

# Every signed planar graph of girth 5 has circular chromatic number strictly less than 4

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## Abstract

For a real number  $r \geq 2$ , a circular  $r$ -colouring of a signed graph  $(G, \sigma)$  is a mapping  $c : V(G) \rightarrow [0, r)$  such that  $|c(x) - c(y)| \in [1, r - 1]$  for each positive edge  $xy$  and  $|c(x) - c(y)| \in [0, r/2 - 1] \cup [r/2 + 1, r)$  for each negative edge  $xy$ . This concept is recently introduced by Naserasr, Wang, and Zhu in 2021, and they show that for any  $\varepsilon > 0$ , there exist signed planar bipartite graphs (of girth 4) which are not circular  $(4 - \varepsilon)$ -colourable. In this paper, we prove that for each signed planar graph  $(G, \sigma)$  of girth at least 5, there exists a real number  $\varepsilon = \varepsilon(G, \sigma) > 0$  such that  $(G, \sigma)$  is circular  $(4 - \varepsilon)$ -colorable. Our proof utilizes a Thomassen-type inductive argument on the dual version in terms of circular flows, which is motivated by a result of Richter, Thomassen, and Younger (2016) on group connectivity of 5-edge-connected planar graphs.

**Mathematics Subject Classifications:** 05C88, 05C89

## 1 Introduction

A *signed graph* is a pair  $(G, \sigma)$ , where  $G$  is a graph and  $\sigma : E(G) \rightarrow \{+, -\}$  is a signature which assigns to each edge of  $G$  a sign. Let  $r \geq 2$  be a real number. As introduced by Naserasr, Wang, and Zhu in [6], a circular  $r$ -colouring of a signed graph  $(G, \sigma)$  is a mapping  $c : V(G) \rightarrow [0, r)$  such that  $|c(x) - c(y)| \in [1, r - 1]$  for each positive edge  $xy$  and  $|c(x) - c(y)| \in [0, r/2 - 1] \cup [r/2 + 1, r)$  for each negative edge  $xy$ . The *circular chromatic number* of a signed graph  $(G, \sigma)$  is defined as

$$\chi_c(G, \sigma) = \inf\{r : G \text{ admits a circular } r\text{-colouring}\}.$$

It is shown in [6] that  $\chi_c(G, \sigma)$  is well-defined and must be a rational number. This concept is a refinement of 0-free  $2k$ -coloring of signed graphs and extends the circular coloring concept introduced by Vince [8] from graphs to signed graphs.

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The classical Grötzsch's theorem states that every triangle-free planar graph is circular 3-colorable. This is no longer true for signed graphs, as it is observed in [6] that  $\chi_c(G, \sigma) \leq 4$  for any signed bipartite graph  $(G, \sigma)$ , and there exists a sequence of signed bipartite planar graphs whose circular chromatic numbers are tending to 4. On the other hand, the circular chromatic number of any signed bipartite planar graph cannot be equal to 4 as proved in [2].

**Theorem 1.** (*Kardos, Narboni, Naserasr, and Wang [2]*) *For every signed bipartite planar graph  $(G, \sigma)$ ,  $\chi_c(G, \sigma) < 4$ .*

In [2], it is also observed that every 2-degenerate signed graph has circular chromatic number strictly less than 4. By Euler's formula, it is straightforward to obtain that every signed planar graph of girth at least 6 is 2-degenerate, and thus has circular chromatic number strictly less than 4. Our main result of this paper shows that this property is still valid for signed planar graphs of girth 5.

**Theorem 2.** *For every signed planar graph  $(G, \sigma)$  of girth at least 5,  $\chi_c(G, \sigma) < 4$ .*

We conjecture that this conclusion is still true for signed planar graphs of girth 4, which, if true, would be best possible as can be evidenced by  $K_4$  with all positive signs.

**Conjecture 3.** *For every signed planar graph  $(G, \sigma)$  of girth 4,  $\chi_c(G, \sigma) < 4$ .*

Note that applying some standard arguments (see Proposition 22 in [6] or Theorem 2.2 in [2]), we may further extend Theorem 2 to show that every  $n$ -vertex signed planar graph  $(G, \sigma)$  of girth at least 5 satisfies  $\chi_c(G, \sigma) \leq 4 - \frac{4}{n+1}$ . But this upper bound still relies on the order of the given signed graph. We propose below a stronger conjecture to suggest a universal upper bound smaller than 4.

**Conjecture 4.** *There exists a constant  $\varepsilon_0 > 0$  such that every signed planar graph  $(G, \sigma)$  of girth at least 5 satisfies  $\chi_c(G, \sigma) \leq 4 - \varepsilon_0$ .*

It is proved in [5] that every signed planar graph  $(G, \sigma)$  of girth at least 7 satisfies  $\chi_c(G, \sigma) \leq 3$ , and so Conjecture 4 is true for that subclass of signed graphs.

## 2 Duality between circular coloring and circular flows in signed graphs

Our proof of Theorem 2 actually uses the dual concept about circular flows of signed graphs recently introduced in [3], which is a natural extension of the same concept on graphs [1].

**Definition 5.** ([3]) Given a signed graph  $(G, \sigma)$  and a real number  $r \geq 2$ , a *circular  $r$ -flow* is a pair  $(D, f)$  where  $D$  is an orientation and  $f : E(G) \rightarrow (-r, r)$  satisfies the following three conditions:

- for each positive edge  $e$ ,  $|f(e)| \in [1, r - 1]$ ;

- for each negative edge  $e$ ,  $|f(e)| \in [0, \frac{r}{2} - 1] \cup [\frac{r}{2} + 1, r)$ ;
- for each vertex  $v$ ,  $\partial_D f(v) = \sum_{e \in \vec{E}(v)} f(e) - \sum_{e \in \overleftarrow{E}(v)} f(e) = 0$ ,

where  $\vec{E}(v)$  is the set of arcs that  $v$  is the tail, and  $\overleftarrow{E}(v)$  is the set of arcs that  $v$  is the head.

The *circular flow number* of a signed graph  $(G, \sigma)$  is defined as

$$\Phi_c(G, \sigma) = \inf\{r : G \text{ admits a circular } r\text{-flow}\}.$$

**Theorem 6.** ([3]) *A signed plane graph  $(G, \sigma)$  admits a circular  $r$ -coloring if and only if its dual signed graph  $(G^*, \sigma^*)$  admits a circular  $r$ -flow, and thus  $\chi_c(G, \sigma) = \Phi_c(G^*, \sigma^*)$ .*

In the study of circular coloring and circular flows, we usually use the discrete form for  $r = \frac{p}{q} \geq 2$ , where we have the following equivalent definition: a circular  $\frac{p}{q}$ -flow is a pair  $(D, f)$  where  $D$  is an orientation and  $f : E(G) \rightarrow \{0, \pm 1, \dots, \pm(p-1)\}$  such that for each edge  $e \in E(G)$ ,  $|f(e)| \in \{q, \dots, p-q\}$  if  $\sigma(e) = +$  and  $|f(e)| \in \{0, \dots, \frac{p}{2} - q\} \cup \{\frac{p}{2} + q, \dots, p-1\}$  if  $\sigma(e) = -$ , and moreover, for each vertex  $v$ ,

$$\partial_D f(v) = \sum_{e \in \vec{E}(v)} f(e) - \sum_{e \in \overleftarrow{E}(v)} f(e) = 0.$$

For convenience, we shall sometimes also use modular flows, whose definition is almost the same except the equality above is taken modulo  $p$ . Some relations between circular flow number and strongly connected orientation are established in [4].

**Theorem 7.** (Li, Thomassen, Wu, and Zhang [4]) *A connected graph has circular flow number strictly less than  $\frac{p}{q}$  if and only if it admits a modular circular  $\frac{p}{q}$ -flow  $(D, f)$  such that  $f : E(G) \rightarrow \{q, q+1, \dots, p-q-1\}$  and  $D$  is strongly connected.*

Our result for signed graphs has a similar flavor. In fact, we prove the following result concerning strongly connected orientations, which implies Theorem 2 as a corollary. We shall need a few more definitions before presenting the following result. Let  $k \geq 2$  be an integer. A mapping  $\alpha : V(G) \rightarrow \mathbb{Z}_k$  is called a  $\mathbb{Z}_k$ -boundary if  $\sum_{v \in V(G)} \alpha(v) \equiv 0$

(mod  $k$ ). For a  $\mathbb{Z}_k$ -boundary  $\alpha$ , a  $(\mathbb{Z}_k, \alpha)$ -flow on a graph  $G$  is a pair  $(D, f)$  where  $D$  is an orientation and  $f : E(G) \rightarrow \{\pm 1, \pm 2, \dots, \pm(k-1)\}$  such that for each vertex  $v \in V(G)$ ,  $\partial_D f(v) = \sum_{e \in \vec{E}(v)} f(e) - \sum_{e \in \overleftarrow{E}(v)} f(e) \equiv \alpha(v) \pmod{k}$ .

**Theorem 8.** *Let  $G$  be a 5-edge-connected planar graph. Then for any  $\mathbb{Z}_4$ -boundary  $\alpha$ , there exists a  $(\mathbb{Z}_k, \alpha)$ -flow  $(D, f)$  on  $G$  such that  $f : E(G) \rightarrow \{1, 2\}$  and  $D$  is strongly connected.*

**Proof of Theorem 2 assuming Theorem 8:** Let  $(G, \sigma)$  be a signed planar graph of girth at least 5, and let  $(H, \sigma^*)$  be its dual signed graph. Then  $H$  is 5-edge-connected. For each vertex  $v \in V(H)$ , denote  $d^-(v)$  to be the number of negative edges incident to  $v$ . Define  $\alpha(v) \equiv 2d^-(v) \pmod{4}$  for every  $v \in V(H)$ . Then

$$\sum_{v \in V(H)} \alpha(v) \equiv 2 \sum_{v \in V(H)} d^-(v) = 4|\{e : \sigma^*(e) = -\}| \equiv 0 \pmod{4},$$

and so  $\alpha$  is a  $\mathbb{Z}_4$ -boundary of  $H$ . By Theorem 8,  $H$  has a  $(\mathbb{Z}_4, \alpha)$ -flow  $(D, f_1)$  such that  $f_1 : E(H) \rightarrow \{1, 2\}$  and  $D$  is strongly connected.

Define another mapping  $f_2 : E(H) \rightarrow \mathbb{Z}_4$  such that for each  $e \in E(H)$ ,

$$f_2(e) = \begin{cases} 2 & \text{if } \sigma^*(e) = -, \\ 0 & \text{if } \sigma^*(e) = +. \end{cases}$$

Let  $f_3 = f_1 + f_2$ . Consider the pair  $(D, f_3)$ . For each edge  $e \in E(H)$ , we have  $f_3(e) \in \{1, 2\}$  if  $e$  is positive and  $f_3(e) \in \{0, 3\}$  otherwise. Moreover, for any vertex  $v \in V(H)$ , we have

$$\partial_D f_3(v) = \partial_D f_1(v) + \partial_D f_2(v) \equiv \alpha(v) + 2d^-(v) \equiv 0 \pmod{4}.$$

Hence  $(D, f_3)$  is a  $(\mathbb{Z}_4, 0)$ -flow on  $H$ .

Since  $D$  is strongly connected, every arc  $a$  in  $D(H)$  is contained in a directed cycle, say  $C_a$ . For every arc  $a \in A(D(H))$ , define  $f_a : E(H) \rightarrow \{0, 1\}$  by setting

$$f_a(e) = \begin{cases} 1 & e \in E(C_a), \\ 0 & \text{otherwise.} \end{cases}$$

Let  $M = |E(H)| + 1$ . Define  $f = Mf_3 + \sum_{a \in A(D(H))} f_a$ . Clearly,  $\partial_D f(v) \equiv 0 \pmod{4M}$  for each  $v \in V(H)$  by definition. Furthermore, for any edge  $e \in E(H)$ , since  $\sum_{a \in A(D(H))} f_a(e) \in \{1, \dots, M-1\}$ , we have

$$f(e) \in \{M+1, \dots, 3M-1\} \text{ if } e \text{ is positive}$$

and

$$f(e) \in \{1, \dots, M-1\} \cup \{3M-1, \dots, 4M-1\} \text{ if } e \text{ is negative.}$$

Therefore,  $(D, f)$  is a modular circular  $\frac{4M}{M+1}$ -flow on the signed graph  $(H, \sigma^*)$ . By duality from Theorem 6, we conclude that  $(G, \sigma)$  admits a circular  $\frac{4M}{M+1}$ -coloring, i.e.,  $\chi_c(G, \sigma) < 4$ . This completes the proof of Theorem 2.  $\blacksquare$

Next, we shall prove Theorem 8 in the rest of this paper. In fact, we utilize a Thomassen-type induction to prove a stronger theorem, which implies Theorem 8. Our technical theorem and proof ideas are mainly motivated by a result of Richter, Thomassen, and Younger [7] on group connectivity of 5-edge-connected planar graphs. However, we

have to make certain modifications for our purpose of searching strongly connected orientations.

For a  $\mathbb{Z}_4$ -boundary  $\beta$  of a graph  $G$  and a subset  $A \subseteq V(G)$ , we define

$$\beta(A) \equiv \sum_{v \in A} \beta(v) \pmod{4}.$$

For a vertex  $v \in V(G)$ , we use  $\delta(v)$  to denote the set of edges incident to  $v$ , and similarly we use  $\delta(A)$  to denote the set of edges with exactly one end in  $A$  for a vertex subset  $A$ . We use  $\deg(v)$  to denote the degree of a vertex  $v$ , and we say  $v$  is a  $k$ -vertex if  $\deg(v) = k$ . For a vertex subset  $A \subseteq V(G)$ , we use  $A^c$  to denote the complement of  $A$  in  $V(G)$ , while  $\bar{A}$  is the complement of  $A$  in the vertex set of a certain subgraph of  $G$ . A  $k$ -cut is an edge cut of size  $k$ . A 2-cut  $[A, A^c]$  is said to be *bad* if  $\beta(A) \equiv \beta(A^c) \equiv 2 \pmod{4}$ .

Now we are ready to state our main theorem below:

**Theorem 9.** *Let  $G$  be a 3-edge-connected planar graph embedded in the plane. Let  $\beta$  be a given  $\mathbb{Z}_4$ -boundary of  $G$ . Suppose that  $G$  has at most two specified vertices  $d$  and  $t$  such that:*

- (i) *if  $d$  exists, then it is in the boundary of the unbounded face, has degree 3, 4, or 5, and has its incident edges oriented and labelled with 1 or 2 satisfying boundary  $\beta(d)$  (i.e., at vertex  $d$  the outflow minus inflow is congruent to  $\beta(d)$  modulo 4);*
- (ii) *if  $t$  exists, then it has degree 3 and is in the boundary of the unbounded face;*
- (iii) *except for possibly  $\delta(d)$  and  $\delta(t)$ , every edge-cut of  $G$  is of size at least 4;*
- (iv) *if  $d$  has degree 5, then  $t$  does not exist;*
- (v) *every vertex not in the boundary of the unbounded face has five edge-disjoint paths to the boundary of the unbounded face;*
- (vi)  *$G - d$  has no bad 2-cut (Note that  $G - d = G$  when  $d$  does not exist).*

*Then the prescription at  $d$  can be extended to a  $(\mathbb{Z}_4, \beta)$ -flow  $(D, f)$  on  $G$  such that:*

- (a)  $\partial_D f \equiv \beta \pmod{4}$ , that is,  $\sum_{e \in \vec{E}(v)} f(e) - \sum_{e \in \overleftarrow{E}(v)} f(e) \equiv \beta(v) \pmod{4}$  for every  $v \in V(G)$ ;
- (b)  $f(e) \in \{1, 2\}$  for every  $e \in E(G)$ ;
- (c)  $D(G - d)$  is strongly connected.

Theorem 8 follows from it when the first specified vertex  $d$  does not exist. The rest of this paper is devoted to a proof of Theorem 9. We first investigate the properties of a potential minimal counterexample in Section 3, especially those about small cuts, which will be frequently used later. In Section 4, we apply some more structural results to find certain local configurations to complete the proof. We will divide our proof into three different cases, determined by whether there exists a copy of contractible structure containing  $t$ , and whether the set of boundary vertices of the unbounded face is sparse enough, and deal with them using proper methods accordingly.

### 3 Properties of the Minimal Counterexample to Theorem 9

With a little abuse of the usage of symbols, in the remaining part of this paper, we always use  $G$  (with a  $\mathbb{Z}_4$ -boundary  $\beta$ ) to denote the minimal counterexample of Theorem 9 in the sense of the lexicographical order  $(|V(G)| + |E(G)|, |E(G - d)|)$ . It is a trivial job to check that  $|V(G)| > 3$ .

We will deal with plenty of cuts and edges in the proof, for the sake of clarity and readability, it is necessary to make a statement about notations at first. When multiple cuts occur in a part of the proof, we name them after  $B$ ,  $F$ ,  $K$  and  $Q$  in sequence, and in this section, we always use  $A$  to represent a cut when stating the content of a theorem. When we need to operate on a single edge, if it is unoriented, we note it  $e$ ; if it is an arc, we note it  $a$ ; the two end vertices are chosen to be  $x$  and  $y$ . When we need to operate on a vertex, we note it  $z$ . These notations are independent between different theorems, propositions and sections, and we suggest the readers to keep this statement in mind.

In this section, we study the necessary properties of  $G$  and list them by a series of propositions. *Prescribing a vertex* means orienting all edges incident to it and labelling each of them with a value. We say the orientation of a vertex is *proper*, or the vertex is *properly oriented* if it is neither a sink nor a source; in other words, the vertex has both indegree and outdegree nonzero.

We start with a lemma in [4], and we provide a proof here for completeness. For convenience, a strongly connected orientation is called *strong* for short in the rest of the paper.

**Lemma 10.** ([4]) *Let  $H$  be a 2-edge-connected graph and  $e = xy$  be an edge of  $H$ . If  $H/e$  has a strong orientation  $D$ , then  $D$  can be extended to a strong orientation of  $H$ .*

*Proof.* If  $D$  is strong on  $H - e$ , just orient  $e$  arbitrarily; if  $D$  is not strong on  $H - e$ , then there exists an arc  $e' \in H - e$  not on a directed cycle, but  $e'$  is on some directed cycle in  $H/e$ . It can be deduced that there is a directed path between  $x$  and  $y$  in  $H - e$ , and that all such paths are directed the same direction, say from  $x$  to  $y$ . Then orient  $e$  from  $y$  to  $x$ . It is now easy to check that there exists a directed path between any pair of vertices of  $H$ , and  $D$  is extended to a strong orientation of  $H$ . ■

Now we begin to present some properties of the minimal counterexample  $G$ . The first one says that  $G$  is essentially a simple graph, which is the basis of the whole proof of Theorem 9.

**Proposition 11.**  $G - d$  contains no multiple edges (which are unoriented).

*Proof.* On the contrary, assume two vertices  $x$  and  $y$  form an unoriented  $sK_2$  ( $s \geq 2$ ) in  $G - d$ . By the induction,  $G/\{x, y\}$  admits a  $(\mathbb{Z}_4, \beta')$ -flow  $(D', f')$  with a strong orientation  $D'(G/\{x, y\} - d)$ , where  $\beta'$  is the  $\mathbb{Z}_4$ -boundary induced from  $\beta$ , with the contraction vertex receiving value  $\beta(x) + \beta(y)$ . As for  $sK_2$  itself,  $f'$  induces a  $\mathbb{Z}_4$ -boundary  $\gamma$  on it, where  $\gamma(x)$  ( $\gamma(y)$ ) is the difference of  $\beta(x)$  ( $\beta(y)$ ) and the total value  $x$  ( $y$ ) receives from  $f'$ .

We claim that there always exists a prescription of  $sK_2$  with a strong orientation to realize  $\gamma(x)$  and  $\gamma(y)$  at  $x$  and  $y$ , except the only case  $s = 2$  and  $\gamma(x) = \gamma(y) = 2$ . This is clear when  $s \geq 3$ ; as for  $s = 2$ , if  $\gamma(x) = \gamma(y) = 0$ , then we label the two edges with 1 and orient them oppositely; if  $\{\gamma(x), \gamma(y)\} = \{1, 3\}$ , then we label the two edges with 1 and 2, and orient them oppositely. In these cases,  $sK_2$  receives a strong orientation, and thus  $(D', f')$  can be extended to a flow  $(D, f)$  on  $G$  naturally such that  $D(G - d)$  is strong, resulting in a contradiction.

When  $s = 2$  and  $\gamma(x) = \gamma(y) = 2$ , the two parallel edges between  $x$  and  $y$  must be oriented to the same direction and labeled with 1. Now we apply Lemma 10, there exists a certain orientation of the two edges to extend  $D'$  to a strong orientation of  $G - d$ , and  $f'$  is extended to a function  $f$  on  $E(G)$  with the desired boundary  $\beta$  by labelling the two edges with 1. A contradiction is obtained too. ■

**Proposition 12.**  $G$  is 2-connected.

*Proof.* Let  $z$  be a cut vertex separating  $G$  into two subgraphs  $G_1$  and  $G_2$  with  $V(G_1) \cup V(G_2) = V(G)$ ,  $V(G_1) \cap V(G_2) = \{z\}$ , and  $d \in V(G_1)$ . We use  $\deg_i(v)$  to denote the degree of a vertex  $v$  in  $G_i$  ( $i = 1, 2$ ). Notice that  $\deg_i(z) = |\delta(G_i \setminus \{z\})|$ , by assumption (iii) of Theorem 9,  $\deg_i(z) \geq 3$  holds for  $i = 1, 2$ .

If  $\deg_i(z) = 3$  for some  $i$ , then  $\delta(G_i \setminus \{z\})$  is a 3-cut,  $G_i \setminus \{z\}$  can only be  $d$  or  $t$ , and by Proposition 11,  $G_i \setminus \{z\} = d$ . Now  $d$  is a pendent vertex of  $G$  whose only neighbour is  $z$ , we can apply the induction on  $G - d$  (setting the prescribed special vertex non-existent) and obtain a flow with a strong orientation, which is also a flow on  $G$  satisfying the conditions in Theorem 9.

If both  $\deg_1(z) \geq 4$  and  $\deg_2(z) \geq 4$ , then by the minimality of  $G$ ,  $G_1$  admits a  $(\mathbb{Z}_4, \beta_1)$ -flow  $(D_1, f_1)$  with a strong orientation  $D_1(G_1 - d)$ , and  $G_2$  admits a  $(\mathbb{Z}_4, \beta_2)$ -flow with a strong orientation  $D_2(G_2)$  (setting the prescribed special vertex non-existent).  $(D_1 \cup D_2, f_1 \cup f_2)$  is a desired  $(\mathbb{Z}_4, \beta)$ -flow on  $G$  with a strong orientation of  $G - d$ , a contradiction. ■

From Proposition 12, we know that the boundary of the unbounded face of  $G$  is a cycle, use  $C$  to represent it. We say a cut  $[A, A^c]$  of  $G$  is *essential* if  $\min\{|A|, |A^c|\} \geq 2$ , and *peripheral* if one of  $|A|$  and  $|A^c|$  is 1.

The next several propositions are concerned with essential 4-cuts and 5-cuts. We use a contraction method in their proofs, that is, contracting the vertex set on one side of an edge cut to a single vertex. Suppose vertex set  $X$  is contracted; the contraction naturally induces a  $\mathbb{Z}_4$ -boundary of  $G/X$  from  $\beta$ , by choosing  $\beta(X)$  as the boundary value of the

vertex contracted from  $X$ . In the remaining text, the phrase “the induced boundary” is used to refer to it. At first we introduce a fact from straightforward observation of the conditions (i)-(vi) in Theorem 9, which implies that contraction always preserves planarity. Recall that a *bond* of a graph  $H$  is a minimal non-empty edge cut. If  $[X, X^c]$  is a bond, then both  $H[X]$  and  $H[X^c]$  are connected induced subgraphs.

**Fact 13.** *In  $G$ , cuts of size at most 5 are bonds; an essential 6-cut is a bond unless it is  $\delta(\{d, t\})$ , on the premise that  $\deg(d) = 3$ , and  $d$  and  $t$  are not adjacent.*

**Proposition 14.** *There does not exist an essential 4-cut  $[A, A^c]$  such that  $d \in A$  and  $G[A^c]$  is 3-edge-connected.*

*Proof.* By contracting  $A^c$  to a single vertex and applying the induction on  $G/A^c$ , we obtain a  $(\mathbb{Z}_4, \beta_1)$ -flow  $(D_1, f_1)$  with a strong orientation  $D_1(G/A^c - d)$ . Next, contract  $A$  to a vertex  $z$  in  $G$ , prescribe  $z$  according to  $f_1$ , and then apply the induction on  $G/A$  by viewing  $z$  as the new “ $d$ ”. There is another  $(\mathbb{Z}_4, \beta_1)$ -flow  $(D_2, f_2)$  with a strong orientation  $D_2(G[A^c] - z)$ . Combining the two flows together, we obtain a flow on  $G$  with a strong orientation of  $G - d$ , a contradiction. ■

**Proposition 15.** *There does not exist an essential 5-cut  $[A, A^c]$  such that  $\{d, t\} \subseteq A$ , and  $G[A^c]$  is 3-edge-connected.*

*Proof.* The proof is very similar to the proof of Proposition 14. Note that, with the assumption  $\{d, t\} \subseteq A$ , we can apply the induction on  $G/A^c$  and  $G/A$  appropriately, as the conditions (i)-(vi) in Theorem 9 can be justified for both  $G/A^c$  and  $G/A$ . By applying the same method as in the proof of Proposition 14, we combine the two flows on  $G/A^c$  and  $G/A$  together to construct a flow on  $G$  with a strong orientation of  $G - d$ . This leads to a contradiction, hence verifying the proposition. ■

We are now able to strengthen the two propositions above by deleting the restriction “3-edge-connected”.

**Proposition 16.** *There is no essential 4-cuts in  $G$ .*

*Proof.* On the contrary, suppose that there exists an essential 4-cut  $[A, A^c]$  with  $d \in A$ , we take the one with  $|A^c|$  minimized. Since  $G$  is 3-edge-connected and the only possible 3-cuts are  $\delta(d)$  or  $\delta(t)$ ,  $G[A^c]$  is 2-edge-connected, and by Proposition 14  $G[A^c]$  must contain a 2-cut. From Proposition 11, it can be deduced that  $|A^c| \geq 3$ , and by the minimality of  $|A^c|$ , this 2-cut can only be  $[t, A^c \setminus \{t\}]$ . As a result,  $\delta(A \cup \{t\})$  is an essential 5-cut.

Among all essential 5-cuts  $[B, B^c]$  with  $A \cup \{t\} \subseteq B$ , choose the one with  $|B^c|$  minimized.  $G[B^c]$  is 2-edge-connected, and by Proposition 15,  $G[B^c]$  has a 2-cut  $[F, \bar{F}]$ , with  $|[B, F]| = 2$  and  $|[B, \bar{F}]| = 3$ , so  $|\delta(F)| = 4$  and  $|\delta(\bar{F})| = 5$ . By minimality of  $|A^c|$  and  $|B^c|$ , both  $F$  and  $\bar{F}$  are singletons linked by parallel double edges, a contradiction to Proposition 11. ■

**Proposition 17.** *There does not exist an essential 5-cut  $[A, A^c]$  such that  $\{d, t\} \subseteq A$ .*

*Proof.* Suppose that there exists such an essential 5-cut, take the one with  $|A^c|$  minimized. Since  $G$  is 3-edge-connected and the only possible 3-cuts are  $\delta(d)$  or  $\delta(t)$ ,  $G[A^c]$  is 2-edge-connected, and by Proposition 15  $G[A^c]$  must contain a 2-cut  $[B, \bar{B}]$ ,  $||[A, B]|| = 2$  and  $||[A, \bar{B}]|| = 3$ , so  $|\delta(B)| = 4$  and  $|\delta(\bar{B})| = 5$ . From Proposition 16 and the minimality of  $|A^c|$ , both  $B$  and  $\bar{B}$  are singletons linked by parallel double edges, a contradiction to Proposition 11. ■

From Proposition 16 and Proposition 17, two corollaries are obtained, which will be helpful in later proofs.

**Corollary 18.**  $G - d$  is 3-edge-connected.

*Proof.* Suppose  $[B, \bar{B}]$  is a cut of  $G - d$  of size at most 2. First we point out that its size is exactly 2. Assume  $||[B, \bar{B}]|| \leq 1$ , since  $G$  is 3-edge-connected and  $\deg(d) \leq 5$ ,  $||[d, B]|| \geq 2$ ,  $||[d, \bar{B}]|| \leq 3$ , and both  $|\delta(B)|$  and  $|\delta(\bar{B})|$  are at most 4. From Proposition 16, both  $B$  and  $\bar{B}$  are singletons, but we have mentioned that  $|V(G)| > 3$  at the beginning of this section.

So  $||[B, \bar{B}]|| = 2$ , and  $||[d, B]|| \geq 1$ ,  $||[d, \bar{B}]|| \geq 1$ . If  $||[d, B]||$  or  $||[d, \bar{B}]||$  equals 1 (assume it is the former), then  $B = \{t\}$ , from condition (iv),  $\deg(d) \leq 4$ , hence  $||[d, \bar{B}]|| \leq 3$  and  $|\delta(\bar{B})| \leq 5$ . By Proposition 17,  $\bar{B}$  is a singleton as well, limiting  $G$  to 3 vertices.

So  $||[d, B]|| \geq 2$ ,  $||[d, \bar{B}]|| \geq 2$ , and since  $\deg(d) \leq 5$ , at least one of the equalities holds (still assume it is  $B$ ). Thus  $\delta(B) = 4$ , by Proposition 16,  $B$  is a singleton.  $|\delta(\bar{B})| = 4$  when  $\deg(d) = 4$  and 5 when  $\deg(d) = 5$ , by Proposition 16 or 17 respectively,  $\bar{B}$  is a singleton too, contradicting Proposition 11. ■

**Corollary 19.** An essential 7-cut of  $G$  is a bond unless it has the form  $\delta(\{x_1, x_2\})$ , where  $x_1, x_2$  are two vertices,  $\deg(x_1) = 3$ ,  $\deg(x_2) = 4$ , and they are not adjacent in  $G$ .

*Proof.* Since  $G$  is 3-edge-connected, an essential 7-cut which is not a bond can only be a union of two cuts, whose size are 3 and 4 respectively. By Proposition 16, both of them are formed by a single vertex. ■

In the following three propositions, we show that the two special vertices  $d$  and  $t$  both exist, and they are not adjacent.

**Proposition 20.**  $d$  exists in  $G$ , and  $\deg(d) = 3$  or  $4$ .

*Proof.* If  $d$  does not exist, then  $G$  belongs to one of the following cases, and in each of them we can apply the induction on a smaller graph with less unoriented edges.

(1) All vertices in the boundary have degree at least 5. Take an edge  $e$  in the boundary and delete it, then essential cuts in  $G - e$  have size at least 4 by Proposition 16. Apply the induction on  $G - e$ .

(2) There is no 3-vertex but a 4-vertex  $z$  in the boundary.  $G - z$  is 2-edge-connected; assume there exists a 2-cut  $[B, \bar{B}]$ . In this case  $G$  has no cut of size at most 3, so  $||[z, B]|| = ||[z, \bar{B}]|| = 2$ . By Proposition 16, both  $B$  and  $\bar{B}$  are singletons with parallel double edges in between, violating Proposition 11, so  $G - z$  is 3-edge-connected. Prescribe  $z$  properly and appoint it as " $d$ " in Theorem 9, then we can apply the induction on  $G - z$ .

(3) There exists a 3-vertex  $z'$  in the boundary. Assume there exists a 2-cut  $[F, \bar{F}]$  in  $G - z'$ , setting  $||[z', F]| = 2$ ,  $||[z', \bar{F}]| = 1$  without loss of generality. By Proposition 16,  $F$  is a singleton linked to  $z'$  by parallel double edges, violating Proposition 11, so  $G - z'$  is 3-edge-connected. Prescribe  $z'$  properly and appoint it as " $d$ " in Theorem 9, then we can apply the induction on  $G - z'$ .

Now we have proved the existence of  $d$ . If  $\deg(d) = 5$ , choose an arc  $a$  in the boundary incident to  $d$ , and apply the induction on  $G - a$ . ■

**Proposition 21.**  $t$  exists.

*Proof.* Presume on the contrary that  $t$  does not exist, then all boundary vertices except  $d$  have degree at least 4.

When  $\deg(d) = 4$ , choose an arc  $a$  in the boundary incident to  $d$ , and apply the induction on  $G - a$ .

When  $\deg(d) = 3$ , by Proposition 17, an essential cut formed by  $d$  and one of its boundary neighbours has size at least 6, and it can be deduced that the minimum degree of  $G - d$  is at least 4. Moreover,  $G - d$  contains no essential cut of size at most 3, since such a cut yields an essential cut of size at most 4 in  $G$ . We apply the induction on  $G - d$ . ■

**Proposition 22.**  $d$  and  $t$  are not adjacent.

*Proof.* If  $d$  and  $t$  are adjacent,  $\delta(\{d, t\})$  is an essential cut of size 4 or 5, contradicting Proposition 16 or Proposition 17. ■

In the following propositions, we make some preliminary descriptions of the structure of  $G$ , which is the basis of our analyses in Section 4. We end this section with a more detailed depiction about essential 5-cuts as a corollary of these descriptions.

**Proposition 23.**  $t$  is not incident to a chord.

*Proof.* On the contrary, suppose  $tx$  is a chord separating  $G$  into two subgraphs  $G_1$  and  $G_2$  with  $G_1 \cup G_2 = G$ ,  $V(G_1) \cap V(G_2) = \{t, x\}$ , and  $d \in V(G_1)$ . We use  $\deg_i$  to denote the degree of a vertex in  $G_i$  ( $i = 1, 2$ ). In  $G_1$ ,  $d$  and  $t$  are not adjacent; in  $G_2$ , there exists a vertex in the boundary other than  $\{t, x\}$ , with degree at least 4. By Proposition 11, each of  $G_i$  has at least four vertices, and it can be deduced from Proposition 16 that  $\deg_i(x) \geq 5$ .

In  $G_1$ , contract  $tx$  to a vertex  $z$ . Denote the  $\mathbb{Z}_4$ -boundary of  $G_1/tx$  by  $\beta_1$ ,  $\beta_1(v) = \beta(v)$  for  $v \in V(G_1) \setminus \{t, x\}$  and set the boundary value  $\beta_1(z)$  so as to make the sum  $\sum_{w \in V(G_1/tx)} \beta_1(w) = 0$ .  $G_1/tx - d$  contains no bad 2-cut under  $\beta_1$ , otherwise yielding a bad 2-cut under  $\beta$  in  $G - d$ . Now apply the induction on  $G_1/tx$ , and we obtain a  $(\mathbb{Z}_4, \beta_1)$ -flow  $(D_1, f_1)$  with a strong orientation  $D_1(G_1/tx - d)$ . We are going to construct a flow on  $G_1$  stemming from it. From the proof of Proposition 11, we can orient  $tx$  to extend  $D_1$  to a strong orientation of  $G_1 - d$ . We choose this orientation, but do not label  $tx$  right now.

Prescribe  $t$  to achieve  $\beta(t)$  in  $G$  with the two unlabelled edges in  $G_2$ , hence  $tx$  has been definitely oriented and labelled so far, a flow on  $G_1$  is obtained. Add an extra arc

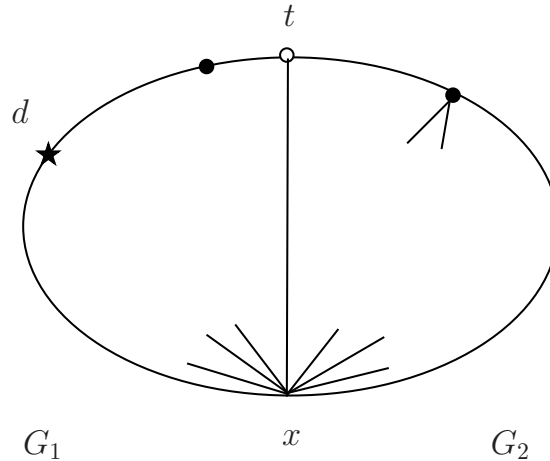


Figure 1

$a = tx$  labelled with a fixed value 1 or 2.  $G_2 + a$  has no essential cut of size at most 3, since such a cut must separate  $t$  and  $x$ , and because  $|[t, V(G_1) \setminus \{t, x\}]| = 1$ , it will yield an essential cut of size at most 4 in  $G$ . Furthermore,  $G_2 - t$  has no 2-cut, otherwise a cut of size at most 3 is induced in  $G$ , which is neither  $\delta(d)$  nor  $\delta(t)$ . Denote the  $\mathbb{Z}_4$ -boundary of  $G_2$  by  $\beta_2$ . Determine  $\beta_2(t)$  according to its prescription in  $G_2 + a$ ,  $\beta_2(v) = \beta(v)$  for  $v \in V(G_2) \setminus \{t, x\}$ , and set  $\beta_2(x)$  so as to make the sum  $\sum_{w \in V(G_2)} \beta_2(w) = 0$ . Now we can

apply the induction on  $G_2 + a$ , a  $(\mathbb{Z}_4, \beta_2)$ -flow  $(D_2, f_2)$  on  $G_2 + a$  (and thus on  $G_2$ ) is obtained, with a strong orientation  $D_2(G_2 - t)$ . Combining the prescription of  $t$ , there arises a  $(\mathbb{Z}_4, \beta)$ -flow on  $G$  with a strong orientation of  $G - d$ , a contradiction. ■

**Proposition 24.** *The degree of boundary neighbours of  $t$  is exactly 4.*

*Proof.* Assume one boundary neighbour of  $t$  has degree at least 5, then  $G - t$  satisfies conditions (i), (ii) and (v) in Theorem 9, and our goal is to check (iii) and (vi) to apply the induction on  $G - t$ . First, since  $\deg(t) = 3$ ,  $G - t$  has no essential cut of size at most 3, otherwise it induces an essential cut of size at most 4 in  $G$ . We will show that (vi) holds if  $t$  has been wisely prescribed by a series of Lemmas and Claims.

Say a *composition* of  $b \in \mathbb{Z}_4$  is a way of writing  $b$  as the sum of an ordered sequence of values in  $\mathbb{Z}_4$ . We start with a preparing lemma:

**Lemma 25.** *Let  $L_1, L_2$  be two subsets of  $L_3 = \{1, -1, 2\} \subseteq \mathbb{Z}_4$ , where  $|L_1| = |L_2| = 2$ . For every value  $b \in \mathbb{Z}_4$ ,  $b$  has a composition  $b = b_1 + b_2 + b_3$ ,  $b_i \in L_i$  ( $i = 1, 2, 3$ ), where either  $2 \in \{b_1, b_2, b_3\}$  or  $\{1, -1\} \subseteq \{b_1, b_2, b_3\}$ .*

**Proof of Lemma 25.** Both  $L_1$  and  $L_2$  can be regarded to be the remaining part after deleting one value from  $L_3$ . We make a table of all possible values of  $L_1 + L_2$ :

Deleted values	$L_1 + L_2$
1, 1	0, 1, 2
-1, -1	0, -1, 2
2, 2	0, 2
1, -1	0, 1, -1
1, 2	0, 1, -1, 2
-1, 2	0, 1, -1, 2

Anyway,  $L_1 + L_2 + L_3 = \{0, 1, -1, 2\}$ . We discuss the composition of  $b$  according to its parity.

- When  $b$  is even, there exists a 2 in any composition of  $b$ ;
- When  $b = \pm 1$ , then if none of 2 is deleted,  $b = 2 + 2 + b$ ; if one of 2 is deleted,  $b = b + 2 + 2$  or  $b = 2 + b + 2$ ; if two of 2 are deleted,  $b = 1 + (-1) + b$ .

**Claim 26.**  $t$  can be prescribed properly without leaving any bad 2-cut in  $G - d - t$ , under the induced  $\mathbb{Z}_4$ -boundary.

*Proof.* We show that the 2-cuts in  $G - d - t$  are too special and rare to make great influence on the prescription of  $t$ . Let  $[B, \bar{B}]$  be a 2-cut in  $G - d - t$ .

*Subclaim 26.1.*  $B$  or  $\bar{B}$  is a single vertex in  $N(t)$ .

**Proof of Subclaim 26.1.**  $[B \cup t, \bar{B}]$  or  $[B, \bar{B} \cup t]$  is a 3-cut of  $G - d$ . We may assume it is  $[B \cup t, \bar{B}]$ , and  $|[t, \bar{B}]| = 1$ .

In  $G$ ,  $|\delta(B \cup t)| > 4$ ,  $|\delta(\bar{B})| > 3$ , and  $\deg(d) = 3$  or 4. It can be deduced from these facts that  $|[d, \bar{B}]| \leq 2$ ,  $|\delta(\bar{B})| = 4$  or 5. By Proposition 16 or 17,  $\bar{B}$  is a single vertex adjacent to  $t$ .

*Subclaim 26.2.* There are at most two 2-cuts in  $G - d - t$ .

**Proof of Subclaim 26.2.** We write  $z$  the single vertex in a 2-cut for convenience. If  $\deg(d) = 3$ ,  $\deg(z) = 4$ ,  $z$  is a boundary neighbour of  $t$  and the subclaim holds. If  $\deg(d) = 4$ , we presume that all three possible 2-cuts coexist, and the three corresponding  $z$  are  $z_1$ ,  $z_2$  and  $z_3$ . The configuration of  $G$  is shown in Figure 2.

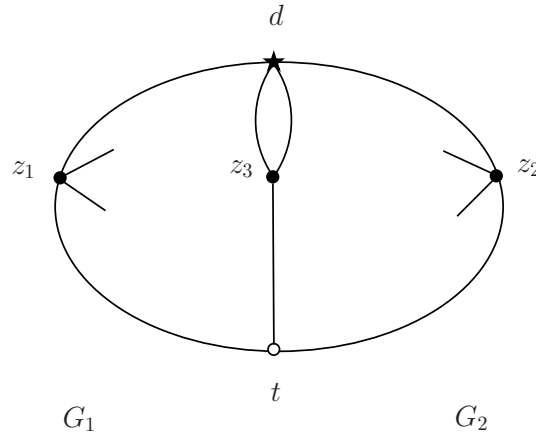


Figure 2

$G$  is separated into two subgraphs  $G_1$  and  $G_2$ , with  $V(G_1) \cap V(G_2) = \{d, z_3, t\}$ ,  $\deg(z_1) = \deg(z_2) = 4$  and  $\deg(z_3) = 5$ . It can be deduced from Proposition 11 that the internal vertex set of  $G_i$  is nonempty, and we use  $U_i$  to represent it ( $i = 1, 2$ ). From condition (v) of Theorem 9,  $|\delta(U_i)| \geq 5$ , but  $|\delta(U_i)| = |\delta(U_i) \cap \delta(z_i)| + |\delta(U_i) \cap \delta(z_3)| \leq 2 + 2 = 4$ , a contradiction.

**Proof of Claim 26:** To find a proper prescription of  $t$ , equivalently, we may make the three edges incident to  $t$  all out-arcs, each of which is equipped with a value list  $\{1, -1, 2\} \in \mathbb{Z}_4$ . Making at most two 2-cuts not bad is equivalent to selecting at most two of the lists, and delete one value from each of them. By Lemma 25, there exists a scheme of labelling containing 2 or  $\{1, -1\}$ . Since the edge labelled with 2 can be oriented arbitrarily, it corresponds a proper prescription of  $t$ . ■

At last, we have proved that  $t$  has a proper prescription making (vi) hold in  $G - t$ . We can apply the induction on  $G - t$ , and gain a flow on  $G - t$  with a strong orientation of  $G - d - t$ , which can be naturally extended to a flow on  $G$  with a strong orientation of  $G - d$ . The proof of Proposition 24 is finished. ■

**Proposition 27.** *A vertex of degree 4 is not incident to an unoriented chord.*

*Proof.* Similarly, let  $x$  be the 4-vertex, and  $xy$  is a chord separating  $G$  into two subgraphs  $G_1$  and  $G_2$  with  $G_1 \cup G_2 = G$ ,  $V(G_1) \cap V(G_2) = \{x, y\}$ , and  $d \in G_1$ . We use  $\deg_i$  to denote the degree of a vertex in  $G_i$  ( $i = 1, 2$ ). In  $G_2$ , there exists a vertex in the boundary other than  $\{x, y\}$ , with degree at least 3, so  $G_2$  has at least four vertices, and by Proposition 16,  $\deg_2(y) \geq 4$ .

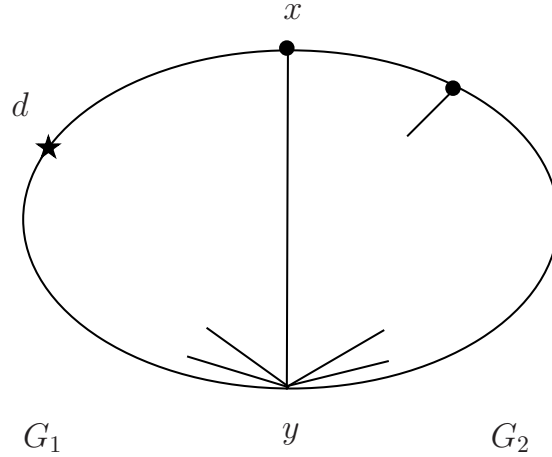


Figure 3

**Case 1.**  $V(G_1) \setminus \{x, y\} = \{d\}$ .

In this case  $\deg_2(x) = 3$ , otherwise there are parallel double arcs between  $d$  and  $x$ , and thus  $\delta(\{d, x\})$  is an essential 3 or 4-cut of  $G$ .

If there is an essential cut of size at most 3 in  $G_2$ , then it separates  $x$  and  $y$ , or the cut induces an essential cut of size at most 3 in  $G$ . Assume it is  $[B, \bar{B}]$  and  $x \in B$ .  $d$  is connected with  $B$  by a single arc  $dx$ , and  $\delta(B)$  is an essential cut of size at most 4 in  $G$ . So essential cuts in  $G_2$  have size at least 4.

Still in  $G_2$ , assume  $[F, \bar{F}]$  is a 2-cut of  $G_2 - x$  and  $y \in F$ .  $|[x, \bar{F}]| = 1$  or 2,  $|\delta(\bar{F})| \leq 4$ , so  $\bar{F}$  is a singleton. If so,  $|[x, \bar{F}]| = 1$  and  $\deg(\bar{F}) = 3$ , hence  $\bar{F} = \{t\}$ , and the possible 2-cut in  $G_2 - x$  is unique. Denote the  $\mathbb{Z}_4$ -boundary in  $G_2$  by  $\beta_2$ ,  $\beta_2(x)$  and  $\beta_2(y)$  are set to achieve  $\beta(x)$  and  $\beta(y)$  under the restriction of  $d$ , and  $\beta_2(w) = \beta(w)$  for  $w \in V(G_2) \setminus \{x, y\}$ . The same as the proof of Claim 26,  $x$  can be prescribed properly so that there is no bad 2-cut in  $G_2 - x$  under the induced boundary from  $\beta_2$ . Applying the induction on  $G_2$ , we obtain a flow  $(D_2, f_2)$ , where  $D_2(G_2 - x)$  is strong, and thus  $D_2(G_2)$  is strong.

**Case 2.**  $V(G_1) \setminus \{x, y\}$  has at least two vertices.

Copy the proof of Claim 23.  $G_1/xy$  admits a flow  $(D_1, f_1)$ , where  $D_1(G_1/xy - d)$  is strong. Add an extra arc  $a = xy$  labelled with a fixed value 1 or 2,  $G_2 + a$  contains no essential cut of size at most 3. Assume  $[K, \bar{K}]$  is a 2-cut of  $G_2 - x$ , and  $y \in K$ .  $|\delta(\bar{K})| \leq 4$ , so  $\bar{K}$  must be a single vertex adjacent to  $x$ . The only possible 2-cut of  $G_2 - x$  is  $[t, V(G_2) \setminus \{x, t\}]$ .

(1) If  $x$  and  $t$  are not adjacent, prescribe  $x$  so that  $D_1$  is extended to a strong orientation of  $G_1 - d$ ; apply the induction on  $G_2 + a$ ,  $G_2$  admits a flow  $(D_2, f_2)$  where  $D_2(G_2 - x)$  is strong. Combining the two parts together, we obtain a flow with a strong orientation of  $G - d$ .

(2) If  $x$  and  $t$  are adjacent and  $\deg_2(x) = 3$ , prescribe  $x$  so that  $D_1$  is extended to a strong orientation of  $G_1 - d$  and  $[t, V(G_2) \setminus \{x, t\}]$  is not bad under the induced  $\mathbb{Z}_4$ -boundary  $\beta_2$ : the orientation of  $xy$  is fixed, and  $xt$  has at most one forbidden value, so this is equivalent to composing a certain value in  $\mathbb{Z}_4$  with three copies of list  $\{1, -1, 2\}$ , at most two of which having one value deleted. By Lemma 25, there exists such a prescription, then apply the induction on  $G_2 + a$  as we have done in (1).

(3) If  $x$  and  $t$  are adjacent and  $\deg_2(x) = 2$ , prescribe  $x$  and  $t$  so that: (i)  $D_1$  is extended to a strong orientation of  $G_1 - d$ ; (ii) the orientation of  $t$  is proper.

$G_2 - x - t$  is 3-edge-connected, and the sizes of essential cuts are at least 4: Let  $[Q, \bar{Q}]$  be a cut of size at most 3 with  $y \in Q$ ,  $t$  is linked to  $\bar{Q}$  by at most two edges, so  $4 \leq |\delta(\bar{Q})| \leq 5$ . The only possible case is that  $\bar{Q}$  is a single vertex in  $N(t) \setminus \{x\}$  of degree 4 in  $G$ , and  $||[Q, \bar{Q}]|| = 3$ .

We apply the induction on  $G_2 - x - t$ : by Proposition 16, the condition  $\deg_2(x) = 2$  implies  $\deg_2(y) \geq 5$ ; remember  $t$  is not incident to a chord,  $G_2 - x - t$  contains at most one 3-vertex. By the induction on  $G_2 - x - t$  (setting the prescribed special vertex in Theorem 9 non-existent), we obtain a flow  $(D_2, f_2)$  where  $D_2(G_2 - x - t)$  is strong. By prescriptions (i) and (ii), we get a flow with boundary  $\beta$  and a strong orientation of  $G - d$  formed by  $D_1$ ,  $D_2$  and  $\{t\}$ . This completes the proof. ■

**Proposition 28.** *There does not exist an essential 5-cut  $[A, A^c]$  such that  $\delta(A) \cap \delta(t)$  is a single edge in the boundary.*

*Proof.* Assume there exists such an  $[A, A^c]$ . By Proposition 17,  $d$  and  $t$  belong to different sides of the cut, say  $d \in A$  and  $t \in A^c$ . Assume, furthermore that among all such cuts  $|A^c|$  is minimum. Prescribe  $t$  in some way and delete it. Denote the boundary edge of  $\delta(A) \cap \delta(t)$  by  $e$ . First contract  $A^c \setminus \{t\}$  and apply the induction on  $(G - t)/(A^c - t)$  (equivalently  $G/A^c - e$ ), then contract  $A$  and apply the induction on  $G/A - t$ . It will be shown that there exists a proper prescription of  $t$  to guarantee the proceeding of the two inductions.

Edge connectivity: essential cuts in  $G/A^c - e$  have size at least 4 because of Proposition 16;  $G/A - t$  contains no essential cut of size at most 3, otherwise yielding an essential cut of size at most 4 in  $G/A$  and thus in  $G$ .

**Claim 29.** *There is at most one 2-cut in  $G/A^c - e - d$ .*

**Proof of Claim 29:** By Corollary 18,  $G - d$  is 3-edge-connected;  $G/A^c - e - d$  is 2-edge-connected, and any 2-cut contains edge  $e$  in  $G$ . Let  $[B, \bar{B}]$  be such a 2-cut, and suppose the vertex contracted from  $A^c$  is in  $\bar{B}$ . By Proposition 16, in  $G$ ,  $||[d, \bar{B}]|| \geq 2$ ,  $||[d, B]|| \leq 2$ ,  $|\delta(B)| \leq 5$ , so  $B$  is a singleton from Proposition 17, which can only be the boundary neighbour of  $t$  contained in  $A$ .

**Claim 30.** *There is at most one 2-cut in  $G[A^c \setminus \{t\}]$ .*

**Proof of Claim 30:** Assume  $[F, \bar{F}]$  is a 2-cut of  $G[A^c \setminus \{t\}]$ .

(1) If  $||[A, F]||$  or  $||[A, \bar{F}]|| = 4$ , we may suppose it is the former without loss of generality (for convenience, we will always choose  $F$  as the special one later in this proof).  $|\delta(\bar{F})| \leq 4$ ,

by Proposition 16,  $\bar{F}$  is a singleton incident to  $t$ , resulting in a 3-vertex  $w \notin \{d, t\}$  or multiple edges.

(2) If  $||[A, F]| = 3$ , and  $||[t, F]|$  or  $||[t, \bar{F}]| = 2$ , then  $|\delta(\bar{F})| = 3$  or  $5$ ,  $\bar{F}$  is a singleton incident to  $t$ , resulting in a 3-vertex  $w \notin \{d, t\}$  or multiple edges too. So  $||[t, F]| = ||[t, \bar{F}]| = 1$ ,  $|\delta(\bar{F})| = 4$ , and  $\bar{F}$  is the boundary neighbour of  $t$  not in  $A$ , see Figure 4.

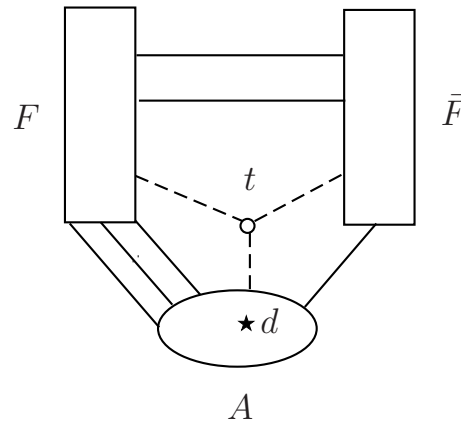


Figure 4

(3) If  $||[A, F]| = ||[A, \bar{F}]| = 2$ , and  $||[t, F]| = ||[t, \bar{F}]| = 1$ , then  $|\delta(F)| = |\delta(\bar{F})| = 5$ , by Proposition 17, both  $F$  and  $\bar{F}$  are singletons with parallel double edges in between. So  $t$  is linked to one of  $F$  and  $\bar{F}$  by two edges (See Figure 5, where we take  $F$  as the special one as usual).  $|\delta(F \cup \{t\})| = 5$ , and  $\delta(t) \cap \delta(F \cup \{t\})$  is a single edge in the boundary of  $G$ , a contradiction to the minimality of  $|A^c|$ .

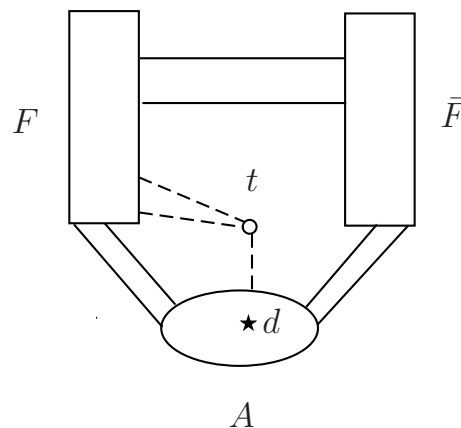


Figure 5

With the two claims above, we make a scheme to prescribe  $t$  to satisfy condition (vi) in Theorem 9, as we have mentioned at the beginning of this proposition.

There are three ways to orient and label  $e$ , of which we choose the one so that: **(I)** the only possible 2-cut of  $G/A^c - e - d$  is not bad under the induced  $\mathbb{Z}_4$ -boundary; **(II)** the induced boundary value of the remaining two edges of  $t$  is not 2. We can apply the induction on  $G/A^c - e$ , and obtain a flow  $(D_1, f_1)$ , where  $D_1(G/A^c - e - d)$  is strong. Transfer this flow to  $G/A - t$ .

If the only possible 2-cut in  $G/A - t$  exists, orient and label the remaining two edges of  $t$  to achieve  $\beta(t)$ , so that: **(1)** the 2-cut is not bad under the induced  $\mathbb{Z}_4$ -boundary, with the restriction of  $(D_1, f_1)$ ; **(2)** the orientation of  $t$  is proper. Because of **(II)**, these can be realized, see the illustration in the proof of Proposition 11. We can apply the induction on  $G/A - t$ , and obtain a flow  $(D_2, f_2)$ , where  $D_2(G[A^c \setminus \{t\}])$  is strong. Combining  $(D_1, f_1)$ ,  $(D_2, f_2)$  as well as  $\{t\}$ , we get a flow with a strong orientation of  $G - d$ . ■

## 4 Proof of Theorem 9

In this section, we denote the two boundary neighbours of  $t$  by  $u$  and  $v$ , while the internal neighbour is  $w$ . The neighbours of  $v$  are  $v_1, v_2, v_3$  and  $t$  in cyclic order, where  $v_1$  is in the boundary. Let  $e_1 = vv_1$  and  $e_2 = vv_2$ .  $d$  and  $t$  cut the boundary cycle  $C$  (We named it in **Section 3** when finishing the proof of Proposition 12) into two segments separating  $u$  and  $v$ . Denote the two boundary neighbours of  $d$  by  $u'$  and  $v'$ , where  $u$  and  $u'$  are on the same side and  $v$  and  $v'$  are on the same side.

With  $v$  specialized, we will perform different operations on the neighbours of  $t$  to create turning points for induction. As a result, a flow with a strong orientation of  $G - d$  will be obtained.

### 4.1 Case 1: $w \in N(u) \cap N(v)$ , and $v$ and $d$ are not adjacent.

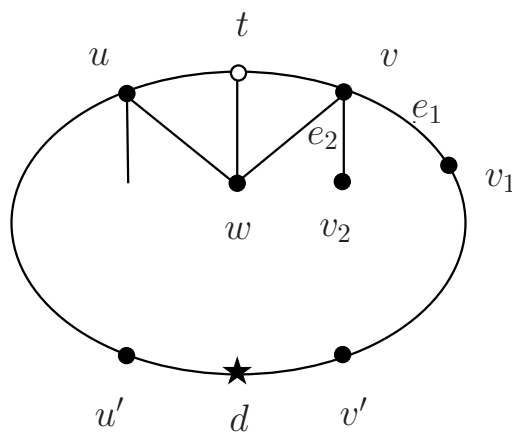


Figure 6: Case I, notice  $v_1$  and  $v'$  may be identical.

Orient and label  $e_1$  and  $e_2$  to realize  $\beta(v)$ . Let  $D^* = \{u, t, v, w\}$ , and denote by  $d^*$  the vertex to which  $D^*$  is contracted. Delete  $e_1$  and  $e_2$  and let  $G' = G/D^* - e_1 - e_2$ . The  $\mathbb{Z}_4$ -boundary  $\beta'$  is induced from  $\beta$  by the contraction and deletion.

**Lemma 31.** *A flow on  $G' - d$  with a strong orientation can be extended to a flow on  $G - d$ , whose orientation is strong too.*

*Proof.* This can be done by prescribing  $u$  and  $t$  wisely within the diamond  $D^*$ . Let  $(D, f)$  be such a flow on  $G' - d$ . First, by Lemma 10, there exists a certain orientation of  $uw$  so that  $D$  is extended to a strong orientation of  $G - d - t - v$ , then we prescribe  $u$  with this orientation of  $uw$  to achieve  $\beta(u)$ . The value and orientation of  $ut$  may be determined but still, we can prescribe  $t$  properly and since the  $\mathbb{Z}_4$ -boundary value of  $v$  in  $D^*$  is 0, we prescribe  $vw$  with the same value and orientation as  $tv$ . As the result,  $utvw$  is a directed path, or  $utw$ ,  $tvw$  are two directed paths. Anyway,  $(D, f)$  is extended to a flow on  $G - d - \{e_1, e_2\}$  with a strong orientation, and thus on  $G - d$ . ■

**Lemma 32.** *There is no essential cut of size at most 3 in  $G'$ .*

*Proof.* Assume  $[B, B^c]$  is such a cut with  $d^* \in B$ . The only possible case is that  $|\delta(B)| = 3$  and  $v_1, v_2 \in B^c$ . But if so, then  $B^c$  together with  $v$  induces an essential 5-cut in  $G$  intersecting  $\delta(t)$  at  $tv$ , a contradiction to Proposition 17 or Proposition 28. ■

We now present the last lemma stating that there is at most one 2-cut in  $G' - d$ . With the three lemmas, we can apply the induction on  $G'$  by making it not bad in the prescription of  $e_1$  and  $e_2$ .

**Lemma 33.** *There is at most one possible 2-cut in  $G' - d$ , and if there is one, it can only be  $\delta(v_1) \cap E(G' - d)$ .*

### The proof of Lemma 33:

Let  $[F, \bar{F}]$  be a 2-cut of  $G' - d$  with  $d^* \in F$ , where  $\bar{F} \neq \{v_1\}$ . We will show that such a cut actually cannot exist.

- If  $|[d, \bar{F}]| \leq 1$ , then  $|\delta(\bar{F})| \leq |[d, \bar{F}]| + |\delta(\bar{F}) \cap \{e_1, e_2\}| + 2 \leq 5$ , so  $\bar{F}$  is a single vertex by Proposition 16 or 17. The only possible case is that  $|\delta(\bar{F})| = 4$  and thus  $\bar{F} = \{v_1\}$ .
- If  $|[d, F]| \leq 1$ , then  $|\delta(F)| \leq 5$ . While by Proposition 16,  $|\delta(F)| \geq 5$ , the only possible case is that  $|\delta(F)| = 5$ , and  $v_1, v_2 \in \bar{F}$ .  $\delta(F \setminus \{v\})$  is an essential 5-cut in  $G$  intersecting  $\delta(t)$  at  $tv$ , a contradiction to Proposition 28.

So  $|[d, F]| = |[d, \bar{F}]| = 2$ , and at least one of  $v_i$  ( $i = 1, 2$ ) must be in  $\bar{F}$ , since if not,  $|\delta(F)| = 4$ , violating Proposition 16.

If exactly one  $v_i$  is in  $\bar{F}$ ,  $|\delta(\bar{F})| = 5$ , so it is  $v_2$  and  $\bar{F} = \{v_2\}$ . Since  $d \neq v_1$ , from planarity, it can be shown from Figure 6 that  $\{d, v, v_2\}$  forms an essential 7-cut which is not a bond, since  $G - \{d, v, v_2\}$  is divided into two non-empty components. By Corollary

19, this will not happen. So both  $v_1$  and  $v_2$  are in  $\bar{F}$ . This last case, see Figure 7, is more subtle than the ones above, requiring further research on the structures of  $F$  and  $\bar{F}$  to eliminate it. Denote the two edges of  $[F, \bar{F}]$  by  $e'_1$  and  $e'_2$ . Contract  $F \cup \{d\}$  to a vertex  $d_1$ , and contract  $\bar{F} \cup \{d\}$  to a vertex  $d_2$  (but not at the same time).

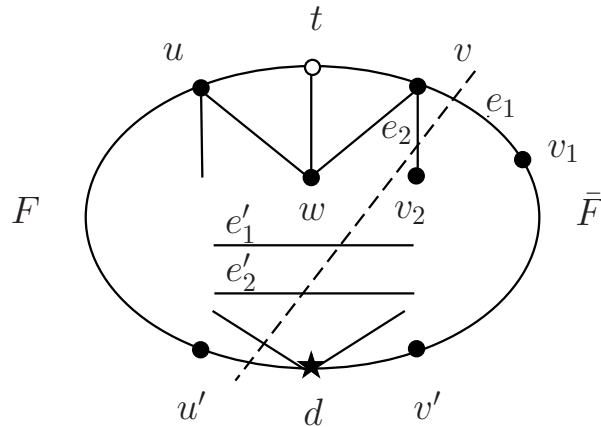


Figure 7:  $||d, F|| = ||d, \bar{F}|| = 2$ , and  $\{v_1, v_2\} \subseteq \bar{F}$ .

**Claim 34.**  $G[\bar{F}]$  has at most one possible 2-cut, and if there is one, it can only be  $\delta(v_1) \cap E(G[\bar{F}])$ ;  $G[F]$  has no 2-cut.

*Proof.* First concentrate on  $G[\bar{F}]$ . Suppose  $[K, \bar{K}]$  is a 2-cut of  $G[\bar{F}]$ . It is obvious that both  $K$  and  $\bar{K}$  are connected with  $d_1$  by at least one edge from  $E(G')$ . For convenience, we call a path  $P$  an *inner path* if internal vertices of  $P$  do not lie on  $C$ .

*Subclaim 34.1.* Vertices on  $C$  from  $v'$  to  $v_1$  in cyclic order are contained in  $\bar{F}$ .

**Proof of Subclaim 34.1:** By Fact 13, both  $\delta(F)$  and  $\delta(\bar{F})$  are bonds of  $G$ , so the boundary vertices belonging to  $F$  and  $\bar{F}$  are consecutive respectively, otherwise  $G[F]$  and  $G[\bar{F}]$  cannot be connected at the same time by planarity.  $F \cup \bar{F} \cup \{d\} = V(G)$ ,  $v \in F$  and  $v_1 \in \bar{F}$ , where it can be deduced that the boundary vertices in  $\bar{F}$  are vertices from  $v'$  to  $v_1$  in cyclic order.

*Subclaim 34.2.* In  $G'$ ,  $||d_1, K|| = ||d_1, \bar{K}|| = 2$  is impossible.

**Proof of Subclaim 34.2:** If so,  $v_1$  and  $v_2$  cannot be separated by the cut, because otherwise  $|\delta(K)| = |\delta(\bar{K})| = 5$  in  $G$ , by Proposition 17,  $K$  and  $\bar{K}$  are singletons with parallel double edges in between. Assume  $v_1, v_2 \in K$  without loss of generality, then  $\bar{K}$  is a boundary 4-vertex of  $G$ . In fact,  $\bar{K} = \{v'\}$ , since  $\delta(K)$  is an essential 6-bond, the boundary vertices within  $K^c = F \cup \{d\} \cup \bar{K}$  are consecutive on  $C$ . What is more,  $|\delta(v') \cap \{e'_1, e'_2\}| = 1$  because there is exactly one edge between  $d$  and  $v'$ .

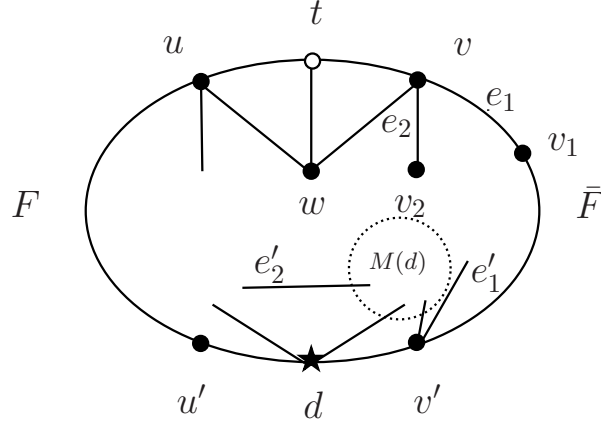


Figure 8: If  $v'$  forms a 2-cut of  $G[\bar{F}]$ .

$\delta(F \setminus \{v\})$  and  $\delta(\bar{F} \cup \{v\})$  are essential 6-bonds, so both  $G[F \setminus \{v\}]$  and  $G[\bar{F} \cup \{v\}]$  are connected. Define  $M(d)$  to be the internal vertex set which  $d$  can reach through an inner path in  $G[\bar{F} \cup \{d, v\}]$ . By planarity,  $d$  does not form a chord with any vertex in  $\{v', \dots, v_1, v\}$ , because one of  $e'_1, e'_2$  is incident to  $v'$ . But  $d$  contributes another edge to  $\delta(\bar{F})$  besides  $dv'$ , so  $M(d) \neq \emptyset$ . Still by planarity, there is no inner path within  $G[\bar{F} \cup \{d, v\}]$  connecting  $d$  with a boundary vertex in  $\bar{F} \cup \{v\}$  except  $v'$ . These are shown in figure 8. As a result,  $|\delta(M(d))| = |\delta(M(d)) \cap \delta(d)| + |\delta(M(d)) \cap \delta(v')| + |\delta(M(d)) \cap \{e'_1, e'_2\}| \leq 1 + 1 + 1 = 3$ , a contradiction.

*Subclaim 34.3.* In  $G'$ ,  $||d_1, K|| = 1$  or 3 is impossible unless  $v' = v_1$  and  $\deg(v_1) = 4$  in  $G$ .

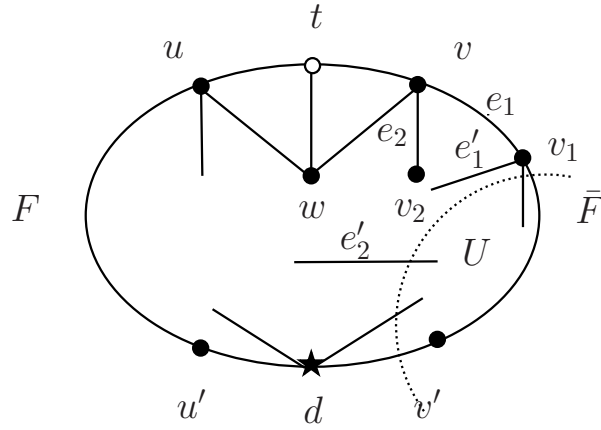


Figure 9: If  $v_1$  forms a 2-cut of  $G[\bar{F}]$ .

**Proof of Subclaim 34.3:** Let  $||d_1, K|| = 3$  without loss of generality.  $|\delta(\bar{K})| \leq 5$ , so  $\bar{K}$  is a singleton. Actually  $\bar{K}$  can only be  $v_1$  with degree 4 in  $G$ . If  $v' \neq v_1$ , then there is no chord between  $d$  and  $v_1$ , because otherwise there exists an essential

4-cut in  $G$ . So  $|\delta(v_1) \cap \{e'_1, e'_2\}| = 1$ . Consider the 6-bond  $\delta(\bar{F} \cup \{v\})$ , by planarity, in  $G[\bar{F} \cup \{v\}]$ , no boundary vertex between  $d$  and  $v_1$  admits an inner path to  $v$ . We define  $U$  to be these boundary vertices as well as the internal vertices they can reach through an inner path in  $G[\bar{F}]$ , see Figure 9.  $\delta(U)$  is a cut not separating  $d$  and  $t$ , while  $|\delta(U)| = |\delta(U) \cap \delta(v_1)| + |\delta(U) \cap \delta(d)| + |\delta(U) \cap \{e'_1, e'_2\}| \leq 2 + 2 + 1 = 5$ . So  $U = \{v'\}$  with degree 4 or 5 in  $G$ , as the result, at least two inequalities of the three terms in the sum hold, causing parallel double edges between  $v'$  and  $d$  or  $v_1$ , neither of which is possible.

The former part of Claim 34 can be deduced from the above three subclaims, and now we turn to  $G[F]$ . Assume  $[Q, \bar{Q}]$  is a 2-cut of  $G[F]$  with  $d^* \in Q$ . In  $G'$ ,  $||[d_2, \bar{Q}]|| \geq 2$  and by Proposition 16,  $||[d_2, Q]|| \geq 1$ . If  $||[d_2, Q]|| = 1$ , then  $|\delta(Q)| = 5$ , and  $\delta(Q \setminus \{v\})$  is an essential 5-cut of  $G$  intersecting  $\delta(t)$  at  $tv$ , violating Proposition 28.

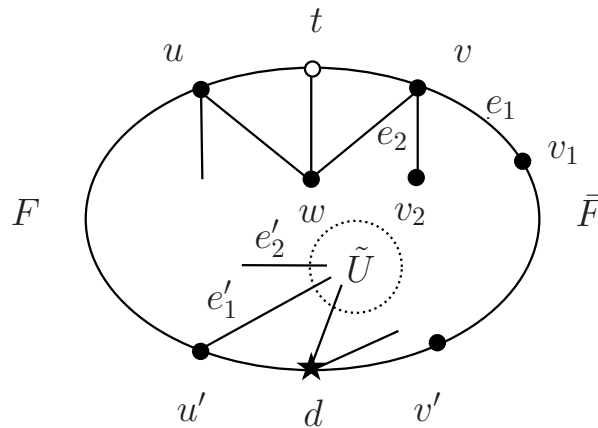


Figure 10: If  $u'$  forms a 2-cut in  $G[F]$ .

So  $||[d_2, Q]|| = ||[d_2, \bar{Q}]|| = 2$ , and  $\bar{Q}$  is a 4-vertex of  $G$ .  $\delta(\bar{Q} \cup \bar{F} \cup \{d\})$  is a 6-bond, so  $\bar{Q}$  can only be  $u'$  with degree 4 in  $G$ . There is exactly one edge between  $d$  and  $u'$ , so  $|\delta(u') \cap \{e'_1, e'_2\}| = 1$ . Consider the 6-bond  $\delta(\bar{F} \cup \{d, v, u'\})$ , by planarity, there is no chord nor an inner path in  $G[\bar{F} \cup \{d, v, u'\}]$  connecting  $u'$  with boundary vertices in  $\{v', \dots, v_1, v\}$  because  $|\delta(d) \cap \delta(F \setminus \{v, u'\})| = 1$ . Define  $\tilde{U}$  to be the set of internal vertices cannot be reached by  $\{v', \dots, v_1, v\}$  through an inner path in  $G[\bar{F} \cup \{v\}]$ , see Figure 10. It can be deduced that  $\tilde{U} \neq \emptyset$  and  $u'$  is linked to  $\tilde{U}$  by one of  $e'_1$  and  $e'_2$ .  $|\delta(\tilde{U})| = |\delta(\tilde{U}) \cap \delta(d)| + |\delta(\tilde{U}) \cap \{e'_1, e'_2\}| \leq 1 + 2 = 3$ , a contradiction. The proof of Claim 34 is completed. ■

In the end, we are able to verify Lemma 33 by constructing a flow on  $G[F]$  and  $G[\bar{F}]$  respectively. Remember we have prescribed  $e_1$  and  $e_2$  to achieve  $\beta(v)$ . Now prescribe  $e'_1$  and  $e'_2$  to achieve  $\beta(F)$  and make sure that the directions of  $\{e_1, e_2, e'_1, e'_2\}$  between  $F$  and  $\bar{F}$  are not identical.  $G[F]$  has no 2-cut, applying the induction on  $F \cup \{d_2\}$  in  $G'$ ,  $G[F]$  admits a flow with a strong orientation.  $G[\bar{F}]$  has one possible 2-cut  $\delta(v_1) \cap E(G[\bar{F}])$  only when  $v' = v_1$  and the degree of  $v_1$  is 4 in  $G$ . When this case arises, we can guarantee the

cut not bad in the prescription of  $e_1$  and  $e_2$ : there are at least two ways to orient and label  $e_1$  to make the cut not bad, of which there exists one so that  $e_2$  is not labelled with 0 to achieve  $\beta(v)$ . So we can apply the induction on  $\bar{F} \cup \{d_1\}$  and obtain a flow on  $G[\bar{F}]$  with a strong orientation too. Combining them together, there is a flow with a strong orientation of  $G' - d$ . By Lemma 31, a flow with a strong orientation of  $G - d$  can be obtained.

#### 4.2 Case 2: $N(v) \cap N(t) = \emptyset$ .

In this case we adopt the lifting method. Consider a vertex  $z$  with two neighbours  $x, y$ . The *lifting* of two edges  $xz$  and  $yz$  means deleting them and adding a new edge  $xy$  (even if one already exists).

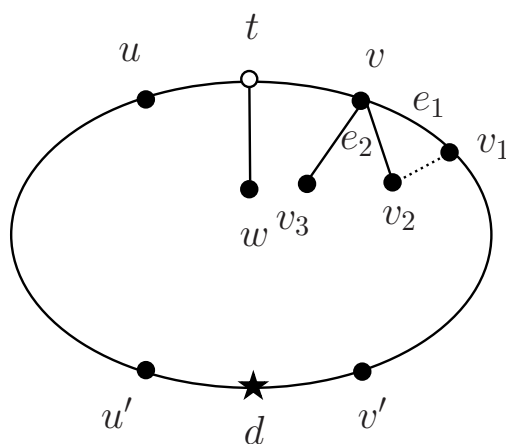


Figure 11: Case II,  $v_1$  and  $d$  may be identical.

In  $G$ , lift  $e_1$  and  $e_2$  at  $v$ , orient and label the remaining two edges to realize  $\beta(v)$ , and properly prescribe  $t$  with the other two edges to realize  $\beta(t)$ . Then delete  $\{v, t\}$  and denote the resulted graph by  $G''$ . Let  $\beta''$  be the induced  $\mathbb{Z}_4$ -boundary of  $G''$ .  $\beta''$  and  $\beta$  differ only at the three vertices  $u, w$  and  $v_3$ , according to the deletion of  $t$  and  $v$ , while for any other vertex  $z \in V(G'')$ ,  $\beta''(z) = \beta(z)$ .

**Lemma 35.** *A flow on  $G'' - d$  with a strong orientation can be extended to a flow on  $G - d$ , whose orientation is strong too.*

*Proof.* If a graph has a flow with a strong orientation, then there is also one if we subdivide an edge of the graph. In  $G''$ , we subdivide the edge  $v_1v_2$  from the lifting with  $v$ ; since  $t$  is properly prescribed, either  $utw$  or  $utv$  is a directed path. So there is a flow on  $G - d$  with a strong orientation. ■

**Lemma 36.** *There is no essential cut of size at most 3 in  $G''$ .*

*Proof.* Let  $[B, \bar{B}]$  be an essential cut of size  $s' \leq 3$  in  $G''$ , with  $d \in B$ . We use this cut to restore an essential cut of  $G$  by adding  $v$  and  $t$  to either  $B$  or  $\bar{B}$ . The rule is to add  $v$  ( $t$ ) to the side containing the larger number of its neighbours. Since three vertices in  $N(v)$  and two vertices in  $N(t)$  are concerned, the resulted cut has size  $s \leq s' + 3 \leq 6$  (do not forget the edge  $tv$ ), which must be a bond. Moreover, by Proposition 16,  $s \geq 5$ , so  $s = s' + 2$  or  $s' + 3$ .

If  $s = s' + 2$ , then  $s' = 3$ ,  $s = 5$ , and thus  $t$  is added to  $\bar{B}$ . Notice that if in the restoration, both  $t$  and  $vt$  contribute an edge to the cut at the same time, we can move  $t$  to  $B$  and obtain an essential cut of size at most 4 in  $G$ , so there are only two possible situations: (1)  $v$  is added to  $B$ ,  $v$  contributes one edge and  $t$  contributes no edge; (2)  $v$  is added to  $\bar{B}$ , and both of them contribute one edge to the cut. As for (1), we will restore an essential cut of size at most 5 intersecting  $\delta(t)$  at  $tv$ , a contradiction to Proposition 28; as for (2),  $v_1$  and  $v_2$  cannot be separated, and  $\delta(t) \cap \delta(B)$  is not in the boundary, so  $\{v_3, w\} \subseteq B$  and  $\{v_1, v_2, u\} \subseteq \bar{B}$ .  $\delta(B \cup \{t\})$  is an essential 6-cut of  $G$  which is not a bond, because by planarity,  $G[B \cup \{t\}]$  and  $G[\bar{B} \cup \{v\}]$  cannot be connected at the same time, which is a contradiction too.

If  $s = s' + 3$ , then  $v$ ,  $t$  and  $vt$  contribute one edge respectively, and moreover,  $v_1$  and  $v_2$  are not separated.  $t$  can be transferred to  $v$ 's side, resulting in an essential cut of size  $s - 1$  in  $G$ , so  $s' = 3$ ,  $s = 6$ ;  $v$  is added to  $\bar{B}$ , and  $t$  is added to  $B$ . What is more, by Proposition 28, it is implied that  $\{v_3, w\} \subseteq B$  and  $\{v_1, v_2, u\} \subseteq \bar{B}$ , which is just the content of (2). ■

**Lemma 37.** *There is at most one 2-cut in  $G'' - d$ , and if there is one, it can only be  $\delta(u) \cap E(G'' - d)$ .*

**Proof of Lemma 37:**

Let  $[F, \bar{F}]$  be a 2-cut of  $G'' - d$ .

(1) If  $||[d, F]||$  or  $||[d, \bar{F}]|| \leq 1$ , since essential cuts in  $G''$  have size at least 4 and the only 3-vertex except  $d$  is  $u$ ,  $F$  or  $\bar{F} = \{u\}$ . This is just the exception we mentioned in the statement of the lemma.

(2) If  $||[d, F]|| = ||[d, \bar{F}]|| = 2$ , write  $T = \{v, t\}$  for convenience. The first thing we need to point out is that  $||[d, F]|| = ||[d, \bar{F}]|| = 2$  implies  $d \neq v_1$ . When  $d = v_1$  happens,  $||[T, F]|| + ||[T, \bar{F}]|| = 3$ , so  $|\delta(F)| \leq 5$  or  $|\delta(\bar{F})| \leq 5$  holds in  $G$ , whence  $F$  or  $\bar{F}$  must be a singleton. Let  $v_2 \in F$  without loss of generality. If the singleton is  $F = \{v_2\}$ , then  $\deg(v_2) = 5$  and  $v_2 \in N(t)$ , contradicting the premise of **Case 2**; if the singleton is  $\bar{F}$ , it can only be  $v_3$  or  $w$  with degree 5 in  $G$ , as well as double arcs between  $d$ . As the result,  $\delta(\{d, w, t\})$  is a bond of size 6, or  $\delta(\{d, v_3, v\})$  is a bond of size 5 in  $G$ . However,  $G$  is separated by three vertices into two disjoint components in both situations.

Moreover,  $||[T, F]|| \geq 1$  and  $||[T, \bar{F}]|| \geq 1$ , otherwise  $\delta(F \cup T)$  or  $\delta(\bar{F} \cup T)$  is an essential 4-cut in  $G$ , violating Proposition 16. We analyse the possible cuts between  $T$  and the two parts, with  $\bar{F}$  specialized to be the side gaining less edges.

- $||[T, \bar{F}]|| = 1$ .  $\bar{F}$  is a singleton with degree 4 or 5 in  $G$ , and degree 4 in  $G''$ , so  $\bar{F} \in \{w, v_1, v_3\}$ . If  $\bar{F} = \{w\}$ , then  $\delta(\{t, w, d\})$  is an essential 5 or 6-cut which

cannot be a bond by planarity, a contradiction to Fact 13; if  $\bar{F} = \{v_1\}$ ,  $\delta(\{d, v_1\})$  is an essential 4-cut of  $G$ , a contradiction to Proposition 16; if  $\bar{F} = \{v_3\}$ ,  $\deg(v_3) = 5$  and  $\delta(\{v, v_3, d\})$  is an essential 7-cut which is not a bond by planarity, a contradiction to Corollary 19.

- $|[T, \bar{F}]|$  is 2. If  $\{w, v\} \subseteq \bar{F}$ , then  $\delta(\bar{F} \cup \{d, t\})$  is an essential 5-cut in  $G$ , a contradiction to Proposition 17.

If two neighbours of  $v$  are in  $\bar{F}$ :  $\delta(\bar{F} \cup \{d, v\})$  is an essential 5-cut intersecting  $\delta(t)$  at  $tv$  if  $v_3 \in \bar{F}$ , violating Proposition 28, so  $\{v_1, v_2\} \subseteq \bar{F}$  and  $v_3 \in F$ .

If one of  $\{w, u\}$  and one of  $\{v_1, v_2, v_3\}$  are in  $\bar{F}$ :  $\delta(\bar{F} \cup \{d, t\})$  is an essential 6 or 7-cut, which must be a bond, so the boundary vertices of  $G$  contained in  $\bar{F} \cup \{d, t\}$  are consecutive.  $v \notin \bar{F}$ , so  $u \in \bar{F}$  and  $w \in F$ . If  $v_1$  or  $v_2 \in \bar{F}$ ,  $\delta(\bar{F})$  is an essential 5-cut in  $G$  not separating  $d$  and  $t$ , violating Proposition 17. Above all, the only possible case is  $\{u, v_3\} \subseteq \bar{F}$ .

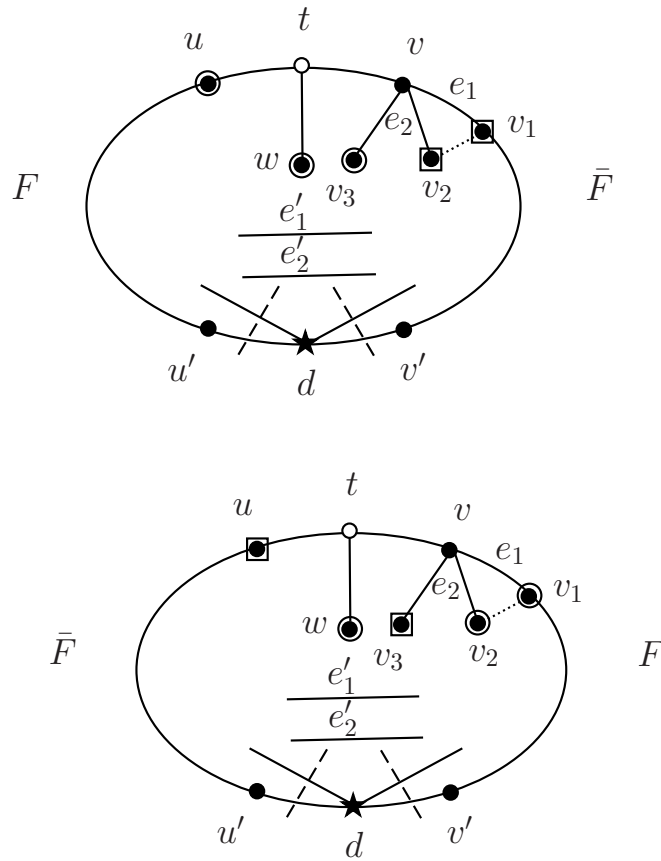


Figure 12: Two bad situations.

Figure 12 shows the configurations of the two possible situations, and for the five concerning vertices  $\{u, w, v_1, v_2, v_3\}$ , we marked each of them with a small circle if it is

in  $F$ , or a small square if it is in  $\bar{F}$ . The same as the proof of **Case 1**, to exclude them, we will work on  $F$  and  $\bar{F}$  respectively. We inherit the notations  $e'_1, e'_2, d_1, d_2$  and  $M(d)$  from **Case 1**. We will discuss the two situations separately with two claims. Actually there is little difference between the proofs of them, and some notations are shared by both proofs.

#### 4.2.1 $\{v_1, v_2\} \subseteq \bar{F}$ .

**Claim 38.**  $G[\bar{F}]$  has no 2-cut;  $G[F]$  has at most one possible 2-cut, and if there is one, it can only be  $\delta(u) \cap E(G[F])$ .

##### Proof of Claim 38:

*Subclaim 38.1.* Boundary vertices from  $v'$  to  $v_1$  in cyclic order are contained in  $\bar{F}$ .

*Proof.* The proof is quite similar to that of Subclaim 34.1. By Fact 13 and Corollary 19, both  $\delta(F)$  and  $\delta(\bar{F})$  are bonds of  $G$ , so the boundary vertices belonging to  $F$  and  $\bar{F}$  are consecutive on  $C$  respectively, otherwise  $G[F]$  and  $G[\bar{F}]$  cannot be connected at the same time by planarity.  $F \cup \bar{F} \cup \{d, t, v\} = V(G)$ ,  $v_1 \in \bar{F}$  and  $u \in F$ , so it can be deduced that the boundary vertices in  $\bar{F}$  are vertices from  $v'$  to  $v_1$  in cyclic order. ■

Suppose  $[K, \bar{K}]$  is a 2-cut of  $G[\bar{F}]$ . By Proposition 16, both  $K$  and  $\bar{K}$  are connected with  $d_1$  by at least one edge.

*Subclaim 38.2.*  $|[d_1, K]| = |[d_1, \bar{K}]| = 2$  is impossible.

*Proof.*  $v_1$  and  $v_2$  cannot be separated by such a cut, otherwise  $|\delta(K)| = |\delta(\bar{K})| = 4$  in  $G$ , by Proposition 16,  $K$  and  $\bar{K}$  are just the two vertices  $v_1$  and  $v_2$ , whence  $\deg(v_2) = 4$  is ridiculous. So assume  $\{v_1, v_2\} \subseteq \bar{K}$  without loss of generality, then  $|\delta(K)| = 4$ ,  $K$  is a single 4-vertex in the boundary;  $|\delta(\bar{K})| = 6$ ,  $\delta(\bar{K})$  is an essential 6-bond, and the boundary vertices in  $\bar{K}$  are consecutive. It can be deduced that  $K = \{v'\}$  distinct from  $v_1$ . There is exactly one edge between  $d$  and  $v'$ , so  $|\delta(v') \cap \{e'_1, e'_2\}| = 1$ .

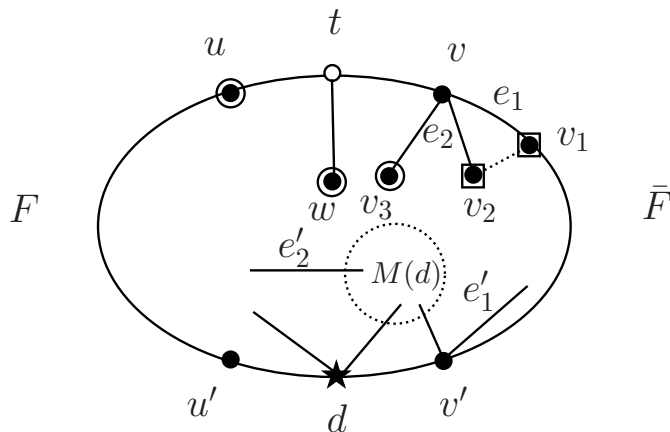


Figure 13: If  $|[d_1, K]| = |[d_1, \bar{K}]| = 2$ , and  $v'$  forms a 2-cut in  $G[\bar{F}]$ .

$\delta(F \cup \{t\})$  and  $\delta(\bar{F} \cup \{v\})$  are essential 6-bonds, so both  $G[F \cup \{t\}]$  and  $G[\bar{F} \cup \{v\}]$  are connected. Define  $M(d)$  to be the internal vertex set which  $d$  can reach through an inner path in  $G[\bar{F} \cup \{d, v\}]$ . By planarity,  $d$  does not form a chord with any vertex in  $\{v', \dots, v_1, v\}$ , because one of  $e'_1, e'_2$  is incident to  $v'$ . But  $d$  contributes another edge to  $\delta(\bar{F})$  besides  $dv'$ , so  $M(d) \neq \emptyset$ . Still by planarity, there is no inner path within  $G[\bar{F} \cup \{d, v\}]$  connecting  $d$  with a boundary vertex in  $\bar{F} \cup \{v\}$  except  $v'$ . These are shown in Figure 13. As a result,  $|\delta(M(d))| = |\delta(M(d)) \cap \delta(d)| + |\delta(M(d)) \cap \delta(v')| + |\delta(M(d)) \cap \{e'_1, e'_2\}| \leq 1 + 1 + 1 = 3$ , a contradiction. ■

*Subclaim 38.3.*  $||[d_1, K]| = 1 \text{ or } 3 \text{ is impossible.}$

*Proof.* By symmetry, we only need to prove  $||[d_1, K]| = 1$  does not hold. If  $||[d_1, K]| = 1$ , then  $|\delta(K)| \leq 5$  in  $G$ , by Proposition 17, it is a single vertex with degree at least 4, so  $||[T, K]| \geq 1$ , and  $K = \{v_1\}$  or  $\{v_2\}$ . Remember we added an extra edge between  $v_1$  and  $v_2$  in the lifting, it is deduced that the degree of  $K$  is actually 3 in  $G$ , which is impossible. ■

We have finished the former part of the claim, now we turn to  $G[F]$ . Assume  $[Q, \bar{Q}]$  is a 2-cut of  $G[F]$ . By Proposition 16 and 17, both  $Q$  and  $\bar{Q}$  are connected with  $d_2$  by at least one edge.

*Subclaim 38.4.*  $||[d_2, Q]| = ||[d_2, \bar{Q}]| = 2 \text{ is impossible.}$

*Proof.* We prove this fact by analysing the edges between  $T$  and the two parts, and suppose  $||[T, Q]| < ||[T, \bar{Q}]|$  without loss of generality. Then there are two possibilities:  $||[T, Q]| = 0$  or  $||[T, Q]| = 1$ .

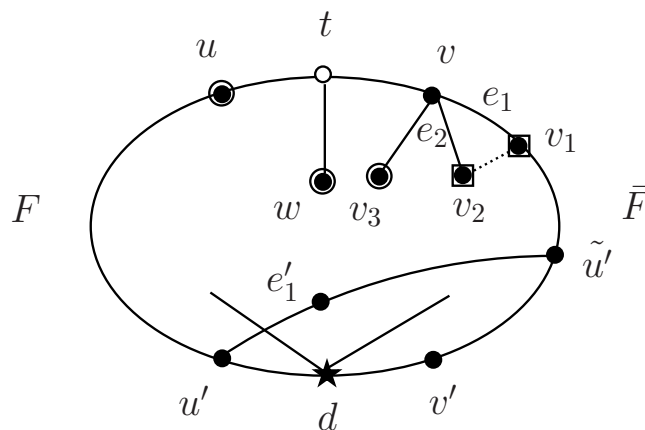


Figure 14: If  $||[T, Q]| = 0$ , and  $u'$  forms a 2-cut in  $G[F]$ .

If  $||[T, Q]| = 0$ ,  $|\delta(Q)| = 4$  in  $G$ , then  $Q$  is a 4-vertex by Proposition 16. Moreover,  $\delta(\bar{F} \cup Q \cup \{d\})$  is an essential 6-cut which must be a bond, so the boundary vertices contained in  $\bar{F} \cup Q \cup \{d\}$  are consecutive in cyclic order. It can be deduced that  $Q = \{u'\}$ ,

and  $u' \neq u$ . Moreover,  $|\delta(u') \cap \{e'_1, e'_2\}| = 1$  because there is exactly one edge between  $d$  and  $u'$ .

Since  $G[\bar{F}]$  is connected, there is an inner path within  $G[\bar{F} \cup \{d, u'\}]$  linking  $u'$  to a boundary vertex of  $\{v', \dots, v_1\}$  (note it  $\tilde{u}'$ , see Figure 14), while  $d$  contributes another edge to  $\delta(\bar{F} \cup \{d, u'\})$ , violating planarity.

If  $|[T, Q]| = 1$ ,  $|\delta(Q)| = 5$  in  $G$ , so  $Q$  is a 5-vertex by Proposition 17. Since  $\deg(u) = 4$ ,  $Q = \{v_3\}$  or  $\{w\}$ .  $Q = \{v_3\}$  is impossible: if so,  $\delta(\bar{Q} \cup \{t\})$  is an essential 5-cut intersecting  $\delta(t)$  at  $tv$ , a contradiction to Proposition 28. So  $Q = \{w\}$ .

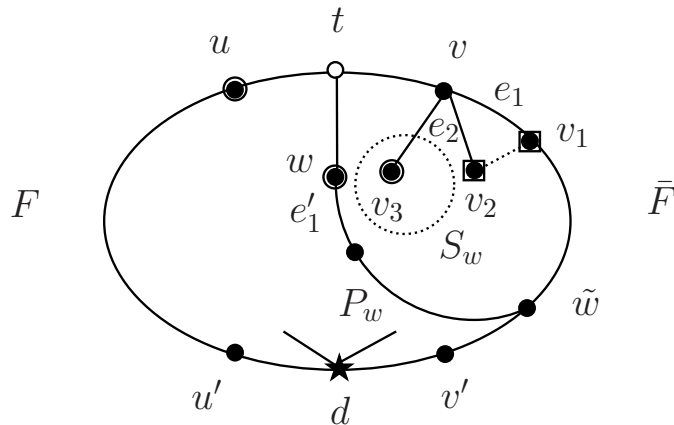


Figure 15: If  $|[T, Q]| = 1$ , and  $w$  forms a 2-cut in  $G[F]$ .

There are two edges in  $\delta(d) \cap \delta(F)$ , one of which is the boundary edge  $du'$ , so  $|\delta(w) \cap \{e'_1, e'_2\}| \geq 1$ . Because  $G[\bar{F}]$  is connected, there is an inner path  $P_w$  linking  $w$  to a boundary vertex of  $\{v', \dots, v_1\}$  (note it  $\tilde{w}$ ), whose internal vertices lie in  $\bar{F}$ .  $wP_w\tilde{w}Ctw$  is a cycle enclosing a non-empty set of internal vertices not in  $\bar{F}$  ( $v_3$  is enclosed), see Figure 15. Use  $S_w$  to denote this set,  $|\delta(S_w)| = |\delta(S_w) \cap \delta(v)| + |\delta(S_w) \cap \delta(w)| + |\delta(S_w) \cap \{e'_1, e'_2\}| \leq 1 + 2 + 1 = 4$ , a contradiction. ■

*Subclaim 38.5.*  $|[d_2, Q]| = 1$  or 3 is impossible unless  $u' = u$  and the degree of  $u$  is 4 in  $G$ .

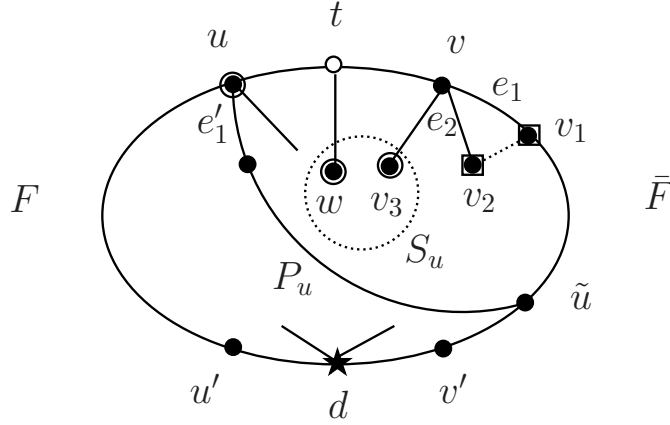


Figure 16: If  $||[d_2, Q]|| = 1$ , and  $u$  forms a 2-cut in  $G[F]$ .

*Proof.* By symmetry, it is enough to prove that the subclaim holds when  $||[d_2, Q]|| = 1$ . obviously  $||[T, Q]|| > 0$ . If  $||[T, Q]|| = 3$ ,  $\delta(Q \cup \{t\})$  is an essential 5-cut intersecting  $\delta(t)$  at  $tv$ , a contradiction to 28. So  $||[T, Q]|| \leq 2$ ,  $|\delta(Q)| \leq 5$ , and thus  $Q$  is a single vertex by Proposition 16 and 17. The only possible case is that  $Q = \{u\}$ , and  $\deg(u) = 4$ .

When  $u' \neq u$ , there is no chord between  $d$  and  $u$ , because otherwise there exists an essential 4-cut in  $G$ . So the edge  $d_2u$  is one of  $e'_1, e'_2$ . There is an inner path  $P_u$  linking  $u$  to some vertex in  $\{v', \dots, v_1\}$  (note it  $\tilde{u}$ ), whose internal vertices lie in  $\bar{F}$ , and  $uP_u\tilde{u}Ctu$  is a cycle enclosing a non-empty set of internal vertices not in  $\bar{F}$  ( $v_3$  and  $w$  are enclosed), see Figure 16. Use  $S_u$  to denote this set,  $|\delta(S_u)| = |\delta(S_u) \cap \delta(v)| + |\delta(S_u) \cap \delta(t)| + |\delta(S_u) \cap \delta(u)| + |\delta(S_u) \cap \{e'_1, e'_2\}| \leq 1 + 1 + 1 + 1 = 4$ , a contradiction. So the only possible case of this configuration is  $u = u'$ , and the only 2-cut is  $\delta(u) \cap E(G[F])$ . This completes the proof of Claim 38. ■

#### 4.2.2 $\{u, v_3\} \subseteq \bar{F}$ .

**Claim 39.**  $G[\bar{F}]$  has at most one possible 2-cut, and if there exists one, it can only be  $\delta(u) \cap E(G[\bar{F}])$ ;  $G[F]$  has no 2-cut.

##### Proof of Claim 39:

*Subclaim 39.1.* Boundary vertices from  $v'$  to  $v_1$  in cyclic order are contained in  $F$ .

*Proof.* By Fact 13 and Corollary 19, both  $\delta(F)$  and  $\delta(\bar{F})$  are bonds of  $G$ , so the boundary vertices belonging to  $F$  and  $\bar{F}$  are consecutive on  $C$  respectively, otherwise  $G[F]$  and  $G[\bar{F}]$  cannot be connected at the same time by planarity. Because  $F \cup \bar{F} \cup \{d, t, v\} = V(G)$ ,  $v_1 \in F$  and  $u \in \bar{F}$ , it can be deduced from these facts that the boundary vertices in  $F$  are vertices from  $v'$  to  $v_1$  in cyclic order. ■

Suppose  $[K, \bar{K}]$  is a 2-cut of  $G[\bar{F}]$ .  $||[T, K]|| + ||[T, \bar{K}]|| = 2$ , both  $K$  and  $\bar{K}$  are connected with  $d_1$  by at least one edge because otherwise there exists an essential 4-cut in  $G$ .

*Subclaim 39.2.*  $||[d_1, K]| = |[d_1, \bar{K}]| = 2$  is impossible.

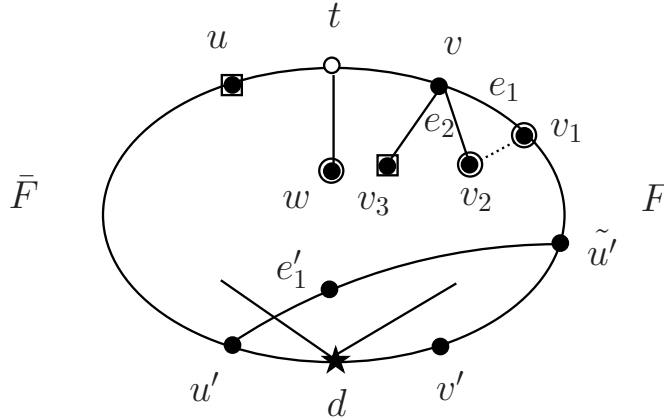


Figure 17: If  $||[d_1, K]| = |[d_1, \bar{K}]| = 2$ , and  $u'$  forms a 2-cut in  $G[\bar{F}]$ .

*Proof.* If so, then  $u$  and  $v_3$  cannot be separated by the cut, because otherwise  $|\delta(K)| = |\delta(\bar{K})| = 5$  in  $G$ , and  $K$  and  $\bar{K}$  are singletons with parallel double edges in between. So assume  $u, v_3 \in K$  without loss of generality, then  $\bar{K}$  is a boundary 4-vertex of  $G$ . In fact,  $\bar{K} = \{u'\}$ , since  $\delta(F \cup \{d, u'\})$  is an essential 7-bond, the boundary vertices in  $F \cup \{d, u'\}$  must be consecutive. Moreover,  $|\delta(u') \cap \{e'_1, e'_2\}| = 1$  because there is exactly one edge between  $d$  and  $u'$ .

As a result, consider the 7-cut  $\delta(F \cup \{d, u'\})$ , there is an inner path within  $G[F \cup \{d, u'\}]$  linking  $u'$  to a boundary vertex of  $\{v', \dots, v_1\}$  (note it  $\tilde{u}'$ , see Figure 17), while  $d$  contributes an edge to  $\delta(F \cup \{d, u'\})$ , violating planarity. ■

*Subclaim 39.3.*  $||[d_1, \bar{K}]| = 1$  or 3 is impossible unless  $u' = u$  and the degree of  $u$  is 4 in  $G$ .

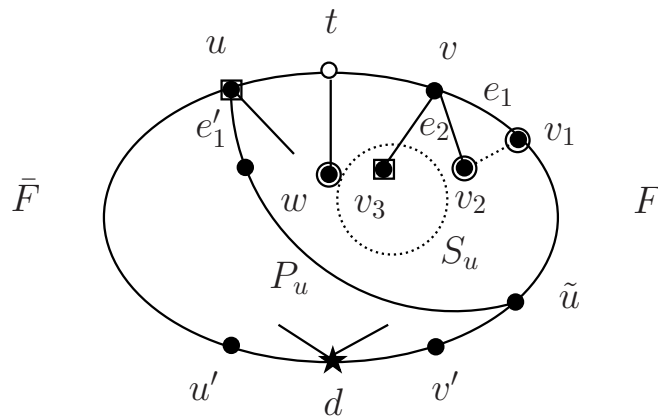


Figure 18: If  $u' \neq u$ , and  $u$  forms a 2-cut in  $G[\bar{F}]$ .

*Proof.* By symmetry, we only need to prove that the subclaim holds when  $||[d_1, \bar{K}]| = 1$ .  $|\delta(\bar{K})| \leq 5$  in  $G$ , so  $\bar{K}$  is a singleton. Actually  $|\delta(\bar{K})| = 4$  and  $\bar{K}$  can only be  $\{u\}$ . If  $u' \neq u$ , then there is no chord between  $d$  and  $u$ , because otherwise there exists an essential 4-cut in  $G$ . So the edge  $d_1u$  is one of  $e'_1, e'_2$ . There is an inner path  $P_u$  linking  $u$  to some vertex in  $\{v', \dots, v_1\}$  (note it  $\tilde{u}$ ), whose internal vertices lie in  $F$ , and  $uP_u\tilde{u}Ctu$  is a cycle enclosing a non-empty set of internal vertices not in  $F$  ( $v_3$  is enclosed), see Figure 18. Use  $S_u$  to denote this set,  $|\delta(S_u)| = |\delta(S_u) \cap \delta(v)| + |\delta(S_u) \cap \delta(u)| + |\delta(S_u) \cap \{e'_1, e'_2\}| \leq 1 + 1 + 1 = 3$ , a contradiction. So  $[K, \bar{K}]$  does not exist on the premise that  $||[d_1, \bar{K}]| = 1$  unless  $u = u'$ , whence the only possible 2-cut is  $\delta(u) \cap E(G[\bar{F}])$ . ■

Now we deal with  $G[F]$ . Suppose  $[Q, \bar{Q}]$  is a 2-cut of  $G[F]$ ,  $||[T, Q]| + |[T, \bar{Q}]| = 3$ , so  $Q$  and  $\bar{Q}$  are connected with  $d_2$  by at least one edge for otherwise in  $G$ , there exists an essential 4-cut, or an essential 5-cut not separating  $d$  and  $t$ .

*Subclaim 39.4.*  $||[d_2, Q]| = |[d_2, \bar{Q}]| = 2$  is impossible.

*Proof.*  $v_1$  and  $v_2$  cannot be separated by such a cut, for otherwise  $Q$  or  $\bar{Q}$  containing  $w$  induces an essential 5-cut not separating  $d$  and  $t$ . There are two possibilities:  $w$  and  $\{v_1, v_2\}$  are separated; or all three vertices are on the same side. We now eliminate them respectively with  $\bar{Q}$  specialized.

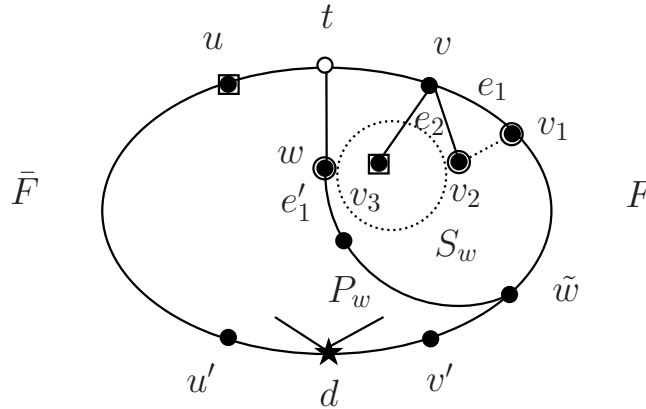


Figure 19: If  $||[d_2, Q]| = |[d_2, \bar{Q}]| = 2$ , and  $w$  forms a 2-cut in  $G[F]$ .

If  $w \in Q$  and  $\{v_1, v_2\} \subseteq \bar{Q}$ , then  $|\delta(Q)| = 5$  in  $G$  and  $Q = \{w\}$ .  $\delta(d) \cap \delta(F)$  contains one boundary edge  $dv'$ , so  $|\delta(w) \cap \{e'_1, e'_2\}| \geq 1$ . Because  $G[F]$  is connected, there is an inner path  $P_w$  linking  $w$  to a boundary vertex of  $\{v', \dots, v_1\}$  (note it  $\tilde{w}$ ), whose internal vertices lie in  $F$ .  $wP_w\tilde{w}Ctw$  is a cycle enclosing a non-empty set of internal vertices not in  $F$  ( $v_3$  is enclosed), see Figure 19. Use  $S_w$  to denote this set,  $|\delta(S_w)| = |\delta(S_w) \cap \delta(v)| + |\delta(S_w) \cap \{e'_1, e'_2\}| \leq 1 + 2 = 3$ , a contradiction.

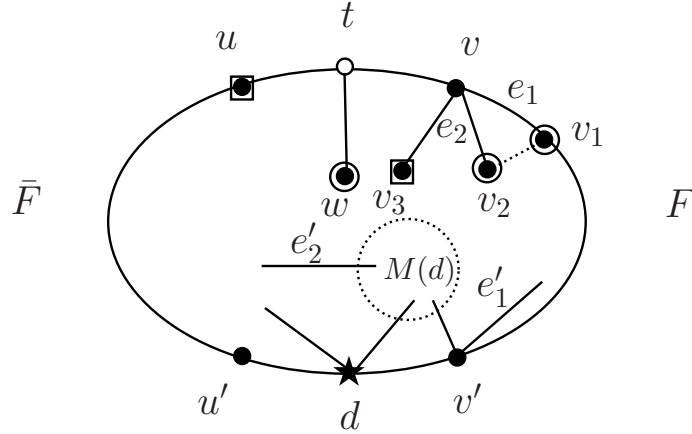


Figure 20: If  $||[d_2, Q]|| = ||[d_2, \bar{Q}]|| = 2$ , and  $v'$  forms a 2-cut in  $G[F]$ .

If  $\{v_1, v_2, w\} \subseteq \bar{Q}$ , then  $Q$  is a boundary 4-vertex of  $G$  in  $F$ . In fact,  $Q = \{v'\}$ , since  $\delta(\bar{F} \cup \{d, Q\})$  is an essential 6-bond, the boundary vertices in  $\delta(\bar{F} \cup \{d, Q\})$  must be consecutive. There is exactly one edge between  $d$  and  $v'$ , so  $|\delta(v') \cap \{e'_1, e'_2\}| = 1$ . Define  $M(d)$  to be the set of internal vertices which  $d$  can reach through an inner path in  $G[F \cup \{d\}]$ . By planarity,  $d$  does not form a chord with any vertex in  $\{v', \dots, v_1\}$ , while  $d$  contributes another edge to  $\delta(F)$  besides  $dv'$ , so  $M(d) \neq \emptyset$ . Still by planarity, there is no inner path within  $G[F \cup \{d\}]$  connecting  $d$  with  $w$  or  $v_2$ , nor a boundary vertex in  $F$  except  $v'$ . These are shown in Figure 20. So  $|\delta(M(d))| = |\delta(M(d)) \cap \delta(d)| + |\delta(M(d)) \cap \delta(v')| + |\delta(M(d)) \cap \{e'_1, e'_2\}| \leq 1 + 1 + 1 = 3$ , a contradiction. ■

*Subclaim 39.5.*  $||[d_2, Q]|| = 1$  or 3 is impossible.

*Proof.* By symmetry, it is enough to prove that the subclaim holds when  $||[d_2, Q]|| = 1$ .  $3 \leq |\delta(Q)| \leq 3 + |[T, Q]| \leq 6$ , by Proposition 16 and 17,  $|[T, Q]| = 1$  or 3 are possible, whence  $|\delta(Q)| = 4$  or 6. If  $|\delta(Q)| = 4$ ,  $Q$  can only be  $\{v_1\}$ , however  $v_1$  and  $v_2$  are separated and thus  $\deg(v_1) = 3$  in  $G$ , which is ridiculous. So  $|\delta(Q)| = 6$  and  $\{v_1, v_2, w\} \subseteq Q$ , then  $\delta(Q \cup \{t, v\})$  is an essential 5-cut intersecting  $\delta(t)$  at  $tu$ , which is a contradiction to Proposition 28. This completes the proof of Claim 39. ■

Having finished the proofs of Claim 38 and Claim 39, now we can eliminate the two situations we mentioned at the beginning of Lemma 37.

By Lemma 25,  $t$  can be prescribed properly so that: when  $d$  and  $u$  are adjacent,  $\delta(u) \cap E(G[F])$  (or  $\delta(u) \cap E(G[\bar{F}])$ ) is not a bad 2-cut under the induced  $\mathbb{Z}_4$ -boundary; the value of arc  $vt$  is not  $\beta(v)$  so that the value of  $vv_3$  is non-zero. Remember we lifted the two edges  $vv_1$  and  $vv_2$ . In  $G$ , orient the 3-path  $v_1vv_2$  as the orientation of edge  $v_1v_2$ , and label the two edges with the same value. Then in  $G$  there is a directed 3-path through  $v$  connecting  $F$  with  $\bar{F}$ , and prescribe  $\{e'_1, e'_2\}$  to make sure that at least one of them has an opposite direction to this path. By the induction, both  $G[F]$  and  $G[\bar{F}]$  admit a flow

with a strong orientation respectively, from which we manage to construct a flow with a strong orientation of  $G - d$ . This completes the proof of Lemma 37.

Having proved Lemma 36 and Lemma 37, we can prescribe  $t$  properly so that the only possible 2-cut in Lemma 37 is not bad under  $\beta''$ , as we just mentioned. Then we can apply the induction on  $G''$  to obtain a flow with a strong orientation of  $G'' - d$ . By Lemma 35, it can be naturally extended to a flow with a strong orientation of  $G - d$ .

### 4.3 Case 3: $w, d \in N(u) \cap N(v)$ .

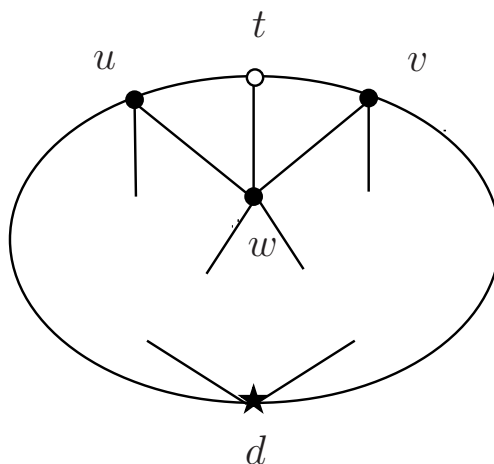


Figure 21

Finally we show that this last case is not possible either. When  $\deg(w) > 5$ , we can still lift  $e_1$  and  $e_2$ , prescribe  $t$  and  $v$  wisely to achieve  $\beta(t)$  and  $\beta(v)$ , then delete them as we have done in **Case 2**. This is actually a special case of **Case 2** with  $v_3 = w$ , so we can repeat the analyses about essential cuts and bad 2-cuts before, and the restriction  $v_3 = w$  will only make things simpler. The method of **Case 2** is still available when  $\deg(w) > 5$ , so  $\deg(w) = 5$ , and this last configuration is shown in Figure 21. Write  $Y = \{d, u, v, w, t\}$ ,  $d$  and  $w$  cannot be adjacent, otherwise  $G$  is separated into two parts by the path  $dwt$ . A cut of size at most 3 would be formed by a set of internal vertices. Obviously  $|Y^c| \geq 2$ . First contract  $Y^c$ , there is a flow  $(D_1, f_1)$  on  $G/Y^c$  and  $D_1(G/Y^c - d)$  is strong; then contract  $Y$  to a vertex  $y$  and delete an edge in  $E(y)$ .  $y$  has degree 5 in the resulted graph  $\tilde{G}$ ,  $\tilde{G}$  neither contains a vertex of degree less than 4 nor an essential cut of size less than 5. What is more,  $\tilde{G} - y$  has no 2-cut, otherwise this will yield double edges or an essential cut of size at most 5 not separating  $d$  and  $t$  in  $G$ . By the induction,  $G/Y$  admits a flow  $(D_2, f_2)$  and  $D_2(Y^c)$  is strong. Combining  $(D_1, f_1)$  and  $(D_2, f_2)$  together, we can obtain a flow  $(D, f)$  of  $G$  and  $D(G - d)$  is strong. This completes the proof of Theorem 9.

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