New bounds on a generalization of Tuza's conjecture

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Abstract

For a k-uniform hypergraph H, let $\nu^{(m)}(H)$ denote the maximum size of a set S of edges of H whose pairwise intersection has size less than m. Let $\tau^{(m)}(H)$ denote the minimum size of a set S of m-sets of $V(H)$ such that every edge of H contains some m-set from S. A conjecture by Aharoni and Zerbib, which generalizes a conjecture of Tuza on the size of minimum edge covers of triangles of a graph, states that for a k-uniform hypergraph H, $\tau^{(k-1)}(H)/\nu^{(k-1)}(H) \leqslant \left\lceil \frac{k+1}{2} \right\rceil$. In this paper, we show that this generalization of Tuza's conjecture holds when $\nu^{(k-1)}(H) \leqslant 3$. As a corollary, we obtain a graph class which satisfies Tuza's conjecture. We also prove various bounds on $\tau^{(m)}(H)/\nu^{(m)}(H)$ for other values of m as well as some bounds on the fractional analogues of these numbers.

Mathematics Subject Classifications: 05C65, 05D15

1 Introduction

1.1 Definitions and Notation

Throughout this paper, unless otherwise specified, we will only be concerned with k-uniform hypergraphs for $k \geqslant 3$. We start by establishing some definitions and notation which will be used throughout the paper.

For a set S with $x \in S$, $y \notin S$, we denote $S \setminus \{x\}$ by $S - x$ and $S \cup \{y\}$ by $S + y$. For a set Z with $|Z| = 2$, when we say $z \in Z$, we will let $\overline{z} = Z - z$. For a hypergraph H, we will use both $E(H)$ and H to mean the edge set of H. Let H be a k-uniform hypergraph with vertex set V and edge set E. A *matching* of H is any collection of disjoint edges of H. We denote the largest matching of H by $\nu(H)$. A *cover* of H is a set $C \subseteq V$ such that for every $e \in E$, there is some $v \in C \cap e$. We denote the size of the smallest cover of H by $\tau(H)$. Clearly, for any k-uniform hypergraph H, $\nu(H) \leqslant \tau(H) \leqslant k\nu(H)$.

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These definitions may be generalized in the following way: for $1 \leqslant m \leqslant k-1$, an *$m$-matching* of H is a collection M of edges of H such that for any $e, e' \in M$, $|e \cap e'| < m$. We denote the size of the largest m-matching of H by $\nu^{(m)}(H)$. Observe that $\nu(H) = \nu^{(1)}(H)$. An *m-cover* of H is a set $C \subseteq \binom{V}{m}$ such that for every $e \in H$, there is some $c \in C$ with $c \subseteq e$. We denote the size of the smallest m-cover of H by $\tau^{(m)}(H)$. Again, observe that $\tau(H) = \tau^{(1)}(H)$. Similar to the inequality above, we trivially have $\nu^{(m)}(H) \leqslant \tau^{(m)}(H) \leqslant \binom{k}{m}\nu^{(m)}(H)$. The main aim of this paper will be to improve the ratio $\tau^{(m)}(H)/\nu^{(m)}(H)$ for various values of m and $\nu^{(m)}(H)$.

We will also study the fractional versions of these parameters. A *fractional m-matching* is a function $f : E(H) \to \mathbb{R}_{\geqslant 0}$ such that for every $S \in \binom{V}{m}$, $\sum_{e \supseteq S} f(e) \leqslant 1$. The *size* of a fractional m-matching is $|f| = \sum_{e \in E(H)} f(e)$. A *fractional m-cover* is a function $c : \binom{V}{m} \to \mathbb{R}_{\geqslant 0}$ such that for every $e \in H$, $\sum_{S \in \binom{e}{m}} c(S) \geqslant 1$. The *size* of a fractional m-cover is $|c| = \sum_{S \in \binom{V}{m}} c(S)$. The fractional m-matching number, $\nu^{*(m)}(H)$, and the fractional m-cover number $\tau^{*(m)}(H)$ are defined to be the maximum size of a fractional m-matching and the minimum size of a fractional m-cover, respectively. We will denote $\nu^{*(1)}(H)$ by $\nu^*(H)$ and $\tau^{*(1)}(H)$ by $\tau^*(H)$. Observe that by LP duality, we always have $\nu^{*(m)}(H) = \tau^{*(m)}(H)$. Also, observe that an m-matching is a fractional m-matching and an m-cover is a fractional m-cover. For any k-uniform hypergraph H and $1 \leqslant m \leqslant k-1$, we have:

$$\nu^{(m)}(H) \leqslant \nu^{*(m)}(H) = \tau^{*(m)}(H) \leqslant \tau^{(m)}(H) \leqslant \binom{k}{m}\nu^{(m)}(H).$$

1.2 A generalization of Tuza's conjecture

We introduce some notation which will be used throughout the paper. Let $\mathcal{H}_k$ denote the family of all k-uniform hypergraphs. Then, define the following functions:

- $h(k,m) = \sup\left\{\frac{\tau^{(m)}(H)}{\nu^{(m)}(H)} : H \in \mathcal{H}_k\right\}$
- $g_i(k,m) = \sup\left\{\frac{\tau^{(m)}(H)}{\nu^{(m)}(H)} : H \in \mathcal{H}_k \text{ and } \nu^{(m)}(H) = i\right\}$
- $h^*(k,m) = \sup\left\{\frac{\tau^{*(m)}(H)}{\nu^{(m)}(H)} : H \in \mathcal{H}_k\right\}$
- $g_i^*(k,m) = \sup\left\{\frac{\tau^{*(m)}(H)}{\nu^{(m)}(H)} : H \in \mathcal{H}_k \text{ and } \nu^{(m)}(H) = i\right\}$.

For reference, some previous papers used $g(k,m)$ for $g_1(k,m)$ and $g^*(k,m)$ for $g_1^*(k,m)$. Observe that by definition, we have:

$$g_i^*(k,m) \leqslant g_i(k,m) \leqslant h(k,m)$$
$$g_i^*(k,m) \leqslant h^*(k,m) \leqslant h(k,m).$$

A famous conjecture of Tuza [7] states that for any graph G, the minimum number of edges needed to intersect every triangle in G ($\tau_t(G)$) is at most twice the maximum number of edge disjoint triangles in G ($\nu_t(G)$). If true, this conjecture is tight as seen e.g., by K_4 or K_5. The conjecture has been shown to be true for various families of graphs (see e.g. [3], [7]). Haxell [6] proved the best known general upper bound of $\tau_t(G) \leqslant \frac{66}{23}\nu_t(G)$.

Note that for a graph G, if we define the triangle graph of G, $T(G)$, to be the hypergraph with edges corresponding to the triangles of G, Tuza's conjecture states that for any graph G, $\tau^{(2)}(T(G))/\nu^{(2)}(T(G)) \leqslant 2$. A conjecture of Aharoni and Zerbib generalizes Tuza's, conjecturing that for all 3-uniform hypergraphs H, $\tau^{(2)}(H)/\nu^{(2)}(H) \leqslant 2$ (i.e. $h(3,2) \leqslant 2$).

Furthermore, they conjectured that a similar bound should hold for hypergraphs of any fixed uniformity:
Conjecture 1 ([1])**.** Let $k \geqslant 3$. Then, $h(k, k-1) \leqslant \left\lceil \frac{k+1}{2} \right\rceil$.

Again, if true, the bound is tight as seen by the following example from [1]: for $H = \binom{[k+1]}{k}$, the k-uniform hypergraph containing all k-subsets of $[k+1]$, one can easily check that $\nu^{(k-1)}(H) = 1$ and $\tau^{(k-1)}(H) = \left\lceil \frac{k+1}{2} \right\rceil$.

1.3 The paper

We begin by studying the function $g_i(k, k-1)$ in section 2. In [1], Aharoni and Zerbib showed that $g_1(k, k-1) \leqslant \left\lceil \frac{k+1}{2} \right\rceil$. We prove the same bound for $g_2(k, k-1)$ and $g_3(k, k-1)$:
Theorem 2. *Let H be a k-uniform hypergraph with $\nu^{(k-1)}(H) = 2$. Then,*

$$\tau^{(k-1)}(H) \leqslant 2 \left\lceil \frac{k+1}{2} \right\rceil.$$

Theorem 3. *Let H be a k-uniform hypergraph with $\nu^{(k-1)}(H) = 3$. Then,*

$$\tau^{(k-1)}(H) \leqslant 3 \left\lceil \frac{k+1}{2} \right\rceil.$$

This immediately implies the following:
Corollary 4. *Let G be a graph with the property that G does not contain 4 edge-disjoint triangles. Then, Tuza's conjecture holds for G.*

In section 3, we study $g_1(k, m)$ for various values of m. We prove the first non-trivial upper bounds for $g_1(k, m)$ when $\frac{k}{2} \leqslant m \leqslant k-2$.
Theorem 5. *Let $k \geqslant 6$ and suppose $\frac{k}{2} \leqslant m \leqslant k-2$. Then, $g_1(k, m) \leqslant \binom{k}{m} - m$.*
Theorem 6. *Let $k \geqslant 3$. Then, we have:*

$$g_1(k, k-2) \leqslant \left\lceil \frac{k^2}{4} \right\rceil = \begin{cases} \frac{1}{4}(k^2+3), & \text{if } k \text{ odd,} \\ \frac{1}{4}k^2, & \text{if } k \text{ even.} \end{cases}$$

Aharoni and Zerbib [1] previously showed that $g_1(k,2) < \binom{k}{2}$ and $g_1(4,2) = 4$. We go on to improve the upper bound of $g_1(5,2)$ (the first remaining open case when $m = 2$) with the best previous bound being $g_1(5,2) \leqslant 9$.

Theorem 7. *We have* $6 \leqslant g_1(5,2) \leqslant 7$.

The lower bound has not been mentioned in previous papers but comes from the 2-cover number of the (unique) symmetric $2-(11,5,2)$ design (an explicit construction can be seen in Table 1.19 in [4]).

In section 4, we study the fractional variants of the problem and prove bounds on $g_1^*(k,m)$ for certain choices of m:

Theorem 8. *For all* $k \geqslant 2$, $g_1^*(2k,k) \leqslant \left(\frac{1}{2} + \frac{1}{2(k+1)}\right)\binom{2k}{k}$.

The proof of this theorem is followed by a lemma, generalizing a result from [2], that allows us to obtain upper bounds on $h^*(k,m)$ from upper bounds on $g_1^*(k,m)$. When $m = k/2$, this gives the following corollary:

Corollary 9. *For all* $k \geqslant 2$, $h^*(2k,k) \leqslant \left(1 - \frac{k}{4(k+2)}\right)\binom{2k}{k}$.

We also prove a fractional upper bound on $g_1^*(k,k-2)$ from which a bound for $h^*(k,k-2)$ may be derived in the same manner as above.

Theorem 10. $g_1^*(k,k-2) \leqslant \frac{1}{6}\binom{k-2}{2} + 2k - 3$.

It should be noted that other fractional variations and results have been shown in [2], [5], among others.

2 $g_i(k,k-1)$

We begin this section with some useful definitions and a few short lemmas.

Definition 11. Let H be a k-uniform hypergraph and M be a maximum $(k-1)$-matching in H. For any vertex $v \in V(H)$, we denote $d_M(v)$ to be the number of edges of M that contain v. For each $e \in M$, define the following two sets:

$$S_e = \{h \in H : |e \cap h| \geqslant k-1 \text{ and } |h \cap f| < k-1 \text{ for all } f \in M - e\}$$
$$T_e = \{h \in H : |e \cap h| \geqslant k-1\}.$$

Lemma 12. *Let H be a k-uniform hypergraph and M a maximum $(k-1)$-matching in H. Then, for any $e, f \in M$, $S_e \cap S_f = \emptyset$. Further, $\nu^{(k-1)}(S_e) = 1$, which implies $\tau^{(k-1)}(S_e) \leqslant g_1(k,k-1)$.*

Proof. This follows directly from the definition of S_e. □

Lemma 13. *Let H be a k-uniform hypergraph and let M be a maximum $(k-1)$-matching in H. If there exists some $e \in M$ such that $\tau^{(k-1)}(T_e) \leqslant \left\lceil \frac{k+1}{2} \right\rceil$, then*

$$\tau^{(k-1)}(H) \leqslant \left\lceil \frac{k+1}{2} \right\rceil + (\nu^{(k-1)}(H) - 1) g_{\nu^{(k-1)}(H)-1}(k,k-1).$$

Proof. Let H be a k-uniform hypergraph and let M be a maximum $(k-1)$-matching in H. Suppose there exists some $e \in M$ such that $\tau^{(k-1)}(T_e) \leqslant \left\lceil \frac{k+1}{2} \right\rceil$. We claim that $H - T_e$ has matching number at most $\nu^{(k-1)}(H) - 1$. Suppose not. Then, there exists some matching M' of $H - T_e$ of size at least $\nu^{(k-1)}(H)$. By definition, all edges of $H - T_e$ intersect e in at most $k-2$ vertices. But then, $M' + e$ is a larger matching than M, a contradiction. Therefore, we have:

$$\tau^{(k-1)}(H) \leqslant \tau^{(k-1)}(T_e) + \tau^{(k-1)}(H - T_e) \leqslant \left\lceil \frac{k+1}{2} \right\rceil + (\nu^{(k-1)}(H) - 1) g_{\nu^{(k-1)}(H)-1}(k, k-1).$$

□

Lemma 14. *Let H be a k-uniform hypergraph and let M be a maximum $(k-1)$-matching in H. If there exists a partition P_1, P_2 of the edges of M such that for all $e \in P_1$ and $e' \in P_2$, $|e \cap e'| < k-2$, then $T_e \cap T_{e'} = \emptyset$ and*

$$\tau^{(k-1)}(H) \leqslant |P_1| g_{|P_1|}(k, k-1) + |P_2| g_{|P_2|}(k, k-1).$$

We call such a matching disconnected.

Proof. Let H be a k-uniform hypergraph and let M be a maximum $(k-1)$-matching in H. Suppose there exists a partition P_1, P_2 of the edges of M such that for all $e \in P_1$ and $e' \in P_2$, $|e \cap e'| < k-2$. Now, let $e \in P_1$, $e' \in P_2$ and suppose $f \in T_e$. Then, f intersects e in $k-1$ vertices and therefore, f can only intersect e' in at most $k-2$ vertices. So, $T_e \cap T_{e'} = \emptyset$. This means that the edges of H are the disjoint union of the sets $H_1 := \bigcup_{e \in P_1} T_e$ and $H_2 := \bigcup_{e' \in P_2} T_{e'}$. Also, because there is no intersection of size $k-1$ between any edge of H_1 and any edge in P_2, $\nu^{(k-1)}(H_1) = |P_1|$. Similarly, $\nu^{(k-1)}(H_2) = |P_2|$. Therefore,

$$\tau^{(k-1)}(H) \leqslant \tau^{(k-1)}(H_1) + \tau^{(k-1)}(H_2) \leqslant |P_1| g_{|P_1|}(k, k-1) + |P_2| g_{|P_2|}(k, k-1).$$ □

Lemma 15. *Let H be a 3-uniform hypergraph and let M be a maximum 2-matching in H. If there exists some $e \in M$ such that $\sum_{v \in e} d_M(v) \leqslant 4$ and $\tau^{(2)}(S_e) = 1$, then*

$$\tau^{(2)}(H) \leqslant 4 + (\nu^{(2)}(H) - 2) g_{\nu^{(2)}(H)-2}(k, k-1).$$

Proof. Let H be a 3-uniform hypergraph and let M be a maximum 2-matching in H. Suppose there exists some $e \in M$ such that $\sum_{v \in e} d_M(v) \leqslant 4$ and $\tau^{(2)}(S_e) = 1$. This means that there are two vertices in e not contained in any other edge of M and at most one vertex of e contained in at most one other edge, say f, of M. Then, it is clear that $(T_e - S_e) \subseteq T_f$. Furthermore, $\nu^{(2)}(H - T_e - T_f) = |M| - 2$. Otherwise, if we may find a 2-matching M' of $H - T_e - T_f$ of size greater than $|M| - 2$, then $M' + e + f$ is larger than M, a contradiction. Now, let S be a 2-set, which 2-covers S_e. Then, since $T_e - S_e \subseteq T_f$ and $S_f \subseteq T_f$, taking $\binom{f}{2}$ to 2-cover T_f, we have found a 2-cover of $T_e \cup T_f$ of size 4. Therefore, we have:

$$\tau^{(2)}(H) \leqslant \tau^{(2)}(T_e \cup T_f) + \tau^{(2)}(H - T_e - T_f) \leqslant 4 + (\nu^{(2)}(H) - 2) g_{\nu^{(2)}(H)-2}(k, k-1).$$

□

We now refine the $\nu^{(k-1)} = 1$ result of Aharoni and Zerbib [1] in order to help with our proof of the $\nu^{(k-1)} \in \{2, 3\}$ cases. First, we reiterate a lemma from [2]:

Lemma 16 (Lemma 2.2 from [2])**.** *Let H be a k-uniform hypergraph with $\nu^{(k-1)}(H) = 1$. Then, either $\tau^{(k-1)}(H) = 1$ or for any edge $e \in E(H)$, there exists a unique vertex $v \in V(H) - V(e)$ such that for all $e' \in E(H) - e$, $e' - e = \{v\}$.*

Now, we are ready to refine the $\nu^{(k-1)} = 1$ result from [1].

Lemma 17. *Suppose H is a k-uniform hypergraph with $\nu^{(k-1)}(H) = 1$. Then, either $\tau^{(k-1)}(H) = 1$ or $\tau^{(k-1)}(H) \leqslant \left\lceil \frac{e(H)}{2} \right\rceil$.*

Proof. Let H be a k-uniform hypergraph with $k \geqslant 3$. Suppose $\nu^{(k-1)}(H) = 1$ and $\tau^{(k-1)}(H) \neq 1$. Let $e \in E(H)$ and let $v \in V(H) - V(e)$ be the vertex described in Lemma 16. Let $e_1, \ldots, e_{e(H)-1}$ denote the edges of $H - e$. Observe that for any $1 \leqslant i \neq j \leqslant e(H) - 1$, $|e_i \cap e_j \cap e| = k - 2$.

Suppose $e(H)$ is odd. For $1 \leqslant i \leqslant \frac{e(H)-1}{2}$, we may cover e_{2i-1}, e_{2i} with the set $(e_{2i-1} \cap e_{2i} \cap e) + v$. Then, we may cover e with any set from $\binom{e}{k-1}$, giving a $(k-1)$-cover of size $\frac{e(H)-1}{2} + 1 = \frac{e(H)+1}{2} = \left\lceil \frac{e(H)}{2} \right\rceil$.

Suppose $e(H)$ is even. For $1 \leqslant i \leqslant \frac{e(H)-2}{2}$, we may cover e_{2i-1}, e_{2i} with the set $(e_{2i-1} \cap e_{2i} \cap e) + v$. Then, we may cover $e_{e(H)-1}, e$ with the set $e_{2i-1} \cap e$, giving a $(k-1)$-cover of size $\frac{e(H)-2}{2} + 1 = \frac{e(H)}{2} = \left\lceil \frac{e(H)}{2} \right\rceil$. □

We obtain the $\nu^{(k-1)} = 1$ result as a corollary:

Corollary 18. *We have $g_1(k, k-1) \leqslant \left\lceil \frac{k+1}{2} \right\rceil$.*

Proof. Let H be a k-uniform hypergraph with $\nu^{(k-1)}(H) = 1$. We may assume $\tau^{(k-1)}(H) > 1$. Let $e \in H$ and let $v \in V(H) - V(e)$ be the unique vertex as described in Lemma 16. Now, aside from e, every other edge of H consists of v together with some $(k-1)$-subset of e. Since e has k different $(k-1)$-subsets, then the total number of edges of H is at most $k + 1$. The result now follows from Lemma 17. □

Next, we prove the case when $\nu^{(k-1)} = 2$.

Proof of Theorem 2. Let H be a k-uniform hypergraph with $k \geqslant 3$. Suppose $\nu^{(k-1)}(H) = 2$. If there exists a $(k-1)$-matching of H, $\{e, f\}$, where $|e \cap f| < k - 2$, then e, f is a disconnected matching and we are done by Lemma 14 together with Lemma 17.

Suppose then that for any maximum $(k-1)$-matching $\{e, f\}$ in H, $|e \cap f| = k - 2$. To this end, let $\{e, f\}$ be a $(k-1)$-matching of H with

$$e = S \cup \{u_1, u_2\}$$

$$f = S \cup \{v_1, v_2\}.$$

Here, $S = e \cap f$ is a $(k-2)$-subset of $V(H)$. Since $\nu^{(k-1)}(S_e) = \nu^{(k-1)}(S_f) = 1$, then as noted before, $\tau^{(k-1)}(S_e) \leqslant \left\lceil \frac{k+1}{2} \right\rceil$ and $\tau^{(k-1)}(S_f) \leqslant \left\lceil \frac{k+1}{2} \right\rceil$. If every edge of S_e contains S, then we may cover T_e with the sets $S + u_1$ and $S + u_2$. Next, we may cover S_f with at most $\left\lceil \frac{k+1}{2} \right\rceil$ $(k-1)$-sets, giving a cover of H of size at most

$$2 + \left\lceil \frac{k+1}{2} \right\rceil \leqslant 2 \left\lceil \frac{k+1}{2} \right\rceil.$$

Similarly, we may find a cover of suitable size if every edge of S_f contains S. Further, if $\tau^{(k-1)}(S_e) = 1$, then we may cover S_e with one $(k-1)$-set and cover the rest of H with elements from $\binom{f}{k-1}$, giving a cover of size at most

$$1 + k \leqslant 2 \left\lceil \frac{k+1}{2} \right\rceil.$$

In a similar manner, we may find a cover of suitable size if $\tau^{(k-1)}(S_f) = 1$. So, we may assume $\tau^{(k-1)}(S_e) \neq 1$, $\tau^{(k-1)}(S_f) \neq 1$ and that there exists $e' \in S_e - e$, $f' \in S_f - f$ such that $S \not\subseteq e'$ and $S \not\subseteq f'$.

If the unique vertex for all edges of $S_e - e$ described in Lemma 16 is not contained in $f - e$, then e', f is a disconnected matching. So, we may assume that for all $e'' \in S_e - e$, $e'' - e = v$ for some $v \in \{v_1, v_2\}$. By a symmetric argument, for all $f'' \in S_f - f$, $f'' - f = u$ for some $u \in \{u_1, u_2\}$.

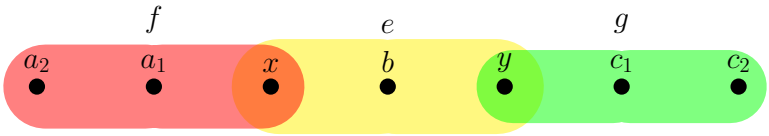
This tells us that every edge in $S_e - e$ is of the form $S' \cup \{u_1, u_2, v\}$ for some $S' \in \binom{S}{k-3}$ (i.e. there are at most $k-1$ edges in S_e). Similarly, every edge in $S_f - f$ is of the form $S'' \cup \{v_1, v_2, u\}$ for some $S'' \in \binom{S}{k-3}$ (i.e. there are at most $k-1$ edges in S_f). By Lemma 17, we may cover every edge in S_e with at most $\left\lceil \frac{k-1}{2} \right\rceil$ $(k-1)$-sets and we may cover every edge in $S_f - f$ with at most $\left\lceil \frac{k-2}{2} \right\rceil$ $(k-1)$-sets. Finally, we may cover $T_f - S_f + f$ with the sets $S + v_1$ and $S + v_2$. This gives us a cover of H of size at most

$$\left\lceil \frac{k-1}{2} \right\rceil + \left\lceil \frac{k-2}{2} \right\rceil + 2 = k + 1 \leqslant 2 \left\lceil \frac{k+1}{2} \right\rceil. \qquad \square$$

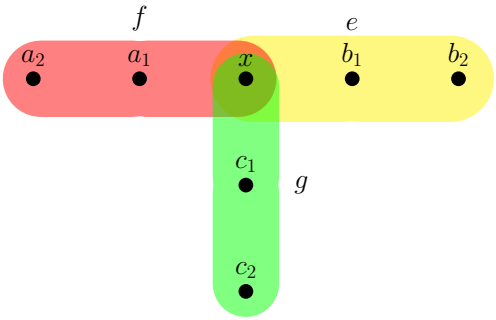
Now, we are ready to prove the $\nu^{(k-1)} = 3$ case:

Proof of Theorem 3. We break the proof into two parts. In the first part, we assume we are dealing with a 3-uniform hypergraph. In the second part, we will deal with an arbitrary k-uniform hypergraph with $k \geqslant 4$.

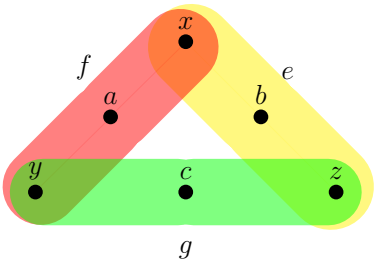
Let H be a 3-uniform hypergraph and let $M = \{e, f, g\}$ be a maximum 2-matching in H. If M is disconnected, then the result follows from Lemma 14. So, suppose M is connected. We may assume $|e \cap f| = 1$ and $|e \cap g| = 1$. Then, M looks like one of the matchings from Figure 1.



(a) $|f \cap g| = 0$



(b) $|f \cap g| = 1$, $f \cap g = e \cap f = e \cap g$



(c) $|f \cap g| = 1$, Intersections between e, f, g disjoint

Figure 1: 2-Matching Types when $\nu^{(k-1)} = 3$

Suppose there is a matching of type 1a. If there is no edge containing $\{a_1, a_2\}$, then we are done by Lemma 13. Similarly, if there is no edge containing $\{c_1, c_2\}$, we are done. So, suppose there are some edges f_1, g_1 with $f_1 = \{a_1, a_2, u\}$, $g_1 = \{c_1, c_2, v\}$. If $u \notin (e \cup g) - x$, then e, f_1, g is a disconnected matching and we are done. Similarly, if $v \notin (e \cup f) - y$, then e, f, g_1 is a disconnected matching and we are done. So, we may assume $u \in (e \cup g) - x$ and $v \in (e \cup f) - y$.

If $\tau^{(k-1)}(S_f) = 1$ or $\tau^{(k-1)}(S_g) = 1$, we are done by Lemma 15. Therefore, we may assume that $|S_f| > 2$ and $|S_g| > 2$. Let $f_2 \in S_f - f_1 - f$ and $g_2 \in S_g - g_1 - g$. So, $f_2 = \{a, x, u\}$, $g_2 = \{c, y, v\}$, where $a \in \{a_1, a_2\}, c \in \{c_1, c_2\}$. Since $f_2 \in S_f$ and $u \in (e \cup g) - x$, then u must be in $g - e$ since otherwise, $|f_2 \cap e| = |f_2 \cap f| = 2$, a contradiction to $f_2 \in S_f$. Similarly, $v \in f - e$. Now, we obtain a 2-cover of H of size exactly 6 as witnessed by $\mathcal{C} = \{\binom{e}{2}, \{u, v\}, \{f - v\}, \{g - u\}\}$.

Observe that for the other cases, if there are 2 disjoint edges in H, we are done. This is because either the union of their 2-sets are a cover of H or we may extend the matching to a matching of the first type or a disconnected matching.

Next, suppose there is a matching of type 1b. By Lemma 13, $\{c_1, c_2\}$ must be contained in some edge other than g, say g_1. But then, either g_1 is disjoint from e or g_1 is disjoint from f. In either case, we are done.

In the final case, because H is assumed to have no disjoint edges, it can be checked that

$$\mathcal{C} = \{\{x, y\}, \{x, z\}, \{y, z\}, \{x, c\}, \{y, b\}, \{z, a\}\}.$$

is a 2-cover of H. This concludes the proof for 3-uniform hypergraphs.

Next, suppose $k \geqslant 4$ and let H be a k-uniform hypergraph with $\nu^{(k-1)}(H) = 3$. Let $M = \{e, f, g\}$ be a maximum $(k-1)$-matching in H. Without loss of generality, suppose $|e \cap f| = k-2$. By Lemma 14, if $|g \cap e| \leqslant k-3$ and $|g \cap f| \leqslant k-3$, we are done. So, again, without loss of generality, suppose $|g \cap e| = k-2$. We now define some notation that will be used throughout the proof. Let $S = e \cap f$, where $|S| = k-2$ and $S' = e \cap f \cap g$. Let $e - f = \{u_1, u_2\}$, $f - e = \{v_1, v_2\}$, and $T = V(g) - e - f$. Now, M will look like one of the matchings from Figure 2.

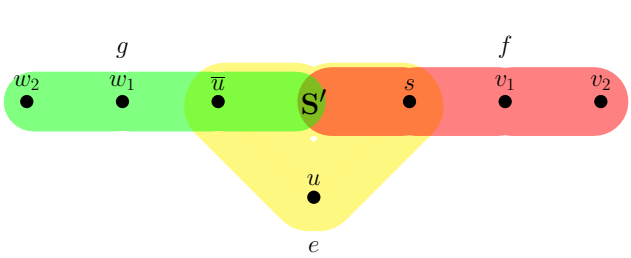
In their respective pictures, $s, s_1, s_2 \in S - S'$, $w, w_1, w_2 \in T$, $\{u, \overline{u}\} = \{u_1, u_2\}$, and $\{v, \overline{v}\} = \{v_1, v_2\}$. Throughout the proof, we will often use the result from Theorem 2 and arguments similar to the proof of the 3-uniform case.

If we have a type 2a matching, then observe that no edge $e' \in S_e - e$ may contain the set $\{\overline{u}, u, s\}$ since then, e', g, f is a disconnected matching and we are done. Therefore, we may $(k-1)$-cover T_e with three sets, namely $S' \cup A$ for each $A \in \binom{\{\overline{u}, u, s\}}{2}$. After covering T_e, $\nu^{(k-1)}(H - T_e) = 2$ with $M - e$ being a maximum $(k-1)$-matching. Now, by Theorem 2, we may find a $(k-1)$-cover of H of size at most

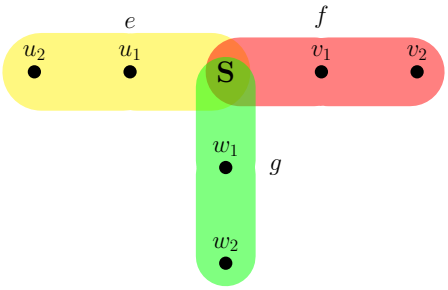
$$3 + 2\left\lceil \frac{k+1}{2} \right\rceil \leqslant 3\left\lceil \frac{k+1}{2} \right\rceil.$$

Suppose there is no type 2a matching. If there is a type 2b matching, then for all $h \in M$, there is no $h' \in S_h - h$ such that h' contains h_1, h_2. (This is because if such an h' existed, then $M - h + h'$ is a disconnected matching or a type 2a matching.) Therefore, for each $h \in M$, we may $(k-1)$-cover T_h with the sets $S + h_1$ and $S + h_2$, giving us a $(k-1)$-cover of H of size 6, which is less than $3\lceil \frac{k+1}{2} \rceil$.

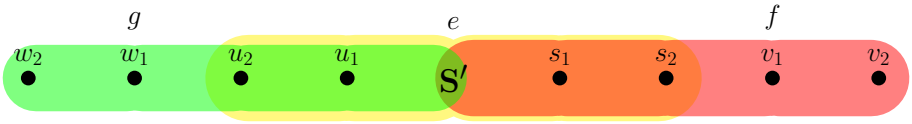
Next, suppose there is no type 2a or 2b matching. If there is a type 2c matching, then notice that no edge in S_e contains the set $\{u_2, u_1, s_1, s_2\}$ since otherwise, we would be able to find a disconnected matching. Therefore, we may $(k-1)$-cover T_e with four sets, namely $S' \cup A$ for each $A \in \binom{\{u_2, u_1, s_1, s_2\}}{3}$. If $\tau^{(k-1)}(S_g) = 2$ or $\tau^{(k-1)}(S_f) = 2$, we are done. Otherwise, for $h \in \{g, f\}$, we know that there exists some $h' \in S_h - h$ such that $h - e \subseteq h'$. This tells us that for all $g' \in S_g - g$, the unique vertex outside of $g' - g$ described in Lemma 16 must be s, where $s \in \{s_1, s_2\}$ (if not, then for any $g' \in S_g - g$ with $g - e \subseteq g'$, $M - g + g'$ is a disconnected matching). Similarly, for all $f' \in S_f - f$, the unique vertex outside of $f' - f$ described in Lemma 16 must be u, where $u \in \{u_1, u_2\}$. Therefore, every uncovered edge in $S_g - g$ has the form $S'' \cup \{w_1, w_2, s\}$, where $S'' \in \binom{S' \cup \{u_1, u_2\}}{k-3}$. By Lemma 17, we may cover these edges as well as g with at most $\left\lceil \frac{k-1}{2} \right\rceil$ $(k-1)$-sets.



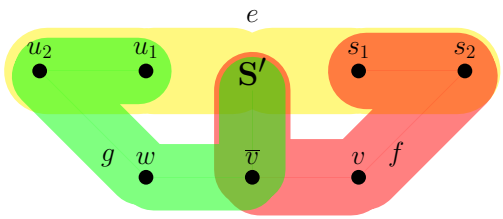
(a) $|S'| = k-3$, $|(g \cap f) - S'| = 0$



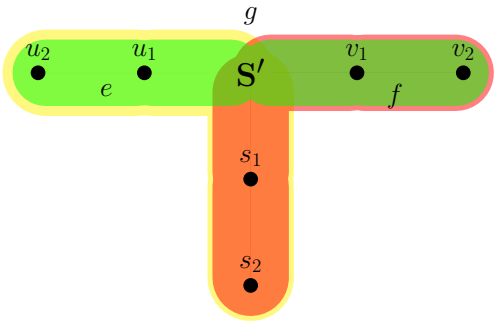
(b) $S = S'$



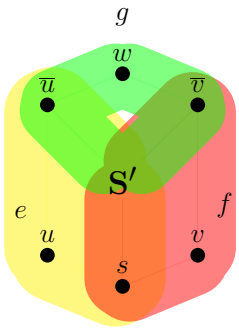
(c) $|S'| = k-4$, $|(g \cap f) - S'| = 0$



(d) $|S'| = k-4$, $|(g \cap f) - S'| = 1$



(e) $|S'| = k-4$, $|(g \cap f) - S'| = 2$



(f) $|S'| = k-3$, $|(g \cap f) - S'| = 1$

Figure 2: $(k-1)$-Matching Types when $\nu^{(k-1)} = 3$, $k \geqslant 4$

A symmetric argument shows that we may cover the remaining uncovered edges of S_f (including f) with at most $\left\lceil \frac{k-1}{2} \right\rceil$ $(k-1)$-sets. Now, we have found a cover of H of size at most

$$4 + 2\left\lceil \frac{k-1}{2} \right\rceil \leqslant 3\left\lceil \frac{k+1}{2} \right\rceil.$$

Now, suppose there is no type 2a - 2c matching and suppose there is a type 2d matching. If $k = 4$, we cover H as follows. First, we add $\{u_1, u_2, w\}, \{u_1, u_2, \overline{v}\}, \{s_1, s_2, v\}, \{s_1, s_2, \overline{v}\}$ to the cover, $\mathcal{C}$. If $\tau^{(k-1)}(S_e) = 1$, we are done. Otherwise, there is a unique vertex x outside of e as described in Lemma 16 such that for all $e' \in S_e - e$, $e' - e = x$. If $x \notin g \cup f$, then for any $e' \in S_e - e$, $M - e + e'$ is a disconnected matching. Otherwise, suppose $x \in g \cup f$ and without loss of generality, suppose $x \in g$. Then, there are at most three edges in S_e that are not already covered. Namely, the edges $\{s_1, s_2, u_1, x\}, \{s_1, s_2, u_2, x\}$, and e. By Lemma 17, we may cover these edges with two additional sets. Now, we wish to show that the edges remaining uncovered in $S_g \cup S_f$ may be covered by at most three 3-sets. By Lemma 17, either we may cover the remaining elements of S_g with one 3-set or we need to cover two edges with a unique vertex outside of g, which may be covered by $\left\lceil \frac{2}{2} \right\rceil = 1$ set and similarly for S_f. In either case, we are done.

Now, suppose $k \geqslant 5$. We begin by adding to our cover the two $(k-1)$-sets contained in g which contain $S' \cup \{u_1, u_2\}$ and the two $(k-1)$-sets contained in f which contain $S' \cup \{s_1, s_2\}$. First, we aim to cover S_e. Either $\tau^{(k-1)}(S_e) = 1$ or there is a unique vertex x outside of e such that for all $e' \in S_e - e$, $e' - e = x$. If $x \neq \overline{v}$, then for any $e' \in S_e - e$, $M - e + e'$ is a disconnected matching. So, we may assume $x = \overline{v}$. Now, any edge $e' \in S_e - e$ which contains all of S' has already been covered. Therefore, all remaining uncovered edges of $S_e - e$ have the form $S'' \cup \{u_1, u_2, v_1, v_2, \overline{v}\}$ for some $S'' \in \binom{S'}{k-5}$. Since e also remains uncovered, we are left to cover at most $k-3$ additional edges, which by Lemma 17, may be done using at most $\left\lceil \frac{k-3}{2} \right\rceil$ $(k-1)$-sets.

We will now make an argument for S_g, which will hold true for S_f by symmetry. The remaining edges of S_g needing to be covered must use both w and $\overline{v}$. Suppose the remaining edges of S_g may not be covered by a single $(k-1)$-set. Then, by Lemma 16, there is a unique vertex y outside of g such that for all $g' \in S_g - g$, $g' - g = y$. This tells us that all edges uncovered in S_g have the form $S'' \cup \{w, \overline{v}, y\}$, where $S'' \in \binom{S' \cup \{u_1, u_2\}}{k-3}$. Specifically, there are at most $k-2$ remaining edges to cover in S_g. By Lemma 17, we may cover these edges with at most $\left\lceil \frac{k-2}{2} \right\rceil$ $(k-1)$-sets. We may make the same argument for the uncovered edges of S_f. All together, we have found a cover for H of size:

$$4 + \left\lceil \frac{k-3}{2} \right\rceil + 2\left\lceil \frac{k-2}{2} \right\rceil \leqslant 3\left\lceil \frac{k+1}{2} \right\rceil.$$

Next, suppose there is no type 2a - 2d matching and suppose there is a type 2e matching. We will make an argument for S_g, which will hold true for S_e, S_f by symmetry. Suppose $\tau^{(k-1)}(S_g) \neq 1$. Then, by Lemma 16, there is a unique vertex x outside of g such that for

all $g' \in S_g - g$, $g' - g = x$. Suppose $x \notin (e \cup f) - g$. Then, for any $g' \in S_g - g$, $M - g + g'$ is either a disconnected matching or a type 2d matching. Therefore, $x \in \{s_1, s_2\}$. Now, if any edge of $S_g - g$ contains S', then this edge is actually an element of $T_g - S_g$. Therefore, every edge in $S_g - g$ has the form $S'' \cup \{u_1, u_2, v_1, v_2, x\}$, where $S'' \in \binom{S'}{k-5}$. Since we wish to also cover g, there are at most $k-3$ edges needed to be covered in S_g. By Lemma 17, this may be done using at most $\left\lceil \frac{k-3}{2} \right\rceil$ $(k-1)$-sets. Similarly, the edges of S_e and S_f may be covered with at most $\left\lceil \frac{k-3}{2} \right\rceil$ $(k-1)$-sets. We are left to cover the edges which intersect more than one of e, f, g in $k-1$ vertices. We cover the edges intersecting both g and e in $k-1$ vertices with the two $(k-1)$-sets contained in e which contain $S' \cup \{u_1, u_2\}$. We cover the edges intersecting both g and f in $k-1$ vertices with the two $(k-1)$-sets contained in f which contain $S' \cup \{v_1, v_2\}$. Finally, we cover the edges intersecting e and f with the two $(k-1)$-sets contained in f which contain $S' \cup \{s_1, s_2\}$. All together, we have found a cover of H of size at most

$$3 \left\lceil \frac{k-3}{2} \right\rceil + 6 \leqslant 3 \left\lceil \frac{k+1}{2} \right\rceil.$$

Finally, suppose there is only a matching of type 2f. We first show that in this case, there are no two edges with intersection size $k-3$. For sake of contradiction, suppose there exists $h, h' \in H$ such that $|h \cap h'| = k-3$. Let us set $A = h \cap h'$. Then, either h, h' may be extended to a matching of size 3 or h, h' is a maximal matching. In the first case, the extended matching must be disconnected or a matching of type 2a or 2d. Suppose then that h, h' is a maximal matching. That is, every edge of H intersects h or h' in $k-1$ vertices. Because $|h \cap h'| = k-3$, then no edge of H can intersect both h and h' in $k-1$ vertices. Now, we construct a suitable cover in this case. First, we cover all edges containing A with the three $(k-1)$-sets contained in h which contain A and the three $(k-1)$-sets contained in h' which contain A. Observe that we have also covered h and h'.

Next, let H_h be the set of uncovered edges intersecting h in $k-1$ vertices and define $H_{h'}$ similarly. We will make an argument for H_h, which will hold true by symmetry for $H_{h'}$. First, observe that $\nu^{(k-1)}(H_h) = 1$. Indeed, otherwise, we may find a disconnected matching of size 3 in H. Also, it is the case that $\nu^{(k-1)}(H_h \cup h) = 1$. This is because by the way H_h is defined, any matching of size two in $H_h \cup h$ does not contain h. Now, suppose $\tau^{(k-1)}(H_h \cup h) > 1$. Then, by Lemma 16, there is a unique vertex v outside of h such that $v \in e$ for all $e \in H_h$. This means that every edge of H_h has the form $(A' \cup h - A) + v$, where $A' \in \binom{A}{k-4}$. This shows that $|H_h| \leqslant k-3$ and so, by Lemma 17, we may find a cover of H_h of size at most $\left\lceil \frac{k-3}{2} \right\rceil$. Similarly, $\tau^{(k-1)}(H_{h'}) \leqslant \left\lceil \frac{k-3}{2} \right\rceil$. Putting this together, we have found a cover of H of size at most

$$6 + 2 \left\lceil \frac{k-3}{2} \right\rceil \leqslant 3 \left\lceil \frac{k+1}{2} \right\rceil.$$

For the remainder of the proof, we may assume that no two edges intersect in exactly $k-3$ vertices. Now, we proceed assuming that there is only a matching of type 2f. We

may cover $(T_e \cup T_f \cup T_g) - (S_e \cup S_f \cup S_g)$ with the three $(k-1)$-sets containing S' and exactly two elements from $\{\overline{u}, \overline{v}, s\}$.
Next, we make an argument for the uncovered edges of S_g, which holds true for S_e, S_f by symmetry. Suppose $\tau^{(k-1)}(S_g) > 1$. Then, by Lemma 16, there is a unique vertex x outside of g such that for all $g' \in S_g - g$, $g' - g = x$. If $x \notin (e \cup f) - g$, then there is an uncovered $g' \in S_g - g$ such that $M - g + g'$ is either a disconnected matching or a matching of type 2a. Therefore, we may assume $x \in (e \cup f) - g$. This tells us all uncovered edges of S_g contain x and w. For any choice of x, there are at most $k-2$ uncovered edges of S_g. By Lemma 17, these uncovered edges of S_e may be covered by at most $\left\lceil \frac{k-2}{2} \right\rceil$ $(k-1)$-sets. Since a symmetric argument is true for S_e and S_f, we have found a cover of H of size at most

$$3 + 3\left\lceil \frac{k-2}{2} \right\rceil \leqslant 3\left\lceil \frac{k+1}{2} \right\rceil. \qquad \square$$

3 Bounds on $g_1(k, m)$

We begin this section with a useful definition and observation.
Definition 19. Let H be a k-uniform hypergraph and let $e \in E(H)$. For $2 \leqslant m \leqslant k-1$, we call an m-set a of e *dispensable* if for every $f \in E(H)$, f intersects e in some m-set other than a. Otherwise, we call a *indispensable*.

For an indispensable m-set a of e, we call any edge $f \in E(H)$ such that $f \cap e = a$ a *witness* to the indispensability of a.
Observation 20. *Let H be a k-uniform hypergraph with m-matching number 1, where $\frac{k}{2} \leqslant m \leqslant k-2$. Let $e \in E(H)$. If there is a pair of indispensable m-sets a, b of e such that $|a \cap b| = 2m - k$, there exist unique witnesses f, g to a, b, respectively. Furthermore, we can m-cover f and g with one m-set.*
Lemma 21. *Let H be a k-uniform hypergraph with m-matching number 1, $m \geqslant 2$. Let $e \in H$ and set $m' = \max\{0, 2m-k\}$. For any set $S \subseteq \binom{e}{m}$ of m-sets of e with $|S| > \frac{1}{2}\binom{k}{m}$, there exists a pair $a, b \in S$ such that $|a \cap b| = m'$.*

Proof. Let G_e be a graph with vertex set $\binom{e}{m}$. For $u, v \in V(G_e)$, $uv \in E(G_e)$ if and only if $|u \cap v| = m'$. Then, G_e is an ℓ-regular graph, where $\ell = \binom{k-m}{m}$ when $m' = 0$ and $\ell = \binom{m}{2m-k}$ when $m' > 0$. Observe that an independent set I in G_e corresponds to a set S of m-sets of e such that for any pair $a, b \in I$, $|a \cap b| \neq m'$. Using the fact that for any graph G', $\alpha(G') \leqslant \frac{|E(G')|}{\Delta(G')}$, we have:

$$\alpha(G_e) \leqslant \frac{|E(G_e)|}{\Delta(G_e)} = \frac{\left(\frac{|V(G_e)|\ell}{2}\right)}{\ell} = \frac{|V(G_e)|}{2} = \frac{1}{2}\binom{k}{m}$$

The result follows. $\square$

We will also need the following inequality in order to prove Theorem 5:

Lemma 22. *For all $k \geqslant 6$, $\frac{k}{2} \leqslant m \leqslant k-2$, $0 \leqslant m' < m$,*

$$\binom{k}{m} > 4m - 2m' - 4$$

In particular,

$$\binom{k}{m} - m' - 2(m - m' - 1) > \frac{1}{2}\binom{k}{m}$$

Proof. Fix $k \geqslant 6$, $\frac{k}{2} \leqslant m \leqslant k-2$, and $0 \leqslant m' < m$. First, observe that

$$4m - 2m' - 4 \leqslant 4m - 4 \leqslant 4(k-2) - 4 = 4(k-3)$$

On the other hand, we have:

$$\binom{k}{m} \geqslant \binom{k}{k-2} = \binom{k}{2}$$

Now, it is left to show the following inequality

$$\binom{k}{2} - 4(k-3) = \frac{1}{2}(k^2 - 9k + 24) > 0$$

Let $f(k) = \frac{1}{2}(k^2 - 9k + 24)$. It can be checked that $f(6) = 3 > 0$. Furthermore, $f'(k) > 0$ for all $k \geqslant 5$. So, f is increasing for all $k \geqslant 5$ and therefore, $f(k) > 0$ for all $k \geqslant 6$. We obtain the second part of the lemma by rearranging the inequality. □

With the help of the above two lemmas, we are able to prove Theorem 5.

Proof of Theorem 5. Let $k \geqslant 6$, $\frac{k}{2} \leqslant m \leqslant k-2$, and let H be a k-uniform hypergraph with m-matching number 1. Fix $e \in E(H)$ with the most dispensable m-sets. Observe that for any non-witnessing edge $f \in E(H)$, f contains at least $m+1$ m-sets of e. If e has at least m dispensable m-sets, then we may delete any m of them and obtain an m-cover of H with the remaining m-sets of e. Suppose then that e has $m' < m$ dispensable m-sets. Denote the set of dispensable m-sets of e by S. So the number of indispensable sets is $\binom{k}{m} - m'$. We wish to find an m-cover of size at most $\binom{k}{m} - m = (\binom{k}{m} - m') - (m - m')$. We do this by deleting S from $\binom{e}{m}$ and then finding $m - m'$ pairs of indispensable m-sets $a_i, b_i \in \binom{e}{m} - S$ such that $|a_i \cap b_i| = 2m - k$ for $1 \leqslant i \leqslant m - m'$.

Note that while $0 \leqslant i - 1 \leqslant m - m' - 1$, $\binom{k}{m} - m' - 2i \geqslant \binom{k}{m} - m' - 2(m - m' - 1)$. Set $i = 0$ and $S' = \binom{e}{m} - S$. While $i \leqslant m - m' - 1$, by Lemmas 21 and 22, there exists a

pair of indispensable m-sets of e, a_i, b_i, with witnessing edges f_i, g_i, respectively, such that $|a_i \cap b_i| = 2m - k$. We may cover f_i, g_i with the m-set $x_i = (a_i \cap b_i) \cup (f_i - e)$. Note that every non-witnessing edge other than e contains either at most one of a and b. Delete a_i, b_i from S', increase i by 1, and repeat. Now, we have the following m-cover $\mathcal{C}$ of H:

$$\begin{aligned}\mathcal{C} &= \left(\binom{e}{m} - S'\right) \cup \left(\bigcup_{i=0}^{m-m'-1} \{x_i\}\right) \\ &= \left(\binom{e}{m} - \left(S \cup \left(\bigcup_{i=0}^{m-m'-1} \{a_i, b_i\}\right)\right)\right) \cup \left(\bigcup_{i=0}^{m-m'-1} \{x_i\}\right).\end{aligned}$$

Now, we can compute $|\mathcal{C}|$:

$$\begin{aligned}|\mathcal{C}| &= \binom{k}{m} - (m' + 2(m - m')) + (m - m') \\ &= \binom{k}{m} - m' - 2(m - m') + (m - m') \\ &= \binom{k}{m} - m' - (m - m') \\ &= \binom{k}{m} - m\end{aligned}$$

Therefore, $g_1(k, m) \leqslant \binom{k}{m} - m$ for all $\frac{k}{2} \leqslant m \leqslant k - 2$. □

Next, we improve the previous upper bound for $g_1(5, 2)$ following a similar argument as the above proof.

Proof of Theorem 7. Let H be a 5-uniform hypergraph with 2-matching number 1. Let $r = \max\{|e \cap f| : e, f \in E(H)\}$. If $r \geqslant 3$, then letting $e, f \in E(H)$ such that $|e \cap f| = r$, we may cover H with the 2-sets $\binom{e \cap f}{2}$ together with the 2-sets containing exactly one element from $e - f$ and one element from $f - e$. This gives a cover of size 7. Suppose then that $r = 2$. That is, every edge intersects every other edge in exactly two vertices. Let e be an edge with the most dispensable sets. Observe that for any dispensable set a of e, there is no edge intersecting e at a. If e has at least 3 dispensable sets, then we are done. Otherwise, we may assume e has $m' \leqslant 2$ dispensable sets and therefore, $10 - m' \geqslant 8$ indispensable sets. Denote the set of dispensable sets by S. Observe that for any pair a, b of indispensable 2-sets of e with $|a \cap b| = 0$, there exist unique witnesses f, g of a, b, respectively. Let $S' = \binom{e}{2} - S$. So, $|S'| = 10 - m'$. Now, by Lemma 21, we may find at least $\left\lceil \frac{|S'|-5}{2} \right\rceil = \left\lceil \frac{5-m'}{2} \right\rceil$ pairs of indispensable 2-sets, a_i, b_i for $1 \leqslant i \leqslant \left\lceil \frac{5-m'}{2} \right\rceil$ such that $|a_i \cap b_i| \geqslant 2$ with witnesses f_i, g_i, respectively. For $1 \leqslant i \leqslant \left\lceil \frac{5-m'}{2} \right\rceil$, we may 2-cover f_i, g_i

with $f_i \cap g_i$. Now, we have the following 2-cover of H:

$$\mathcal{C} = \left(S' - \bigcup_{i=1}^{\left\lceil \frac{5-m'}{2} \right\rceil} \{a_i, b_i\} \right) \cup \bigcup_{i=1}^{\left\lceil \frac{5-m'}{2} \right\rceil} (f_i \cap g_i).$$

The size of this 2-cover is:

$$|\mathcal{C}| = \left((10 - m') - 2 \cdot \left\lceil \frac{5-m'}{2} \right\rceil \right) + \left\lceil \frac{5-m'}{2} \right\rceil = 10 - m' - \left\lceil \frac{5-m'}{2} \right\rceil \leqslant 7. \qquad \square$$

We next improve the bound given by Theorem 5 for the case when $m = k - 2$. We will need the following lemma:

Lemma 23. *Let $k \geqslant 5$ and let G be a graph with vertex set $\binom{[k]}{k-2}$ and for $A, B \in V(G)$, $AB \in E(G)$ if and only if $|A \cap B| = k - 4$. Then, G has a perfect matching if $\binom{k}{2}$ is even and G has a matching with one unsaturated vertex when $\binom{k}{2}$ is odd.*

Proof. By Theorem 1.2 from [8], G has a maximum matching such that any pair of unsaturated vertices have no common neighbors. Therefore, if every pair of vertices have a common neighbor, we are done. When $k \geqslant 6$, by inclusion-exclusion, it is easy to check that for any $x, y \in V(G)$, $|N(x) \cap N(y)| > 0$. When $k = 5$, let M be a maximum matching of G such that any pair of unsaturated vertices have no common neighbors. Suppose A, B are unsaturated by M. Then, $AB \notin E(G)$ as this would contradict that M is a maximum matching. This implies that $|A \cap B| = 2$. But then, the vertex $C = \{A - B, B - A, [k] - (A \cup B)\}$ is a common neighbor of A and B, a contradiction. $\square$

Lemma 24. *Let $k \geqslant 5$ and let H be a k-uniform hypergraph with $\nu^{(k-2)}(H) = 1$. If there exists an edge that intersects every other edge in exactly $k - 2$ vertices, then*

$$\tau^{(k-2)}(H) \leqslant \left\lceil \frac{\binom{k}{k-2} + 1}{2} \right\rceil = \left\lceil \frac{\binom{k}{2} + 1}{2} \right\rceil.$$

Proof. Let $k \geqslant 5$ and let H be a k-uniform hypergraph with $\nu^{(k-2)}(H) = 1$. Suppose there exists an edge $e \in E(H)$ that intersects every other edge in exactly $k - 2$ vertices. Using the graph G_e from Lemma 21 which satisfies the properties of the graph in Lemma 23, there exists a matching M of G_e of size $\left\lfloor \frac{|V(G_e)|}{2} \right\rfloor$.

For each $uv \in M$, if there are witnessing edges f_u, f_v of u and v, respectively, these witnessing edges are unique and their intersection has size exactly $k - 2$. We may cover this pair of edges with the $(k-2)$-set $f_u \cap f_v$. If there is only one of the two witnessing edges, say f_u, then v is a dispensable $(k-2)$-set and we may cover all edges intersecting e in u by the $(k-2)$-set u. Doing this for all edges of M, we arrive at collection of $(k-2)$-sets covering all edges of $H - e$ with the exception of the witnessing edges of at

most one $(k-2)$-set. We may cover the remaining edges with at most 1 $(k-2)$-set, giving a $(k-2)$-cover of H of size

$$|M|+1 = \left\lfloor \frac{\binom{k}{k-2}}{2} \right\rfloor + 1 = \left\lceil \frac{\binom{k}{2}+1}{2} \right\rceil. \qquad \square$$

We are now ready to prove Theorem 6.

Proof of Theorem 6. We will prove the odd and even case separately by induction. First, suppose k is odd. It is not hard to show that $g_1(3,1) = 3 = \frac{1}{4}(k^2+3)$. Now, let H be a k-uniform hypergraph with $k \geqslant 5$, k odd, where $\nu^{(k-2)}(H) = 1$. Furthermore, we will assume $g_1(k-2,k-4) \leqslant \frac{1}{4}((k-2)^2+3)$. If there is an edge e of H such that every other edge of H intersects e in exactly $k-2$ vertices, then by Lemma 24, we may find an $(k-2)$-cover of H of size $\left\lceil \frac{\binom{k}{k-2}+1}{2} \right\rceil = \left\lceil \frac{\binom{k}{2}+1}{2} \right\rceil \leqslant \frac{1}{4}(k^2+3)$.

Suppose then that there is a pair of edges e, f such that $|e \cap f| = k-1$. Let us denote $e \cap f$ by S and suppose $e - S = u$, $f - S = v$. Observe that all edges intersect S in at least $k-3$ vertices. We may $(k-2)$-cover all edges intersecting S in at least $k-2$ vertices by the $k-1$ $(k-2)$-sets $\binom{S}{k-2}$. Now, observe that the uncovered edges all intersect S in $k-3$ vertices. Therefore, they must contain both u and v since H has $(k-2)$-matching number 1. Take H' to be the $(k-2)$ uniform hypergraph with vertex set $V(H) - \{u,v\}$ and edge set $E(H') = \{g - \{u,v\} : g \in E(H), |g \cap S| = k-3\}$. Now, H' has $(k-4)$ matching number 1. Otherwise, there exist edges $h_1', h_2' \in H'$ such that $|h_1 \cap h_2| \leqslant k-5$. But then, setting $h_1 = h_1' \cup \{u,v\}, h_2 = h_2' \cup \{u,v\}$, we find that h_1, h_2 is a $(k-2)$-matching in H, a contradiction. By induction, we have:

$$\tau^{(k-2)}(H') \leqslant g_1(k-2,k-4) \leqslant \frac{1}{4}((k-2)^2+3).$$

Letting C' be a $(k-4)$ cover of H' of size $\tau^{(k-2)}(H')$, then the following is a cover of H:

$$C = \{T \cup \{u,v\} : T \in C'\} \cup \binom{S}{k-2}.$$

We compute the size of C to be:

$$|C| = \tau^{(k-2)}(H') + (k-1) \leqslant \frac{1}{4}((k-2)^2+3) + (k-1) = \frac{1}{4}(k^2+3).$$

The proof for k even is almost the exact same. We include it here for completeness. Suppose k is now even. It was shown in [1] that $g_1(4,2) = 4 = \frac{1}{4}4^2$. Now, let H be a k-uniform hypergraph with $k \geqslant 6$, k even, where $\nu^{(k-2)}(H) = 1$. We will assume $g_1(k-2,k-4) \leqslant \frac{1}{4}(k-2)^2$. If there is an edge e of H such that every other edge of H

intersects e in exactly $k-2$ vertices, then by Lemma 24, we may find an $(k-2)$-cover of H of size $\left\lceil \frac{\binom{k}{k-2}+1}{2} \right\rceil = \left\lceil \frac{\binom{k}{2}+1}{2} \right\rceil \leqslant \frac{1}{4}k^2$.

Suppose then that there is a pair of edges e, f such that $|e \cap f| = k-1$. Let us denote $e \cap f$ by S and suppose $e - S = u, f - S = v$. Observe that all edges intersect S in at least $k-3$ vertices. We may $(k-2)$-cover all edges intersecting S in at least $k-2$ vertices by the $k-1$ $(k-2)$-sets $\binom{S}{k-2}$. Now, observe that the uncovered edges all intersect S in $k-3$ vertices. Therefore, they must contain both u and v since H has $(k-2)$-matching number 1. Take H' to be the $(k-2)$ uniform hypergraph with vertex set $V(H) - \{u, v\}$ and edge set $E(H') = \{g - \{u, v\} : g \in E(H), |g \cap S| = k-3\}$. Now, H' has $(k-4)$-matching number 1. Otherwise, there exist edges $h_1', h_2' \in H'$ such that $|h_1 \cap h_2| \leqslant k-5$. But then, setting $h_1 = h_1' \cup \{u, v\}, h_2 = h_2' \cup \{u, v\}$, we find that h_1, h_2 is a $(k-2)$-matching in H, a contradiction. By induction, we have:

$$\tau^{(k-2)}(H') \leqslant g_1(k-2, k-4) \leqslant \frac{1}{4}(k-2)^2.$$

Letting C' be a $(k-4)$ cover of H' of size $\tau^{(k-2)}(H')$, then the following is a cover of H:

$$C = \{T \cup \{u, v\} : T \in C'\} \cup \binom{S}{k-2}.$$

We compute the size of C to be:

$$|C| = \tau^{(k-2)}(H') + (k-1) \leqslant \frac{1}{4}(k-2)^2 + (k-1) = \frac{1}{4}k^2. \qquad \square$$

4 Fractional Results

We begin this section by proving Theorem 8:

Proof of Theorem 8. Let $k \geqslant 2$ and H be a $2k$-uniform hypergraph with k-matching number 1 and take $e \in H$. Begin by assigning every m-set contained in e a weight of $\frac{1}{k+1}$. In doing this, every edge intersecting e in at least $k+1$ vertices is fractionally k-covered. The remaining uncovered edges intersect e in exactly k vertices and currently have weight $\frac{1}{k+1}$. Observe that for any k-set S of e, there is a unique k-set T of e such that $S \cup T = e$ and $S \cap T = \emptyset$. There are exactly $\frac{1}{2}\binom{2k}{k}$ such pairs of k-sets of e. Let us label them as $\{(S_i, T_i) : 1 \leqslant i \leqslant \frac{1}{2}\binom{2k}{k}\}$. Now, for each pair S_i, T_i, either there is a unique pair of edges f, g intersecting S_i, T_i, respectively or there are multiple edges intersecting one of these k-sets and no edges intersecting the other k-set. In either case, we may find a single k-set and assign it weight $\frac{k}{k+1}$ in order to fractionally k-cover all uncovered edges intersecting e at S_i and T_i. Now, we have covered all edges with a total weight of:

$$\frac{1}{k+1}\binom{2k}{k} + \frac{k}{k+1}\frac{\binom{2k}{k}}{2} = \left(\frac{1}{k+1} + \frac{k}{2(k+1)}\right)\binom{2k}{k} = \left(\frac{1}{2} + \frac{1}{2(k+1)}\right)\binom{2k}{k}. \qquad \square$$

We may obtain bounds on $h^*(k,m)$ from $g_1^*(k,m)$ using the following lemma. This generalizes the upper bound proof strategy of Proposition 14 in [2] to work for all choices of k and m.

Lemma 25. *For all $2 \leqslant m < k$, we have $h^*(k,m) \leqslant \frac{1}{2}\left(\binom{k}{m} + g_1^*(k,m)\right)$.*

Proof. Let H be a k-uniform hypergraph and fix $2 \leqslant m < k$. Suppose H has m-matching number ν and let $M = \{e_1, \ldots, e_\nu\}$ be a maximum m-matching in H. Begin by assigning weight $1/2$ to all of the m-sets in $\bigcup_{i=1}^{\nu} \binom{e_i}{m}$. Any edge which intersects at least 2 edges of the matching in m vertices is now fractionally m-covered as well as any edge which intersects a matching edge in more than m vertices. The uncovered edges now intersect exactly 1 matching edge in exactly m vertices. For $1 \leqslant i \leqslant \nu$, let $S_{e_i} = \{f \in H : |f \cap e_i| = m \text{ and } f \text{ is uncovered}\}$. Clearly, all uncovered edges are contained in some S_{e_i}. Furthermore, for any i, the subgraph of H with edge set S_{e_i} has m-matching number 1. Otherwise, we may find an m-matching of H of size larger than M. So, for each i, we may fractionally m-cover the uncovered edges in S_{e_i} with a total weight of at most $\frac{1}{2}g_1^*(k,m)$ (We only need $\frac{1}{2}g_1^*(k,m)$ since each m-set of a matching edge was intially given a weight of $\frac{1}{2}$). Now, we have fractionally m-covered H with a total weight of at most $\frac{1}{2}\left(\binom{k}{m} + g_1^*(k,m)\right)\nu$, giving us

$$h^*(k,m) \leqslant \frac{1}{2}\left(\binom{k}{m} + g_1^*(k,m)\right).$$

□

As mentioned in the introduction, using Lemma 25 together with Theorem 8, we obtain Corollary 9.

Lastly, we improve the upper bound on $g_1^*(k,k-2)$ by proving Theorem 10:

Proof of Theorem 10. Let H be a k-uniform hypergraph with $(k-2)$-matching number 1. If there exists some edge e of H such that every other edge of H intersects e in $k-1$ vertices, then assigning weight $\frac{1}{k-1}$ to every $(k-2)$-set of e, we obtain a fractional $(k-2)$-cover of size $\frac{k}{2}$. Otherwise, we may find two edges e, f of H such that $|e \cap f| = k-2$. Let $S = e \cap f$. Then, for any other edge $g \in H - e - f$, $|g \cap S| \in \{k-2, k-3, k-4\}$. We fractionally cover all edges intersecting S in $k-2$ vertices (including e, f) by assigning weight 1 to S. Now, the edges which intersect S in $k-3$ vertices also intersect $e-S$ and $f-S$ in at least 1 vertex. Assigning weight $\frac{1}{k-3}$ to every $(k-2)$-set of the form $S' \cup \{x,y\}$, where $S' \in \binom{S}{k-4}$, $x \in e-S$, $y \in f-S$, we fractionally $(k-2)$-cover all edges intersecting S in $k-3$ vertices. Also, all edges intersecting S in $k-4$ vertices are partially covered (each have weight $\frac{4}{k-3}$). Now, for every edge g intersecting S in $k-4$ vertices, $(e \cup f) - S \subseteq g$. So, assigning weight $\left(1 - \frac{4}{k-3}\right)\frac{1}{\binom{k-4}{2}}$ to every $(k-2)$-set of the form $S'' \cup ((e \cup f) - S)$, where $S'' \in \binom{S}{k-6}$, we fractionally $(k-2)$-cover the edges intersecting

S in $k-4$ vertices and we have now covered all edges of H. The weight of this cover is:

$$\begin{aligned}
&1+\frac{1}{k-3}\left(4\binom{k-2}{k-4}\right)+\left(1-\frac{4}{k-3}\right)\frac{1}{\binom{k-4}{2}}\binom{k-2}{k-6}\\
&=1+\frac{4\binom{k-2}{2}}{k-3}+\frac{k-7}{k-3}\frac{1}{\binom{k-4}{2}}\binom{k-2}{4}\\
&=1+2(k-2)+\frac{k-7}{6(k-3)}\binom{k-2}{2}\\
&\leqslant\frac{1}{6}\binom{k-2}{2}+2k-3.
\end{aligned}$$

□

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References

[1] R. Aharoni and S. Zerbib. A generalization of Tuza's conjecture. *Journal of Graph Theory*, 94(3):445–462, 2020. doi:10.1002/jgt.22533

[2] A. Basit, D. McGinnis, H. Simmons, M. Sinnwell, and S. Zerbib. Improved bounds on a generalization of Tuza's conjecture. *Electronic Journal of Combinatorics*, 29(4):#P4.14, 2022. doi:10.37236/10829

[3] F. Botler, C. G. Fernandes, and J. Gutiérrez. On Tuza's conjecture for triangulations and graphs with small tree width. *Electronic Notes in Theoretical Computer Science*, 346:171–183, 2019. doi:10.1016/j.entcs.2019.08.016

[4] C. J. C. J. Colbourn and J. H. Dinitz. *The CRC handbook of combinatorial designs*. CRC Press, 1996.

[5] V. Guruswami and S. Sandeep. Approximate hypergraph vertex cover and generalized Tuza's conjecture. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 927–944. SIAM, 2022. doi:10.1137/1.9781611977073.40

[6] P. E. Haxell. Packing and covering triangles in graphs. *Discrete Mathematics*, 195:251–254, 1999. doi:10.1016/S0012-365X(98)00183-6

[7] Z. Tuza. A conjecture on triangles of graphs. *Graphs and Combinatorics*, 6:373–380, 1990. doi:10.1007/BF01787705

[8] D. Ye. Maximum matchings in regular graphs. *Discrete Mathematics*, 341:1195–1198, 2018. doi:10.1016/j.disc.2018.01.016