New structures and their applications to variants of zero forcing and propagation time

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Abstract

We introduce a generalization of the concept of a chronological list of forces, called a relaxed chronology. This concept is used to introduce a new way of formulating the zero forcing process, which we refer to as parallel increasing path covers. The combinatorial properties of parallel increasing path covers are utilized to identify bounds comparing standard zero forcing propagation time to positive semidefinite (PSD) propagation time. A set of paths within a set of PSD forcing trees, called a path bundle, is used to identify the PSD forcing analog of the reverse of a standard zero forcing set, as well as to draw a connection between PSD forcing and rigid linkage forcing.

Mathematics Subject Classifications: 05C57, 05C69, 05C70, 68R10

1 Introduction

Zero forcing is a dynamic coloring process on (finite, simple, and undirected) graphs. It has several applications and has been introduced independently in multiple fields. Examples of these applications include bounding the maximum nullity and minimum rank of graphs in combinatorial linear algebra [2], efficient placement of monitors in an electrical grid

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through power domination in [11] (with the role of zero forcing evident in [6]), and the study of control of quantum systems in [7].

In zero forcing and its variants, one starts with a graph whose vertices have all been colored either blue or white, and then an iterative process occurs during which white vertices become blue according to some color change rule. For most variants, one goal is to identify forcing sets, which are initial colorings for which the entire graph will eventually become blue after sufficiently many applications of the associated color change rule. The variants considered in this paper include standard zero forcing, positive semidefinite (PSD) zero forcing as introduced in [5], power domination as introduced in [11], and rigid linkage forcing as introduced in [10]. For these variants, the minimum size of a forcing set is called the standard zero forcing number, PSD forcing number, power domination number, and RL-forcing number of a graph, respectively. Our work will also involve the minimum number of time-steps needed to color the entire graph blue (while forcing all possible vertices at each time-step), which is known as the propagation time. This was introduced formally as a graph parameter for standard zero forcing in [8] and [13], for PSD forcing in [16], and for power domination in [1].

Our work introduces several new structures and applies them to the study of zero forcing. We first introduce a special family of path covers for a graph called *parallel increasing path covers* (see Definitions 12 and 14 for formal definitions). Given a path cover $\mathcal{Q} = \{Q_i\}_{i=1}^m$ of a graph G, express each path as $Q_i = v_{i,0}v_{i,1}\dots v_{i,n_i-1}$ (so $|V(Q_i)| = n_i$). We say $\mathcal{Q}$ is a parallel increasing path cover if there exists some integer K and assignment of some set $A_{i,j} \subseteq \{0, 1, 2, \dots, K\}$ to each vertex $v_{i,j}$ with all of the following properties.

- For each $i = 1, \dots, m$, the sets $A_{i,0}, \dots, A_{i,n_i-1}$ partition the set $\{0, 1, 2, \dots, K\}$.
- For each $i = 1, \dots, m$ and $0 \leqslant j_1 < j_2 \leqslant n_i - 1$, each element in A_{i,j_1} is less than each element in A_{i,j_2}.
- If v_{i_1,j_1} and v_{i_2,j_2} are adjacent in G with $i_1 \neq i_2$, then $A_{i_1,j_1} \cap A_{i_2,j_2} \neq \emptyset$.

In this case, we call the multiset $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^m$ a *witness* of $\mathcal{Q}$. An example is shown in Figure 1.

The relationship between zero forcing and parallel increasing path covers on a graph is not immediately clear. One connection is through the path covers of G that arise from zero forcing, which are called chain sets. We will show that parallel increasing path covers arise naturally as these chain sets. However, a much deeper connection exists between parallel increasing path covers and zero forcing. We introduce a generalization of the zero forcing process called a *relaxed chronology* (Definition 9), where any number of the permitted forces can be performed at each time-step. This generalizes both chronological lists of forces, where exactly one vertex is forced at each time-step, and propagating forces, where every possible vertex is forced at each time-step (see Section 2.3 for precise definitions of these forcing processes). Our first main result shows that parallel increasing path covers of G are precisely the path covers resulting from relaxed chronologies of zero

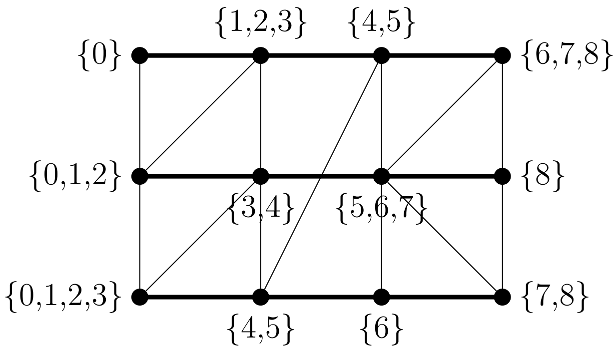


Figure 1: A graph with a parallel increasing path cover $\mathcal{Q}$ shown in bold. Vertices have been labeled using a witness of $\mathcal{Q}$.

forces. Its complete version, including the underlying connections via block partitions and active sets, appears as Theorem 15.

Theorem 1. *Let G be a graph. Then $\mathcal{Q}$ is a parallel increasing path cover of G if and only if $\mathcal{Q}$ is a chain set for some relaxed chronology $\mathcal{F}$ of a standard zero forcing set B of G.*

We also introduce a parameter called the *parallel increasing path cover number* of G, denoted $\mathrm{PIP}(G)$ (Definition 14). This is the smallest number m such that G has a parallel increasing path cover consisting of m paths. The preceding theorem then implies the following (which is Corollary 16):

Corollary 2. *For any graph G,* $\mathrm{PIP}(G) = \mathrm{Z}(G)$.

Though parallel increasing path covers directly correspond to relaxed chronologies in standard zero forcing, we also apply them to the study of PSD forcing and structural graph properties. Using properties of a witness and the correspondence in Theorem 1, we show that the set of active vertices at any time-step is a PSD forcing set, as well as a vertex cut when the resulting set of vertices does not consist only of endpoints of the path cover. See Figure 1 for examples, as well as Lemma 25 and Lemma 28 for precise statements of these results. Letting $\mathrm{Z}_+(G)$ denote the PSD forcing number of G, we establish upper bounds for PSD propagation time $\mathrm{pt}_+(G, m)$ in terms of standard propagation time $\mathrm{pt}(G, m)$ on forcing sets of size m (see Sections 2.2 and 2.3 for formal definitions, as well as Theorem 31 and Corollary 34 for the results and their proofs). Note that the m in $\mathrm{pt}_+(G, m)$ and $\mathrm{pt}(G, m)$ is suppressed when $m = \mathrm{Z}_+(G)$ or $m = \mathrm{Z}(G)$, respectively.

Theorem 3. *Let G be a graph and $m \in \mathbb{N}$ such that $m \geqslant \mathrm{Z}(G)$. Then*

$$\mathrm{pt}_+(G, m) \leqslant \left\lceil \frac{\mathrm{pt}(G, m)}{2} \right\rceil$$

Corollary 4. *For any graph G such that $\mathrm{Z}_+(G) = \mathrm{Z}(G)$,*

$$\mathrm{pt}_+(G) \leqslant \left\lceil \frac{\mathrm{pt}(G)}{2} \right\rceil.$$

Relaxed chronologies also allow for a natural way to restrict zero forcing on a graph to its subgraphs. We introduce a special case called *path bundles* (Definition 41), where the restriction of PSD forcing aligns with standard zero forcing. We use path bundles to construct a lower bound on PSD propagation time using standard zero forcing propagation time and establish a PSD analog of the reversal of standard chain sets. We apply the latter result to show that given a set of PSD forcing trees in G and any fixed vertex v, one can find a PSD forcing set of the same cardinality that contains v and preserves the PSD forcing trees (stated as Corollary 52).

Theorem 5. *Let G be a graph with $\mathcal{T}$ being the PSD forcing trees for some PSD forcing set of size k. For any $v \in V(G)$, there exists a PSD forcing set B of size k containing v and a relaxed chronology of forces $\mathcal{F}$ for B with $\mathcal{T}$ as its induced forcing trees.*

Though our results stated thus far involve only standard and PSD forcing, we also apply our techniques to other variants of zero forcing, including power domination (Theorem 36) and rigid linkage zero forcing (Theorem 59). Furthermore, our new structures and techniques may have applications to additional parameters, such as skew forcing; we leave further exploration of these applications as directions for future work.

We start in Section 2 with preliminaries. In Section 3, we define relaxed chronologies and parallel increasing path covers, and we establish a correspondence between parallel increasing path covers and relaxed chronologies for standard zero forcing. In Section 4, we apply our results on parallel increasing path covers to establish results for several variants of zero forcing. Finally, in Section 5, we consider restrictions of relaxed chronologies to subgraphs and establish results on path bundles. Our results on power domination and rigid linkages are included in appendices.

2 Preliminaries

In this section we provide precise definitions for graphs, zero forcing and variants, and propagation. Additional background can be found in [14, Part 4].

2.1 Graph terminology

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is the set of *vertices* and $E(G)$ is a set of 2-element sets of vertices called *edges*. To help differentiate between subsets of $V(G)$ of cardinality two and edges, given two distinct vertices $u, v \in V(G)$, the subset of $V(G)$ composed of these two vertices will be denoted $\{u, v\}$ while the edge between u and v will be denoted uv (or vu). All graphs are assumed to be finite and *simple*, that is, to have neither loops (edges between between a vertex and itself) nor multiple edges between any two distinct vertices.

If G and H are graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G. If H is a subgraph of G and for any two vertices $u, v \in V(H)$ we have $uv \in E(H)$ if and only if $uv \in E(G)$, then H is an *induced subgraph* of G. Given a subset of vertices $S \subseteq V(G)$, the induced subgraph H of G with vertex set $V(H) = S$ will be denoted $G[S]$. In addition, for a given set of vertices $B \subseteq V(G)$, the notation $G - B$ will be used to denote $G[V(G) \setminus B]$.

Given a vertex $v \in V(G)$, the *open neighborhood* of v in G, denoted by $N_G(v)$, is the set of vertices u such that $vu \in E(G)$. The *closed neighborhood* of a vertex v is $N_G[v] = N_G(v) \cup \{v\}$. Likewise, given a set of vertices $S \subseteq V(G)$, the open (respectively, closed) neighborhood of S is defined to be the union of the open (respectively, closed) neighborhoods of the vertices of S.

A *path* is a sequence of distinct vertices $v_1, v_2, \dots, v_m$ such that for each i with $1 \leqslant i \leqslant m - 1$ we have $v_i v_{i+1} \in E(G)$. Given a pair of vertices $u, v \in V(G)$, a *uv-path* is a path $v_1, v_2, \dots, v_m$ such that $u = v_1$, $v = v_m$. A graph G is said to be *connected* if for each pair of vertices $u, v \in V(G)$ there exists a uv-path. A graph is *disconnected* if it is not connected. The maximal connected subgraphs of G are its *components*. The set of these components is denoted $\mathrm{comp}(G)$, and given a vertex v, the component of G that contains the vertex v is denoted $\mathrm{comp}(G, v)$. If $S \subsetneq V(G)$ such that $G - S$ has more components than G does, then S is a *vertex cut* of G. We also view a path as a graph: the *path* P_n is the graph with $V(P_n) = \{v_1, \dots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : i = 1, \dots, n - 1\}$. A *path cover* of G is a set of induced paths in G with the property that every vertex of G is in exactly one path.

2.2 Zero forcing

In standard zero forcing and each of the variants discussed here, one starts with a graph and colors every vertex of the graph either blue or white. Given a set of blue vertices (with the other vertices colored white), a process is started during which vertices cause white vertices to become blue. Each variant is defined by its *color change rule*, which governs under what circumstances a vertex can cause a white vertex to become blue during this process; the color change rule is customarily denoted by *CCR-X*, where X is the associated parameter. When a vertex u causes a white vertex v to become blue, this is referred to as *forcing* and denoted $u \to v$. There is often a choice as to which vertex is chosen to force v among those that can, but only one vertex can force v. It is worth noting that once a vertex is blue, it will remain blue. In the variants discussed, the goal is for every vertex in the graph to become blue, so the color change rule will be applied until no further forces are possible. Two color change rules that we study are as follows.

- Standard zero forcing color change rule (CCR-Z): If a blue vertex u has a unique white neighbor v, then u can force v to become blue.

- Positive semidefinite (PSD) zero forcing color change rule (CCR-Z_+): If B is the set of currently blue vertices, C is a component of $G - B$, and u is a blue vertex such that $N_G(u) \cap V(C) = \{v\}$, then u can force v to become blue.

Power domination on graphs was introduced in [11] before zero forcing was defined as a separate parameter. Following the introduction of zero forcing in [2] and [7] and the work of Brueni and Heath simplifying power domination in [6], power domination can be viewed as the process of starting with a set of initially blue vertices B, coloring $N_G[B]$ blue during time-step 1, and for time-steps $k \geqslant 2$, applying CCR-Z. We will also be discussing rigid linkage forcing in Appendix B, but we defer the definition of its color change rule to that section.

A *standard zero forcing set* of a graph G is a set of vertices B such that if B is the set of initially blue vertices and CCR-Z is applied a sufficient number of times, then all vertices of G become blue. The *standard zero forcing number* of a graph G, denoted $\mathrm{Z}(G)$, is the minimum cardinality of a standard zero forcing set. A *minimum standard zero forcing set* B of a graph G is a standard zero forcing set of G such that $|B| = \mathrm{Z}(G)$. A *(minimum) PSD forcing set* and the *PSD forcing number* $\mathrm{Z}_+(G)$ of a graph are defined analogously using CCR-Z_+ as the color change rule. The terms *(minimum) power dominating set* and *power domination number* are defined analogously by applying CCR-Z to $N_G[B]$ where B is the set of initially blue vertices.

2.3 Forcing and propagation

The process of coloring vertices blue has been viewed from various perspectives, including performing only one force at a time or as many forces as are independently possible in each step. In a forcing process applied to a forcing set B where exactly one white vertex is forced blue in each time-step, a list of these forces in the order in which they occur is known as a *chronological list of forces*. Note that a chronological list of forces of B contains $K = |V(G)| - |B|$ forces. Let $F^{(k)}$ be the set containing the one force $u \to v$ that occurs during time-step k. Then this chronological list of forces can also be viewed as an ordered set and denoted by $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$. Given a set of initially blue vertices, a great deal of choice may occur in creating a chronological list of forces that colors all vertices blue, because at each time-step there may be multiple vertices capable of being forced and multiple vertices capable of forcing each such vertex. However, it is well-known that for standard zero forcing and PSD forcing, the choice and order of forces does not affect what vertices can be colored blue by a given set of initially blue vertices [2].

There are also times when considering a set of forces without reference to a specific order may be useful; this is called a *set of forces* and denoted by F. For example, if $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ is a chronological list of forces, then $F = \bigcup_{k=1}^{K} F^{(k)}$ is a set of forces.

A *propagation process* records the sequence of sets of vertices in which the forces take place, assuming the set of blue vertices is fixed until the next round and all possible forces occur simultaneously. More formally, for any fixed color change rule, for any graph G, and for any initial set B of blue vertices, let $B^{[0]} = B^{(0)} = B$. For $k \geqslant 1$, define $B^{(k)}$ to be the set of vertices that can be forced blue during time-step k of propagation, i.e., $B^{(k)} = \{v : v \text{ can be forced by some } u, \text{ given the set of blue vertices is } B^{[k-1]}\}$. Define $B^{[k]}$ to be the set of vertices that are blue after time-step k, i.e., $B^{[k]} = B^{[k-1]} \cup B^{(k)}$. Suppose B is a forcing set. Define the *round function* by $\mathrm{rd}(v) = k$ where k is the unique index such that $v \in B^{(k)}$. Consider the propagation process for B, i.e., the sequence of blue sets

$B^{(k)}$. Let t be the least k such that $B^{[k]} = V(G)$. For $k = 1, \dots, t$ and for each vertex v in $B^{(k)}$, choose a vertex u_v such that u_v can force v (in accordance with the color change rule being used). Let $F^{(k)} = \{u_v \to v : \mathrm{rd}(v) = k\}$. The ordered set $\mathcal{F}$ of sets $F^{(k)}$ is a *propagating family of forces* and $F = \bigcup_{k=1}^{t} F^{(k)}$ is a *propagating set of forces*. For a given propagating set of forces, a vertex v is *active* at time-step k if v is blue after time-step k but v has not yet performed a force. Note that in a propagation process, the sets $B^{(k)}$ and $B^{[k]}$ are uniquely determined by B. Furthermore, each of the sets $B^{(k)}$ and $F^{(k)}$ is nonempty for $k = 1, \dots, t$.

Consider the standard zero forcing color change rule. For a set of forces F of a forcing set B of G, each vertex $v \in B$ defines a *forcing chain* $C_v = (v = v_0, v_1, \dots, v_k)$ where $v_{i-1} \to v_i \in F$ for $i = 1, \dots k$ and v_k does not perform a force. Every vertex of G appears in exactly one forcing chain defined from F (note $k = 0$ is allowed with the forcing chain (v_0)). The *chain set* defined by F is the set of forcing chains $\mathcal{C} = \{C_v\}_{v \in B}$. The subgraph of G induced by the vertices of a forcing chain is a path. For a set of PSD forces F of B, similar sets of vertices are constructed by F, but in this case each vertex might force multiple vertices, so rather than forming induced paths the process constructs induced trees. For this reason, a set of PSD forces constructs forcing trees: For a vertex b in a PSD forcing set B and a set of PSD forces F of B, define V_b to be the set of all vertices w such that there is a sequence of forces $b = v_0 \to v_1 \to \cdots \to v_k = w$ in F (the empty sequence of forces is permitted, i.e., $b \in V_b$). The *forcing tree* T_b is the induced subgraph $T_b = G[V_b]$. The *forcing tree cover* (for a set of forces F) is $\mathcal{T} = \{T_b : b \in B\}$ and every vertex of G is a vertex of some tree in the forcing tree cover. Note that each vertex in a PSD forcing set is the first vertex of a forcing tree, so the number of forcing trees is the cardinality of the PSD forcing set [9].

Let B be a standard zero forcing set of a graph G. The *standard propagation time of* B is $\mathrm{pt}(G, B) = t$ where t is the least k such that $B^{[k]} = V(G)$. The *standard k-propagation time* of G is

$$\mathrm{pt}(G, k) = \min\{\mathrm{pt}(G, B) : B \text{ is a standard zero forcing set of } G \text{ and } |B| = k\}.$$

Finally, the *standard propagation time* of a graph G is

$$\mathrm{pt}(G) = \min\{\mathrm{pt}(G, B) : B \text{ is a standard zero forcing set of } G \text{ and } |B| = \mathrm{Z}(G)\}.$$

A set of vertices $B \subseteq V(G)$ is said to be an *efficient standard zero forcing set* if B is a standard zero forcing set such that $|B| = \mathrm{Z}(G)$ and $\mathrm{pt}(G, B) = \mathrm{pt}(G)$. Similarly B is said to be a *k-efficient standard zero forcing set* if B is a standard zero forcing set such that $|B| = k$ and $\mathrm{pt}(G, B) = \mathrm{pt}(G, k)$. For a PSD forcing set B of a graph G, the *PSD propagation time of* B, denoted by $\mathrm{pt}_+(G, B)$, the *PSD k-propagation time* of G, denoted by $\mathrm{pt}_+(G, k)$, and the *PSD propagation time* of G, denoted by $\mathrm{pt}_+(G)$, are defined analogously. The concepts of *efficient PSD forcing sets* and *k-efficient PSD forcing sets* are also defined analogously. The terms *power propagation time of* B, denoted by $\mathrm{ppt}(G, B)$, *power k-propagation time* $\mathrm{ppt}(G, k)$, and *power propagation time* $\mathrm{ppt}(G)$ are defined analogously by setting $B^{[0]} = B$, $B^{[1]} = N_G[B]$, and for time-steps $k \geqslant 2$ applying CCR-Z.

3 Relaxed chronologies and parallel increasing path covers

In this section we formally define two related new structures, relaxed chronologies, which generalize both chronological lists of forces and propagating families of forces, and parallel increasing path covers, which model chain sets and carry additional information.

3.1 Relaxed chronologies

We start by presenting a flexible framework for discussing zero forcing, called a relaxed chronology. To help introduce and motivate this framework, we offer the next example.

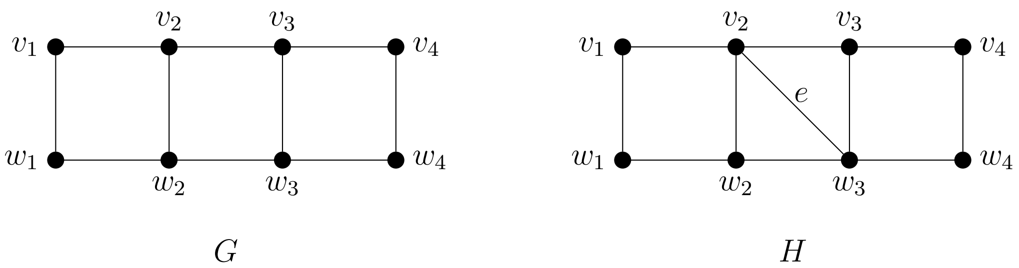


Figure 2: A zero forcing example.

Example 6. Consider the graphs G and H in Figure 2, which are identical except that $e = v_2w_3 \in E(H)$. $B = \{v_1, w_1\}$ is a standard zero forcing set for both G and H, and the ordered set $(v_1 \to v_2, w_1 \to w_2, w_2 \to w_3, v_2 \to v_3, v_3 \to v_4, w_3 \to w_4)$ is a chronological list of forces for both graphs. Note that $\mathcal{F} = (F^{(1)} = \{v_1 \to v_2, w_1 \to w_2\}, F^{(2)} = \{w_2 \to w_3\}, F^{(3)} = \{v_2 \to v_3\}, F^{(4)} = \{v_3 \to v_4, w_3 \to w_4\})$ is a propagating family of forces on H. We can also view $\mathcal{F}$ as an ordered set of sets of forces in G with the forces in $F^{(k)}$ all occurring in the k-th time-step. We particularly note that performing those forces at the specified time-steps is consistent with the standard zero forcing color change rule. Indeed, this would hold if $\mathcal{F}$ were any other propagating family of forces on H not involving a force along e.

It is clear that neither a chronological list of forces nor a propagating family of forces describes the process of $\mathcal{F}$ forcing in G in the preceding example, since $\mathcal{F}$ is neither one with respect to G. In order to discuss the observations of Example 6 formally, we therefore introduce a more general forcing process.

We define the process in terms of a fairly-generic color change rule CCR-X, with correspondingly-named X-forces. The X-forcing color change rule CCR-X can be any color change rule satisfying two specific properties outlined in the next definition.

Definition 7 (Consistent color change rule)**.** A color change rule CCR-X is *consistent* if it satisfies the following conditions for every graph G and set of all currently blue vertices B:

1. The validity of the force $u \to v$ under CCR-X depends only on B and G.

2. If $u \to v$ is a valid X-force and $B \subseteq B'$ with $v \notin B'$, then $u \to v$ is a valid X-force when exactly the vertices in B' are blue.

Examples of consistent color change rules include those for standard zero forcing, PSD forcing, skew zero forcing [15], or k-forcing [3], although we focus primarily on standard and PSD forcing in this article. Neither the rigid linkage color change rule nor the minor monotone floor of zero forcing color change rule [4] are consistent color change rules, however.

Definition 8 (Set of possible forces)**.** Let G be a graph, let CCR-X be a consistent color change rule, and let B be any set of blue vertices. The *set of possible X-forces* for B in G is

$$S_X(G, B) = \{u \to v : u \to v \text{ is a valid } X\text{-force in } G \text{ with exactly the vertices in } B \text{ blue}\}.$$

Definition 9 (Relaxed chronology)**.** Let G be a graph, let CCR-X be a consistent color change rule, and let B be an X-forcing set of G. Consider an ordered family $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$, where each $F^{(k)}$ is a set of X-forces. Define

$$E_{\mathcal{F}}^{[0]} = E_{\mathcal{F}}^{(0)} = B,\ E_{\mathcal{F}}^{(k)} = \{v : u \to v \in F^{(k)} \text{ for some } u\},\ \text{and } E_{\mathcal{F}}^{[k]} = \bigcup_{j=0}^{k} E_{\mathcal{F}}^{(j)}$$

for $k = 1, 2, \ldots, K$. Then $\mathcal{F}$ is a *relaxed chronology* of X-forces for B on G if

1. $F^{(k)} \subseteq S_X(G, E_{\mathcal{F}}^{[k-1]})$ for $k = 1, 2, \ldots, K$,
2. $u_1 \to v, u_2 \to v \in F^{(k)}$ implies $u_1 = u_2$, and
3. $E_{\mathcal{F}}^{[K]} = V(G)$.

In this case, we call $\{E_{\mathcal{F}}^{[k]}\}_{k=0}^{K}$ the *expansion sequence* of B induced by $\mathcal{F}$, each individual $E_{\mathcal{F}}^{[k]}$ the *k-th expansion* of B induced by $\mathcal{F}$, and K the *completion time* of $\mathcal{F}$, which we denote by $\operatorname{ct}(\mathcal{F})$. When the color change rule is clear from context, the X can be dropped.

Returning briefly to Example 6, observe that $\mathcal{F}$ is a relaxed chronology for B in both G and H. More generally, any relaxed chronology for B in H that does not contain a force along e is a relaxed chronology for B in G since $G = H - e$.

Notice that given an X-forcing set, we can inductively construct a relaxed chronology by choosing a subset of the possible X-forces at each time-step. Condition (2) of Definition 9 ensures that multiple vertices do not force the same vertex, while condition (3) ensures that we do not terminate until some time after all vertices of G are blue. A particularly unusual feature of this process is that we will allow for $F^{(k)}$ to be the empty set since we allow any subset $F^{(k)}$ of the valid X-forces at each step to be performed. Hence, $\operatorname{ct}(\mathcal{F})$ is not necessarily the first time-step when all of $V(G)$ is blue.

Observe that relaxed chronologies generalize both propagating families of forces and chronological lists of forces. If $F^{(k)}$ is maximal for all k, then the expansion sequence $\{E_{\mathcal{F}}^{[k]}\}$

reduces to $\{B^{[k]}\}$, the set of vertices that are blue after round k using the customary X-propagation process. Additionally, if $|F^{(k)}| = 1$ for all k, then $\mathcal{F}$ reduces to a chronological list of forces. For standard or PSD forcing, any set of forces of a relaxed chronology is also the set of forces of some chronological list of forces, but is not necessarily a propagating set of forces.

In the special case of the standard zero forcing color change rule, we define analogs of two definitions from zero forcing.

Definition 10 (Active times and chain set)**.** Let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ be a relaxed chronology of forces for the standard zero forcing color change rule on a graph G.

1. For $v \in V(G)$, the $\mathcal{F}$-*active times* $\operatorname{act}_{\mathcal{F}}(v) \subseteq \{0, 1, 2, \ldots, K\}$ are the time-steps when v is active with respect to $\mathcal{F}$, that is, $k \in \{0, 1, 2, \ldots, K\}$ is in $\operatorname{act}_{\mathcal{F}}(v)$ if and only if v is blue after time-step k and has not performed a force (the $\mathcal{F}$ can be omitted when it is clear from context).
2. The *chain set* defined by $\mathcal{F}$ is the chain set of the underlying set of forces $\bigcup_{k=1}^{K} F^{(k)}$.

Note that we use the term 'active' for a vertex that is blue but has not yet performed a force, but for a consistent color change rule, one does not need to know whether a vertex is active to determine whether it can perform a force. This is in contrast to the use of 'active' in forcing for minor monotone floors and rigid linkage forcing, where in some cases one must know whether or not a vertex is active to determine whether it can perform a force [4, 10].

3.2 Parallel increasing path covers

Using the framework of relaxed chronologies, we can now develop a model of chain sets from a global perspective. The key objects in this new perspective are structures called parallel increasing path covers. To help motivate and introduce this model and these structures, we consider another example.

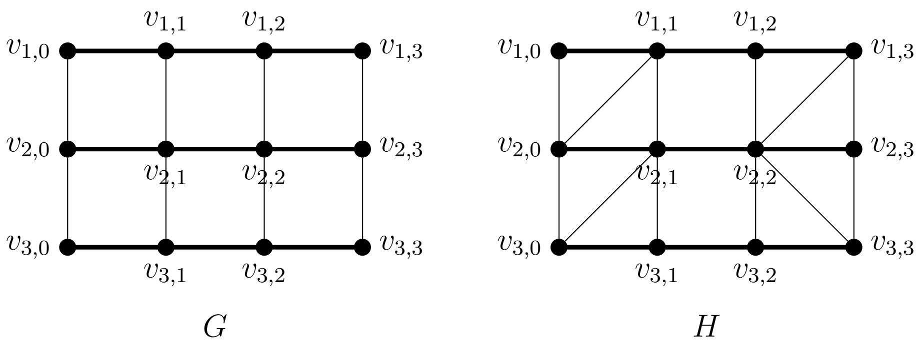


Figure 3: Another zero forcing example.

Example 11. Consider the graphs in Figure 3. $B = \{v_{1,0}, v_{2,0}, v_{3,0}\}$ is a standard zero forcing set for each, and $\mathcal{F} = \{F^{(k)}\}_{k=1}^{8} = (\{v_{1,0} \to v_{1,1}\}, \emptyset, \{v_{2,0} \to v_{2,1}\}, \{v_{1,1} \to v_{1,2},$

$v_{3,0} \to v_{3,1}\}$, $\{v_{2,1} \to v_{2,2}\}, \{v_{1,2} \to v_{1,3}, v_{3,1} \to v_{3,2}\}, \{v_{3,2} \to v_{3,3}\}, \{v_{2,2} \to v_{2,3}\})$ is a relaxed chronology for both graphs. For each vertex $v_{i,j}$, let $A_{i,j} = \text{act}_{\mathcal{F}}(v_{i,j})$. Then we obtain the following:

$$\begin{array}{llll} A_{1,0} = \{0\} & A_{1,1} = \{1,2,3\} & A_{1,2} = \{4,5\} & A_{1,3} = \{6,7,8\} \\ A_{2,0} = \{0,1,2\} & A_{2,1} = \{3,4\} & A_{2,2} = \{5,6,7\} & A_{2,3} = \{8\} \\ A_{3,0} = \{0,1,2,3\} & A_{3,1} = \{4,5\} & A_{3,2} = \{6\} & A_{3,3} = \{7,8\}. \end{array}$$

From these sets, we observe some interesting properties, which hold for both G and H:

1. Only one vertex per forcing chain is active after a given time-step, so for each $i \in \{1,2,3\}$, the multiset of sets $\{A_{i,j}\}_{j=0}^{3}$ partitions the set $\{0,1,2,\ldots,8\}$.

2. If two vertices v_{i_1,j_1} and v_{i_2,j_2} are adjacent but are not contained in the same forcing chain, then
$$A_{i_1,j_1} \cap A_{i_2,j_2} \neq \emptyset.$$

To further explore these two properties we provide the following definitions. Note that if S_1 and S_2 are sets of integers such that $x_1 < x_2$ for all $x_1 \in S_1$ and $x_2 \in S_2$, then we write $S_1 < S_2$.

Definition 12 (Block partition)**.** Let $K \in \mathbb{N}$, and let $\mathcal{A} = \{A_j\}_{j=0}^{n_{\mathcal{A}}-1}$ be a partition of the set $\{0,1,2,\ldots,K\}$ into $n_{\mathcal{A}}$ parts. If $j_1 < j_2$ implies that $A_{j_1} < A_{j_2}$ for each pair $j_1, j_2 \in \{0,1,2,\ldots,n_{\mathcal{A}}-1\}$, then we say that $\mathcal{A} = (A_j)_{j=0}^{n_{\mathcal{A}}-1}$ is a *block partition* of $\{0,1,2,\ldots,K\}$.

Example 13. Consider the sets $A_1 = \{0\}$, $A_2 = \{1,2,3\}$, and $A_3 = \{4,5\}$. Then $\mathcal{A} = \{A_1, A_2, A_3\}$ is a block partition of $\{0,1,\ldots,5\}$. Also the set $\mathcal{A}_i = \{A_{i,j}\}_{j=0}^{3}$ in Example 11 is a block partition of $\{0,1,\ldots,8\}$ for each $i \in \{1,2,3\}$. However, the sets $B_1 = \{0,2\}$, $B_2 = \{1,3\}$, and $B_3 = \{4,5\}$ do not form a block partition of $\{0,1,\ldots,5\}$.

Definition 14 (Parallel increasing path cover)**.** Let G be a graph, and let $\mathcal{Q} = \{Q_i\}_{i=1}^{m}$ be a path cover of G with $n_i = |V(Q_i)|$ for each $i \in \{1,2,\ldots,m\}$. Label $V(G)$ so that the vertices of the path Q_i are $\{v_{i,j}\}_{j=0}^{n_i-1}$ in path order (i.e., $v_{i,j_1}v_{i,j_2} \in E(G)$ if and only if $|j_1 - j_2| = 1$). Choose $K \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,m\}$, let $\mathcal{A}_i = (A_{i,j})_{j=0}^{n_i-1}$ be a block partition of $\{0,1,2,\ldots,K\}$, where we say $A_{i,j}$ *corresponds* to vertex $v_{i,j}$. If for distinct $i_1, i_2 \in \{1,2,\ldots,m\}$,

$$v_{i_1,j_1}v_{i_2,j_2} \in E(G) \text{ implies } A_{i_1,j_1} \cap A_{i_2,j_2} \neq \emptyset, \tag{1}$$

then $\mathcal{Q}$ is a *parallel increasing path cover* of G, with the multiset of block partitions $\{\mathcal{A}_i\}_{i=1}^{m}$ as a *witness* or *witnessing* that fact. Define the *parallel increasing path cover number* to be

$$\text{PIP}(G) = \min\big\{\, |\mathcal{Q}| : \mathcal{Q} \text{ is a parallel increasing path cover of } G\big\}.$$

If $\mathcal{Q}$ is a parallel increasing path cover of G such that $|\mathcal{Q}| = \text{PIP}(G)$, then we say that $\mathcal{Q}$ is a *minimum parallel increasing path cover* of G.

Note that in the previous definition, one can always choose sufficiently large $K \in \mathbb{N}$

and define a multiset of block partitions $\{\mathcal{A}_i\}_{i=1}^m$ of $\{0, 1, 2, \ldots, K\}$ where each $A_{i,j}$ corresponds to a vertex $v_{i,j}$, but the key property of parallel increasing path covers is that a multiset of block partitions can be chosen that satisfies the property in (1).

For the graphs in Figure 3, the bold edges form a path cover $\mathcal{Q}$, and the block partitions $\mathcal{A}_i = (A_{i,j})_{j=0}^3$ in Example 11 witness that $\mathcal{Q}$ is a parallel increasing path cover of those graphs. As another example, for the tree in Figure 4(a), a path cover $\mathcal{Q}$ is indicated by bold edges. The following multiset of block partitions $\{\mathcal{A}_i\}_{i=1}^3$ is a witness that $\mathcal{Q}$ is a parallel increasing path cover of that tree:

$$
\begin{array}{lllll}
\mathcal{A}_1: & A_{1,0} = \{0\} & A_{1,1} = \{1\} & A_{1,2} = \{2,3,4\} & \\
\mathcal{A}_2: & A_{2,0} = \{0,1\} & A_{2,1} = \{2\} & A_{2,2} = \{3,4\} & \\
\mathcal{A}_3: & A_{3,0} = \{0\} & A_{3,1} = \{1,2\} & A_{3,2} = \{3\} & A_{3,3} = \{4\}.
\end{array}
$$

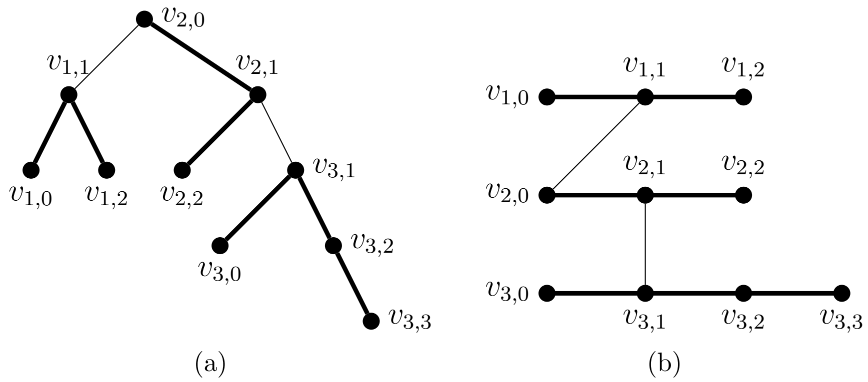


Figure 4: A parallel increasing path cover shown (a) with the tree drawn naturally, and (b) redrawn with the paths horizontal.

The name parallel increasing path cover has been chosen because given a parallel increasing path cover $\mathcal{Q}$, for each $Q_i, Q_j \in \mathcal{Q}$ distinct, $G[V(Q_i) \cup V(Q_j)]$ has the familiar structure of a graph on two parallel paths. However, even though the structure of two parallel paths might be very familiar and thus be a desirable property for a definition, a set of paths that taken pairwise are parallel paths does not necessarily form a parallel increasing path cover, as seen in Figure 5.

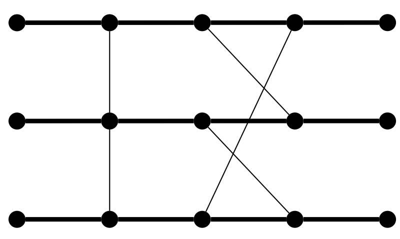


Figure 5: A set of paths that is not a parallel increasing path cover, but where the vertices of any two paths induce a graph on two parallel paths.

Rather, an additional property is needed, which is the requirement that if two elements of distinct block partitions correspond to two adjacent vertices, then these elements must have a nonempty intersection, as stated in (1). This property ensures that in some sense the paths progress in the same direction and in a manner consistent with zero forcing.

3.3 The relationship between relaxed chronologies, zero forcing, and parallel increasing path covers

The next theorem explicitly establishes the connection between parallel increasing path covers and relaxed chronologies in standard zero forcing.

Theorem 15. *Let G be a graph. Then $\mathcal{Q}$ is a parallel increasing path cover of G if and only if $\mathcal{Q}$ is a chain set for some relaxed chronology $\mathcal{F}$ of a standard zero forcing set B of G. Under this correspondence, a multiset of block partitions $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^m$ that is a witness of $\mathcal{Q}$ as a parallel increasing path cover corresponds to a relaxed chronology $\mathcal{F}$ where $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^m$ records the active time-steps for vertices $\{v_{i,j}\}_{j=0}^{n_i-1}{}_{i=1}^{m}$ with $\{v_{i,0}\}_{i=1}^m$ as the zero forcing set.*

Proof. Let $\mathcal{Q}$ be a parallel increasing path cover of G, with the vertices being labeled $\{v_{i,j}\}_{j=0}^{n_i-1}{}_{i=1}^{m}$ as in Definition 14 and the multiset of block partitions $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^m$ of $\{0, 1, 2, \ldots, K\}$ as a witness. For $k = 0, 1, \ldots, K$, define $E^{[k]} = \{v_{i,j} : \min A_{i,j} \leqslant k\}$. We show that when $E^{[k]}$ is blue, then the remaining white vertices in $E^{[k+1]}$ can be forced blue with forces along the paths using the standard zero forcing color change rule. This implies that $\{E^{[k]}\}_{k=0}^K$ is an expansion sequence of $B = E^{[0]} = \{v_{i,0}\}_{i=1}^m$ for some relaxed chronology of forces $\mathcal{F}$ where all forces occur along the paths in $\mathcal{Q}$.

Suppose the vertices in $E^{[k]} = \{v_{i,j} : \min A_{i,j} \leqslant k\}$ are blue, and consider $v_{a,b} \in E^{[k+1]} \setminus E^{[k]}$. So Q_a is the unique element of $\mathcal{Q}$ containing $v_{a,b}$ and there is a unique vertex $v_{a,b-1}$ preceding $v_{a,b}$ in Q_a. Observe that $\max A_{a,b-1} = k$. Then after time-step k, $v_{a,b-1}$ is blue, and $v_{a,b}$ is the only white neighbor of $v_{a,b-1}$ in Q_a. Also, since $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^m$ is a multiset of block partitions of $\{0, 1, 2, \ldots, K\}$ witnessing that $\mathcal{Q}$ is a parallel increasing path cover, if $v_{a,b-1}$ is adjacent to a vertex $v_{a',b'} \notin Q_a$, then $A_{a,b-1} \cap A_{a',b'} \neq \emptyset$. Then $v_{a',b'} \in E^{[k]}$ because $\max A_{a,b-1} = k$. Thus $v_{a,b}$ is the unique neighbor of $v_{a,b-1}$ in G that is not blue, and $v_{a,b-1} \to v_{a,b} \in S(G, E^{[k]})$. Hence, the remaining white vertices in $E^{[k+1]}$ can be colored blue by $E^{[k]}$ performing forces along the paths in $\mathcal{Q}$. Thus we can select $F^{(k)} \subseteq S(G, E^{[k]})$ so that $\mathcal{F} = \{F^{(k)}\}_{k=0}^K$ is a relaxed chronology of standard forces. By construction, $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^m$ records precisely the active time-steps of the corresponding vertices $\{v_{i,j}\}_{j=0}^{n_i-1}{}_{i=1}^{m}$.

We now prove the reverse direction. Let $\mathcal{C} = \{C_i\}_{i=1}^m$ be the chain set given by the relaxed chronology of forces $\mathcal{F} = \{F^{(k)}\}_{k=1}^K$ acting on the standard zero forcing set $B = \{v_{i,0}\}_{i=1}^m$ of G. For each forcing chain C_i, label the vertices $\{v_{i,j}\}_{j=0}^{n_i-1}$ so that $v_{i,j} \to v_{i,j+1} \in \bigcup_{k=1}^K F^{(k)}$, and define $\{A_{i,j}\}_{j=0}^{n_i-1}$ by $A_{i,j} = \operatorname{act}_{\mathcal{F}}(v_{i,j})$. By construction, $(A_{i,j})_{j=0}^{n_i-1}$ is a block partition of $\{0, 1, 2, \ldots, K\}$ for all i. Now suppose there exists $v_{i_1,j_1}v_{i_2,j_2} \in E(G)$ such that $i_1 \neq i_2$ and $A_{i_1,j_1} \cap A_{i_2,j_2} = \emptyset$. Each $A_{i,j}$ is a set of consecutive integers, so without loss of generality we assume that $A_{i_1,j_1} < A_{i_2,j_2}$. However, this implies $K \notin A_{i_1,j_1}$, so there exists $k \in \{1, 2, \ldots, K\}$ and a vertex v_{i_1,j_1+1} such that $v_{i_1,j_1} \to v_{i_1,j_1+1}$ during time-step k of $\mathcal{F}$. Then $k-1 \in A_{i_1,j_1}$, so $k-1 < y$ for all $y \in A_{i_2,j_2}$. In particular, v_{i_2,j_2} is white when v_{i_1,j_1} forces v_{i_1,j_1+1}. Since $v_{i_2,j_2} \notin C_{i_1}$, we know $v_{i_2,j_2} \neq v_{i_1,j_1+1}$. We conclude that when v_{i_1,j_1} forces v_{i_1,j_1+1}, it has two white neighbors, which is a contradiction. Thus, if $v_{i_1,j_1}v_{i_2,j_2} \in E(G)$ with $i_1 \neq i_2$, then $A_{i_1,j_1} \cap A_{i_2,j_2} \neq \emptyset$, and $\mathcal{C}$ is a parallel increasing path cover. □

Corollary 16. *For any graph G,* $\mathrm{PIP}(G) = \mathrm{Z}(G)$.

Proof. Let $\mathcal{Q}$ be a minimum parallel increasing path cover of G with the multiset of block partitions $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{m}$ as a witness. Let $B = \{v_{i,0}\}_{i=1}^{m}$ be the set of vertices corresponding to $\{A_{i,0}\}_{i=1}^{m}$. By Theorem 15, B is a zero forcing set of G, and thus $\mathrm{Z}(G) \leqslant |B| = |\mathcal{Q}| = \mathrm{PIP}(G)$.

Now let B be a minimum zero forcing set of G, $\mathcal{F}$ be a relaxed chronology of forces of B on G, and $\mathcal{C} = \{C_i\}_{i=1}^{m}$ be the chain set defined by $\mathcal{F}$ with the vertices of the path C_i labeled $\{v_{i,j}\}_{j=0}^{n_i-1}$ in path order. Let $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{m}$ be such that each $A_{i,j} = \mathrm{act}_{\mathcal{F}}(v_{i,j})$. By Theorem 15, $\mathcal{C}$ is a parallel increasing path cover of G with the multiset of block partitions $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{m}$ as a witness. Thus, $\mathrm{PIP}(G) \leqslant |\mathcal{C}| = |B| = \mathrm{Z}(G)$. □

It is known that given a graph G and a zero forcing set B of G, a chronological list of forces acting on B will create a chain set of G, which is a partition of the vertices of G. However, given a graph G, a parallel increasing path cover $\mathcal{Q}$ of G, and a multiset of block partitions $\{\mathcal{A}_i\}_{i=1}^{m}$ that is a witness for $\mathcal{Q}$, Definition 14 shows that the elements of the block partitions have a direct relationship with the edge set of G:

$$\text{For } i_1, i_2 \text{ distinct, } v_{i_1,j_1}v_{i_2,j_2} \in E(G) \Longrightarrow A_{i_1,j_1} \cap A_{i_2,j_2} \neq \emptyset.$$

Due to this, not only can one find parallel increasing path covers of a graph G, but one can also reverse this process. Given a number $K \in \mathbb{N}$ and a multiset of block partitions $\{\mathcal{A}_i\}_{i=1}^{m}$ of $\{0, 1, 2, \ldots, K\}$, one can construct the family of graphs for which $\{\mathcal{A}_i\}_{i=1}^{m}$ outlines a relaxed chronology of forces. The specifics of this fact are laid out in the next definition and corollary.

Definition 17 (Family of graphs induced by $\{\mathcal{A}_i\}_{i=1}^{m}$)**.** Let $K, m \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, m\}$ let $\mathcal{A}_i = (A_{i,j})_{j=0}^{n_i-1}$ be a block partition of $\{0, 1, 2, \ldots, K\}$. Let $V = \{v_{i,j}\}_{j=0}^{n_i-1}{}_{i=1}^{m}$, $E_1 = \{v_{i,j_1}v_{i,j_2} : |j_1 - j_2| = 1\}$, and

$$E_2 = \left\{v_{i_1,j_1}v_{i_2,j_2} : i_1 \neq i_2 \text{ and } A_{i_1,j_1} \cap A_{i_2,j_2} \neq \emptyset\right\}.$$

For each $E \subseteq E_2$, define $G(E)$ to be the graph with vertex set V and edge set $E_1 \cup E$. Furthermore, define $\mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^{m}}$ to be the set of all graphs of the form $G(E)$ for some $E \subseteq E_2$. We refer to $\mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^{m}}$ as the *family of graphs induced by* $\{\mathcal{A}_i\}_{i=1}^{m}$.

Example 18. Consider the multiset of block partitions $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ given by

$\mathcal{A}_1$:	$A_{1,0} = \{0\}$	$A_{1,1} = \{1, 2, 3\}$	$A_{1,2} = \{4\}$		
$\mathcal{A}_2$:	$A_{2,0} = \{0, 1, 2, 3, 4\}$				
$\mathcal{A}_3$:	$A_{3,0} = \{0\}$	$A_{3,1} = \{1\}$	$A_{3,2} = \{2\}$	$A_{3,3} = \{3\}$	$A_{3,4} = \{4\}$.

Recall from Definition 17 that the vertices $v_{i,j}$ are associated with the sets $A_{i,0}, \ldots, A_{i,n_i-1}$, so define the set of vertices $V = \{v_{1,0}, v_{1,1}, v_{1,2}, v_{2,0}, v_{3,0}, v_{3,1}, v_{3,2}, v_{3,3}, v_{3,4}\}$. Define the sets of edges E_1 and E_2 as follows:

$$E_1 = \{v_{1,0}v_{1,1}, v_{1,1}v_{1,2}, v_{3,0}v_{3,1}, v_{3,1}v_{3,2}, v_{3,2}v_{3,3}, v_{3,3}v_{3,4}\},$$

$$\begin{aligned} E_2 = \{&v_{1,0}v_{2,0}, v_{2,0}v_{3,0}, v_{1,0}v_{3,0}, v_{1,1}v_{2,0}, v_{2,0}v_{3,1}, v_{1,1}v_{3,1}, \\ &v_{2,0}v_{3,2}, v_{1,1}v_{3,2}, v_{2,0}v_{3,3}, v_{1,1}v_{3,3}, v_{1,2}v_{2,0}, v_{2,0}v_{3,4}, v_{1,2}v_{3,4}\}. \end{aligned}$$

Then $G(E_2)$ is the graph with the largest edge set among graphs in $\mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^3}$, and is shown in Figure 6. The graph $G(\emptyset)$ has the smallest edge set among graphs in $\mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^3}$. In Figure 6, the bold edges are the edges of $G(\emptyset)$. If we choose any H that is a subgraph of $G(E_2)$ and contains $G(\emptyset)$, then we obtain another graph in $\mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^3}$ such that $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ defines a multiset of block partitions for the parallel increasing path cover induced by the edges in E_1.

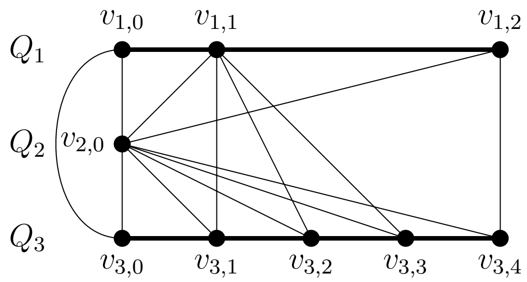


Figure 6: The graph $G(E_2)$, with the edges in E_1 shown in bold.

Theorem 19. *Let G be a graph.*

1. *If $G \in \mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^m}$ for some multiset of block partitions $\{\mathcal{A}_i\}_{i=1}^m$ of $\{0, 1, 2, \ldots, K\}$ with $\mathcal{A}_i = (A_{i,j})_{j=0}^{n_i-1}$, then $\mathcal{C} = \{C_i\}_{i=1}^m$ with $C_i = G\big[\{v_{i,j}\}_{j=0}^{n_i-1}\big]$ forms a chain set induced by a relaxed chronology of forces $\mathcal{F}$ acting on the zero forcing set $\{v_{i,0}\}_{i=1}^m$ of G.*

2. $\mathrm{Z}(G) = \min\{m : G \in \mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^m}$ *for some multiset of block partitions* $\{\mathcal{A}_i\}_{i=1}^m\}$.

Proof. For (1), since $G \in \mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^m}$, we have that $V(G) = \{v_{i,j}\}_{j=0}^{n_i-1}{}_{i=1}^{m}$ and $E(G) = E_1 \cup E$, where $E_1 = \big\{v_{i,j_1}v_{i,j_2} : |j_1 - j_2| = 1\big\}$, and $E \subseteq E_2 = \big\{v_{i_1,j_1}v_{i_2,j_2} : i_1 \neq i_2 \text{ and } A_{i_1,j_1} \cap A_{i_2,j_2} \neq \emptyset\big\}$. Since $E_1 \subseteq E(G)$, there exist m path subgraphs $\{C_i\}_{i=1}^m$ of G whose vertex sets $V(C_i) = \{v_{i,j}\}_{j=0}^{n_i-1}$ partition $V(G)$. Letting each set $A_{i,j}$ correspond to each vertex $v_{i,j}$, the fact that $E(G) \setminus E_1 = E \subseteq E_2$ ensures that $\mathcal{C}$ is a parallel increasing path cover of G as witnessed by $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^m$. The result then follows by Theorem 15.

For (2), it follows from Corollary 16 and (1) that $\mathrm{Z}(G) = \mathrm{PIP}(G) \leqslant m'$, where m' is the minimum defined in the statement of the corollary. Now let B be a minimum zero forcing set of G, let $\mathcal{F}$ be a chronological list of forces of B on G, and $\mathcal{C} = \{C_i\}_{i=1}^{|B|}$ be the chain set defined by $\mathcal{F}$, with each forcing chain C_i having vertex set $\{v_{i,j}\}_{j=0}^{n_i-1}$. Next, define the multiset of block partitions $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{|B|}$ by setting $A_{i,j} = \mathrm{act}_{\mathcal{F}}(v_{i,j})$. By Theorem 15,

$\mathcal{C}$ is a parallel increasing path cover of G as witnessed by $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{|B|}$. Then, if we let $E_1 = \{v_{i,j_1}v_{i,j_2} : |j_1 - j_2| = 1\}$, and $E_2 = \{v_{i_1,j_1}v_{i_2,j_2} : i_1 \neq i_2 \text{ and } A_{i_1,j_1} \cap A_{i_2,j_2} \neq \emptyset\}$, it follows that $E(G) = E_1 \cup E$, where $E \subseteq E_2$. Thus, $G \in \mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^{|B|}}$, and so $m' \leqslant |B| = \mathrm{Z}(G)$, completing the proof. □

Corollary 20. *Let G be a graph, B be a standard zero forcing set of G, $\mathcal{F}$ be a propagating family of forces of B on G, and let $\{\mathcal{A}_i\}_{i=1}^{|B|} = \{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{|B|}$ be a multiset of block partitions that record the $\mathcal{F}$-active time-steps of the vertices $\{v_{i,j}\}_{j=0}^{n_i-1}{}_{i=1}^{|B|}$ of the chain set $\mathcal{C} = \{C_i\}_{i=1}^{|B|}$ defined by $\mathcal{F}$. Then for each $H \in \mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^{|B|}}$,*

$$\mathrm{Z}(H) \leqslant |B| \qquad \text{and} \qquad \mathrm{pt}(H, B) \leqslant \mathrm{pt}(G, B).$$

If in addition, B is an efficient standard zero forcing set of G, then

$$\mathrm{Z}(H) \leqslant \mathrm{Z}(G) \qquad \text{and} \qquad \mathrm{pt}(H, B) \leqslant \mathrm{pt}(G).$$

Proof. Since $H \in \mathcal{G}_{\{\mathcal{A}_i\}_{i=1}^{|B|}}$, it follows that B is a zero forcing set of H and $\mathcal{F}$ is a relaxed chronology of forces of B on H. Thus $\mathrm{Z}(H) \leqslant |B|$ and $\mathrm{pt}(H, B) \leqslant \mathrm{ct}(\mathcal{F}) = \mathrm{pt}(G, B)$ because $\mathcal{F}$ is a propagating family of forces. If in addition B is an efficient zero forcing set, then B is a minimum zero forcing set and $\mathrm{Z}(H) \leqslant \mathrm{Z}(G)$. Furthermore, $\mathrm{pt}(G, B) = \mathrm{pt}(G)$. □

Since $G(\emptyset)$ consists of m disjoint paths by Definition 17, it follows that $\mathrm{Z}(G(\emptyset)) = m$. We can use Theorem 19 to determine $\mathrm{Z}(G(E_2))$. It is well known that $\mathrm{Z}(G) \geqslant \delta(G)$ where $\delta(G)$ denotes the minimum degree among vertices of G. A set $A \subseteq V(G)$ is a *clique* if u and v are adjacent for $u \neq v$, $u, v \in A$.

Proposition 21. *Let $K, m \in \mathbb{N}$ with $m \geqslant 1$. Then $\delta(G(E_2)) = \mathrm{Z}(G(E_2)) = m$ unless $n_i = 1$ for all $i = 1, \dots, m$, in which case $G(E_2) \cong K_m$ and $\mathrm{Z}(G(E_2)) = m - 1$.*

Proof. If $n_i = 1$ for all $i = 1, \dots, m$, i.e., $E_1 = \emptyset$, then $G(E_2) \cong K_m$ and $\mathrm{Z}(G(E_2)) = m - 1$. So assume E_1 has an edge, which implies $|V| \geqslant m + 1$. Consider the chain set $C_i = G\big[\{v_{i,j}\}_{j=0}^{n_i-1}\big]$ in Corollary 19. If $n_i = 1$, then $\mathrm{act}_{\mathcal{F}}(v_{i,0}) = \{0, \dots, K\}$, so $v_{i,0}$ is adjacent to every other vertex and therefore has at least m neighbors. Now assume $n_i > 1$. For $k = 1, \dots, K$, let $A_k = \{v_{i,j} : k \in \mathrm{act}_{\mathcal{F}}(v_{i,j})\}$, and note that A_k induces a clique for $k = 0, \dots, K$. Since every vertex $v_{i,j}$ is in some A_k, it has $m - 1$ neighbors not in its chain. And since its chain has at least two vertices, it has at least one neighbor in its chain. Thus $m \leqslant \delta(G(E_2) \leqslant \mathrm{Z}(G(E_2))$. Since A_0 is a standard zero forcing set of size m, $\mathrm{Z}(G(E_2)) = m$. □

Remark 22. For readers familiar with the minor monotone floor of zero forcing [4, 14], we note that one can modify Definition 14 (and necessary prior definitions) to define a structure analogous to parallel increasing path covers for which corresponding versions of Theorem 15 and Corollary 16 hold. Chain sets for the minor monotone floor of zero forcing in a graph G correspond to chain sets for standard zero forcing in some supergraph of G

on the same vertex set. Thus, to create an analog of Definition 14 for the minor monotone floor of zero forcing, we simply allow each $\mathcal{Q}$ to be a path cover of some supergraph of G on the same vertex set. Observe that property (1) is unchanged, as this is not related to $\mathcal{Q}$ being a path cover. The corresponding change for Definition 17 is to consider graphs of the form $G(E) = (V, E)$ with $E \subseteq E_1 \cup E_2$.

4 Applications of parallel increasing path covers

We now apply the results of the previous section to standard and positive semidefinite zero forcing. Notice that in a parallel increasing path cover, the relationship between the multisets of block partitions of time-steps and the edge set is based on nonempty intersections as described in (1). This property is preserved under certain operations, which we can use to generalize known results and establish new ones.

Hogben et al. defined the *terminus* and *reversal* of a set of forces in [13]. They then compared the propagation time for a reversal with the original set of forces. We adapt these definitions to relaxed chronologies of forces.

Definition 23 (Terminus and reversal)**.** Let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ be a relaxed chronology of forces for a standard zero forcing set B of a graph G.

1. The *terminus* of $\mathcal{F}$, denoted $\mathrm{Term}(\mathcal{F})$, is the set of vertices of G that do not perform any forces in $\mathcal{F}$.
2. The *reversal* of $\mathcal{F}$, denoted $\mathrm{Rev}(\mathcal{F})$, is the result of reversing the forces and time-steps in $\mathcal{F}$, i.e., $\mathrm{Rev}(\mathcal{F}) = \{F^{(k)}_{\mathrm{Rev}}\}_{k=1}^{K}$ with
$$F^{(k)}_{\mathrm{Rev}} = \{v \to u : u \to v \in F^{(K-k+1)}\}.$$

Observe that the terminus depends only on the set of forces in the relaxed chronology. The next definition is equivalent to the definition of the standard propagation time of a set of standard forces in [13] and extends the definition to other types of zero forcing.

Definition 24 (Propagation time of forces)**.** Let CCR-X be a consistent color change rule, G be a graph, and B be an X-forcing set of G. For a set of X-forces F of B, define $E^{(0)} = B$ and for $t \geqslant 0$, define $E^{(t+1)}$ to be the set of vertices w such that $v \to w \in F \cap S_X(G, \bigcup_{i=0}^{t} E^{(i)})$ for some v. The *X-propagation time* of F in G, denoted $\mathrm{pt}_X(G, F)$, is the least t_0 such that $V(G) = \bigcup_{t=0}^{t_0} E^{(t)}$. For a relaxed chronology $\mathcal{F}$, the *X-propagation time* of $\mathcal{F}$ in G, denoted $\mathrm{pt}_X(G, \mathcal{F})$, is the propagation time of the underlying set of forces in $\mathcal{F}$.

It is clear that $\mathrm{pt}_X(G, \mathcal{F}) \leqslant \mathrm{ct}(\mathcal{F})$. The next lemma restates [5, Theorem 2.6], generalizes [13, Observation 2.4] to relaxed chronologies in the obvious manner, and improves [13, Corollary 2.10].

Lemma 25. *Let G be a graph, B be a standard zero forcing set of G, and $\mathcal{F}$ be a relaxed chronology of forces for B. Then*

1. [5] $\mathrm{Term}(\mathcal{F})$ *is a standard zero forcing set of* G,

2. $\mathrm{Rev}(\mathcal{F})$ *is a relaxed chronology for* $\mathrm{Term}(\mathcal{F})$, *and*

3. $\mathrm{pt}(G, \mathrm{Rev}(\mathcal{F})) = \mathrm{pt}(G, \mathcal{F})$.

Proof. For (2), let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$. By Theorem 15, $\mathcal{F}$ induces a parallel increasing path cover $\mathcal{Q}$ with corresponding multiset of block partitions $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{m}$ recording active time-steps. Define the multiset of block partitions $\{(R_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{m}$ by

$$N \in A_{i,j} \iff K - N \in R_{i,n_i-j-1}.$$

Then $\{(R_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{m}$ is a multiset of block partitions of $\{0, 1, 2, \ldots, K\}$ that is also a witness that $\mathcal{Q}$ is a parallel increasing path cover of G. The vertices of $\mathrm{Term}(\mathcal{F})$ correspond precisely to $\{R_{i,0}\}_{i=1}^{m}$, so Theorem 15 implies $\mathrm{Rev}(\mathcal{F})$ is a relaxed chronology of forces for $\mathrm{Term}(\mathcal{F})$ with $\mathrm{ct}(\mathrm{Rev}(\mathcal{F})) = \mathrm{ct}(\mathcal{F})$.

For (3), note that although [13, Corollary 2.10] is stated for sets of forces of minimum zero forcing sets, it is a consequence of Lemma 2.9, which does not assume minimality. Thus $\mathrm{pt}(G, \mathrm{Rev}(\mathcal{F})) \leqslant \mathrm{pt}(G, \mathcal{F})$. Equality follows by noting that $\mathrm{Rev}(\mathrm{Rev}(\mathcal{F})) = \mathcal{F}$. □

Remark 26. Note that relaxed chronologies can give a simpler proof that $\mathrm{pt}(G, \mathrm{Rev}(\mathcal{F})) \leqslant \mathrm{pt}(G, \mathcal{F})$ than that given in Lemma 2.9 and Corollary 2.10 in [5] (the reader is invited to consult the proof of [5, Lemma 2.9], where a new parameter $Q_t(\mathcal{F})$ is introduced for a set $\mathcal{F}$ of forces). Let $\mathcal{F}'$ be the relaxed chronology obtained by propagating the forces in $\mathcal{F}$. Since $\mathrm{ct}(\mathrm{Rev}(\mathcal{F}')) = \mathrm{ct}(\mathcal{F}')$ and $\mathcal{F}$ and $\mathcal{F}'$ have the same underlying set of forces (as do $\mathrm{Rev}(\mathcal{F})$ and $\mathrm{Rev}(\mathcal{F}')$), we see that

$$\mathrm{pt}(G, \mathrm{Rev}(\mathcal{F})) = \mathrm{pt}(G, \mathrm{Rev}(\mathcal{F}')) \leqslant \mathrm{ct}(\mathrm{Rev}(\mathcal{F}')) = \mathrm{ct}(\mathcal{F}') = \mathrm{pt}(G, \mathcal{F}') = \mathrm{pt}(G, \mathcal{F}).$$

We can take further advantage of the structure of parallel increasing path covers and block partitions to obtain additional results by focusing on when vertices are active. Of particular note, we utilize this additional information provided by parallel increasing path covers to compare each of PSD propagation time and power propagation time to standard propagation time. Specifically, we are able to show that if G is a graph and $m \in \mathbb{N}$ such that $m \geqslant \mathrm{Z}(G)$, then

$$\mathrm{pt}_+(G, m) \leqslant \left\lceil \frac{\mathrm{pt}(G, m)}{2} \right\rceil \text{ and } \mathrm{ppt}(G, m) \leqslant \left\lceil \frac{\mathrm{pt}(G, m)}{2} \right\rceil$$

(see Theorems 31 and 36).

Definition 27 (Sets $V_{\mathcal{F}}^{N-}$, $V_{\mathcal{F}}^{N}$, $V_{\mathcal{F}}^{N+}$, $V_{\mathcal{F}}^{[M,N]}$)**.** Let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ be a relaxed chronology of forces for a graph G and some standard zero forcing set. For any $N \in \{0, 1, \ldots, K\}$, define the following partition of $V(G)$:

$$\begin{aligned} V_{\mathcal{F}}^{N-} &= \{v \in V(G) \mid \mathrm{act}_{\mathcal{F}}(v) \subseteq \{0, 1, \ldots, N-1\}\}, \\ V_{\mathcal{F}}^{N} &= \{v \in V(G) \mid N \in \mathrm{act}_{\mathcal{F}}(v)\}, \text{ and} \\ V_{\mathcal{F}}^{N+} &= \{v \in V(G) \mid \mathrm{act}_{\mathcal{F}}(v) \subseteq \{N+1, N+2, \ldots, K\}\}. \end{aligned}$$

Now for any $0 \leqslant M \leqslant N \leqslant K$, define

$$V_{\mathcal{F}}^{[M,N]} = (V_{\mathcal{F}}^{M} \cup V_{\mathcal{F}}^{M+}) \cap (V_{\mathcal{F}}^{N-} \cup V_{\mathcal{F}}^{N}).$$

Note that $V_{\mathcal{F}}^{N}$ consists of the vertices active after time-step N, while $V_{\mathcal{F}}^{N-}$ and $V_{\mathcal{F}}^{N+}$ partition the remaining vertices into ones that have performed a force during some time-step $N' < N$ and those who do not become blue until some time-step $N' > N$, respectively. Likewise, $V_{\mathcal{F}}^{[M,N]}$ consists of the vertices which are active at some time-step k with $M \leqslant k \leqslant N$.

Lemma 28. *Let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ be a relaxed chronology of forces for a graph G and some standard zero forcing set B, and let $N \in \{0, 1, \ldots, K\}$.*

1. *B is a zero forcing set of $H = G[V_{\mathcal{F}}^{[0,N]}]$ with $\operatorname{pt}(H, B) \leqslant N$.*
2. *$B' = V_{\mathcal{F}}^{N}$ is a zero forcing set of $H' = G[V_{\mathcal{F}}^{[N,K]}]$ with $\operatorname{pt}(H', B') \leqslant K - N$.*

Proof. By Theorem 15, $\mathcal{F}$ induces a parallel increasing path cover $\mathcal{Q}$ with witness $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{m}$, where $A_{i,j} = \operatorname{act}_{\mathcal{F}}(v_{i,j})$. Denote the vertices in $V_{\mathcal{F}}^{N}$ as $\{v_{i,j_i}\}_{i=1}^{m}$, and note that these vertices correspond to the members of the block partitions A_{i,j_i} containing N.

For (1), form a path cover $\mathcal{Q}'$ of H by restricting $\mathcal{Q}$ to the vertices $\{v_{i,j}\}_{j=0}^{j_i}{}_{i=1}^{m}$. The multiset of block partitions $\{(A'_{i,j})_{j=0}^{j_i}\}_{i=1}^{m}$ formed by starting with $\{(A_{i,j})_{j=0}^{j_i}\}_{i=1}^{m}$ and removing all elements in $\{N+1, N+2, \ldots, K\}$ produces a multiset of block partitions of $\{0, 1, \ldots, N\}$. Notice that $A'_{i_1,j_1} \cap A'_{i_2,j_2} \neq \emptyset$ precisely when $A_{i_1,j_1} \cap A_{i_2,j_2} \neq \emptyset$. Hence, $\{(A'_{i,j})_{j=0}^{j_i}\}_{i=1}^{m}$ is a witness that $\mathcal{Q}'$ is a parallel increasing path cover of H. The conclusions in (1) then follow from Theorem 15.

For (2), form $\mathcal{Q}'$ by restricting $\mathcal{Q}$ to the vertices $\{v_{i,j}\}_{j=j_i}^{n_i-1}{}_{i=1}^{m}$ in H'. Now, start with $\{(A_{i,j})_{j=j_i}^{n_i-1}\}_{i=1}^{m}$, remove all elements in $\{0, 1, \ldots, N-1\}$, and subtract N from all remaining elements in each partition to obtain a multiset of block partitions of $\{0, 1, \ldots, K-N\}$. By similar reasoning as given above, this is a witness that $\mathcal{Q}'$ is a parallel increasing path cover of H', and Theorem 15 implies (2). ☐

Corollary 29. *Let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ be a relaxed chronology of forces for a graph G for some standard zero forcing set. For any $0 \leqslant M < N \leqslant K$, $V_{\mathcal{F}}^{M}$ and $V_{\mathcal{F}}^{N}$ are zero forcing sets of $G[V_{\mathcal{F}}^{[M,N]}]$, and both sets have propagation time at most $N - M$.*

Proof. The results for $V_{\mathcal{F}}^{M}$ follow from first applying part (1) of Lemma 28 with N and then applying part (2) with M. Lemma 25 then implies the corresponding results for $V_{\mathcal{F}}^{N}$, as this is $\operatorname{Term}(\mathcal{F}')$ for the relaxed chronology $\mathcal{F}' = \{F^{(k+M)}\}_{k=1}^{N-M}$ on the graph $G[V_{\mathcal{F}}^{[M,N]}]$ with zero forcing set $V_{\mathcal{F}}^{M}$. ☐

Using these techniques we also obtain results for positive semidefinite zero forcing and propagation time.

Lemma 30. *Let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ be a relaxed chronology of forces for a graph G and some standard zero forcing set. Then no vertices in $V_{\mathcal{F}}^{N-}$ are adjacent to vertices in $V_{\mathcal{F}}^{N+}$. In particular, if both sets are nonempty, then $V_{\mathcal{F}}^{N}$ is a vertex cut of G.*

Proof. Using Theorem 15, $\mathcal{F}$ induces a parallel increasing path cover $\mathcal{Q}$ with corresponding multiset of block partitions $\{(A_{i,j})_{j=0}^{n_i-1}\}_{i=1}^{m}$. Consider $v_{i_1,j_1} \in V_{\mathcal{F}}^{N-}$ and $v_{i_2,j_2} \in V_{\mathcal{F}}^{N+}$ with corresponding sets of active time-steps A_{i_1,j_1} and A_{i_2,j_2}. By Definition 14, if $v_{i_1,j_1}v_{i_2,j_2} \in E(G)$ for $i_1 \neq i_2$, then it must be that $A_{i_1,j_1} \cap A_{i_2,j_2} \neq \emptyset$, and if $v_{i_1,j_1}v_{i_1,j_2} \in E(G)$ with $i_1 = i_2$ and $j_1 < j_2$, it must be that $\max(A_{i_1,j_1}) + 1 = \min(A_{i_1,j_2})$. Neither of these situations can occur since $A_{i_1,j_1} \subseteq \{0, 1, \ldots, N-1\}$ and $A_{i_2,j_2} \subseteq \{N+1, N+2, \ldots, K\}$. $\square$

Theorem 31. *Let G be a graph and $m \in \mathbb{N}$ such that $m \geqslant \mathrm{Z}(G)$. Then*

$$\mathrm{pt}_+(G, m) \leqslant \left\lceil \frac{\mathrm{pt}(G, m)}{2} \right\rceil.$$

Proof. Let B be a zero forcing set of G of size $m \geqslant \mathrm{Z}(G)$ such that $\mathrm{pt}(G, B) = \mathrm{pt}(G, m)$, with corresponding propagating family of forces $\mathcal{F} = \{F^{(k)}\}_{k=1}^{\mathrm{pt}(G,m)}$ (where $F^{(k)}$ denotes the set of forces during time-step k). Using Theorem 15, this induces a parallel increasing path cover $\mathcal{Q} = \{Q_i\}_{i=1}^{m}$, where each $Q_i = \{v_{i,j}\}_{j=0}^{n_i-1}$ corresponds to a block partition $(A_{i,j})_{j=1}^{n_i-1}$ of $\{0, 1, 2, \ldots, \mathrm{pt}(G, m)\}$. Now set $N = \left\lceil \frac{\mathrm{pt}(G,m)}{2} \right\rceil$, and let $B' = V_{\mathcal{F}}^{N}$. If either $V_{\mathcal{F}}^{N-}$ or $V_{\mathcal{F}}^{N+}$ is empty then $\mathrm{pt}(G, m) \leqslant 1$ and the result is immediate, so suppose that $V_{\mathcal{F}}^{N-}$ and $V_{\mathcal{F}}^{N+}$ are nonempty sets of vertices. Then by Lemma 30, B' is a vertex cut of G, and as a result $H_1 = G[V_{\mathcal{F}}^{N-}]$ and $H_2 = G[V_{\mathcal{F}}^{N+}]$ are disjoint sets of components of $G - B'$. By the definition of the PSD color change rule, PSD forcing in $H_1 = G[V_{\mathcal{F}}^{N-}]$ and $H_2 = G[V_{\mathcal{F}}^{N+}]$ with B' blue will occur independently and simultaneously. Using Corollary 29 on H_1 and H_2, we conclude that

$$\mathrm{pt}_+(G, B') \leqslant \max\{\mathrm{pt}(H_1, B), \mathrm{pt}(H_2, B)\} \leqslant \max\{\mathrm{pt}(G, B) - N, N\} = \left\lceil \frac{\mathrm{pt}(G, m)}{2} \right\rceil. \quad \square$$

Remark 32. In the preceding proof, B' is chosen to be $V_{\mathcal{F}}^{N}$ for $N = \left\lceil \frac{\mathrm{pt}(G,m)}{2} \right\rceil$, but any choice of N such that $0 \leqslant N \leqslant \mathrm{pt}(G, m)$ will yield a PSD forcing set of G of size m with propagation time bounded above by $\max\{\mathrm{pt}(G, m) - N, N\}$. In particular, when $\mathrm{pt}(G, m)$ is odd, the choice of $N = \left\lfloor \frac{\mathrm{pt}(G,m)}{2} \right\rfloor$ also establishes the bound in Theorem 31.

Throttling minimizes the sum of the resources used to accomplish a task (number of blue vertices) and the time needed to complete that task (propagation time); see [14, Chapter 10] for more information. Define $\mathrm{th}_+(G, m) = m + \mathrm{pt}_+(G, m)$, the minimum throttling that can be achieved with m vertices, and $\mathrm{th}_+(G) = \min_{\mathrm{Z}_+(G) \leqslant m \leqslant |V(G)|} \mathrm{th}_+(G, m)$.

Corollary 33. *For any graph G,*

$$\mathrm{th}_+(G) \leqslant \min_{m \geqslant \mathrm{Z}(G)} \left(m + \left\lceil \frac{\mathrm{pt}(G, m)}{2} \right\rceil \right).$$

Corollary 34. *For any graph G such that $\mathrm{Z}_+(G) = \mathrm{Z}(G)$,*

$$\mathrm{pt}_+(G) \leqslant \left\lceil \frac{\mathrm{pt}(G)}{2} \right\rceil.$$

Remark 35. In [16], it was shown that for $t \in \{1,2,3\}$, $\mathrm{Z}_+(P_s \,\Box\, P_t) = \mathrm{Z}(P_s \,\Box\, P_t) = \min\{s,t\}$, where $P_s \,\Box\, P_t$ denotes the Cartesian product of the path graphs P_s and P_t, i.e., the graph with vertex set $V(P_s) \times V(P_t)$ such that (u,v) is adjacent to (u',v') if and only if (1) $u = u'$ and $vv' \in E(P_t)$, or (2) $v = v'$ and $uu' \in E(P_s)$. It was also shown that $\mathrm{pt}(P_s \,\Box\, P_t) = \max\{s,t\} - 1$, and $\mathrm{pt}_+(P_s \,\Box\, P_t) = \left\lceil \frac{\max\{s,t\}-1}{2} \right\rceil$. Thus the infinite class of graphs $P_s \,\Box\, P_t$ establishes that the bound in Corollary 34 is sharp.

Note that in the proof of Theorem 31, we started with a relaxed chronology $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ and chose a single time-step of vertices $V_{\mathcal{F}}^{N}$ as the PSD forcing set. One can generalize this approach and choose multiple time-steps. For example, choosing $B = V_{\mathcal{F}}^{\lceil K/4 \rceil} \cup V_{\mathcal{F}}^{\lfloor 3K/4 \rfloor}$ would construct a PSD forcing set with $\mathrm{pt}_+(G,B) \leqslant \left\lceil \frac{K}{4} \right\rceil$. In general, this technique multiplies the size of the zero forcing set by some positive integer ℓ, and reduces propagation time by approximately a factor of 2ℓ.

Letting $\mathrm{ppt}(G,m)$ denote the power propagation time for sets of size m, we can also establish power domination versions of the results above. While one can do this by directly generalizing the techniques of this section, we will see that the results of the next section simplify arguments significantly. Hence, we present the proof of the following theorem in Appendix A.

Theorem 36. *Let G be a graph and $m \in \mathbb{N}$ such that $m \geqslant \mathrm{Z}(G)$. Then*

$$\mathrm{ppt}(G,m) \leqslant \left\lceil \frac{\mathrm{pt}(G,m)}{2} \right\rceil.$$

5 Path bundles

In this section, we introduce path bundles (see Definition 41). These are sets of paths contained in PSD forcing trees that are parallel increasing path covers for the induced subgraph on the vertices in the bundle. We apply path bundles to compare PSD and standard propagation times, and we establish a PSD analog of the reversal of standard zero forcing. We start by showing that relaxed chronologies provide us a convenient method of restricting forces to subgraphs.

Definition 37 (Restriction)**.** Let G be a graph with induced subgraph H, and let CCR-X be a consistent color change rule.

1. Given a set of X-forces F between the vertices in G, define its *restriction* to H, denoted $F|_H$, to be $\{v \to u \in F : u, v \in V(H)\}$.

2. Given a relaxed chronology $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ for some X-forcing set B of G, define its *restriction* to H, denoted $\mathcal{F}|_H$, to be $\{F^{(k)}|_H\}_{k=1}^{K}$.

Notice that in $\mathcal{F}|_H$, we preserve the time-step of each force. If $\mathcal{F}|_H$ is a relaxed chronology for some forcing set B' of H, we have that $\mathrm{ct}(\mathcal{F}|_H) = \mathrm{ct}(\mathcal{F})$. Additionally, it is possible that for some k, $F^{(k)} \neq \emptyset$ and $F^{(k)}|_H = \emptyset$.

Definition 38 (Forcing subtree)**.** Let G be a graph, H be an induced subgraph of G, and B be a PSD forcing set of G. Let $\mathcal{F}$ be a relaxed chronology of PSD forces of B on G with expansion sequence $\{E^{[k]}\}_{k=0}^{K}$ and PSD forcing trees $\mathcal{T} = \{T_i\}_{i=1}^{|B|}$. A component T of $T_i \cap H$ is a *forcing subtree* of T_i in H, and $u \in V(T)$ is an *initial vertex* if either $u \in B$ or there exists $k \in \mathbb{N}$ such that $u \in E_{\mathcal{F}}^{(k)}$ and $v \to u \in F^{(k)}$ but $v \notin V(T)$. When a forcing subtree T is a path, we also call T a *forcing subpath.*

Lemma 39. *Let G be a graph, H be an induced subgraph of G, B be a PSD forcing set of G, and $\mathcal{F}$ be a relaxed chronology of PSD forces of B on G with forcing trees $\mathcal{T}$. Let B' be the set of initial vertices in the forcing subtrees of $\mathcal{T}$ in H. Then B' is a PSD forcing set of H, and $\mathcal{F}|_H$ defines a relaxed chronology of PSD forces for B' in H with* $\mathrm{pt}_+(H, \mathcal{F}|_H) \leqslant \mathrm{pt}_+(G, \mathcal{F})$.

Proof. Let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ with corresponding expansion sequence $\{E_{\mathcal{F}}^{[k]}\}_{k=0}^{K}$. We first show that the PSD forces in $F^{(k)}|_H$ are valid when $E_{\mathcal{F}}^{[k-1]} \cap V(H)$ is blue. Consider $v \to u \in F^{(k)}|_H$. Since $v \to u \in F^{(k)}$, u is the unique neighbor of v in some component of $G - E_{\mathcal{F}}^{[k-1]}$. The components of $H - E_{\mathcal{F}}^{[k-1]}$ are formed from induced subgraphs of $G - E_{\mathcal{F}}^{[k-1]}$, and hence $v \to u$ is a valid PSD force in H during time-step k.

We now claim that if $(E_{\mathcal{F}}^{[k-1]} \cap V(H)) \cup B'$ is blue, then all vertices in $(E_{\mathcal{F}}^{[k]} \cap V(H)) \cup B'$ will be blue after the forces in $F^{(k)}|_H$ are performed. Consider $v \to u \in F^{(k)}$ with $u \in (E_{\mathcal{F}}^{[k]} \cap V(H)) \cup B'$. If $v \notin V(H)$, then u is an initial vertex in some forcing subtree. By definition, $u \in B'$, so we conclude that u is initially blue, and hence also blue after time-step k. Otherwise, $v \in V(H)$, so the preceding paragraph implies that $v \to u \in F^{(k)}|_H$ is a valid PSD force in H, and u will be blue after time-step k. Combining these two results with induction on k, we conclude that $\mathcal{F}|_H$ is a relaxed chronology of PSD forces for B' in H with expansion sequence $\{(E_{\mathcal{F}}^{[k]} \cap V(H)) \cup B'\}_{k=0}^{K}$. □

In the case that $\mathcal{F}$ is a propagating set of PSD or standard forces, then by Lemmas 39 and 40, it follows that

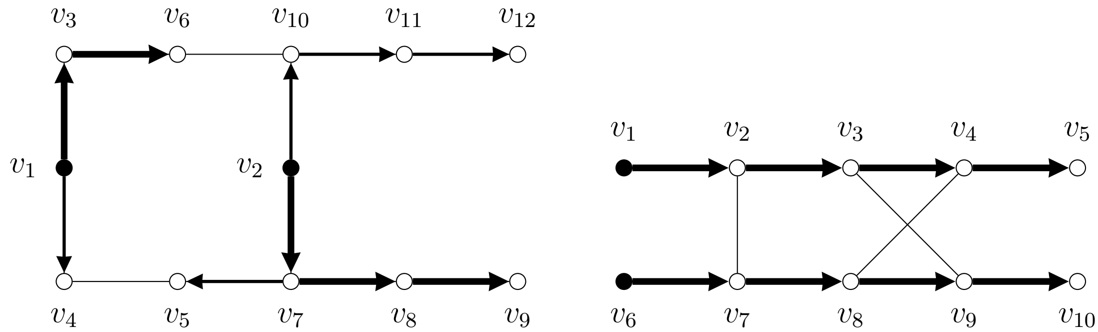
$$\mathrm{pt}_+(H, B') \leqslant \mathrm{pt}_+(G, B) \text{ and } \mathrm{pt}(H, B') \leqslant \mathrm{pt}(G, B).$$

The proof of Lemma 39 can be adapted to establish the next lemma.

Lemma 40. *Let G be a graph, H be an induced subgraph of G, B be a standard zero forcing set of G, and $\mathcal{F}$ be a relaxed chronology of standard forces of B on G with chain set $\mathcal{C}$. Let B' be the set of initial vertices in the forcing subpaths of $\mathcal{C}$ in H. Then B' is a standard zero forcing set of H, and $\mathcal{F}|_H$ defines a relaxed chronology of standard forces for B' in H with* $\mathrm{pt}(H, \mathcal{F}|_H) \leqslant \mathrm{pt}(G, \mathcal{F})$.

We now turn our attention to the case when the restriction of PSD forcing results in standard zero forcing and the resulting zero forcing set has the same size as the original PSD forcing set.

Definition 41 (Path bundle)**.** Let G be a graph, B be a PSD forcing set of G, and let $\mathcal{F}$ be a relaxed chronology of PSD forces of B on G with associated PSD forcing tree cover $\mathcal{T} = \{T_i\}_{i=1}^{|B|}$. Let $\mathcal{Q} = \{Q_i\}_{i=1}^{|B|}$ be a set of paths such that $V(Q_i) \subseteq V(T_i)$. Define $H = G[\bigcup_{Q_i \in \mathcal{Q}} V(Q_i)]$. We say that $\mathcal{Q}$ is a *path bundle* of $\mathcal{F}$ if $\mathcal{F}|_H$ is a relaxed chronology of standard forces in H for the initial vertices in $\mathcal{Q}$. We abuse notation and use $\mathcal{F}|_{\mathcal{Q}}$ to also denote $\mathcal{F}|_H$.



(a) G (b) H

Figure 7: (a) The graph G in Example 42 and (b) the graph H in Example 43.

Example 42. Consider the graph G in Figure 7(a) and let $\mathcal{F} = (F^{(1)}, F^{(2)}, F^{(3)})$, where

$$F^{(1)} = \{v_1 \to v_3, v_1 \to v_4, v_2 \to v_7, v_2 \to v_{10}\}$$
$$F^{(2)} = \{v_7 \to v_5, v_7 \to v_8, v_3 \to v_6, v_{10} \to v_{11}\}$$
$$F^{(3)} = \{v_8 \to v_9, v_{11} \to v_{12}\}.$$

Then $\mathcal{F}$ is a relaxed chronology of PSD forces of $B = \{v_1, v_2\}$ on G and forces are indicated in the figure with arrows. If $W_1 = \{v_1, v_3, v_4, v_6\}$ and $W_2 = \{v_2, v_5, v_7, v_8, \dots, v_{12}\}$, then the associated PSD forcing tree cover is $\mathcal{T} = \{G[W_1], G[W_2]\}$ and one of the path bundles of $\mathcal{F}$ is $\mathcal{Q} = \{G[\{v_1, v_3, v_6\}], G[\{v_2, v_7, v_8, v_9\}]\}$. The path bundle $\mathcal{Q}$ is indicated with bold edges in Figure 7(a).

Example 43. Consider the graph H in Figure 7(b). The set $B = \{v_1, v_6\}$ is a PSD forcing set of H with relaxed chronology

$$\mathcal{F} = (\{v_1 \to v_2, v_6 \to v_7\}, \{v_2 \to v_3, v_7 \to v_8\}, \{v_3 \to v_4, v_8 \to v_9\}, \{v_4 \to v_5, v_9 \to v_{10}\})$$

and forcing trees given by the paths $v_1, \dots, v_5$ and $v_6, \dots, v_{10}$. Since the forcing trees, denoted with bold edges, are themselves paths, they are forcing subpaths. However, since $\mathrm{Z}(H) = 3$, they cannot form a path bundle.

Observe that by definition, the chain set of $\mathcal{F}|_{\mathcal{Q}}$ will be $\mathcal{Q}$ itself. Additionally, any path bundle $\mathcal{Q}$ is a parallel increasing path cover of the induced subgraph H. When $\mathcal{Q}$ contains the vertices of B and $|\mathcal{Q}| = |B|$, then the set of initial vertices of $\mathcal{F}$ in H is precisely B, and $\operatorname{pt}(H,B) \leqslant \operatorname{pt}_+(G,B)$, see Corollary 48. There are trivial examples of path bundles, such as the paths consisting of just the vertices of B or the set consisting of a single path $Q_1 \subseteq T_1$. However, there are also many nontrivial cases where $\mathcal{F}|_{\mathcal{Q}}$ provides us information about $\mathcal{F}$ and the original graph G. Since $\mathcal{F}|_{\mathcal{Q}}$ is a relaxed chronology of standard forces for the subgraph H induced by the vertices of the paths in $\mathcal{Q}$, we can consider its terminus $\operatorname{Term}(\mathcal{F}|_{\mathcal{Q}})$.

Definition 44 (Vertex-induced path bundles)**.** Let G be a graph, B be a PSD forcing set of G, and $\mathcal{F}$ be a relaxed chronology of PSD forces of B on G. Fix a vertex $x \in V(G)$. For $k = 0, 1, 2, \ldots, \operatorname{rd}(x) - 1$, define $C_x^k = \operatorname{comp}(G - E_{\mathcal{F}}^{[k]}, x)$ to be the component of $G - E_{\mathcal{F}}^{[k]}$ containing x. Construct sets of paths $\mathcal{Q}^{[k]}$ as follows:

1. Let $\{v_i^0\}_{i=1}^{|B|}$ be the vertices of B, and let $\mathcal{Q}^{[0]} = \{Q_i^{[0]}\}_{i=1}^{|B|} = \{\{v_i^0\}\}_{i=1}^{|B|}$ be the set of single-vertex paths on the vertices of B.

2. For each i, if v_i^k forces some vertex $w \in C_x^k$ at time-step $k+1$ of $\mathcal{F}$ (i.e., $v_i^k \to w \in F^{(k+1)}$), then define $v_i^{k+1} = w$, and construct $Q_i^{[k+1]}$ by adding v_i^{k+1} to the end of the path $Q_i^{[k]}$. For the remaining paths $Q_i^{[k]}$, define $Q_i^{[k+1]} = Q_i^{[k]}$ and $v_i^{k+1} = v_i^k$.

3. Finally, let $\mathcal{Q}^{[k+1]} = \{Q_i^{[k+1]}\}_{i=1}^{|B|}$.

We call $\mathcal{Q} = \mathcal{Q}^{[\operatorname{rd}(x)]}$ the *path bundle of $\mathcal{F}$ induced by x*. If we wish to speak of this type of path bundle in general rather than a specific instance of one, then we will refer to them as *vertex-induced path bundles* or *path bundles induced by a vertex*.

Example 45. Refer to G, B, and $\mathcal{F}$ in Example 42. The path bundle induced by v_9 is $\mathcal{Q} = \{G[\{v_1, v_4\}], G[\{v_2, v_7, v_8, v_9\}]\}$, but the path bundle in Example 42 is not induced by a vertex.

Lemma 46. *Let G be a graph, B be a PSD forcing set of G, $\mathcal{F}$ be a relaxed chronology of PSD forces of B on G, and $\mathcal{T}$ be the PSD forcing tree cover defined by $\mathcal{F}$. Then at each time-step k, given a component C of $G - E_{\mathcal{F}}^{[k]}$ and a PSD forcing tree $T \in \mathcal{T}$, there is at most one vertex $v \in V(T) \cap E_{\mathcal{F}}^{[k]}$ such that v has white neighbors in C.*

Proof. We proceed by contradiction. Suppose there are distinct $v, w \in V(T) \cap E_{\mathcal{F}}^{[k]}$ respectively adjacent to $v', w' \in V(C)$. Then at time k, there exists a path of white vertices $v', \ldots, w'$ entirely within C. Let $u \in T$ be the vertex such that the forces $u \to v_1 \to \cdots \to v$ and $u \to w_1 \to \cdots \to w$ are in $\mathcal{F}$ with $v_1 \neq w_1$. Without loss of generality, assume v_1 is forced at the same time as or before w_1. Letting j be the time when v_1 is forced, we see that the path $v_1, \ldots, v, v', \ldots, w', w, \ldots, w_1$ consists entirely of white vertices in a single component of $G - E_{\mathcal{F}}^{[j-1]}$. Then at time j, vertex u is adjacent to two white vertices v_1 and w_1 in the same component of $G - E_{\mathcal{F}}^{[j-1]}$, which contradicts the fact that $u \to v_1$ at this time-step. ☐

Lemma 47. *Let G be a graph, B be a PSD forcing set of G, and $\mathcal{F}$ be a relaxed chronology of forces of B on G. For any $x \in V(G)$, the path bundle $\mathcal{Q}$ of $\mathcal{F}$ induced by x is a path bundle that contains B and x. Furthermore, if $\mathcal{F}$ is a propagating family of forces, then in $\mathcal{F}|_{\mathcal{Q}}$, there is at least one force in each round until all vertices of $G[\bigcup_{P\in\mathcal{Q}} V(P)]$ are blue.*

Proof. We prove by induction that each $\mathcal{Q}^{[k]}$ is a path bundle. Label $B = \{v_i^0\}_{i=1}^m$, and note that $\mathcal{Q}^{[0]}$ is the set of trivial paths on B, so this is a path bundle. Now suppose that we have constructed the path bundle $\mathcal{Q}^{[k]} = (Q_1^{[k]}, \ldots, Q_m^{[k]})$ with corresponding induced subgraph H_k. If no forces occur into C_x^k at time-step $k+1$ of $\mathcal{F}$, then $\mathcal{Q}^{[k+1]} = \mathcal{Q}^{[k]}$ is again a path bundle. Otherwise, we let $S \subseteq \{1, 2, \ldots, m\}$ contain all indices such that for each $i \in S$, there exists $v_i \in T_i \cap E_{\mathcal{F}}^{[k]}$ and $w_i \in C_x^k$ with $v_i \to w_i \in F^{(k+1)}$, where $\mathcal{T} = \{T_i\}_{i=1}^m$ is the PSD forcing tree cover induced by $\mathcal{F}$.

If $v_i = v_i^0$, then $v_i^0 \in Q_i^{[k]}$, and by definition of the PSD forcing rule, $v_i \to w_i$ is a valid standard force in $G[V(H_k) \cup V(C_x^k)]$ with the vertices v_i and w_i forming a forcing chain, and thus a path, of length 1. If $v_i \neq v_i^0$, then there exists some sequence of forces in $\mathcal{F}$ from v_i^0 to v_i consisting of vertices in T_i. Some vertex v in this sequence is the last one in $Q_i^{[k]}$, as we have that the vertex $v_i^0 \in Q_i^{[k]}$. Lemma 46 implies that this v was the unique vertex in T_i with a white neighbor in $C_x^{k'}$ for some $k' \leqslant k$. Since C_x^k is contained in $C_x^{k'}$, our construction of $\mathcal{Q}^{[k]}$ implies that if $k' < k$ and v forced a vertex v' in $C_x^{k'}$ during $F^{(k'+1)}$, then v' must also be in $Q_i^{[k]}$. Since we chose v as the last vertex in $Q_i^{[k]}$ in the chain from v_i^0 to v_i, we conclude that v has not forced any vertex in $C_x^{k'}$, and in particular $v = v_i$. Again, we see that $v_i \to w_i$ is a valid standard force in $G[V(H_k) \cup V(C_x^k)]$. Hence, if we define $Q_i^{[k+1]} = Q_i^{[k]}$ when $i \notin S$ and $Q_i^{[k+1]}$ to be $Q_i^{[k]}$ with w_i appended when $i \in S$, then $(v_i^0, \ldots, v_i, w_i)$ is a forcing chain, and thus a path, in $T_i[V(H_{k+1})]$, and we obtain a strictly larger path bundle $\mathcal{Q}^{[k+1]}$ from $\mathcal{Q}^{[k]}$ by adding the vertices in C_x^k forced during time-step $k+1$.

This process terminates at $\mathrm{rd}(x)$, and induction implies that $\mathcal{Q}^{[\mathrm{rd}(x)]}$ is a path bundle. By construction, it contains both B and x. Additionally, if $\mathcal{F}$ is propagating, then there cannot be time-steps where no forces occur in C_x^k, as PSD forcing in each component occurs independently. Hence, $\mathcal{Q}^{[k]} \neq \mathcal{Q}^{[k+1]}$ for all $0 \leqslant k \leqslant \mathrm{rd}(x) - 1$, and at least one force occurs at each time-step of $\mathcal{F}|_{\mathcal{Q}}$ until we reach $\mathrm{rd}(x)$. ☐

Corollary 48. *Let G be a graph, B be a PSD forcing set of G of size k, and $\mathcal{F}$ be a propagating family of PSD forces for B on G. Let $\mathcal{H}$ be the set of subgraphs H of G such that $H = G\left[\bigcup_{P\in\mathcal{Q}} V(P)\right]$ for some path bundle $\mathcal{Q}$ of $\mathcal{F}$ induced by some $x \in V(G)$. Then*

$$\max_{H\in\mathcal{H}} \mathrm{pt}(H, B) \leqslant \mathrm{pt}_+(G, B).$$

Moreover, if $x \in B^{\left(\mathrm{pt}_+(G,B)\right)}$, then

$$\mathrm{pt}_+(G, B) \leqslant \min_{H\in\mathcal{H}} |V(H)| - k.$$

Proof. Since $\mathcal{F}$ is a propagating family of PSD forces for B on G, we have that

$$\operatorname{pt}(H, B) \leqslant \operatorname{ct}(\mathcal{F}|_{\mathcal{Q}}) = \operatorname{pt}_+(G, \mathcal{F}) = \operatorname{pt}_+(G, B),$$

and the first result follows from maximizing on the left. Using Lemma 47, any such H with corresponding B and $\mathcal{F}$ satisfies

$$\operatorname{pt}_+(G, \mathcal{F}) \leqslant |V(H)| - k.$$

Since $\operatorname{pt}_+(G, \mathcal{F}) = \operatorname{pt}_+(G, B)$, the second result follows from minimizing on the right. □

If $\mathcal{F}$ is a relaxed chronology of forces for a PSD forcing set B, then the set of vertices that do not perform a force in $\mathcal{F}$ need not have the same size as B. However, vertex-induced path bundles allow us to produce PSD forcing sets of the same size using the terminus of the resulting standard zero forcing set.

Theorem 49. *Let G be a graph and B be a PSD forcing set of G. Let $\mathcal{F}$ be a relaxed chronology of forces for B on G, and let $\mathcal{Q}$ be the path bundle of $\mathcal{F}$ induced by $x \in V(G)$. Then* $\operatorname{Term}(\mathcal{F}|_{\mathcal{Q}})$ *is a PSD forcing set of G. Furthermore, a relaxed chronology of PSD forces for* $\operatorname{Term}(\mathcal{F}|_{\mathcal{Q}})$ *can be constructed by reversing the forces between vertices in $\mathcal{Q}$ and preserving all remaining forces.*

Proof. Since $\mathcal{Q}$ is a path bundle, $\mathcal{F}|_{\mathcal{Q}}$ is a relaxed chronology of standard forces. By Theorem 15, one can use $\mathcal{F}|_{\mathcal{Q}}$ to construct a multiset of block partitions witnessing that $\mathcal{Q}$ is a parallel increasing path cover of $H = G[\bigcup_{Q_i \in \mathcal{Q}} V(Q_i)]$ and thus the sets of vertices $\{V_{\mathcal{F}|_{\mathcal{Q}}}^{N}\}_{N=0}^{\operatorname{rd}(x)}$ as described in Definition 27. Define the sets of vertices $\{R^k\}_{k=0}^{\operatorname{rd}(x)}$ such that for each k, $R^k = \bigcup_{i=0}^{k} V_{\mathcal{F}|_{\mathcal{Q}}}^{\operatorname{rd}(x)-i} = V_{\mathcal{F}|_{\mathcal{Q}}}^{[\operatorname{rd}(x)-k, \operatorname{rd}(x)]}$. First note that $\operatorname{Term}(\mathcal{F}|_{\mathcal{Q}}) = R^0$. We will now show inductively that for k with $0 \leqslant k < \operatorname{rd}(x)$, if R^k is blue, then R^{k+1} can be forced blue using only the reverses of PSD forces contained in $\mathcal{F}|_{\mathcal{Q}}$. This inductive process terminates at $R^{\operatorname{rd}(x)}$, which contains the PSD forcing set B of G. Thus any remaining white vertices can be forced using only PSD forces found in $\mathcal{F}$, and this will show that $\operatorname{Term}(\mathcal{F}|_{\mathcal{Q}})$ is a PSD forcing set of G.

Let $k \in \{0, 1, \ldots, \operatorname{rd}(x) - 1\}$, and suppose R^k is currently blue. Since $\mathcal{F}|_{\mathcal{Q}}$ is a relaxed chronology of standard forces in H, it follows by Lemma 25 that $\operatorname{Rev}(\mathcal{F}|_{\mathcal{Q}})$ is a relaxed chronology of standard forces for $\operatorname{Term}(\mathcal{F}|_{\mathcal{Q}})$ in H. Due to this, for each $u \in R^{k+1} \setminus R^k$, there exists a vertex $v \in R^k$ such that u is the only white neighbor of v in H. We claim that u is the only white neighbor of v in $\operatorname{comp}(G - R^k, u)$, the component of $G - R^k$ that contains u, and hence $v \to u \in S_+(G, R^k)$.

Let $k' = \operatorname{rd}(x) - k$. Since $u \to v$ at time k', $v \in V(C_x^{k'-1}) \subseteq V(C_x^{t-1})$ for any $t \leqslant k'$. Now, suppose by way of contradiction that there exists $u' \in (N(v) \cap V(\operatorname{comp}(G - R^k, u))) \setminus V(H)$. Then there exists a path $u' = p_0, p_1, \ldots, p_m = u$ with $p_i \notin R^k$. Let $j = \min\{i \colon p_i \in V(H)\} > 0$, which is well-defined since $u \in V(H)$. Finally, let $y = p_j$.

It is asserted that for all $0 \leqslant i < j$, $\operatorname{rd}_{\mathcal{F}}(p_i) > k'$. Otherwise, when the first such p_i was forced by some vertex z at time $t \leqslant k'$, it was in the same component of $G - E_{\mathcal{F}}^{[t-1]}$

as v. Thus, $p_i \in V(\mathrm{comp}(G - E_{\mathcal{F}}^{[t-1]}, v)) = V(C_x^{t-1})$. This would imply, though, that $z \to p_i \in \mathcal{F}|_{\mathcal{Q}}$, and so $p_i \in V(H)$.

Thus, for all $0 \leqslant i < j$, $p_i \in V(\mathrm{comp}(G - E_{\mathcal{F}}^{[k'-1]}, v)) = V(C_x^{k'-1})$. Since $y \in V(H)$, but $y \notin R^k$, there must be some time-step $t \leqslant k'$ and some vertex $y' \in V(C_x^{t-1})$ such that $y \to y' \in \mathcal{F}|_{\mathcal{Q}}$ at time t. However, this implies that $\{p_{j-1}, y'\} \subseteq N_G(y) \setminus E_{\mathcal{F}}^{[t-1]}$, so $y \to y'$ is not a valid PSD force at time t, a contradiction. □

Example 50. Refer to H in Figure 7(b). Even though the vertices v_5, v_{10} in H are the endpoints of forcing trees that are paths, the set $\{v_5, v_{10}\}$ cannot be the terminus of any vertex-induced path bundle as it is not a PSD forcing set.

Remark 51. Alternatively, it is worth noting that the process of multiple-vertex migration introduced in [12] by Hogben et al. could also be used inductively for the purposes of the proof. In some sense the restriction to the induced subgraph given by vertex-induced path bundles allows the concept of parallel increasing path covers to be useful in the PSD forcing setting. Together, parallel increasing path covers and vertex-induced path bundles do globally in the graph G what multiple-vertex migration does locally.

Corollary 52. *Let G be a graph with $\mathcal{T}$ being the PSD forcing trees for some PSD forcing set of size k. For any $v \in V(G)$, there exists a PSD forcing set B of size k containing v and a relaxed chronology of forces $\mathcal{F}$ for B with $\mathcal{T}$ as its induced forcing trees.*

Proof. Let $\mathcal{F}$ be a relaxed chronology of forces inducing $\mathcal{T}$, and let $\mathcal{Q}$ be the path bundle of $\mathcal{F}$ induced by v. Theorem 49 implies that $\mathrm{Term}(\mathcal{F}|_{\mathcal{Q}})$ is a PSD forcing set of G containing v, and a corresponding relaxed chronology $\mathcal{F}'$ can be constructed by reversing the forces between vertices in $\mathcal{Q}$ and preserving the other forces. Observe that $\mathcal{F}$ and $\mathcal{F}'$ contain forces between the same pairs of vertices, albeit possibly in different directions. Hence, they induce the same forcing trees $\mathcal{T}$. □

Our results on path bundles have connections with *rigid linkages* studied by Ferrero et al. [10]. We detail the explicit relationship in Appendix B.

6 Concluding remarks

In this paper, we introduced the concept of a relaxed chronology as a generalization of a chronological list of forces. This framework permitted the development of parallel increasing path covers, an alternative formulation of a zero forcing process from a more global perspective. These parallel increasing path covers allowed for the construction of families of graphs with predetermined chain sets, as well as the establishment of certain relations between standard zero forcing and other zero forcing variants.

Relaxed chronologies also were used to discuss the restriction of forcing to subgraphs. A special case of such restrictions, called path bundles, was introduced for which the restriction of PSD forcing is standard zero forcing. These path bundles allowed the construction of PSD forcing sets containing a chosen vertex that have the same PSD forcing trees as a chosen initial PSD forcing set. Connections between path bundles and rigid linkage forcing were established in the appendices.

Looking forward, it is hoped that these new frameworks and concepts will prove useful outside the confines of this particular paper. It is intended that these ideas and definitions will help serve as a foundation for a broader and more robust discussion of zero forcing and its variants.

A Proof of Theorem 36

In this section, we generalize the results in Section 4 to establish that

$$\mathrm{ppt}(G,m) \leqslant \left\lceil \frac{\mathrm{pt}(G,m)}{2} \right\rceil.$$

Note that we use the results of Section 5 extensively, further demonstrating their usefulness. We start with generalizing the sets from Definition 27.

Definition 53 (Sets $V_{\mathcal{F}}^{(M,N]}$, $V_{\mathcal{F}}^{[M,N)}$, $V_{\mathcal{F}}^{(M,N)}$, $V_{\mathcal{F}}^{\mathrm{bd}(M+)}$, $V_{\mathcal{F}}^{\mathrm{bd}(N-)}$)**.** Let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ be a relaxed chronology of standard forces for a graph G and some standard zero forcing set. For any $M, N \in \{0, 1, \ldots, K\}$ with $M \leqslant N$, define

$$\begin{aligned} V_{\mathcal{F}}^{[M,N)} &= V_{\mathcal{F}}^{[M,N]} \setminus V_{\mathcal{F}}^{N}, \\ V_{\mathcal{F}}^{(M,N]} &= V_{\mathcal{F}}^{[M,N]} \setminus V_{\mathcal{F}}^{M}, \text{ and} \\ V_{\mathcal{F}}^{(M,N)} &= V_{\mathcal{F}}^{[M,N]} \setminus (V_{\mathcal{F}}^{M} \cup V_{\mathcal{F}}^{N}). \end{aligned}$$

Finally, define $V_{\mathcal{F}}^{\mathrm{bd}(M+)}$ to be the initial vertices of the chain set for $\mathcal{F}$ in $G[V_{\mathcal{F}}^{(M,N]}]$ and $V_{\mathcal{F}}^{\mathrm{bd}(N-)}$ to be the initial vertices for $\mathrm{Rev}(\mathcal{F})$ in $G[V_{\mathcal{F}}^{[M,N)}]$.

Lemma 54. *Let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ be a relaxed chronology of standard forces for a graph G and some standard zero forcing set B and let $M \in \{0, 1, 2, \ldots, K-1\}$ and $N \in \{1, 2, 3, \ldots, K\}$. Then*

1. *$B_1 = V_{\mathcal{F}}^{\mathrm{bd}(N-)}$ is a zero forcing set of $H_1 = G[V_{\mathcal{F}}^{N-}] = G[V_{\mathcal{F}}^{[0,N)}]$ with* $\mathrm{pt}(H_1, B_1) \leqslant N - 1$.
2. *$B_2 = V_{\mathcal{F}}^{\mathrm{bd}(M+)}$ is a zero forcing set of $H_2 = G[V_{\mathcal{F}}^{M+}] = G[V_{\mathcal{F}}^{(M,K]}]$ with* $\mathrm{pt}(H_2, B_2) \leqslant K - M - 1$.

Proof. Lemma 28 implies $V_{\mathcal{F}}^{0}$ is a zero forcing set of $H = G[V_{\mathcal{F}}^{[0,N-1]}]$ with propagation time at most $N-1$. Lemma 40 then implies $V_{\mathcal{F}}^{0}$ is a zero forcing set of the subgraph of H given by $H_1 = G[V_{\mathcal{F}}^{[0,N)}]$, where the propagation time is again at most $N-1$. Since B_1 is the terminus of $\mathcal{F}|_{H_1}$, Lemma 25 implies (1). Applying (1) on $\mathrm{Rev}(\mathcal{F})$ and $\mathrm{Term}(\mathcal{F}) = V_{\mathcal{F}}^{K}$ then implies claim (2). ☐

Corollary 55. *Let $\mathcal{F} = \{F^{(k)}\}_{k=1}^{K}$ be a relaxed chronology of forces for a graph G for some standard zero forcing set. For any $0 \leqslant M < N \leqslant K$,*

- $V_{\mathcal{F}}^{\mathrm{bd}(M+)}$ *and* $V_{\mathcal{F}}^{\mathrm{bd}(N-)}$ *are zero forcing sets of* $G[V_{\mathcal{F}}^{(M,N)}]$*, and both sets have propagation time at most* $N - M - 2$*.*
- $V_{\mathcal{F}}^{M}$ *and* $V_{\mathcal{F}}^{\mathrm{bd}(N-)}$ *are zero forcing sets of* $G[V_{\mathcal{F}}^{[M,N)}]$*, and both sets have propagation time at most* $N - M - 1$*.*
- $V_{\mathcal{F}}^{\mathrm{bd}(M+)}$ *and* $V_{\mathcal{F}}^{N}$ *are zero forcing sets of* $G[V_{\mathcal{F}}^{(M,N]}]$*, and both sets have propagation time at most* $N - M - 1$*.*

Proof. Each claim follows immediately from applying Lemma 40 with either Corollary 29 or the preceding lemma. □

We now use these results to establish our power propagation time bound.

Proof of Theorem 36. We can assume $\mathrm{pt}(G, m) > 0$, as otherwise the result is trivial. As in the proof of Theorem 31, we let B be an m-efficient standard zero forcing set of G with corresponding relaxed chronology $\mathcal{F} = \{F^{(k)}\}_{k=1}^{\mathrm{pt}(G,m)}$. Letting $N = \left\lceil \frac{\mathrm{pt}(G,m)}{2} \right\rceil$, we have that $N_G[V_{\mathcal{F}}^{N}]$ contains $V_{\mathcal{F}}^{\mathrm{bd}(N-)}$ and $V_{\mathcal{F}}^{\mathrm{bd}(N+)}$. By Lemma 54, $V_{\mathcal{F}}^{\mathrm{bd}(N-)}$ is a standard zero forcing set of $G[V_{\mathcal{F}}^{N-}]$ with $\mathrm{pt}\left(G[V_{\mathcal{F}}^{N-}], V_{\mathcal{F}}^{\mathrm{bd}(N-)}\right) \leqslant N - 1$. Likewise, $V_{\mathcal{F}}^{\mathrm{bd}(N+)}$ is a standard zero forcing set of $G[V_{\mathcal{F}}^{N+}]$ with $\mathrm{pt}\left(G[V_{\mathcal{F}}^{N+}], V_{\mathcal{F}}^{\mathrm{bd}(N+)}\right) \leqslant K - N - 1 \leqslant N - 1$, since $K = \mathrm{pt}(G, m)$. Since Lemma 30 implies that $V_{\mathcal{F}}^{N}$ separates $V_{\mathcal{F}}^{N-}$ and $V_{\mathcal{F}}^{N+}$ and every vertex in $V_{\mathcal{F}}^{N}$ is blue, the forcing process can proceed independently in $G[V_{\mathcal{F}}^{N-}]$ and $G[V_{\mathcal{F}}^{N+}]$. It thus follows that

$$\mathrm{ppt}(G, m) \leqslant 1 + \max\left\{\mathrm{pt}\left(G[V_{\mathcal{F}}^{N-}], V_{\mathcal{F}}^{\mathrm{bd}(N-)}\right), \mathrm{pt}\left(G[V_{\mathcal{F}}^{N+}], V_{\mathcal{F}}^{\mathrm{bd}(N+)}\right)\right\} \leqslant \left\lceil \frac{\mathrm{pt}(G, m)}{2} \right\rceil.$$

□

B Rigid linkages and path bundles

In this section, we discuss a connection between path bundles and rigid linkages. We establish that the path bundle induced by a relaxed chronology and a vertex forms a rigid linkage. We refer the reader to Ferrero et al. [10] for further details of rigid linkages and their applications.

Let G be a graph. A *linkage* in G is a subgraph whose connected components are paths. Note that a linkage need not contain all vertices of G. Let $\alpha, \beta \subseteq V(G)$. A linkage $\mathcal{P}$ is an (α, β)-*linkage* if α consists of one endpoint of each path in $\mathcal{P}$ and β consists of the other endpoint of each path. In the case that a path is a single vertex, the vertex is in both α and β. A linkage $\mathcal{P}$ is (α, β)-*rigid* if $\mathcal{P}$ is the unique (α, β)-linkage in G. A linkage $\mathcal{P}$ is *rigid* if there exist α and β such that $\mathcal{P}$ is an (α, β)-linkage and $\mathcal{P}$ is (α, β)-rigid.

For $X \subseteq V(G)$, define the boundary $\partial_G(X)$ of X to be the set of vertices not in X that have at least one neighbor in X. When C is a subgraph of G, define $\partial_G(C) = \partial_G(V(C))$. Rigid linkage forcing is defined by the *rigid linkage color change rule (CCR-RL)*. Given a current set of blue vertices $B^{[k]}$, an application of CCR-RL (to go from time-step k to time-step $k+1$) consists of the following:

- Choose a component C of $G - B^{[k]}$, such that $\partial_G(C)$ does not contain any inactive blue vertices (that is, those which have previously performed a force).
- Select an active blue vertex u such that w is the only white neighbor of u in C:
  - Let u force w, so that $B^{[k+1]} = B^{[k]} \cup \{w\}$.
  - Update the active vertices (w becomes active, u becomes inactive).

For a given rigid linkage forcing process $\mathcal{F}$ on G with r steps:

- A rigid linkage forcing chain is a path $(v_0, v_1, \ldots, v_\ell)$ such that $v_0 \in B^{[0]}$, $v_i \to v_{i+1}$ for all $i = 0, \ldots, \ell - 1$, and v_ℓ is active after time-step r.
- The rigid linkage chain set is the set of all rigid linkage forcing chains (for the given forcing process).

Ferrero et al. established that zero forcing is a special type of rigid linkage forcing and identified the close connection between rigid linkage forcing and rigid linkages.

Proposition 56. [10, Proposition 2.3] *Any zero forcing process on a graph G is a rigid linkage forcing process.*

Theorem 57. [10, Theorem 2.10] *Let G be a graph and $\mathcal{P}$ be a linkage in G. Then $\mathcal{P}$ is a rigid linkage if and only if $\mathcal{P}$ is a rigid linkage chain set under some rigid linkage forcing process.*

Given a graph G and a set of blue vertices B, since PSD forcing works independently in different components of $G - B$, we have the observation below.

Observation 58. *Let G be a graph, B be a PSD forcing set of G, $\mathcal{F}$ be a chronological list of PSD forces of B on G, $x \in V(G)$, $\mathcal{Q}$ be the path bundle of $\mathcal{F}$ induced by x, and $H = G\left[\bigcup_{P \in \mathcal{Q}} V(P)\right]$. Construct $\mathcal{F}^*$ by performing the forces in $\mathcal{F}|_{\mathcal{Q}}$ first (in order) and afterwards performing the remaining forces in $\mathcal{F}$ (in order). Then $\mathcal{F}^*$ is a chronological list of PSD forces for G.*

We now establish the connection between rigid linkages and vertex-induced path bundles.

Theorem 59. *Let G be a graph, B be a PSD forcing set of G, $\mathcal{F}$ be a relaxed chronology of PSD forces of B on G, $x \in V(G)$, $\mathcal{Q}$ be the path bundle of $\mathcal{F}$ induced by x, and $H = G\left[\bigcup_{P \in \mathcal{Q}} V(P)\right]$. Then $\mathcal{Q}$ is a rigid linkage of G.*

Proof. Let $K = |V(H)| - |B|$. By Observation 58, we can suppose without loss of generality that $\mathcal{F}$ is a chronological list of forces where the K forces in $\mathcal{F}|_{\mathcal{Q}}$ are performed first. Since $\mathcal{Q}$ is a linkage, we proceed by first showing that $\{F^{(k)}|_{\mathcal{Q}}\}_{k=1}^{K}$ is a valid rigid linkage forcing process and then applying Theorem 57 to complete the proof. Prior to application of the first force, all vertices in B are active, so the first force in $\mathcal{F}$ is a valid rigid linkage force.

We proceed by induction. Fix $k \in \{0, 1, \ldots, K-1\}$, and suppose that $B^{[k]}$ is blue and every force in $\{F^{(i)}|_{\mathcal{Q}}\}_{i=1}^{k}$ is a valid rigid linkage force. Additionally, let $u_{k+1} \to v_{k+1} \in F^{(k+1)}|_{\mathcal{Q}}$. We assert that this force is a valid rigid linkage force. Suppose $u_j \to v_j \in F^{(j)}|_{\mathcal{Q}}$ for some $j \leqslant k$. Then v_j was the only neighbor of u_j in C_x^{j-1}. Since $V(C_x^k) \subseteq V(C_x^{j-1}) \backslash B^{[k]}$ and $v_j \in B^{[k]}$, it follows that u_j has no neighbors in C_x^k. Thus $\partial_G(C_x^k)$ contains no inactive blue vertices. Since v_{k+1} is the only white neighbor of u_{k+1} in C_x^k, $u_{k+1} \to v_{k+1}$ is a valid rigid linkage force. Finally, $\{F^{(k)}|_{\mathcal{Q}}\}_{k=1}^{K}$ is a valid rigid linkage forcing process with rigid linkage chain set $\mathcal{Q}$, so applying Theorem 57 completes the proof. □

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