{\sum_{j=1}^m \tilde{\phi}_j^2(\lambda_i)}}, \text{ for for } 2 \leqslant i \leqslant m. \tag{11}$$

$$\epsilon_i \frac{\sum_{j=1}^m \varphi_j(\lambda_i)}{\sqrt{\sum_{j=1}^m \phi_j^2(\lambda_i)}} = \tilde{\tau}_i \tilde{\epsilon}_i \frac{\sum_{j=1}^m \tilde{\varphi}_j(\lambda_i)}{\sqrt{\sum_{j=1}^m \tilde{\phi}_j^2(\lambda_i)}}, \text{ for } 2 \leqslant i \leqslant m. \tag{12}$$

By dividing Eq. (12) by Eq. (11), one obtains $\frac{\sum_{j=1}^m \phi_j(\lambda_i)}{\sum_{j=1}^m \varphi_j(\lambda_i)} = \frac{\tau_i}{\tilde{\tau}_i} \frac{\sum_{j=1}^m \tilde{\phi}_j(\lambda_i)}{\sum_{j=1}^m \tilde{\varphi}_j(\lambda_i)}$, together with Eq. (10), we get the conclusion that $\tau_i = \tilde{\tau}_i = \pm 1$ for $2 \leqslant i \leqslant m$. □

Claim 14. *If $e_m^{\mathrm{T}} u_1 = \sigma_1 e_m^{\mathrm{T}} \tilde{v}_1$ and $e_m^{\mathrm{T}} v_1 = \sigma_1 e_m^{\mathrm{T}} \tilde{u}_1$ with $\sigma_1 \in \{1, -1\}$, then $e_m^{\mathrm{T}} u_i = \sigma_i e_m^{\mathrm{T}} \tilde{v}_i$ and $e_m^{\mathrm{T}} v_i = \sigma_i e_m^{\mathrm{T}} \tilde{u}_i$ for some $\sigma_i \in \{1, -1\}$, for all $2 \leqslant i \leqslant m$.*

Proof. This follows by using the same argument as Claim 14; we omit the details here. □

Write

$$U = [u_1, u_2, \ldots, u_m],\ V = [v_1, v_2, \ldots, v_m],$$
$$\tilde{U} = [\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_m],\ \tilde{V} = [\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_m].$$

If the condition of Claim 14 holds, we may replace u_i and v_i with $-u_i$ and $-v_i$ respectively, whenever $\tau_i = -1$ for $1 \leqslant i \leqslant m$. Then we have $e_m^{\mathrm{T}} u_i = e_m^{\mathrm{T}} \tilde{u}_i$ and $e_m^{\mathrm{T}} v_i = e_m^{\mathrm{T}} \tilde{v}_i$ for $1 \leqslant i \leqslant m$. Let $R = \frac{1}{\sqrt{2}} \begin{bmatrix} U & U \\ V & -V \end{bmatrix}$ and $\tilde{R} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{U} & \tilde{U} \\ \tilde{V} & -\tilde{V} \end{bmatrix}$. Define

$$Q := R\tilde{R}^{\mathrm{T}} = \begin{bmatrix} U\tilde{U}^{\mathrm{T}} & O \\ O & V\tilde{V}^{\mathrm{T}} \end{bmatrix}. \tag{13}$$

Then Q is an orthogonal matrix and

$$R^{\mathrm{T}} A R = \tilde{R}^{\mathrm{T}} \tilde{A} \tilde{R} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m, -\lambda_1, \ldots, -\lambda_m).$$

Thus, $Q^{\mathrm{T}} A Q = \tilde{A}$. Next, it remains to show that Q is regular, i.e., $Qe_{2m} = e_{2m}$, which is equivalent to $\tilde{U}^{\mathrm{T}} e_m = U^{\mathrm{T}} e_m$ and $\tilde{V}^{\mathrm{T}} e_m = V^{\mathrm{T}} e_m$. That is, $e_m^{\mathrm{T}} u_i = e_m^{\mathrm{T}} \tilde{u}_i$, $e_m^{\mathrm{T}} v_i = e_m^{\mathrm{T}} \tilde{v}_i$, for $1 \leqslant i \leqslant m$, which are precisely that we have obtained before, as desired.

If the condition of Claim 13 holds, similarly, we may replace u_i and v_i with $-u_i$ and $-v_i$ respectively, whenever $\sigma_i = -1$. Then $e_m^{\mathrm{T}} u_i = e_m^{\mathrm{T}} \tilde{v}_i$ and $e_m^{\mathrm{T}} v_i = e_m^{\mathrm{T}} \tilde{u}_i$, for $1 \leqslant i \leqslant m$. Now let $R = \frac{1}{\sqrt{2}} \begin{bmatrix} U & U \\ V & -V \end{bmatrix}$ and $\tilde{R} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{U} & -\tilde{U} \\ \tilde{V} & \tilde{V} \end{bmatrix}$. Define

$$Q := R\tilde{R}^{\mathrm{T}} = \begin{bmatrix} O & U\tilde{V}^{\mathrm{T}} \\ V\tilde{U}^{\mathrm{T}} & O \end{bmatrix}. \tag{14}$$

Then Q is an orthogonal matrix and still $Q^{\mathrm{T}} A Q = \tilde{A}$ holds. Moreover, it is easy to verify that $Qe_{2m} = e_{2m}$. So Q is regular.

The proof is complete. □

4 Proof of Theorem 2

In this section, we present the proof of Theorem 2.

Recall that for a monic polynomial $f(x) \in \mathbb{Z}[x]$ with degree n, the *discriminant* of $f(x)$ is defined as:

$$\Delta(f) = \prod_{1 \leqslant i < j \leqslant n} (\alpha_i - \alpha_j)^2,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all the roots of $f(x)$.

Then it is clear that $\Delta(f)$ is always an integer for $f(x) \in \mathbb{Z}[x]$, and $\Delta(f) = 0$ if and only if f has a multiple root. Define the *discriminant* of a matrix A, denoted by $\Delta(A)$, as the discriminant of its characteristic polynomial, i.e., $\Delta(A) := \Delta(\det(xI - A))$. The *discriminant* of a graph G, denoted by $\Delta(G)$, is defined to be the discriminant of its adjacency matrix.

In [25], Wang and Yu give the following theorem, which is our main tool in proving Theorem 2.

Theorem 15 ([25])**.** *Let A be a symmetric integral matrix. Suppose there exists a rational orthogonal matrix Q such that $Q^{\mathrm{T}}AQ$ is an integral matrix. If $\Delta(A)$ is odd and square-free, then Q must be a signed permutation matrix.*

However, Theorem 15 cannot be used directly, since the $\Delta(\Gamma)$ is always a perfect square for a signed bipartite graph Γ with an equal size of bipartition, as shown by the following lemma.

Lemma 16. *Let Γ be a signed bipartite graph with bipartite-adjacency matrix M, where M is a square matrix of order $m := n/2$. Then $\Delta(\Gamma) = 2^n \det^2(M)\Delta^2(M^{\mathrm{T}}M)$.*

Proof. Let the eigenvalues of Γ be $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_m$. By Lemma 12, the eigenvalues of $M^{\mathrm{T}}M$ are $\lambda_1^2, \lambda_2^2, \ldots, \lambda_m^2$. Thus, we have

$$\begin{aligned}
\Delta(\Gamma) &= \prod_{1 \leqslant i < j \leqslant m} (\lambda_i - \lambda_j)^2 \prod_{1 \leqslant i, j \leqslant m} (\lambda_i + \lambda_j)^2 \prod_{1 \leqslant i < j \leqslant m} (-\lambda_i + \lambda_j)^2 \\
&= 2^n \lambda_1^2 \lambda_2^2 \cdots \lambda_m^2 \prod_{1 \leqslant i < j \leqslant m} (\lambda_i^2 - \lambda_j^2)^4 \\
&= 2^n \det(M^{\mathrm{T}}M)\Delta^2(M^{\mathrm{T}}M) \\
&= 2^n (\det(M))^2 \Delta^2(M^{\mathrm{T}}M).
\end{aligned}$$

This completes the proof. □

Let a_0 be the constant term of the characteristic polynomial of the underlying graph G defined as above. Then

$$a_0 = (-1)^m \det(M^{\mathrm{T}}M) = (-1)^m \det{}^2(M).$$

Note that for a tree with an irreducible characteristic polynomial $\phi(x)$, the constant term of $\phi(x)$ is always ± 1. Thus we have

Corollary 17. *Let T be a tree with an irreducible characteristic polynomial. Then $\Delta(T) = 2^n\Delta^2(M^{\mathrm{T}}M)$.*

Finally, we are ready to present the proof of Theorem 2.

Proof of Theorem 2. Let $\tilde{\Gamma}$ be any signed graph that is generalized cospectral with $\Gamma = (T, \sigma)$. We shall show that $\tilde{\Gamma}$ is isomorphic to Γ. Note that $\tilde{\Gamma}$ has the same number of edges as Γ and moreover, the assumption that $\phi(\tilde{\Gamma}) = \phi(\Gamma)$ is irreducible forces $\tilde{\Gamma}$ to be connected. Thus, $\tilde{\Gamma}$ is signed graph whose underlying graph is a tree (say $\tilde{T}$), and $\tilde{\Gamma} = (\tilde{T}, \tilde{\sigma})$.

Note that both T and $\tilde{T}$ are balanced as signed graphs, we have $\phi(T) = \phi(\Gamma)$ and $\phi(\tilde{T}) = \phi(\tilde{\Gamma})$. Let $A(\Gamma) = D_1A(T)D_1$ and $A(\tilde{\Gamma}) = D_2A(\tilde{T})D_2$, where D_1 and D_2 are diagonal matrices whose diagonal entries are ± 1.

By Theorem 5, the fact that Γ and $\tilde{\Gamma}$ are generalized cospectral implies that there exists a regular rational orthogonal matrix Q such that

$$Q^{\mathrm{T}}A(\Gamma)Q = A(\tilde{\Gamma}), \tag{15}$$

i.e., $Q^{\mathrm{T}}(D_1A(T)D_1)Q = D_2A(\tilde{T})D_2$, which is equivalent to $\hat{Q}^{\mathrm{T}}A(T)\hat{Q} = A(\tilde{T})$, where $\hat{Q} = D_1QD_2$ is a rational orthogonal matrix.

Let

$$A(T) = \begin{bmatrix} O & M \\ M^{\mathrm{T}} & O \end{bmatrix}, A(\tilde{T}) = \begin{bmatrix} O & \tilde{M} \\ \tilde{M}^{\mathrm{T}} & O \end{bmatrix}.$$

By Theorem 7, assume without loss of generality that $Q = \begin{bmatrix} Q_1 & O \\ O & Q_2 \end{bmatrix}$ and $\hat{Q} = \begin{bmatrix} \hat{Q}_1 & O \\ O & \hat{Q}_2 \end{bmatrix}$. Then we have $\hat{Q}_1^{\mathrm{T}}M\hat{Q}_2 = \tilde{M}$. It follows that

$$\hat{Q}_1^{\mathrm{T}}MM^{\mathrm{T}}\hat{Q}_1 = \tilde{M}\tilde{M}^{\mathrm{T}} \text{ and } \hat{Q}_2^{\mathrm{T}}M^{\mathrm{T}}M\hat{Q}_2 = \tilde{M}^{\mathrm{T}}\tilde{M}.$$

Note that $\Delta(M^{\mathrm{T}}M) = \Delta(MM^{\mathrm{T}}) = 2^{-n/2}\sqrt{\Delta(T)}$, which is odd and square-free. Thus, according to Theorem 15, both $\hat{Q}_1$ and $\hat{Q}_2$ are signed permutation matrices. It follows that $Q = D_1\hat{Q}D_2$ is a signed permutation matrix. Moreover, note that Q is regular. Therefore, Q is a permutation matrix, and by Eq. (15), we conclude that $\tilde{\Gamma}$ is isomorphic to Γ. The proof is complete. □

Remark 18. The condition of Theorem 2 is tight in the sense that Theorem 2 is no longer true if $2^{-\frac{n}{2}}\sqrt{\Delta(T)}$ has a multiple odd prime factor. Let the signed bipartite-adjacency matrices of two signed trees T and $\tilde{T}$ (see Figure 2) be given as follows, respectively:

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}, \tilde{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Then

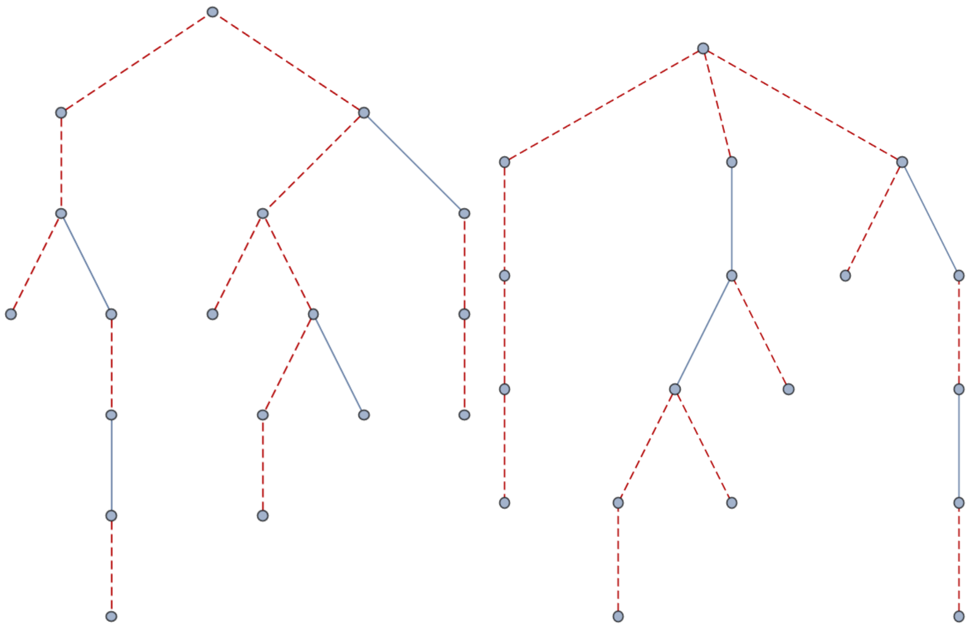

Figure 2: Two signed trees Γ and $\tilde{\Gamma}$, where the dashed lines represent the edges with negative signs.

$$\phi(T) = \phi(\tilde{T}) = -1+22x^2-162x^4+538x^6-897x^8+809x^{10}-410x^{12}+116x^{14}-17x^{16}+x^{18},$$

which is irreducible over $\mathbb{Q}$. However, $2^{-9}\sqrt{\Delta(T)} = 7^2 \times 347 \times 357175051$, i.e., $2^{-9}\sqrt{\Delta(T)}$ has a multiple factor 7 and the condition of Theorem 2 is not satisfied. Actually, there indeed exists a regular rational orthogonal matrix $Q \in \mathcal{Q}(G)$ such that $\tilde{A} = Q^{\mathrm{T}}AQ$, where $Q = \mathrm{diag}(Q_1, Q_2)$ and Q_1 and Q_2 are given as follows respectively.

$$Q_1 = \frac{1}{7}\begin{pmatrix} -1 & -1 & -2 & -2 & 4 & 3 & 3 & 2 & 1 \\ -2 & -2 & 3 & 3 & 1 & -1 & -1 & 4 & 2 \\ 2 & 2 & 4 & -3 & -1 & 1 & 1 & 3 & -2 \\ 4 & -3 & 1 & 1 & -2 & 2 & 2 & -1 & 3 \\ -3 & 4 & 1 & 1 & -2 & 2 & 2 & -1 & 3 \\ 3 & 3 & -1 & -1 & 2 & -2 & -2 & 1 & 4 \\ 1 & 1 & 2 & 2 & 3 & 4 & -3 & -2 & -1 \\ 2 & 2 & -3 & 4 & -1 & 1 & 1 & 3 & -2 \\ 1 & 1 & 2 & 2 & 3 & -3 & 4 & -2 & -1 \end{pmatrix}$$

$$Q_2 = \frac{1}{7}\begin{pmatrix} 2 & 2 & 4 & -3 & -1 & 1 & 1 & 3 & -2 \\ 2 & 2 & -3 & 4 & -1 & 1 & 1 & 3 & -2 \\ -2 & -2 & 3 & 3 & 1 & -1 & -1 & 4 & 2 \\ 4 & -3 & 1 & 1 & -2 & 2 & 2 & -1 & 3 \\ 1 & 1 & 2 & 2 & 3 & 4 & -3 & -2 & -1 \\ -3 & 4 & 1 & 1 & -2 & 2 & 2 & -1 & 3 \\ 3 & 3 & -1 & -1 & 2 & -2 & -2 & 1 & 4 \\ -1 & -1 & -2 & -2 & 4 & 3 & 3 & 2 & 1 \\ 1 & 1 & 2 & 2 & 3 & -3 & 4 & -2 & -1 \end{pmatrix}.$$

Remark 19. Theorem 7 does not hold without the assumption that the characteristic polynomial of Γ is irreducible over $\mathbb{Q}$, even if Γ is controllable. Let Γ and $\tilde{\Gamma}$ be two signed trees (see Figure 3) with bipartite-adjacency matrices M and $\tilde{M}$ given as follows respectively:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}, \tilde{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$Q = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & -2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 3 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & -1 & -1 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

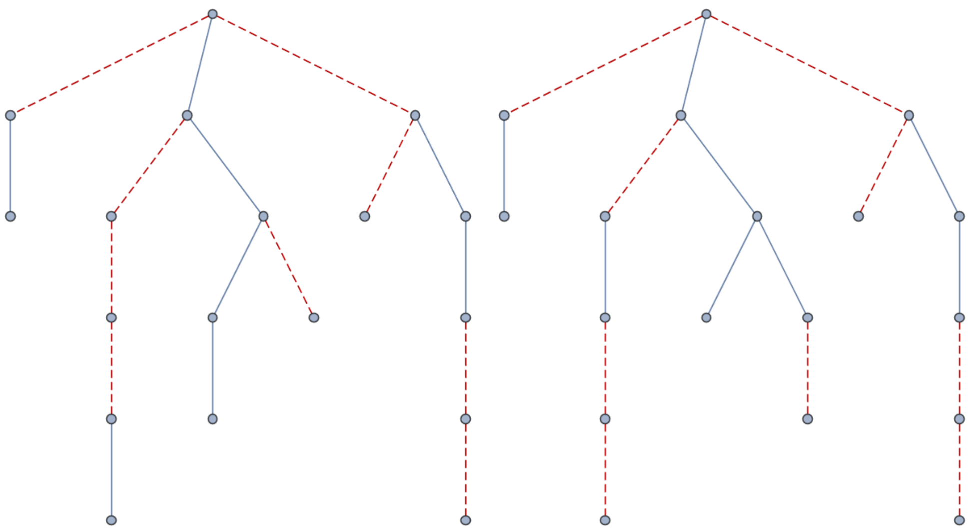

Figure 3: Two signed trees Γ and $\tilde{\Gamma}$, where the dashed lines represent the edges with negative signs.

It is easy to verify that

$$\phi(\Gamma; x) = (-1+x)(1+x)(-1-x+x^2)(-1+x+x^2)(1-21x^2+95x^4-119x^6+60x^8-13x^{10}+x^{12}),$$

which is reducible over $\mathbb{Q}$ and Γ is controllable. Nevertheless, the unique regular rational orthogonal matrix Q (shown as above) such that $Q^{\mathrm{T}}A(\Gamma)Q = A(\tilde{\Gamma})$ is not the form as in Theorem 7.

5 Conclusions and Future Work

In this paper, we have given a simple arithmetic condition on a tree T with an irreducible characteristic polynomial, under which every signed tree with underlying tree T is DGS. This is a little bit surprising in contrast with Schwenk's remarkable result stating almost every tree has a cospectral mate.

However, there are several questions that remain to be answered. We end the paper by proposing the following questions:

Question 20. How can Theorem 2 be generalized to signed bipartite graphs?

Question 21. Is it true that every tree with an irreducible characteristic polynomial is DGS?

Question 22. Is Theorem 7 true for controllable bipartite graphs?

For Question 20, the difficulty lies in the fact that for a signed bipartite graph Γ, a signed graph $\tilde{\Gamma}$ generalized cospectral with Γ is not necessarily bipartite. For Question 21, we know that it is not true for signed trees. For Question 22, we know that it is not true for controllable signed bipartite graphs. But generally we do not know any single counterexample to Questions 21 and 22. The above questions need further investigations in the future.

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