A random coloring process gives improved bounds for the Erdős-Gyárfás problem on generalized Ramsey numbers

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Abstract

The Erdős-Gyárfás number $f(n, p, q)$ is the smallest number of colors needed to color the edges of the complete graph K_n so that all of its p-clique spans at least q colors. In this paper we improve the best known upper bound on $f(n, p, q)$ for many fixed values of p, q and large n. Our proof uses a randomized coloring process, which we analyze using the so-called differential equation method to establish dynamic concentration.

Mathematics Subject Classifications: 05C55,05D40

1 Introduction

Ramsey theory is a branch of combinatorics that seeks to understand the emergence of ordered substructures within an otherwise unordered large structure. One of the more well-known and difficult parameters in Ramsey Theory, the ***k*-color diagonal Ramsey number** $R_k(p)$, is the least integer n such that any k-coloring of the edges of the complete graph K_n contains a monochromatic clique on p vertices. The current best general bounds on $R_k(p)$ are still relatively far apart, namely,

$$2^{kp/4} \leqslant R_k(p) \leqslant k^{kp}.$$

The lower bound above was proved by Lefmann [31] and the upper bound follows in a straightforward manner using a "neighborhood chasing" technique, perhaps first employed by Erdős and Szekeres [21]. (For better bounds in special cases see, e.g., [14, 15, 35, 36, 39]).

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In 1975, Erdős and Shelah [20] proposed a natural generalization of Ramsey numbers. In particular, given integers n, p and q, with $n > p \geqslant 3$ and $q \in [\binom{p}{2}]$, a **(p, q)-coloring** of K_n is an edge coloring that has the property that every p-clique of K_n sees at least q colors, and the **generalized Ramsey number**, $f(n, p, q)$, denotes the least number of colors necessary to give a (p, q)-coloring of K_n. This generalization did not receive significant attention in the literature until Erdős and Gyárfás began a more systematic study of these Ramsey numbers in 1997 [22]. Due to their treatment, $f(n, p, q)$ is sometimes referred to as the **Erdős-Gyárfás function**. It is worth noting that

$$R_k(p) = \min_n \{n : f(n, p, 2) > k\}$$

so the function $f(n, p, q)$ is in general at least as hard to determine as the multicolor diagonal Ramsey numbers.

1.1 Bounds on $f(n, p, q)$ and our main theorem

Erdős and Gyárfás proved the following upper bound on $f(n, p, q)$ for fixed p and q in [22] using a short and straightforward application of the Lovász Local Lemma:

$$f(n, p, q) = O\left(n^{\frac{p-2}{\binom{p}{2}-q+1}}\right). \tag{1}$$

They also showed that this upper bound is sharp at the so-called linear and quadratic thresholds, i.e. if $q = \binom{p}{2} - p + 3$, then $f(n, p, q) = \Theta(n)$. Similarly, if $q = \binom{p}{2} - \lfloor \frac{p}{2} \rfloor + 2$, then $f(n, p, q) = \Theta(n^2)$. There are many pairs p, q for which (1) has been improved, but no improvement has been made in the general case. We summarize such improvements here. Conlon, Fox, Lee and Sudakov [16] improved (1) for $q \leqslant p - 1$ by showing that $f(n, p, p - 1) = n^{o(1)}$. Trivially, (1) can be improved to $O(n^2)$ whenever $q > \binom{p}{2} - \frac{p}{2} + 2$. Other than the cases mentioned in the last two sentences, there are only a finite number of pairs p, q for which the bound (1) has been improved (of which we will mention some and omit a few). Of particular interest is the diagonal case, when $p = q$. It is straightforward to see that a $(3, 3)$-coloring is simply a proper edge coloring, and so $f(n, 3, 3) = n - 1$ or n depending on if n is even or odd. For larger values, the bound in (1), along with a simple inductive argument [22] gives us that

$$n^{\frac{1}{p-2}} - 1 \leqslant f(n, p, p) = O(n^{\frac{2}{p-1}})$$

For $p = 4$, Mubayi [32] gave an explicit algebraic construction that shows $f(n, 4, 4) \leqslant n^{\frac{1}{2}+o(1)}$, matching the lower bound up to sub-polynomial terms. Cameron and Heath extended this idea to also show that $f(n, 5, 5) \leqslant n^{\frac{1}{3}+o(1)}$ [12], again matching the lower bound up to subpolynomial factors. Subsequentially, Cameron and Heath were also able to show $f(n, 6, 6) \leqslant n^{\frac{1}{3}+o(1)}$ and $f(n, 8, 8) \leqslant n^{\frac{1}{4}+o(1)}$ [13], which does not match the lower bound, but is better than (1). Worth noting, the $(5, 5)$, $(6, 6)$ and $(8, 8)$ colorings beating the lower bound also made use of a $(p, p-1)$-coloring with $n^{o(1)}$ colors, given by Conlon, Fox, Lee and Sudakov [16].

We briefly discuss some additional lower bounds. Fox and Sudakov [24] proved that $f(n, 4, 3) = \Omega(\log n)$, which was an improvement on a bound given by Kostochka and Mubayi [29]. Subsequently, Conlon, Fox, Lee and Sudakov [16] also showed $f(n, p, p-1) = \Omega(\log n)$ for all $p \geqslant 5$. Pahoata and Sheffer [33] adapted the idea of additive energy to obtain some new lower bounds on $f(n, p, q)$ when we are above the linear threshold. Fish, Pahoata and Sheffer [23] used similar methods to obtain several more lower bounds, as did Balogh, English, Heath and Krueger [2].

In this work, we improve upon (1) for a wide range of fixed p and q, and large n. Our main theorem is the following.

Theorem 1. *For fixed p, q with $q \leqslant \frac{p^2-26p+55}{4}$, we have*

$$f(n, p, q) = O\left(n^{\frac{p-2}{\binom{p}{2}-q+1}} \log^{-\frac{1}{\binom{p}{2}-q+1}} n\right).$$

As a corollary we obtain a new bound for an extremal problem first studied by Brown, Erdő and Sös [11]. Let $F^{(r)}(n; k, s)$ be the smallest number of edges m such that every r-uniform hypergraph with n vertices and m edges contains some k vertices inducing at least s edges. In terms of classical extremal numbers, $F^{(r)}(n; k, s) = 1 + \text{ex}_r(n, \mathcal{G}_{k,s})$ where $\mathcal{G}_{k,s}$ is the family of all r-uniform hypergraphs on k vertices and s edges. Before we state our corollary we discuss the history of the problem. Their original paper [11] contained two unanswered questions that attracted particular interest. First, they asked whether $F^{(3)}(n; 6, 3) = o(n^2)$. This was answered in the positive by Ruzsa and Szemerédi [34], and their proof featured the first ever appearance of the triangle removal lemma. The question was generalized to the conjecture that $F^{(3)}(n; s+3, s) = o(n^2)$ for all $s \geqslant 3$ (often called the Brown-Erdős-Sós conjecture). Very recently Conlon, Gishboliner, Levanzov and Shapira [17] gave the best-known result for this problem, which says

$$F^{(3)}(n; s + O(\log s/ \log\log s), s) = o(n^2).$$

The second conjecture from [11] was that

$$\lim_{n\to\infty} n^{-2} F^{(3)}(n; k, k-2)$$

exists for all $k \geqslant 4$. This conjecture was proved for $k = 4$ by Brown, Erdős and Sös [37], for $k = 5$ by Glock [25], and for $k = 6$ very recently by Glock, Joos, Kim, Kühn, Lichev and Pikhurko [27] and in all of these results the limit was explicitly found. The full conjecture was proved very recently by Delcourt and Postle [19] (without finding the limit). A key ingredient in the proof is a constrained random process, which produces asymptotically optimal constructions.

Using a straightforward alterations approach, Brown, Erdős and Sós [11] proved that

$$F^{(r)}(n; k, s) = \Omega\left(n^{\frac{rs-k}{s-1}}\right),$$

which for many fixed r, k, s is still the best known lower bound. When $r = 2$, as a corollary to our Theorem 1 we obtain an improvement for many fixed k, s. This follows from the following observation. In a $\left(k, \binom{k}{2} - s + 2\right)$-coloring of K_n, no set of k vertices can have s edges of the same color. Thus $F^{(2)}(n; k, s)$ is at least the size of any color class in such a coloring, so

$$F^{(2)}(n; k, s) \geqslant \frac{\binom{n}{2}}{f(n, k, \binom{k}{2} - s + 2)}.$$

In light of Theorem 1, we have the following bound.

Corollary 2. *For fixed k, s with $s \geqslant \frac{k^2+24k-47}{4}$, we have*

$$F^{(2)}(n; k, s) = \Omega\left(n^{\frac{2s-k}{s-1}} \log^{\frac{1}{s-1}} n\right).$$

1.2 The coloring procedure and differential equation method

Now we will give some specifics behind the process which we will need throughout the proof. If C' is a multiset of colors, say $C' = \{c_1 * r_1, \ldots c_k * r_k\}$ with $r_1, \ldots r_k \geqslant 1$ we say that C' has $\sum_k (r_k - 1)$ **repeated colors** or **repeats**. Note that an edge colored p-clique sees at least q distinct colors if and only if the multiset of colors that appear on the clique has at most $\binom{p}{2} - q$ repeats.

The coloring procedure

Let V be a set of n vertices. Our coloring procedure will have two phases, each of which uses randomness. Phase 1 will consist of a random coloring process that colors one edge at a time according to a rule we will describe shortly. For Phase 1 we will use a set C of

$$\kappa n^{\frac{p-2}{\binom{p}{2}-q+1}} \log^{-\frac{1}{\binom{p}{2}-q+1}} n$$

colors for some large constant κ depending only on p. If Phase 1 is successful (which we will prove it is w.h.p. [1]), it colors almost all the edges. Phase 2 consists of a much simpler random coloring, where we use a fresh set C' of colors, where $|C'| = |C|$ and each edge that was left uncolored by Phase 1 gets a uniform random color from C' (independently for all edges).

We start our Phase 1 process at step 0 with all edges uncolored, and at each step i we will randomly color one random edge (details to follow). Let U_i be the set of uncolored edges at the beginning of step i. At step i, we choose an edge $e_i \in U_i$ uniformly at random. We then assign e_i a color c_i chosen uniformly at random from all colors that are *available* at e_i (definition follows). For technical reasons, to make our process amenable to analysis,

[1] We say an event dependent on n occurs *with high probability* (abbreviated w.h.p.) if the probability of that event tends to one as n tends to infinity.

we will actually avoid a family of colored subgraphs. This family will of course include p-cliques with more than $\binom{p}{2} - q$ repeats, but it will also include s-cliques, $3 \leqslant s \leqslant p$, with more than $R(s)$ repeats, where

$$R(s) := \left\lceil \frac{(s-2)\left(\binom{p}{2} - q + 1\right)}{p-2} \right\rceil - 1. \tag{2}$$

Roughly speaking, the motivation for our choice of $R(s)$ is as follows. First note that for each fixed edge $e \in E(K_n)$, (heuristically, using the fact that our coloring "looks random") the expected number of extensions of e to a set $P \supseteq e$ of size p with $\binom{p}{2} - q + 1$ repeats is

$$\Theta\left(n^{p-2}|C|^{-\left(\binom{p}{2}-q+1\right)}\right) = \Theta\left(\log n\right).$$

Suppose we did have some set S of size $s \geqslant 3$ with $R(s) + 1$ repeats. Then (heuristically, using the fact that our coloring "looks random") the expected number of extensions of S to a set $P \supseteq S$ of size p with $\binom{p}{2} - q + 1$ repeats is

$$\Theta\left(n^{p-s}|C|^{-\left(\binom{p}{2}-q-R(s)\right)}\right) \geqslant \tilde{\Theta}\left(n^{p-s} n^{-\frac{p-2}{\binom{p}{2}-q+1}\left(\binom{p}{2}-q+1-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}\right)}\right) = \tilde{\Theta}\left(1\right).$$

We would like our process to be "driven" by the colors that are available at each edge, and we do not want to concern ourselves with larger sets like this S which could cause problems. So we do not like to see some set S which presents nearly as much danger (in terms of the number of possible extensions to a bad p-clique) as a single edge. So we avoid such S.

A color c is called **available** at e if assigning c to e does not cause any set of s vertices, $3 \leqslant s \leqslant p$, to have more than $R(s)$ repeats among its colored edges. Note that $R(p) = \binom{p}{2} - q$ and so our definition of availability enforces that we have a partial (p,q)-coloring. If c is not available at e we say c is **forbidden** at e. Of course, c is forbidden at e if and only if there exists some set $S \supseteq e$ with $|S| \leqslant p$ such that the colored edges in S have $R(|S|)$ repeats and c appears among the colors. For such a set S we say that **c is forbidden at e through S.**

We note that Phase 1 of our coloring process uses ideas similar to those used by Guo, Patton and Warnke [28]. In particular they use an edge-coloring process which colors edges one at a time where each color is chosen randomly and uniformly from all "available" colors (for some appropriate definition of "available"). Also, the first two authors together with Cushman and Pralat [3] used a strategy somewhat similar to Phases 1 and 2 to bound $f(n, 4, 5)$.

For a formal description of Phase 1, see Algorithm 1. Note that for technical reasons having to do with ensuring dynamic concentration of our random variables, we stop our coloring process at a predetermined step i_{max} (defined later), even if the process could continue.

Input : Vertex set V, color set C
Output: Partial (p, q)-coloring
while $i \leqslant i_{max}$ *and every edge in* U_i *has an available color* **do**
  Choose an edge e_i uniformly at random from U_i;
  Choose a color c uniformly at random from all colors in C available at e_i;
  Assign edge e_i the color c;
  Update $U_{i+1} = U_i \setminus \{e_i\}$;
  Increment i;
end

Algorithm 1: Partial (p, q)-Coloring Process

Proof methods: The differential equation method, dynamic concentration and the local lemma

Most of the work necessary to prove Theorem 1 is in showing that w.h.p. Algorithm 1 does not terminate before step i_{max} (i.e. until then, every uncolored edge has some available color), and that the partial (p, q)-coloring we obtain at the end of Phase 1 has certain nice properties that allow us to finish the coloring.

To analyze Phase 1 we use the so-called **differential equation method** to establish **dynamic concentration** of a large family of random variables. We say a random variable is dynamically concentrated if there exists a deterministic function (which typically depends on i), which we call the **trajectory** for that random variable, such that w.h.p. the random variable stays within a narrow window of its trajectory as the process evolves. Of course technically this trajectory is not unique, since our random variable would also stay close to a slightly perturbed trajectory (as long as the perturbation is small compared to the width of the window we require our random variable to stay in). However all such trajectories are close to each other so we will take one canonical representative and call it "the" trajectory. See [5] for a gentle introduction to the differential equation method. This method evolved from work done at least as early as 1970 (see Kurtz [30]). In the 1990's Wormald [40, 41] developed the method into a quite general tool. Indeed, he proved a "black box" theorem, which guarantees dynamic concentration under some relatively simple conditions. Warnke [38] recently proved a stronger version of Wormald's "black box" theorem. For our purposes it seems the existing theorems are insufficient, but we are nevertheless able to analyze our process using some standard arguments that resemble previous analyses of other processes. The tools (and style) we use for the differential equation method resemble the work of Bohman [6], Bohman and Keevash [9, 10], and Bohman, Frieze and Lubetzky [7, 8]. The gentle introduction in [5] uses similar tools and style.

In proving that w.h.p. Algorithm 1 gets to step i_{max}, we will obtain some nice properties that the partial edge coloring enjoys when the algorithm terminates. These properties will help us use the Lovász Local Lemma (see for example [1]) to show that the final uniform random coloring on the remaining uncolored edges is suitable to complete the (p, q)-coloring.

1.3 Organization of the paper and notation

In Section 2, we will introduce a framework which allows us to describe a family of random variables relevant to the evolution of our edge coloring process. In Section 3, we heuristically derive the trajectories of our random variables. In Section 4, we give the bounds on our random variable which we will prove hold w.h.p. throughout the process. In Sections 5-7, we prove that our trajectories indeed approximate our random variables. In Section 9, we prove a few elementary, but technical lemmas which are used in earlier sections, and in Section 8, we show that we can color the remaining $\binom{n}{2} - i_{max}$ edges that are not colored by Algorithm 1 with a small set of new colors, completing a (p, q)-coloring. In Section 10 we discuss an extension of Theorem 1 by the first author, Delcourt, Li and Postle [4] which came out shortly after the first draft of this paper.

All asymptotics given throughout the paper will be with n going to infinity and all parameters that do not depend on n will be assumed to be constant unless otherwise specified. Given two functions $f = f(n)$ and $g = g(n)$, we will write $f = O(g)$ if there exists a constant c such that $f \leqslant cg$, and $f = o(g)$ if $\frac{f}{g} \to 0$. We write $f = \Omega(g)$ if $g = O(f)$, and $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. We also will write $f = \tilde{O}(g)$ if there exists some constant c such that $f \leqslant g \cdot \log^c n$, $f = \tilde{\Omega}(g)$ if $g = \tilde{O}(f)$, and $f = \tilde{\Theta}(g)$ if $f = \tilde{O}(g)$ and $f = \tilde{\Omega}(g)$.

When working with a variable $X = X(i)$, where i is a timestep from Algorithm 1, we will write ΔX to mean $X(i+1) - X(i)$ (where the specific value of i we are considering should be clear from context).

2 Framework

In this section we introduce the framework we will use to analyze the coloring process. The process keeps going as long as each uncolored edge has some available color, and so we are naturally interested in following random variables.

Definition 3. Let $U_i \subseteq \binom{V}{2}$ be the set of uncolored edges at step i. For each $e \in U_i$, let $\textsc{Ava}(e) \subseteq C$ be the set of colors available at e.

Of course, the availability of a color c at an edge e depends on sets of vertices $S \supseteq e$, how many repeats each such set S already has, and whether assigning c to e would be an additional repeat. To analyze this process we therefore need a model describing the evolving state of the coloring in terms of sets S of $s \leqslant p$ vertices and how the edges in S may be colored.

From now on we call the elements of V **real vertices**, and call the elements of C **real colors**. Let $\overline{V}$ be a set (disjoint from V) of $2p$ vertices we will call **Platonic**[2] **vertices**, and let $\overline{C}$ be a set (disjoint from C) of $\binom{2p}{2}$ **Platonic colors**. An edge $e \in \binom{V}{2}$ is called

[2]We use the word Platonic in an allusion to the Platonic theory of forms, which holds that there exist abstract objects that represent real objects in an idealized way, but that are not themselves part of the physical world.

a **real edge**, and an edge $\overline{e} \in \binom{\overline{V}}{2}$ is called a **Platonic edge**. We will use the Platonic vertices to represent sets of real vertices (we are mostly interested in sets of $s \leqslant p$ real vertices but sometimes we will consider the union of two such sets which is why we let $|\overline{V}| = 2p$). We will use colorings of the Platonic edges to represent the coloring of the corresponding real edges. In such a coloring of the Platonic edges, we may use some Platonic color on some subset of the edges to indicate that they all have the same real color without specifying what that real color is. This use of Platonic colors is crucial for our analysis for reasons we will explain later. We will also use the framework of Platonic vertices, edges and colors to indicate when a color is available at an uncolored edge. More formally, define the following:

Definition 4. For a set $\overline{S} \subseteq \overline{V}$ we define a **type on $\overline{S}$** to be a labeling z of $\binom{\overline{S}}{2}$ where each label is of one of the following forms:

(i) $(\text{COLORED}, c)$ for some $c \in C \cup \overline{C}$,

(ii) $(\text{AVAILABLE}, c)$ for some $c \in C \cup \overline{C}$,

(iii) $(\text{AVAILABLE}, ?)$,

(iv) $(?)$.

We say a type z on $\overline{S}$ is **legal** if for every $\overline{S}' \subseteq \overline{S}$ with $|\overline{S}'| \leqslant p$, z has at most $R(|\overline{S}'|)$ repeats among the colors of colored edges in $\binom{\overline{S}'}{2}$ (i.e. the colors c such that z labels some edge in $\binom{\overline{S}'}{2}$ the label $(\text{COLORED}, c)$).

Roughly speaking (formally stated in the next definition), the meaning of the labels is as follows. $(\text{COLORED}, c)$ specifies a color on a colored edge. $(\text{AVAILABLE}, c)$ specifies an available color on an uncolored edge. $(\text{AVAILABLE}, ?)$ just means an edge is uncolored, and $(?)$ does not specify anything. We abuse notation and write $\overline{S}(z) = \overline{S}$ when z is a type on $\overline{S}$ (i.e. when we write $\overline{S}(z)$ we mean the set of vertices such that z is a labeling of $\binom{\overline{S}}{2}$).

Definition 5. For $\overline{D} \subseteq \overline{V}$ we call an injection $\phi : \overline{D} \to V$ a **partial embedding** and say that the **order** of ϕ is $|\overline{D}|$. We abuse notation and denote by $\overline{D}(\phi)$ the domain $\overline{D}$ of ϕ. For a superset $\overline{S} \supseteq \overline{D}$ we say $\phi' : \overline{S} \to V$ is an **extension** of ϕ if ϕ' agrees with ϕ on $\overline{D}$. For a type z on $\overline{S}$ we say that the extension ϕ' **fits** z if there exists an injection ψ from the set of colors that are either used on colored edges or specified as available colors in z to the set of real colors C such that

(i) $\psi(c) = c$ for any real color $c \in C$,

(ii) for each $\overline{e} \in \binom{\overline{S}}{2}$, if $z(\overline{e}) = (\text{COLORED}, c)$, then $\phi'(\overline{e})$ is colored $\psi(c)$,

(iii) for each $\overline{e} \in \binom{\overline{S}}{2}$, if $z(\overline{e}) = (\text{AVAILABLE}, c)$, then the edge $\phi'(\overline{e})$ has not been assigned a color, and $\psi(c)$ is available at $\phi'(\overline{e})$, and

(iv) for each $\overline{e} \in \binom{\overline{S}}{2}$, if $z(\overline{e}) = (\text{AVAILABLE}, ?)$, then the edge $\phi'(\overline{e})$ has not been assigned a color.

We call ψ a **color map** for ϕ'. For a type z let $\text{Ext}(z, \phi) = \text{Ext}(z, \phi, i)$ be the set of extensions ϕ' of ϕ that fit z at step i. We also say that $S \subseteq V$ **fits** z if there exists some ϕ' fitting z whose image is S. We note that whether an extension fits a type or not may change as we color more edges throughout the process.

We will sometimes write the name of a set when we mean the cardinality of that set. In particular, we will use this convention when we will consider one-step changes, for example, we write $\mathbb{E}[\Delta\text{Ext}(z, \phi)|\mathcal{H}_i]$ instead of $\mathbb{E}[\Delta|\text{Ext}(z, \phi)||\mathcal{H}_i]$.

We now explain the importance of Platonic colors. We will define a family of random variables, including certain variables of the form $\text{Ext}(z, \phi)$, which we will track (i.e. give sharp estimates that hold w.h.p.) as the process evolves. In order to show we have the necessary concentration it is important that our tracked variables all be sufficiently large (like a positive power of n). We would like to track many extension variables $\text{Ext}(z, \phi)$ including ones for types z assigning colors to almost all their edges. If we used only real colors to represent such a type z, then $\text{Ext}(z, \phi)$ may even have expectation going to 0 since extensions with so many specified real colors are unlikely. By using some Platonic colors in a type z instead of real colors, we get to keep the important information describing the way colors are repeated on the edges while discarding the unimportant information that specifies which real colors correspond to which Platonic colors. In turn there will be more extensions that fit the type z which allows us to track $\text{Ext}(z, \phi)$.

We use labels of the form $(\text{AVAILABLE}, c)$ and $(\text{AVAILABLE}, ?)$ because sometimes we would like to specify uncolored edges and colors that are available there. However other times we will discard that information and replace those labels with the default label (?). More specifically we define the following:

Definition 6. Let z be a type. We define the **colored portion** of z, $\text{col}(z)$, to be the type on $\overline{S}(z)$ such that for every edge $\overline{e}' \in \binom{\overline{S}(z)}{2}$,

$$\text{col}(z)(\overline{e}') = \begin{cases} z(\overline{e}') & \text{if } z(\overline{e}') = (\text{COLORED}, c) \text{ for any } c \in C \cup \overline{C}, \\ (?) & \text{otherwise.} \end{cases}$$

If $\text{col}(z) = z$, i.e. z does not assign any available colors to uncolored edges, then we say that z is a **colors-only type**.

To analyze the evolution of the process, we must describe how an extension ϕ' that fits type z at step i may no longer fit type z at step $i+1$ (and how it might now fit some other type). Towards that goal we define the following:

Definition 7. Let z be a type on $\overline{S}$, suppose $\overline{e} \in \binom{\overline{S}}{2}$, and $z(\overline{e}) = (\text{COLORED}, c)$. If c is a real color or if c is a Platonic color that is repeated among the colored edges of z, then we let $\text{Pre}(z, \overline{e}) = \{z'\}$ where z' is the unique type on $\overline{S}(z)$ that agrees with z except that $z'(\overline{e}) = (\text{AVAILABLE}, c)$. Otherwise (i.e. when c is a Platonic color that appears only once among the colored edges of z), we let $\textsc{Pre}(z, \overline{e})$ be the set of all types z' on $\overline{S}$ such that for all $\overline{e}' \in \binom{\overline{S}}{2}$,

$$z'(\overline{e}') = \begin{cases} (\text{AVAILABLE}, c') \text{ if } \overline{e}' = \overline{e} \text{ or } z(\overline{e}') = (\text{AVAILABLE}, c) \\ z(\overline{e}') \text{ otherwise} \end{cases}$$

for some real color c' which is not already specified as available or used on any edge. Note that in this case $|\textsc{Pre}(z, \overline{e})| = |C| - O(1)$ (the number of choices for c'). If $z' \in \textsc{Pre}(z, \overline{e})$ for some edge $\overline{e}$ we call z' a **predecessor** of z.

In our analysis we will consider ways to find extensions $\phi' \in \textsc{Ext}(z, \phi)$, which generally involves finding suitable real vertices for the image of ϕ' such that certain edges are colored appropriately. In such a situation, certain Platonic colors may already be associated with real colors while other Platonic colors may not (in particular, any Platonic color assigned to an edge $\overline{e}' \subseteq \overline{D}(\phi)$ must correspond to the real color given to its image $\phi(\overline{e}')$). Thus, when we count our extensions ϕ', we expect to find fewer of them when there are more requirements stipulating that the colors of edges must be repeats. The next two definitions give us some tools to (heuristically for now, formally later) count extensions ϕ' based on the considerations in this paragraph.

Definition 8. Suppose z is a type on $\overline{S}$ and let $\overline{E} \subseteq \overline{E}' \subseteq \binom{\overline{S}}{2}$ be sets of Platonic edges. Let $M = M(\overline{E}', \overline{E}, z)$ be the set of edges in $\overline{E}' \setminus \overline{E}$ that are colored by z. Let $\textsc{Free} = \textsc{Free}(\overline{E}', \overline{E}, z)$ be the set of Platonic colors that z assigns to edges of $\overline{E}' \setminus \overline{E}$ but does not assign to any edge of $\overline{E}$. Define[3]

$$\textsc{EdgCoi}(\overline{E}', \overline{E}, z) := |M| - |\textsc{Free}|. \tag{3}$$

We will also define a version of the above for vertex sets $\overline{S} \subseteq \overline{S}'$:

$$\textsc{Coi}(\overline{S}', \overline{S}, z) := \textsc{EdgCoi}\left(\binom{\overline{S}'}{2}, \binom{\overline{S}}{2}, z\right).$$

Clearly $|\textsc{Free}| \leqslant |M|$ so we always have that $\textsc{EdgCoi}(\overline{E}', \overline{E}, z)$ and $\textsc{Coi}(\overline{S}', \overline{S}, z)$ are nonnegative. To motivate the definitions of $\textsc{EdgCoi}$ and $\textsc{Coi}$, we note that we often want to calculate $\textsc{EdgCoi}$ and $\textsc{Coi}$ based on repeats in situations where $\overline{E}$ (from (3)) contains no colored edges. In this case, note that $|M| - |\textsc{Free}|$ equals the number of repeats in $\overline{E}'$ plus the number of real colors assigned to edges in $\overline{E}'$.

Say we would like a crude heuristic estimate of the number of extensions $\phi' \in \textsc{Ext}(z, \phi)$ where z is a type on $\overline{S}$ and ϕ has domain $\overline{D}$. Assume for now that z does not specify any available colors at uncolored edges. There are $|\overline{S} \setminus \overline{D}|$ many Platonic vertices to embed into V, and we must embed them in such a way that there are $\textsc{Coi}(\overline{S}, \overline{D}, z)$ many color coincidences among the newly embedded edges (i.e. if we inspect the newly embedded

[3] $\textsc{Coi}$ here is short for "coincidence"

edges one by one there are $\text{Coi}(\overline{S}, \overline{D}, z)$ that are required to have the same color as a previously embedded or inspected edge). Heuristically, since our coloring should in many ways resemble a uniform random coloring, each color coincidence has a probability of $O(1)/|C| = \tilde{\Theta}\left(n^{-\frac{p-2}{\binom{p}{2}-q+1}}\right)$. Thus we heuristically predict that (ignoring log powers and everything else besides the power of n)

$$\text{Ext}(z, \phi) \approx n^{|\overline{S} \setminus \overline{D}| - \frac{p-2}{\binom{p}{2}-q+1}\text{Coi}(\overline{S}, \overline{D}, z)}.$$

This motivates the following definition:

Definition 9. For $\overline{S} \subseteq \overline{S}' \subseteq \overline{V}$ we let

$$\text{Pow}(\overline{S}', \overline{S}, z) := |\overline{S}' \setminus \overline{S}| - \frac{p-2}{\binom{p}{2} - q + 1}\text{Coi}(\overline{S}', \overline{S}, z).$$

Note that $\text{Pow}(\overline{S}', \overline{S}, z)$ can always be written as a rational number whose denominator is $\binom{p}{2} - q + 1$. Thus, for example, the smallest positive value Pow can take is $\frac{1}{\binom{p}{2}-q+1}$. Next we define restrictions of types, which we will use in the next paragraph.

Definition 10. Let z be a type on $\overline{S}$, and let $\overline{S} \subseteq \overline{S}'$. The **restriction of z to $\overline{S}$**, denoted $z|_{\overline{S}}$, is the type on $\overline{S}$ that agrees with z on all edges of $\overline{S}$.

While the Pow function will be very useful, it is not always true that it gives the correct power of n when we count extensions. Indeed, we will have to deal with situations where Pow gives a negative number, which does not necessarily mean that the extensions in question do not exist. This can complicate the situation when a type may have an "unlikely part" (i.e. a subset such that there are probably no extensions to that subset) even if the whole type has a positive Pow. A bit more formally, suppose again that ϕ has domain $\overline{D}$ and we want to count extensions $\phi' \in \text{Ext}(z, \phi)$ where z is a type on $\overline{S}'$. If there is some $\overline{S}$ with $\overline{D} \subseteq \overline{S} \subseteq \overline{S}'$ such that $\text{Pow}(\overline{S}, \overline{D}, z) < 0$ then it would seem that there is probably no extension in $\text{Ext}(z|_{\overline{S}}, \phi)$ and therefore none in $\text{Ext}(z|_{\overline{S}'}, \phi)$. However, if there is even one extension in $\text{Ext}(z|_{\overline{S}}, \phi)$, then it seems there should be about $n^{\text{Pow}(\overline{S}', \overline{S}, z)}$ extensions in $\text{Ext}(z|_{\overline{S}}, \phi)$. This motivates the following definition.

Definition 11. For $\overline{S} \subseteq \overline{S}'' \subseteq \overline{V}$ we let

$$\text{MaxPow}(\overline{S}'', \overline{S}, z) := \max\left\{\text{Pow}(\overline{S}'', \overline{S}', z) : \overline{S} \subseteq \overline{S}' \subseteq \overline{S}''\right\}.$$

Note that MaxPow is always nonnegative since $\text{Pow}(\overline{S}, \overline{S}, z) = 0$. To analyze our process, it will be very important to pay attention to how colors become forbidden at edges. To model that using Platonic vertices we define the following.

Definition 12. Let $\overline{e}$ and $\overline{e}'$ be distinct Platonic edges and c, c' be colors (real or Platonic). A **$(\overline{e}, c, \overline{e}', c')$-preforbidder** is a type y on $\overline{S} \supseteq \overline{e} \cup \overline{e}'$ such that the following all hold.

(i) y does not assign a color to $\overline{e}$ nor to $\overline{e}'$.

(ii) Let $y_{\overline{e}}$ be the type that agrees with y except $y_{\overline{e}}(\overline{e}) = (\text{COLORED}, c)$, and similarly let $y_{\overline{e}'}$ agree with y except $y_{\overline{e}'}(\overline{e}') = (\text{COLORED}, c')$. Then $y_{\overline{e}}$ and $y_{\overline{e}'}$ are both legal.

(iii) Let $y_{\overline{e},\overline{e}'}$ be the type that agrees with y except we have both $y_{\overline{e},\overline{e}'}(\overline{e}) = (\text{COLORED}, c)$ and $y_{\overline{e},\overline{e}'}(\overline{e}') = (\text{COLORED}, c')$. Then $y_{\overline{e},\overline{e}'}$ is illegal.

(iv) the restriction of $y_{\overline{e},\overline{e}'}$ to any set $\overline{S}'$ with $\overline{e} \cup \overline{e}' \subseteq \overline{S}' \subsetneq \overline{S}$ is legal, and

(v) there are at most three real colors appearing on the colored edges of y.

We call y a **preforbidder** if it is a $(\overline{e}, c, \overline{e}', c')$-preforbidder for some $(\overline{e}, c, \overline{e}', c')$.

The motivation for condition (v) is that we would like our framework to use Platonic colors whenever possible while still retaining enough information to say what is happening to the real colors. In particular, we will use these preforbidders when c, c' are real colors in order to keep track of which edges have c, c' available (and so we need our preforbidders to at least include the colors c, c' if they are real). We will also have to consider such a preforbidder overlapping with another trackable type which may contribute a third real color.

Note that the above definition has some symmetry, and in particular we have the following:

Observation 13. *A type y is a $(\overline{e}, c, \overline{e}', c')$-preforbidder if and only if it is a $(\overline{e}', c', \overline{e}, c)$-preforbidder.*

Recall the definition (2):

$$R(s) := \left\lceil \frac{(s-2)\left(\binom{p}{2} - q + 1\right)}{p-2} \right\rceil - 1.$$

Since the argument of the ceiling function above is a fraction with denominator $p-2$, we have

$$\frac{(s-2)\left(\binom{p}{2} - q + 1\right)}{p-2} - 1 \leqslant R(s) \leqslant \frac{(s-2)\left(\binom{p}{2} - q + 1\right)}{p-2} - \frac{1}{p-2}. \tag{4}$$

Next we use the above to bound the Pow of preforbidders:

Observation 14. *Suppose y is a $(\overline{e}, c, \overline{e}', c')$-preforbidder on $\overline{S}$ and there is at least one real color in $\{c, c'\}$ (including the possibility that $c = c'$ is real). Then*

$$\text{Pow}(\overline{S}, \overline{e}, y) \leqslant \frac{p-2}{\binom{p}{2} - q + 1}.$$

Proof. Since y does not assign a color to $\overline{e}$ or $\overline{e}'$ we have $\textsc{Pow}(\overline{S}, \overline{e}, y) = \textsc{Pow}(\overline{S}, \overline{e}', y)$. Thus WLOG we may assume that c is real.

Recall that by Definition 12 (ii) and (iii) we have that $y_{\overline{e},\overline{e}'}$ is illegal, while $y_{\overline{e}}$ is legal. Therefore $y_{\overline{e},\overline{e}'}$ must have exactly $R(|\overline{S}|) + 1$ repeats and $y_{\overline{e}}$ has exactly $R(|\overline{S}|)$ repeats. Therefore, either the multiset of colors assigned by y has $R(|\overline{S}|)$ repeats and does not include c, or else it has $R(|\overline{S}|) - 1$ repeats and includes c. In either case we have $\textsc{Coi}(\overline{S}, \overline{e}, y) \geqslant R(|\overline{S}|)$ and so

$$\textsc{Pow}(\overline{S}, \overline{e}, y) \leqslant |\overline{S}| - 2 - \frac{p-2}{\binom{p}{2} - q + 1} R(|\overline{S}|) \leqslant \frac{p-2}{\binom{p}{2} - q + 1}.$$

□

We will be especially interested in modeling the mechanism through which a particular real color c becomes forbidden at an edge. Thus we define the following:

Definition 15. For a Platonic edge $\overline{e}$ and a real color c, we let $\textsc{PreFbd}(\overline{e}, c)$ be the set of all triples $(z, \overline{e}', c')$ such that z is a $(\overline{e}, c, \overline{e}', c')$-preforbidder with the following properties:

(i) $z(\overline{e}') = (\textsc{available}, c')$,

(ii) either $c' = c$ or c' is Platonic,

(iii) other than c, all colors used on colored edges of z are Platonic,

(iv) other than the edge $\overline{e}$, z does not use the labels $(\textsc{available}, ?)$ or $(?)$,

(v) replacing all of the labels $(\textsc{available}, c'')$ with $(\textsc{colored}, c'')$ except at $\overline{e}$ would result in a legal type, and

(vi) if z has any edge label $(\textsc{available}, c'')$ for a Platonic color c'' then z also has some edge label $(\textsc{colored}, c'')$.

To estimate the probability that an available color c becomes forbidden at e at step i, we would like to estimate the number of ways that can happen. Thus we will track $\textsc{Ext}(z, \phi)$ for all z, ϕ such that $(z, \overline{e}', c') \in \textsc{PreFbd}(\overline{e}, c)$ for some $\overline{e}', c', c$ and $\overline{D}(\phi) = \overline{e}$. But in order to track those extension variables, we will also need to track extensions of all their predecessors. More formally we will track the following family of variables:

Definition 16. We say that $(y, \overline{e})$ is a **rooted trackable type** if y can be obtained by iteratively taking predecessors of some type z such that $(z, \overline{e}', c') \in \textsc{PreFbd}(\overline{e}, c)$ for some $\overline{e}', c', c$. (This includes the possibility of not taking any predecessors, i.e. $y = z$). We say that y is a **trackable type** if there exists some $\overline{e}$ such that $(y, \overline{e})$ is a rooted trackable type. In that case we may also say that y is a **trackable type with root** $\overline{e}$. We call $\overline{e}$ the **root** of $(y, \overline{e})$.

When y is a trackable type, we will abuse notation and write $\overline{e}(y)$ to denote an edge $\overline{e}$ such that $(y, \overline{e})$ is a rooted trackable type, if there is a choice of edges $\overline{e}$, $\overline{e}(y)$ will be one chosen arbitrarily. We will track $\textsc{Ext}(y, \phi)$ for all trackable types y and all ϕ of order 2 such that $(y, \overline{D}(\phi))$ is a rooted trackable type. For other pairs (y, ϕ) we will use only a crude upper bound on $\textsc{Ext}(y, \phi)$. We now list some useful properties of trackable types.

Observation 17. *Suppose $(y, \overline{e})$ is a rooted trackable type on $\overline{S}$. Then we have the following:*

(i) replacing all of the labels $(\textsc{available}, c'')$ *with* $(\textsc{colored}, c'')$ *except at $\overline{e}$ would result in a legal type,*

(ii) $\textsc{Pow}(\overline{S}', \overline{e}, y) \geqslant \frac{1}{\binom{p}{2}-q+1}$ *for all* $\overline{e} \subsetneq \overline{S}' \subseteq \overline{S}$,

(iii) there is at most one real color c'' such that y labels any edge $(\textsc{colored}, c'')$,

(iv) other than possibly at the edge $\overline{e}$, y does not use the labels $(\textsc{available}, ?)$ *or* $(?)$,

(v) if y has any edge label $(\textsc{available}, c'')$ *for a Platonic color c'' then y also has some edge label* $(\textsc{colored}, c'')$.

Proof. Parts (i) and (iii)-(v) follow from Definitions 7, 15 and 16. Indeed, if $(z, \overline{e}', c') \in \textsc{PreFbd}(\overline{e}, c)$ then by Definition 15 z satisfies (i) and (iii)-(v). Furthermore, each of properties (i) and (iii)-(v) is preserved when we take predecessors by Definition 7.

Now we prove (ii). Since this property is also preserved under taking predecessors, we will assume that $(y, \overline{e}', c') \in \textsc{PreFbd}(\overline{e}, c)$. Suppose $\overline{e} \subsetneq \overline{S}' \subseteq \overline{S}$. Since y is legal, there are at most $R(|\overline{S}'|)$ repeats among the colored edges in $\binom{\overline{S}'}{2}$. If there is no real color appearing on a colored edge in $\binom{\overline{S}'}{2}$ then we have $\textsc{Coi}(\overline{S}', \overline{e}, y) \leqslant R(|\overline{S}'|)$ and hence

$$\textsc{Pow}(\overline{S}', \overline{e}, y) \geqslant |\overline{S}'| - 2 - \frac{p-2}{\binom{p}{2} - q + 1} R(|\overline{S}'|) \geqslant \frac{1}{\binom{p}{2} - q + 1}$$

so we are done. If on the other hand there is a real color appearing on a colored edge in $\binom{\overline{S}'}{2}$ then that color must be c. We claim that in this case there can only be at most $R(|\overline{S}'|) - 1$ repeats among the colored edges in $\binom{\overline{S}'}{2}$. Indeed, if there were $R(|\overline{S}'|)$ repeats then the type y' formed by replacing the label of $\overline{e}$ with $(\textsc{colored}, c)$ would be illegal, which is a contradiction to Definition 12 (ii). Thus $\overline{S}'$ has at most $R(|\overline{S}'|) - 1$ repeats and one real color, so again we have $\textsc{Coi}(\overline{S}', \overline{e}, y) \leqslant R(|\overline{S}'|)$ and we are done. ∎

Section 9 will be devoted to proving three technical lemmas about preforbidders and other types. We state them here since we will use them before proving them.

Lemma 39. Let y be a $(\overline{e}', c', \overline{e}'', c'')$-preforbidder on $\overline{S}$. Let $\alpha = \alpha(\overline{S}')$ be the number of Platonic colors in $\{c', c''\}$ that appear in $\overline{S}$ but not in $\overline{S}'$. Then for all $\overline{S}'$ with $\overline{e}' \cup \overline{e}'' \subseteq \overline{S}' \subsetneq \overline{S}$,

$$\textsc{Pow}(\overline{S}, \overline{S}', y) \leqslant \frac{p-2}{\binom{p}{2} - q + 1}\alpha - \frac{1}{\binom{p}{2} - q + 1}$$

The other two technical lemmas are about unions of types, which we define before stating the lemmas:

Definition 18. Let y_1, y_2 be types on $\overline{S}_1, \overline{S}_2$ respectively. We say y_1 and y_2 are **compatible** if for every edge $\overline{e}' \subseteq \overline{S}_1 \cap \overline{S}_2$, we have that

$$y_1(\overline{e}') = (\text{COLORED}, c) \iff y_2(\overline{e}') = (\text{COLORED}, c).$$

In other words, $\text{col}(y_1)|_{\overline{S}_1 \cap \overline{S}_2} = \text{col}(y_2)|_{\overline{S}_1 \cap \overline{S}_2}$.

If y_1 and y_2 are compatible, we define the **union of types** $y_1 \cup y_2$ to be the unique colors-only type y such that $y|_{\overline{S}_1} = \text{col}(y_1)$, $y|_{\overline{S}_2} = \text{col}(y_2)$, and y only assigns colors to edges within $\overline{S}_1$ and to edges within $\overline{S}_2$.

Lemma 42. Let y_1 be a trackable type on $\overline{S}_1$ with root $\overline{e}_1$, and let y_2 be a $(\overline{e}_2, c_2, \overline{e}_3, c_3)$-preforbidder on $\overline{S}_2$ that is compatible with y_1. Assume $\overline{e}_2, \overline{e}_3 \subseteq \overline{S}_1$ and that $y_1(\overline{e}_2) = (\text{AVAILABLE}, c_2), y_1(\overline{e}_3) = (\text{AVAILABLE}, c_3)$. Assume $\overline{S}_2 \setminus \overline{S}_1 \neq \emptyset$. Then

$$\text{MaxPow}(\overline{S}_1 \cup \overline{S}_2, \overline{e}_1, y_1 \cup y_2) \leqslant \text{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \frac{1}{\binom{p}{2} - q + 1}.$$

Lemma 43. Let y_1 be a trackable type on $\overline{S}_1$ with root $\overline{e}_1$, and let y_2 be a $(\overline{e}_2, c_2, \overline{e}_3, c_3)$-preforbidder on $\overline{S}_2$ that is compatible with y_1. Assume $\overline{e}_3 \subseteq \overline{S}_1$, $\overline{e}_3 \neq \overline{e}_1$, and $\overline{e}_2 \not\subseteq \overline{S}_1$. Assume $y_1(\overline{e}_3) = (\text{AVAILABLE}, c_3)$. Assume c_2 is a real color. Then

$$\text{MaxPow}(\overline{S}_1 \cup \overline{S}_2, \overline{e}_1 \cup \overline{e}_2, y_1 \cup y_2) \leqslant \text{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \frac{1}{\binom{p}{2} - q + 1}.$$

3 Trajectories of our random variables

In this section we describe the trajectories for the variables $\text{Ava}(e)$ and $\text{Ext}(z, \phi)$ for trackable types z and partial embeddings ϕ of order 2. Rather than simply giving formulas we attempt to provide a heuristic derivation for them.

Say we have colored $i = \binom{n}{2}t$ edges so far. Our heuristic assumptions are as follows: t is the probability that an edge has been colored, $1/|C|$ is the probability that a colored edge has been assigned any particular color, and some number $a = a(t)$ is the probability that a particular color is available at a particular uncolored vertex. We also generally assume that events are approximately independent if we see no significant reason why they should be dependent. Under those heuristic assumptions we anticipate that

$$\text{Ava}(e) = \text{Ava}(e, i) \approx \text{ava}(t) := |C|a(t)$$

for all $e \in U_i$ and for some deterministic function $a(t)$.

Assume ϕ has order 2, and z is a trackable type with $|\overline{S}(z)| = s$ that has ℓ colored edges, r repeats, and $k \in \{0, 1\}$ real colors among its colored edges. Then we heuristically predict

$$\begin{aligned}\textsc{Ext}(z,\phi) \approx \mathrm{ext}_z(t) :=& n^{s-2}|C|^{\ell-r-k}t^{\ell}\left(\frac{1}{|C|}\right)^{\ell}(1-t)^{\binom{s}{2}-\ell-1}a(t)^{\binom{s}{2}-\ell-1}\\ &= n^{s-2}|C|^{-r-k}a(t)^{\binom{s}{2}-\ell-1}t^{\ell}(1-t)^{\binom{s}{2}-\ell-1}. \end{aligned} \tag{5}$$

We will now derive a differential equation for $a(t)$ by heuristically estimating the one-step change in $\textsc{Ava}(e)$. We estimate the probability that a fixed real color c becomes forbidden at an edge e in a single step (assuming that c was available at e one step previously). First, note that the probability that any particular edge e' gets any assigned any particular color $c' \in \textsc{Ava}(e')$ at step i is approximately $1/[(1-t)\binom{n}{2}|C|a]$. So now we count pairs (e', c') such that $c' \in \textsc{Ava}(e')$ and assigning c' to e' would forbid c at e. Let $(z, \overline{e}', c^*) \in \textsc{PreFbd}(\overline{e}, c)$. Each extension $\phi' \in \textsc{Ext}(z, \phi)$ counts the pair (e', c') where $e' = \phi'(\overline{e}')$ and c' is the color corresponding to c^* (i.e. $c' = c^*$ if $c^* \in C$ or otherwise c' is the real color representing c^* in the extension ϕ'). But each such pair (e', c') is counted about $\mathrm{equ}(z, \overline{e}')\mathrm{ava}(t)^{\binom{s}{2}-\ell-2}$ times, where $\mathrm{equ}(z, \overline{e}')$ counts the number of equivalent ways to represent a clique fitting type z as a preforbidder in $\textsc{PreFbd}(c, \overline{e})$. More precisely, $\mathrm{equ}(z, \overline{e}')$ is the number of ways to permute the Platonic colors in z times the number of bijections from $\overline{S}(z)$ to itself fixing both vertices in $\overline{e}$ and fixing the edge $\overline{e}'$ (but possibly swapping the endpoints of $\overline{e}'$) and preserving the colors of colored edges (not necessarily preserving the available colors at uncolored edges). Thus our estimate for the number of pairs (e', c') is (below, every term is indexed by some type z, and $\ell = \ell(z)$ is always the number of edges colored by that z, $r = r(z)$ is the number of repeats in z, and $k = k(z) \in \{0, 1\}$ is the number of real colors on colored edges in z)

$$\sum_{(z,\overline{e}',c^*)\in\textsc{PreFbd}(\overline{e},c)} \frac{\mathrm{ext}_z(t)}{\mathrm{equ}(z,\overline{e}')\mathrm{ava}(t)^{\binom{s}{2}-\ell-2}} \tag{6}$$

$$= \sum_{3\leqslant s\leqslant p}\ \sum_{\substack{(z,\overline{e}',c^*)\in\textsc{PreFbd}(\overline{e},c)\\ |\overline{S}(z)|=s}} \frac{n^{s-2}|C|^{-r-k}a(t)^{\binom{s}{2}-\ell-1}t^{\ell}(1-t)^{\binom{s}{2}-\ell-1}}{\mathrm{equ}(z,\overline{e}')(|C|a(t))^{\binom{s}{2}-\ell-2}}$$

$$= \sum_{3\leqslant s\leqslant p}\ \sum_{\substack{(z,\overline{e}',c^*)\in\textsc{PreFbd}(\overline{e},c)\\ |\overline{S}(z)|=s}} \frac{n^{s-2}|C|^{\ell-r-k-\binom{s}{2}+2}a(t)t^{\ell}(1-t)^{\binom{s}{2}-\ell-1}}{\mathrm{equ}(z,\overline{e}')} \tag{7}$$

Now we claim the following:

Claim 19. *The number of triples $(z, \overline{e}', c^*) \in \textsc{PreFbd}(\overline{e}, c)$ such that $|\overline{S}(z)| = s, \ell(z) = \ell$ and $\mathrm{equ}(z, \overline{e}', c^*) = \eta$ is $\mu(\ell, \eta, s) \cdot |C|^{\binom{s}{2}-\ell-2}$ for some $\mu(\ell, \eta, s)$ not depending on n.*

Indeed, for fixed ℓ, η, s there are some constant number of ways to choose the edges that are colored and the colors on those edges (note that the only real color that can show up on a preforbidder in $\textsc{PreFbd}(\overline{e}, c)$ is c), a constant number of ways to choose c^* (this color must be Platonic or must be c), and then for each uncolored edge that is not $\overline{e}$ or $\overline{e}'$, there are $|C|$ choices for the color available there, giving us the claim.

In addition, note that

$$\text{for all } (z, \overline{e}', c^*) \in \textsc{PreFbd}(\overline{e}, c) \text{ we have that } r + k = R(s). \tag{8}$$

Therefore (7) becomes

$$\begin{aligned}
&\sum_{\substack{3 \leqslant s \leqslant p \\ 1 \leqslant \ell \leqslant \binom{s}{2}-2 \\ 1 \leqslant \eta \leqslant s!|\overline{C}|!}} \mu(\ell, \eta, s) \cdot |C|^{\binom{s}{2}-\ell-2} \cdot \frac{n^{s-2}|C|^{\ell-\binom{s}{2}+2-R(s)} a(t) t^{\ell} (1-t)^{\binom{s}{2}-\ell-1}}{\eta} \\
&= \sum_{\substack{3 \leqslant s \leqslant p \\ 1 \leqslant \ell \leqslant \binom{s}{2}-2 \\ 1 \leqslant \eta \leqslant s!|\overline{C}|!}} \frac{\mu(\ell, \eta, s) n^{s-2} |C|^{-R(s)} a(t) t^{\ell} (1-t)^{\binom{s}{2}-\ell-1}}{\eta}
\end{aligned} \tag{9}$$

Now for a value s such that $p-2$ divides $(s-2)\left(\binom{p}{2}-q+1\right)$, we have $R(s) = \frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2} - 1$ and so

$$\begin{aligned}
n^{s-2}|C|^{-R(s)} &= n^{s-2}|C| \left(\kappa n^{\frac{p-2}{\binom{p}{2}-q+1}} \log^{-\frac{1}{\binom{p}{2}-q+1}} n \right)^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}} \\
&= \kappa^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}} |C| \log^{\frac{s-2}{p-2}} n.
\end{aligned}$$

On the other hand if $p-2$ does not divide $(s-2)\left(\binom{p}{2}-q+1\right)$, then $R(s) \geqslant \frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2} - 1 + \frac{1}{p-2}$ and we get

$$n^{s-2}|C|^{-R(s)} = \tilde{O}\left(n^{s-2}|C| \left(n^{\frac{p-2}{\binom{p}{2}-q+1}} \right)^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2} - \frac{1}{p-2}} \right) = \tilde{O}\left(n^{\frac{p-3}{\binom{p}{2}-q+1}} \right).$$

Thus, (9) becomes

$$\sum_{\substack{3\leqslant s\leqslant p \\ (p-2)\,\mid\,(s-2)\left(\binom{p}{2}-q+1\right) \\ 1\leqslant \ell\leqslant \binom{s}{2}-2 \\ 1\leqslant \eta\leqslant s!|\overline{C}|!}} \frac{\mu(\ell,\eta,s)\kappa^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}}|C|\log^{\frac{s-2}{p-2}} n\cdot a(t)t^{\ell}(1-t)^{\binom{s}{2}-\ell-1}}{\eta}$$
$$+\tilde{O}\left(n^{\frac{p-3}{\binom{p}{2}-q+1}}\right)$$
$$=|C|a(t)(1-t)\sum_{\substack{3\leqslant s\leqslant p \\ (p-2)\,\mid\,(s-2)\left(\binom{p}{2}-q+1\right) \\ 1\leqslant \ell\leqslant \binom{s}{2}-2 \\ 1\leqslant \eta\leqslant s!|\overline{C}|!}} \frac{\mu(\ell,\eta,s)\kappa^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}}t^{\ell}(1-t)^{\binom{s}{2}-\ell-2}}{\eta}\log^{\frac{s-2}{p-2}} n$$
$$+\tilde{O}\left(n^{\frac{p-3}{\binom{p}{2}-q+1}}\right)$$
$$=|C|a(t)(1-t)\sum_{\substack{3\leqslant s\leqslant p \\ (p-2)\,\mid\,(s-2)\left(\binom{p}{2}-q+1\right)}} h_s(t)\log^{\frac{s-2}{p-2}} n+\tilde{O}\left(n^{\frac{p-3}{\binom{p}{2}-q+1}}\right) \tag{10}$$
$$=|C|a(t)(1-t)h(t)+\tilde{O}\left(n^{\frac{p-3}{\binom{p}{2}-q+1}}\right) \tag{11}$$

where on line (10) we use the definition

$$h_s(t):=\sum_{\substack{1\leqslant \ell\leqslant \binom{s}{2}-2 \\ 1\leqslant \eta\leqslant s!|\overline{C}|!}} \frac{\mu(\ell,\eta,s)\kappa^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}}t^{\ell}(1-t)^{\binom{s}{2}-\ell-2}}{\eta}$$

and on line (11) we use

$$h(t):=\sum_{\substack{3\leqslant s\leqslant p \\ (p-2)\,\mid\,(s-2)\left(\binom{p}{2}-q+1\right)}} h_s(t)\log^{\frac{s-2}{p-2}} n. \tag{12}$$

Note that $h_s(t)$ is a polynomial in t with constant coefficients (not depending on n).

We derive a differential equation by estimating the one-step change in $\textsc{Ava}(e)$ in two ways. First, since $\textsc{Ava}(e)\approx |C|a(t)$ we should have

$$\Delta\textsc{Ava}(e)\approx |C|a'(t)\Delta t=\frac{1}{\binom{n}{2}}|C|a'(t).$$

Second, we can estimate $\Delta\textsc{Ava}(e)$ by considering each available color and the probability that it becomes forbidden in one step. Since there are $\textsc{Ava}(e) \approx |C|a(t)$ available colors and each one becomes forbidden with probability about

$$\frac{1}{(1-t)\binom{n}{2}|C|a(t)} \cdot |C|a(t)(1-t)h(t),$$

since for each $c \in \textsc{Ava}(e)$ there are about $|C|a(t)(1-t)h(t)$ pairs (e', c') such that assigning c' to e' would forbid c at e, and each such pair has a probability of about $1/\left[(1-t)\binom{n}{2}|C|a(t)\right]$ of being the choice made in one step. Thus, we heuristically expect

$$\frac{1}{\binom{n}{2}}|C|a'(t) = -|C|a(t) \cdot \frac{1}{(1-t)\binom{n}{2}|C|a(t)} \cdot |C|a(t)(1-t)h(t)$$

or

$$a'(t) = -a(t)h(t).$$

Since $a(0) = 1$ we have

$$a(t) = \exp\{-H(t)\} \tag{13}$$

where

$$H(t) := \int_0^t h(\tau)d\tau \tag{14}$$

Note that we will take (13) as the definition of $a(t)$.

Let

$$\epsilon := 10^{-3}p^{-6}. \tag{15}$$

(this choice will appear arbitrary at the moment. The reader can safely think of ϵ as just a small positive constant). Note that for $3 \leqslant s \leqslant p$ we have

$$h_s(t) \leqslant \kappa^{-\frac{\binom{p}{2}-q+1}{p-2}} \sum_{\substack{1 \leqslant \ell \leqslant \binom{s}{2}-2 \\ 1 \leqslant \eta \leqslant s!|\overline{C}|!}} \frac{\mu(\ell, \eta, s)}{\eta}$$

and so

$$\begin{aligned} h(t) &= \sum_{3 \leqslant s \leqslant p} h_s(t) \log^{\frac{s-2}{p-2}} n \\ &\leqslant \log n \cdot \sum_{3 \leqslant s \leqslant p} h_s(t) \\ &\leqslant \log n \cdot \kappa^{-\frac{\binom{p}{2}-q+1}{p-2}} \sum_{\substack{3 \leqslant s \leqslant p \\ 1 \leqslant \ell \leqslant \binom{s}{2}-2 \\ 1 \leqslant \eta \leqslant s!|\overline{C}|!}} \frac{\mu(\ell, \eta, s)}{\eta} \\ &\leqslant \epsilon \log n \end{aligned} \tag{16}$$

where on the last line we have used the (as yet unstated) assumption that we will choose κ so that

$$\kappa \geqslant \left(\frac{\epsilon}{\displaystyle\sum_{\substack{3 \leqslant s \leqslant p \\ 1 \leqslant \ell \leqslant \binom{s}{2} - 2 \\ 1 \leqslant \eta \leqslant s! |\overline{C}|!}} \frac{\mu(\ell, \eta, s)}{\eta}} \right)^{-\frac{p-2}{\binom{p}{2} - q + 1}} \tag{17}$$

Note that by (16) and (14) we have

$$H(t) \leqslant \epsilon t \log n \qquad \text{for all } 0 \leqslant t \leqslant 1$$

and so

$$a(t) \geqslant \exp\{-\epsilon t \log n\} \geqslant n^{-\epsilon} \qquad \text{for all } 0 \leqslant t \leqslant 1. \tag{18}$$

4 The good event

In this section we define the good event $\mathcal{E}_i$, which among other things stipulates that every uncolored edge still has plenty of available colors. If we manage to show $\mathcal{E}_{i^*}$ holds w.h.p. then that means the process manages to color at least i^* edges before terminating. More specifically, $\mathcal{E}_i$ will stipulate that all of our tracked variables are within a small window of their respective trajectories we derived in Section 3. $\mathcal{E}_i$ will also stipulate some crude upper bounds on certain other extension variables.

We define our **error functions**

$$f_{\text{AVA}}(t) := n^{-\frac{1/2}{\binom{p}{2} - q + 1} + 20p^2\epsilon} (1-t)^{-5p^2} e^{10p^4 \epsilon t \log n} \tag{19}$$

$$f_{\text{EXT}}(t) := n^{-\frac{1/2}{\binom{p}{2} - q + 1} + 20p^2\epsilon} (1-t)^{-5p^2+1} a(t)^{-1} e^{10p^4 \epsilon t \log n} \tag{20}$$

$$f_z(t) := a(t)^{\binom{s}{2} - 1 - \ell} f_{\text{EXT}}(t) \tag{21}$$

where in the last line z is a type on s vertices with ℓ colored edges.

Definition 20. The **good event** $\mathcal{E}_i$ to be the event that for all $i' \leqslant i$ we have the conditions below. We let $t' = i'/\binom{n}{2}$.

(i) For all $e \in U_{i'}$ we have

$$|\text{AVA}(e, i') - \text{ava}(t')| \leqslant |C| f_{\text{AVA}}(t')$$

(ii) For all rooted trackable types $(z, \overline{e})$ and for all ϕ of order 2 mapping $\overline{e}$ to an edge $e \in U_{i'}$, we have

$$|\text{EXT}(z, \phi, i') - \text{ext}_z(t')| \leqslant n^{s-2} |C|^{-r-k} f_z(t')$$

(iii) Let z be a type. Let $\overline{S}' := \overline{S}(z)$ and ϕ be a partial embedding with domain $\overline{D} := \overline{D}(\phi)$. Let $s' := |\overline{S}'|, d := |\overline{D}|$ and let m be the number of edges not inside $\overline{D}$ that are assigned colors by z. Suppose further that we have the property:

$$\text{for all } \overline{S} \text{ such that } \overline{D} \subsetneq \overline{S} \subseteq \overline{S}' \text{ we have that } \textsc{Pow}(\overline{S}', \overline{S}, z) \leqslant 0. \tag{22}$$

(Note that the above property implies that $\textsc{MaxPow}(\overline{S}', \overline{D}, z) = \textsc{Pow}(\overline{S}', \overline{D}, z)$). Then

$$\textsc{Ext}(z, \phi, i') \leqslant n^{\textsc{MaxPow}(\overline{S}', \overline{D}, z) + 10(m+s'-d)\epsilon}.$$

We also define

$$i_{max} := \left(1 - n^{-\epsilon}\right)\binom{n}{2}, \quad t_{max} := i_{max}/\binom{n}{2} = 1 - n^{-\epsilon}.$$

Observation 21. *We have*

$$\frac{|C| f_{\textsc{Ava}}(t)}{\text{ava}(t))} = \frac{f_{\textsc{Ava}}(t)}{a(t)} = \frac{f_{\textsc{Ext}}(t)}{1-t} \leqslant n^{-\frac{1/2}{\binom{p}{2}-q+1} + (10p^4+25p^2+1)\epsilon} = o(1).$$

for all $t \leqslant t_{max}$. In particular, the bound in Condition (i) *gives an asymptotically tight estimate of* $\textsc{Ava}(e)$ *for all $t \leqslant t_{max}$.*

Proof. The fact that $\frac{|C| f_{\textsc{Ava}}(t)}{\text{ava}(t))} = \frac{f_{\textsc{Ava}}(t)}{a(t)} = \frac{f_{\textsc{Ext}}(t)}{1-t}$ is by definition. Using the fact that $a(t) \geqslant n^{-\epsilon}$ (from (18)) and $1 - t \geqslant n^{-\epsilon}$, we have

$$\frac{f_{\textsc{Ava}}(t)}{a(t)} = \frac{n^{-\frac{1/2}{\binom{p}{2}-q+1} + 20p^2\epsilon}(1-t)^{-5p^2} e^{10p^4 \epsilon t \log n}}{a(t)} \leqslant n^{-\frac{1/2}{\binom{p}{2}-q+1} + (10p^4+25p^2+1)\epsilon} = o(1)$$

where for the last equality we have used that ϵ is small (recall (15)). $\square$

We will prove the following:

Theorem 22. *Fix p, let $\epsilon = 10^{-3}p^{-6}$ and choose κ large enough so that* (17) *holds. Then $\mathcal{E}_{i_{max}}$ holds w.h.p.. In particular, the coloring process colors at least i_{max} many edges.*

A proof outline of the theorem is as follows. We will show that the complement of $\mathcal{E}_{i_{max}}$ is contained in the event that some member of a large family of supermartingales increases by some amount that is very unlikely. The probability that one of our supermartingales misbehaves can be bounded by Freedman's inequality, which we now state:

Lemma 23. *Let $Y(0), Y(1), \ldots$ be a supermartingale with respect to the filtration $\mathcal{H}_0, \mathcal{H}_1, \ldots$, i.e. $\mathbb{E}[\Delta Y(i)|\mathcal{H}_i] \leqslant 0$. Assume $\Delta Y(i) \leqslant D$ for all i, and let $V(i) := \sum_{k \leqslant i} \mathbf{Var}[\Delta Y(k)|\mathcal{H}_k]$. Then,*

$$\mathbb{P}\left[\exists i : V(i) \leqslant b, Y(i) - Y(0) \geqslant \lambda\right] \leqslant \exp\left(-\frac{\lambda^2}{2(b + D\lambda)}\right).$$

The probability that $\mathcal{E}_{i_{max}}$ fails for any reason (i.e. due to any member of our family of supermartingales misbehaving) can then be bounded using the union bound.

5 The crude bounds on Ext variables

In this section we bound the probability that $\mathcal{E}_{i_{max}}$ fails due to Condition (iii). We also prove some bounds that are implied by Condition (iii). In particular, the following lemma will give us that in the good event, the conclusion of Condition (iii) holds for all extension variables, even those that do not satisfy (22).

Lemma 24. *Suppose z is a type on $\overline{S}$ and ϕ is a partial embedding with domain $\overline{D}$. Let $s := |\overline{S}|, d := |\overline{D}|$, and suppose z assigns colors to m edges that are not in $\overline{D}$. Then in the good event we (deterministically) have*

$$\textsc{Ext}(z,\phi) \leqslant n^{\textsc{MaxPow}(\overline{S},\overline{D},z)+10(m+s-d)\epsilon}$$

Proof. We will define a sequence of sets $\overline{D} = \overline{D}_1 \subseteq \overline{D}_2 \subsetneq \ldots \subsetneq \overline{D}_j = \overline{S}$, and let $z_{j'} := z|_{\overline{D}_{j'}}$. We will then bound $\textsc{Ext}(z,\phi)$ by multiplying the number of ways to extend an embedding of $\overline{D}_{j'-1}$ fitting $z_{j'-1}$ to an embedding of $\overline{D}_{j'}$ fitting $z_{j'}$ for $j' = 2,\ldots,j$. We let

$$\overline{D}_2 \in \arg\min_{\overline{D}'}\{\textsc{Pow}(\overline{D}',\overline{D},z) : \overline{D} \subsetneq \overline{D}' \subseteq \overline{S}\}$$

(note here that $\textsc{Pow}(\overline{D}',\overline{D},z)$ is defined even if z is a type on $\overline{S}$ properly containing $\overline{D}'$, and in particular $\textsc{Pow}(\overline{D}',\overline{D},z) = \textsc{Pow}(\overline{D}',\overline{D},z|_{\overline{D}'}))$. For $j' \geqslant 3$

$$\overline{D}_{j'} \in \arg\min_{\overline{D}'}\{\textsc{Pow}(\overline{D}',\overline{D}_{j'-1},z) : \overline{D}_{j'-1} \subsetneq \overline{D}' \subseteq \overline{S}\}.$$

We claim that for all $2 \leqslant j' \leqslant j$ the triple $(\overline{D}_{j'-1},\overline{D}_{j'},z_{j'})$ has the property on line (22). Indeed, let $\overline{D}_{j'-1} \subsetneq \overline{D}' \subseteq \overline{D}_{j'}$ and suppose that $\textsc{Pow}(\overline{D}_{j'},\overline{D}',z_{j'}) = \textsc{Pow}(\overline{D}_{j'},\overline{D}',z) > 0$. Then

$$\textsc{Pow}(\overline{D}',\overline{D}_{j'-1},z) = \textsc{Pow}(\overline{D}_{j'},\overline{D}_{j'-1},z) - \textsc{Pow}(\overline{D}_{j'},\overline{D}',z) < \textsc{Pow}(\overline{D}_{j'},\overline{D}_{j'-1},z)$$

contradicting our choice of $\overline{D}_{j'}$. Thus $\textsc{Pow}(\overline{D}_{j'},\overline{D}',z_{j'}) \leqslant 0$, and so the triple $(\overline{D}_{j'-1},\overline{D}_{j'},z_{j'})$ has Property (22).

Now (recall the comment after (22)) we have

$$\textsc{MaxPow}(\overline{D}_{j'},\overline{D}_{j'-1},z_{j'}) = \textsc{Pow}(\overline{D}_{j'},\overline{D}_{j'-1},z)$$

for all $j' \geqslant 3$. Similarly, we have $\textsc{MaxPow}(\overline{D}_2,\overline{D}_1,z_2) = 0$. Indeed, for $\overline{D}_1 \subseteq \overline{D}' \subseteq \overline{D}_2$ we have $\textsc{Pow}(\overline{D}_2,\overline{D}',z_2) = \textsc{Pow}(\overline{D}_2,\overline{D}_1,z_2) - \textsc{Pow}(\overline{D}',\overline{D}_1,z_2) \leqslant 0$ by our choice of D_2.

Putting this together, we multiply the upper bound given by Condition (iii) on the number of embeddings of $\overline{D}_{j'}$ given an embedding of $\overline{D}_{j'-1}$ for $j' = 2,\ldots,j$. Let $d_{j'} := |\overline{D}_{j'}|$ and $m_{j'}$ be the number of edges not inside $\overline{D}_{j'-1}$ that $z_{j'}$ assigns colors to. Then

$|\textsc{Ext}(z,\phi)|$ is at most

$$\begin{aligned}&\prod_{j'=2}^{j} n^{\textsc{MaxPow}(\overline{D}_{j'},\overline{D}_{j'-1},z_{j'})+10(m_{j'}+d_{j'}-d_{j'-1})\epsilon}\\ &= n^{0+\textsc{Pow}(\overline{D}_3,\overline{D}_2,z)+\ldots+\textsc{Pow}(\overline{D}_j,\overline{D}_{j-1},z)+10(m_2+\ldots+m_j+d_j-d_1)\epsilon}\\ &= n^{\textsc{Pow}(\overline{S},\overline{D}_2,z)+10(m+s-d)\epsilon}.\end{aligned}$$

Since by our choice of $\overline{D}_2$, we have that $\textsc{MaxPow}(\overline{S},\overline{D},z) = \textsc{Pow}(\overline{S},\overline{D}_2,z)$ we are done. ☐

Recall that in Lemma 24, m was the number of colored edges in a type, s was the number of vertices and d was the size of the domain of a partial embedding. In particular, the next corollary follows from $m+s-d \leqslant \binom{p}{2}+p-0 < 2p^2$.

Corollary 25. *Suppose z is a type on $\overline{S}$ and ϕ is a partial embedding with domain $\overline{D}$. Then in the good event we (deterministically) have*

$$\textsc{Ext}(z,\phi) \leqslant n^{\textsc{MaxPow}(\overline{S},\overline{D},z)+20p^2\epsilon}$$

Lemma 26. *Suppose z is a type on $\overline{S}$ and that ϕ is a partial embedding with domain $\overline{D} := \overline{D}(\phi) \subseteq \overline{S}$ such that ϕ fits the type $z|_{\overline{D}}$. Let $s := |\overline{S}|, d := |\overline{D}|$ and let m be the number of edges not inside $\overline{D}$ that are assigned colors by z. Suppose further that we have the property:*

$$\textit{for all } \overline{S}' \textit{ such that } \overline{D} \subsetneq \overline{S}' \subseteq \overline{S} \textit{ we have that } \textsc{Pow}(\overline{S},\overline{S}',z) \leqslant 0.$$

Then the probability that there exists a step i such that $\mathcal{E}_{i-1}$ holds and then at step i we have

$$\textsc{Ext}(z,\phi) > n^{\textsc{MaxPow}(\overline{S},\overline{D},z)+10(m+s-d)\epsilon}$$

(i.e. we do not have $\mathcal{E}_i$ due to Condition (iii) *failing for this z,ϕ) is at most*

$$\exp\{-\Omega(n^{\epsilon})\}$$

Proof. First we handle the trivial case where $m=0$, i.e. z does not color any edges outside $\overline{D}$. Then $\textsc{MaxPow}(\overline{S},\overline{D},z) = s-d$ and trivially

$$\textsc{Ext}(z,\phi) \leqslant n^{s-d}$$

so the lemma holds. Assume henceforth that $m \geqslant 1$. We define the variable $\widetilde{\textsc{Ext}}(z,\phi) = \widetilde{\textsc{Ext}}(z,\phi,i)$ to be the cumulative (up to step i) number of positive contributions to $\textsc{Ext}(z,\phi)$, i.e. the total number of instances of an extension ϕ' of ϕ becoming a member of $\textsc{Ext}(z,\phi,j)$ at some step $j \leqslant i$.

$$\begin{aligned}&\textsc{Ext}^+(z,\phi) = \textsc{Ext}^+(z,\phi,i)\\ &:= \begin{cases}\widetilde{\textsc{Ext}}(z,\phi,i) - n^{\textsc{MaxPow}(\overline{S},\overline{D},z)-2+10(m+s-d)\epsilon}\cdot i & \text{if } \mathcal{E}_{i-1} \text{ holds and}\\ & \phi(\overline{D}) \text{ fits type } z|_{\overline{D}},\\ \textsc{Ext}^+(z,\phi,i-1) & \text{otherwise.}\end{cases}\end{aligned}$$

We will show that $\text{Ext}^+(z,\phi)$ is a supermartingale. Note that $\Delta\widetilde{\text{Ext}}(z,\phi)$ is bounded above by a constant times the maximum possible value of $\text{Ext}(z,\phi')$, where ϕ' is an extension of the partial embedding ϕ whose domain is $\overline{D}' := \overline{D} \cup \overline{e}_i$ (recall e_i is the edge recieving a color in step i. Here $\overline{e}_i$ is meant to be a Platonic edge corresponding to e_i). If $\overline{e}_i \subseteq \overline{D}$ then $\Delta\text{Ext}^+(z,\phi) \leqslant 0$ since no extension of ϕ can fit the type z any longer. Thus we assume $\overline{D}' \supsetneq \overline{D}$ and so for any $\overline{D}' \subseteq \overline{S}' \subseteq \overline{S}$ we have that $\overline{D} \subsetneq \overline{S}'$ and so $\text{Pow}(\overline{S},\overline{S}',z) \leqslant 0$ and so in the good event we have

$$\text{Ext}(z,\phi') \leqslant n^{\text{MaxPow}(\overline{S},\overline{D}',z)+10(m+s-d-1)\epsilon} = n^{10(m+s-d-1)\epsilon}$$

and so

$$\Delta\widetilde{\text{Ext}}(z,\phi) \leqslant n^{10(\ell+s-d-1)\epsilon}. \tag{23}$$

Now we bound $\mathbb{E}[\Delta\widetilde{\text{Ext}}(z,\phi)|\mathcal{H}_i]$. A positive contribution to $\text{Ext}(z,\phi)$ arise from the newly colored edge e_i and some extension of ϕ to a clique of type z' where $z' = \text{Pre}(z,\overline{e}_i)$ is some predecessor of z formed by uncoloring the Platonic edge $\overline{e}_i$ corresponding to e_i.

If the color of $\overline{e}_i$ under z is Platonic and no other colored edge has the same color, then the edge could be recolored with almost any color and still fit the type z. In that case, for any $\overline{S}'$ with $\overline{D} \subseteq \overline{S}' \subseteq \overline{S}$ we have $\text{Pow}(\overline{S},\overline{S}',z') = \text{Pow}(\overline{S},\overline{S}',z)$. Thus in the good event we have

$$\text{Ext}(z',\phi) \leqslant n^{\text{MaxPow}(\overline{S},\overline{D},z')+10(m+s-d-1)\epsilon} = n^{\text{MaxPow}(\overline{S},\overline{D},z)+10(m+s-d-1)\epsilon}.$$

Now, in order for any particular extension in $\text{Ext}(z',\phi)$ to become an extension in $\text{Ext}(z,\phi)$, the colored edge e_i must be a particular edge. Thus the expected positive contribution to $\text{Ext}(z,\phi)$ is at most (recalling that $1-t \geqslant n^{-\epsilon}$)

$$n^{\text{MaxPow}(\overline{S},\overline{D},z)+10(m+s-d-1)\epsilon} \cdot \frac{1}{(1-t)\binom{n}{2}} \leqslant n^{\text{MaxPow}(\overline{S},\overline{D},z)-2+[10(m+s-d-1)+1]\epsilon}.$$

The other case we need to consider is where the color of $\overline{e}_i$ under z is a real color or a repeated Platonic color. In this case, for any $\overline{S}'$ with $\overline{D} \subseteq \overline{S}' \subseteq \overline{S}$ we have $\text{Pow}(\overline{S},\overline{S}',z') \leqslant \text{Pow}(\overline{S},\overline{S}',z) + \frac{p-2}{\binom{p}{2}-q+1}$. Thus in the good event we have

$$\text{Ext}(z',\phi) \leqslant n^{\text{MaxPow}(\overline{S},\overline{D},z')+10(m+s-d-1)\epsilon} \leqslant n^{\text{MaxPow}(\overline{S},\overline{D},z)+\frac{p-2}{\binom{p}{2}-q+1}+10(m+s-d-1)\epsilon}.$$

Now, in order for any particular extension in $\text{Ext}(z',\phi)$ to become an extension in $\text{Ext}(z,\phi)$, the colored edge e_i must be a particular edge and it must receive a particular color. Thus the expected positive contribution to $\text{Ext}(z,\phi)$ is at most (recalling that $a(t) \geqslant n^{-\epsilon}$ and $1-t \geqslant n^{-\epsilon}$)

$$n^{\text{MaxPow}(\overline{S},\overline{D},z)+\frac{p-2}{\binom{p}{2}-q+1}+10(m+s-d-1)\epsilon} \cdot \frac{1}{(1-t)\binom{n}{2}|C|a} \leqslant n^{\text{MaxPow}(\overline{S},\overline{D},z)-2+[10(m+s-d-1)+3]\epsilon}.$$

Thus in both cases we have

$$\mathbb{E}[\Delta\widetilde{\textsc{Ext}}(z,\phi)|\mathcal{H}_i] \leqslant n^{\textsc{MaxPow}(\overline{S},\overline{D},z)-2+[10(m+s-d-1)+3]\epsilon}. \tag{24}$$

Recall that we let $\Delta\textsc{Ext}^+(z,\phi) = 0$ if $\phi(\overline{D})$ no longer fits the type $z|_{\overline{D}}$. Now by line (24) we have that $\mathbb{E}[\Delta\textsc{Ext}^+(z,\phi)|\mathcal{H}_i] \leqslant 0$, i.e. $\textsc{Ext}^+(z,\phi)$ is a supermartingale. We also have

$$\Delta\textsc{Ext}^+(z,\phi) \leqslant \Delta\widetilde{\textsc{Ext}}(z,\phi) \leqslant n^{10(m+s-d-1)\epsilon}$$

by line (23). Thus for our application of Freedman's inequality we can use $D = n^{10(m+s-d-1)\epsilon}$. Also we have

$$\begin{aligned}
\mathbf{Var}[\textsc{Ext}^+(z,\phi,k)|\mathcal{H}_k] &= \mathbf{Var}[\widetilde{\textsc{Ext}}(z,\phi,k)|\mathcal{H}_k] \\
&\leqslant \mathbb{E}[\left(\Delta\widetilde{\textsc{Ext}}(z,\phi,k)\right)^2|\mathcal{H}_k] \\
&\leqslant \mathbb{E}[n^{10(m+s-d-1)\epsilon}\cdot\Delta\widetilde{\textsc{Ext}}(z,\phi,k)|\mathcal{H}_k] \\
&\leqslant n^{10(m+s-d-1)\epsilon}\cdot n^{\textsc{MaxPow}(\overline{S},\overline{D},z)-2+[10(m+s-d-1)+3]\epsilon} \\
&= n^{\textsc{MaxPow}(\overline{S},\overline{D},z)-2+[20(m+s-d-1)+3]\epsilon}
\end{aligned}$$

and so for all $i \leqslant i_{max} < n^2$ we have

$$V(i) = \sum_{0\leqslant k\leqslant i}\mathbf{Var}[\textsc{Ext}^+(z,\phi,k)|\mathcal{H}_k] \leqslant n^{\textsc{MaxPow}(\overline{S},\overline{D},z)+[20(m+s-d-1)+3]\epsilon},$$

so for our application of Freedman's inequality we will use the value $b = n^{\textsc{MaxPow}(\overline{S},\overline{D},z)+[20(m+s-d-1)+3]\epsilon}$. We will choose $\lambda = n^{\frac{1}{2}\textsc{MaxPow}(\overline{S},\overline{D},z)+[10(m+s-d-1)+2]\epsilon}$. Note that since z colors at least one edge outside $\overline{D}$, $\textsc{Ext}^+(z,\phi,0) = \widetilde{\textsc{Ext}}(z,\phi,0) = 0$. Then, Freedmann's inequality (Lemma 23) tells us that

$$\begin{aligned}
&\mathbb{P}\left[\exists i : \textsc{Ext}^+(z,\phi,i) \geqslant \lambda\right] \\
&\leqslant \exp\left(-\frac{\lambda^2}{2(b+D\lambda)}\right) \\
&= \exp\left(-\frac{n^{\textsc{MaxPow}(\overline{S},\overline{D},z)+[20(m+s-d-1)+4]\epsilon}}{2\left(n^{\textsc{MaxPow}(\overline{S},\overline{D},z)+[20(m+s-d-1)+3]\epsilon}+n^{\frac{1}{2}\textsc{MaxPow}(\overline{S},\overline{D},z)+[20(m+s-d-1)+2]\epsilon}\right)}\right) \\
&= \exp\left(-\Omega\left(n^{\epsilon}\right)\right).
\end{aligned}$$

But if the good event $\mathcal{E}_i$ holds and $\textsc{Ext}^+(z,\phi,i) < \lambda$ then

$$\begin{aligned}
\textsc{Ext}(z,\phi,i) &\leqslant \widetilde{\textsc{Ext}}(z,\phi,i) \\
&< n^{\textsc{MaxPow}(\overline{S},\overline{D},z)-2+10(m+s-d)\epsilon}\cdot i + n^{\frac{1}{2}\textsc{MaxPow}(\overline{S},\overline{D},z)+[10(m+s-d-1)+2]\epsilon} \\
&< n^{\textsc{MaxPow}(\overline{S},\overline{D},z)+10(m+s-d)\epsilon}.
\end{aligned}$$

This completes the proof of the lemma.

□

6 Dynamic concentration of $\textsc{Ava}(e)$

In this section we bound the probability that $\mathcal{E}_{i_{max}}$ fails due to Condition (i). We define a family of random variables that we will show are supermartingales, to which we will then apply Freedman's inequality. For each edge e we define variables

$$\textsc{Ava}^{\pm}(e) = \textsc{Ava}^{\pm}(e,i) := \begin{cases} \textsc{Ava}(e,i) - \mathrm{ava}(t) \mp |C| f_{\textsc{Ava}}(t) & \text{if } \mathcal{E}_{i-1} \text{ holds and } e \in U_i, \\ \textsc{Ava}^{\pm}(e, i-1) & \text{otherwise.} \end{cases}$$

Note that in the above definition we are using "±" to simultaneously define $\textsc{Ava}^{+}(e)$ and $\textsc{Ava}^{-}(e)$. We will establish that the variables $\textsc{Ava}^{+}(e)$ are supermartingales. We claim that $\textsc{Ava}^{-}(e)$ are submartingales (or equivalently, $-\textsc{Ava}^{-}(e)$ are supermartingales), but we will not separately justify it since it is very similar to showing that the $\textsc{Ava}^{+}(e)$ are supermartingales.

6.1 Establishing that $\textsc{Ava}^{+}(e)$ is a supermartingale

We need to show that $\mathbb{E}[\Delta\textsc{Ava}^{+}(e)|\mathcal{H}_i] \leqslant 0$. First we will estimate $\mathbb{E}[\Delta\textsc{Ava}(e)|\mathcal{H}_i]$. Let $\mathcal{F}(e,c,i)$ be the event that c is forbidden at e at step i (this event excludes the possibility that $e_i = e$). Then we have

$$\mathbb{E}[\Delta\textsc{Ava}(e)|\mathcal{H}_i] = -\sum_{c\in\textsc{Ava}(e)} \mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i]. \tag{25}$$

and

$$\mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i] = \frac{(1-t)\binom{n}{2} - 1}{(1-t)\binom{n}{2}} \cdot \mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i \text{ and } e_i \neq e]$$

where conditioning on $\mathcal{H}_i$ and an event $\mathcal{E}$ means that we are given the history of the process up to step i and that we are in the event $\mathcal{E}$.

Fix a color $c \in \textsc{Ava}(e)$. Let $F(e,c)$ be the number of pairs (e',c') such that $e' \neq e$ is uncolored, c' is available at e', and assigning c' to e' would forbid c at e. Then

$$\begin{aligned} \mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i \text{ and } e_i \neq e] &= \sum_{e'\in U_i\setminus\{e\}} \frac{1}{(1-t)\binom{n}{2} - 1} \cdot \mathbb{P}[\mathcal{F}(e,c,i) \mid e_i = e' \text{ and } \mathcal{H}_i] \\ &= \frac{1}{(1-t)\binom{n}{2} - 1} \sum_{e'\in U_i\setminus\{e\}} \frac{|\{c' : (e',c') \in F(e,c)\}|}{\textsc{Ava}(e')} \\ &\geqslant \frac{|F(e,c)|}{\left[(1-t)\binom{n}{2} - 1\right] \cdot (\mathrm{ava} + |C| f_{\textsc{Ava}})} \end{aligned} \tag{26}$$

and so

$$\begin{aligned} \mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i] &\geqslant \frac{|F(e,c)|}{(1-t)\binom{n}{2} \cdot (\mathrm{ava} + |C| f_{\textsc{Ava}})} \\ &\geqslant \left(1 - \frac{f_{\textsc{Ava}}}{a}\right) \frac{|F(e,c)|}{(1-t)\binom{n}{2} \cdot \mathrm{ava}} \end{aligned} \tag{27}$$

where on the last line we used that

$$\frac{1}{1+\frac{f_{\text{AvA}}}{a}} \geqslant 1-\frac{f_{\text{AvA}}}{a}$$

(recall from Observation 21 that $f_{\text{AvA}}/a = o(1)$).

Note that for each pair $(e',c') \in F(e,c)$ there exists at least one $S \subseteq V$ such that coloring e' the color c' at step i would cause c to be forbidden at e through S. We define the following notation.

Definition 27. We let $S_{e,c}(e',c') = S_{e,c}(e',c',i)$ be the family of minimal sets $S \subseteq V$ such that assigning c' to e' at step i would forbid c at e through S.

We would like to claim (with some explicitly bounded error term) that

$$|F(e,c)| = \left|\Big\{(e',c') : (e',c') \in F(e,c)\Big\}\right| \approx \left|\Big\{(S,e',c') : (e',c') \in F(e,c), S \in S_{e,c}(e',c')\Big\}\right|.$$

In other words, we claim that for almost all pairs $(e',c') \in F(e,c)$, $S_{e,c}(e',c')$ is just a singleton and that there are not too many pairs (e',c') where $S_{e,c}(e',c')$ is too large. We have the precise identity

$$\left|\Big\{(S,e',c') : (e',c') \in F(e,c), S \in S_{e,c}(e',c')\Big\}\right| = |F(e,c)| + \sum_{\substack{(e',c')\in F(e,c)\\ |S_{e,c}(e',c')|\geqslant 2}} \Big(|S_{e,c}(e',c')|-1\Big) \tag{28}$$

so we try to bound the sum on the right. First we bound the number of terms, which we denote by B_1. Fix an edge e and a real color c. We want to bound the number of pairs (e',c') such that there exist distinct sets $S_1, S_2 \in S_{e,c}(e',c')$. The number of such pairs (e',c') is at most a constant times the number of possible sets $S = S_1 \cup S_2$ so we will bound the latter instead.

We will use Lemma 42. For $j = 1, 2$ say S_j fits the type y_j on the set of Platonic vertices $\overline{S}_j$, where y_j is a $(\overline{e}, c, \overline{e}', c'')$-preforbidder (here $\overline{e}$ corresponds to e, $\overline{e}'$ corresponds to e', and c'' corresponds to c' (meaning that $c'' = c$ if $c' = c$ or else c'' is Platonic if $c' \neq c$)). Choose y_1, y_2 to be compatible, so $S = S_1 \cup S_2$ fits the type $y_1 \cup y_2$. To bound the number of possible sets $S = S_1 \cup S_2$, we bound $\textsc{Ext}(y_1 \cup y_2, \phi)$ where ϕ is of order 2 (mapping $\overline{e}$ to e). Applying Lemma 42 with $\overline{e}_1 = \overline{e}_2 = \overline{e},\ \overline{e}_3 = \overline{e}',\ c_2 = c,\ c_3 = c''$ we have

$$\textsc{MaxPow}(\overline{S}_1 \cup \overline{S}_2, \overline{e}, y_1 \cup y_2) \leqslant \textsc{Pow}(\overline{S}_1, \overline{e}, y_1) - \frac{1}{\binom{p}{2}-q+1} \leqslant \frac{p-3}{\binom{p}{2}-q+1},$$

where the last inequality follows from Observation 14. Thus by Corollary 25 we have in the good event that

$$|\textsc{Ext}(y_1 \cup y_2, \phi)| = O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\right).$$

Of course $|\textsc{Ext}(y_1 \cup y_2, \phi)|$ only counts the contribution to B_1 due to some fixed y_1, y_2. But since there are only a constant number of choices for the colored portions $\mathrm{col}(y_1), \mathrm{col}(y_2)$ there are only a constant number of relevant types $y_1 \cup y_2$. Thus the number of terms in the sum on line (28) is

$$B_1 = \left|\left\{(e', c') \in F(e,c) : |S_{e,c}(e',c')| \geqslant 2\right\}\right| = O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\right).$$

Now we will bound the maximum possible size of any term in the sum on line (28). Suppose e, e' are fixed real edges and c, c' are fixed real colors. We bound $|S_{e,c}(e',c')|$ by the sum of a constant number of terms of the form $|\textsc{Ext}(y, \phi)|$ where ϕ has domain $\overline{e} \cup \overline{e}'$ sending $\overline{e}$ to e and $\overline{e}'$ to e', and y is a $(\overline{e}, c, \overline{e}', c')$-preforbidder containing no real colors outside of $\{c, c'\}$. Consider any $\overline{S}'$ with $\overline{e} \cup \overline{e}' \subseteq \overline{S}' \subseteq \overline{S}$. If $\overline{S}' \neq \overline{S}$ then $\textsc{Pow}(\overline{S}, \overline{S}', y) < 0$ by Lemma 39. On the other hand if $\overline{S}' = \overline{S}$ then of course $\textsc{Pow}(\overline{S}, \overline{S}', y) = 0$. Altogether we have $\textsc{MaxPow}(\overline{S}, \overline{e} \cup \overline{e}', y) = 0$ and so by Corollary 25 we have

$$|\textsc{Ext}(y, \phi)| \leqslant n^{20p^2\epsilon},$$

and so the maximum possible size of any term in the sum on line (28) is at most B_2 where

$$B_2 = O\left(n^{20p^2\epsilon}\right).$$

We return to our calculation of $\mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i]$ on (27). Since the sum on line (28) is at most $B_1 B_2 = O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+40p^2\epsilon}\right)$ we have

$$|F(e,c)| = \left|\left\{(S, e', c') : (e', c') \in F(e,c), S \in S_{e,c}(e',c')\right\}\right| + O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+40p^2\epsilon}\right). \quad (29)$$

Now for each triple (S, e', c') counted above we have that before we colored e', there was an embedding ϕ with image S fitting a type z such that $(z, \overline{e}', c') \in \textsc{PreFbd}(\overline{e}, c)$, where $\phi(\overline{e}') = e'$. But each such S is counted by at most $\mathrm{equ}(z, \overline{e}') \cdot (\mathrm{ava}(t) + |C| f_{\textsc{Ava}}(t))^{\binom{s}{2}-\ell-2}$ types. So (29) is at least (using $\ell = \ell(z)$, $r = r(z)$, $k = k(z)$, $s = |\overline{S}(z)|$)

$$\begin{aligned} &\sum_{(z,\overline{e}',c^*) \in \textsc{PreFbd}(\overline{e},c)} \frac{|\textsc{Ext}(z,\phi)|}{\mathrm{equ}(z,\overline{e}')(\mathrm{ava} + |C| f_{\textsc{Ava}})^{\binom{s}{2}-\ell-2}} + O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+40p^2\epsilon}\right) \\ \geqslant &\sum_{(z,\overline{e}',c^*) \in \textsc{PreFbd}(\overline{e},c)} \frac{\mathrm{ext}_z - n^{s-2}|C|^{-r-k} f_z}{\mathrm{equ}(z,\overline{e}')(\mathrm{ava} + |C| f_{\textsc{Ava}})^{\binom{s}{2}-\ell-2}} + O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+40p^2\epsilon}\right) \quad (30) \end{aligned}$$

We estimate the sum in (30) by splitting the numerator. For the first term in the numerator we write

$$\begin{aligned}
&\sum_{(z,\overline{e}',c^*)\in \textsc{PreFbd}(\overline{e},c)} \frac{\mathrm{ext}_z}{\mathrm{equ}(z,\overline{e}')(\mathrm{ava}+|C|f_{\textsc{Ava}})^{\binom{s}{2}-\ell-2}} \\
&\geqslant \left(1-p^2\frac{f_{\textsc{Ava}}}{a}\right) \sum_{(z,\overline{e}',c^*)\in \textsc{PreFbd}(\overline{e},c)} \frac{\mathrm{ext}_z}{\mathrm{equ}(z,\overline{e}')\mathrm{ava}^{\binom{s}{2}-\ell-2}} \\
&= \left(1-p^2\frac{f_{\textsc{Ava}}}{a}\right)|C|a\cdot(1-t)h + \tilde{O}\left(n^{-\frac{p-3}{\binom{p}{2}-q+1}}\right) \\
&= |C|a\cdot(1-t)h - p^2|C|(1-t)hf_{\textsc{Ava}} + \tilde{O}\left(n^{-\frac{p-3}{\binom{p}{2}-q+1}}\right). \qquad (31)
\end{aligned}$$

Indeed, on the second line we used that since $f_{\textsc{Ava}}/a = o(1)$ for all $t \leqslant t_{max}$, we have that for any $\ell \geqslant 0$

$$\frac{1}{\left(1+\frac{f_{\textsc{Ava}}}{a}\right)^{\binom{s}{2}-2-\ell}} \geqslant \frac{1}{\left(1+\frac{f_{\textsc{Ava}}}{a}\right)^{\binom{p}{2}-2}} \geqslant 1-p^2\frac{f_{\textsc{Ava}}}{a}.$$

On the third line we have replaced the expression from (6) with (11). Now for the second term in the numerator of (30) we recall (see (8)) that all terms have $r+k = R(s) \geqslant \frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2} - 1$ so we have

$$\begin{aligned}
n^{s-2}|C|^{-r-k}f_z &\leqslant n^{s-2}|C|^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}+1}f_z \\
&= n^{s-2}|C|\left[\kappa n^{\frac{p-2}{\binom{p}{2}-q+1}}\log^{-\frac{1}{\binom{p}{2}-q+1}} n\right]^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}} \cdot a(t)^{\binom{s}{2}-1-\ell}f_{\textsc{Ext}} \\
&\leqslant \kappa^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}}|C|\log n\cdot a(t)^{\binom{s}{2}-1-\ell}f_{\textsc{Ext}}
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{(z,\overline{e}',c^*)\in \textsc{PreFbd}(\overline{e},c)} \frac{n^{s-2}|C|^{-r-k} f_z}{\mathrm{equ}(z,\overline{e}')(\mathrm{ava}+|C|f_{\textsc{Ava}})^{\binom{s}{2}-\ell-2}} \\
&\leqslant \sum_{(z,\overline{e}',c^*)\in \textsc{PreFbd}(\overline{e},c)} \frac{\kappa^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}} |C|\log n \cdot a(t)^{\binom{s}{2}-1-\ell} f_{\textsc{Ext}}}{\mathrm{equ}(z,\overline{e}')\mathrm{ava}^{\binom{s}{2}-\ell-2}} \\
&= \sum_{\substack{3\leqslant s\leqslant p \\ 1\leqslant \ell \leqslant \binom{s}{2}-2 \\ 1\leqslant \eta\leqslant s!|\overline{C}|!}} \mu(\ell,\eta,s)|C|^{\binom{s}{2}-\ell-2}\cdot \frac{\kappa^{-\frac{(s-2)\left(\binom{p}{2}-q+1\right)}{p-2}} |C|\log n \cdot a(t)^{\binom{s}{2}-1-\ell} f_{\textsc{Ext}}}{\eta \mathrm{ava}^{\binom{s}{2}-\ell-2}} \qquad (32)\\
&\leqslant |C|\log n\cdot a(t) f_{\textsc{Ext}}\cdot \kappa^{-\frac{\binom{p}{2}-q+1}{p-2}} \sum_{\substack{3\leqslant s\leqslant p \\ 1\leqslant \ell \leqslant \binom{s}{2}-2 \\ 1\leqslant \eta\leqslant s!|\overline{C}|!}} \frac{\mu(\ell,\eta,s)}{\eta} \\
&\leqslant \epsilon |C|\log n\cdot a(t) f_{\textsc{Ext}}. \qquad (33)
\end{aligned}$$

where (32) follows from Claim 19, and (33) follows from (17). Putting (31) and (33) together, line (30) is at least

$$|C|a(t)(1-t)h(t) - p^2|C|(1-t)h(t)f_{\textsc{Ava}} - \epsilon|C|\log n\cdot a(t) f_{\textsc{Ext}} + O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+40p^2\epsilon}\right). \qquad (34)$$

Returning to line (27), we see that

$$\begin{aligned}
&\mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i] \\
&\geqslant \frac{1-\frac{f_{\textsc{Ava}}}{a}}{(1-t)\binom{n}{2}\cdot \mathrm{ava}} \\
&\cdot \left[|C|a(t)(1-t)h(t) - p^2|C|(1-t)h(t)f_{\textsc{Ava}} - \epsilon|C|\log n\cdot a(t) f_{\textsc{Ext}} + O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+40p^2\epsilon}\right)\right] \\
&\geqslant \frac{1-\frac{f_{\textsc{Ava}}}{a}}{\binom{n}{2}} \left[h(t) - p^2\epsilon\log n\cdot \frac{f_{\textsc{Ava}}}{a} - \epsilon \log n\cdot \frac{f_{\textsc{Ext}}}{1-t} + O\left(n^{-\frac{1}{\binom{p}{2}-q+1}+50p^2\epsilon}\right)\right] \\
&\geqslant \frac{1}{\binom{n}{2}} \left[h(t) - 2p^2\epsilon\log n\cdot \frac{f_{\textsc{Ava}}}{a} - \epsilon \log n\cdot \frac{f_{\textsc{Ext}}}{1-t}\right] + O\left(n^{-2-\frac{1}{\binom{p}{2}-q+1}+50p^2\epsilon}\right), \qquad (35)
\end{aligned}$$

where the last line follows since the extra $p^2\epsilon\log n\cdot \frac{f_{\textsc{Ava}}}{a}$ term absorbs $\frac{f_{\textsc{Ava}}}{a}\cdot h(t)$ (recall $h(t)\leqslant \epsilon\log n$), and also absorbs the $f_{\textsc{Ava}}^2$ and $f_{\textsc{Ava}}\cdot f_{\textsc{Ext}}$ terms by Observation 21.

Similarly, one can obtain an asymptotically matching upper bound on $\mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i]$. Indeed, to do so we would start with an upper bound analogous to (26), but using $(\mathrm{ava}-|C|f_{\textsc{Ava}})$ in place of $(\mathrm{ava}+|C|f_{\textsc{Ava}})$ to make the inequality go in the right direction.

Analogously to (27) we would have $\mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i] \leqslant \left(1+\frac{2f_{\text{Ava}}}{a}\right)\frac{|F(e,c)|}{(1-t)\binom{n}{2}\cdot \text{ava}}$. After that we would obtain an upper bound on $|F(e,c)|$ analogous to (34), i.e.

$$\begin{aligned}&|F(e,c)|\\ &\leqslant |C|a(t)(1-t)h(t)+p^2|C|(1-t)h(t)f_{\text{Ava}}+\epsilon|C|\log n\cdot a(t)f_{\text{Ext}}+O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+40p^2\epsilon}\right).\end{aligned}$$

Of course, this involves using bounds on several of our random variables, and in each case we use whichever bound makes the inequality correct. In the calculations, this essentially just swaps some "+" and "−" signs. Bounding error terms similarly to (35), we would obtain the upper bound

$$\begin{aligned}\mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i] &\leqslant \frac{1}{\binom{n}{2}}\left[h(t)+2p^2\epsilon\log n\cdot\frac{f_{\text{Ava}}}{a}+\epsilon\log n\cdot\frac{f_{\text{Ext}}}{1-t}\right]+O\left(n^{-2-\frac{1}{\binom{p}{2}-q+1}+50p^2\epsilon}\right).\\ &\leqslant \frac{2\epsilon\log n}{\binom{n}{2}}, \end{aligned}\tag{36}$$

which we will need later. Note that for the last line we have used $h(t)\leqslant \epsilon\log n$ and Observation 21.

Returning to line (25) we see that

$$\begin{aligned}&\mathbb{E}[\Delta\text{Ava}(e)|\mathcal{H}_i] = -\sum_{c\in\text{Ava}(e)}\mathbb{P}[\mathcal{F}(e,c,i)|\mathcal{H}_i]\\ &\leqslant -\Big(\text{ava}(t)-|C|f_{\text{Ava}}(t)\Big)\\ &\cdot\left\{\frac{1}{\binom{n}{2}}\left[h(t)-2p^2\epsilon\log n\cdot\frac{f_{\text{Ava}}}{a}-\epsilon\log n\cdot\frac{f_{\text{Ext}}}{1-t}\right]+O\left(n^{-2-\frac{1}{\binom{p}{2}-q+1}+50p^2\epsilon}\right)\right\}\\ &\leqslant -\frac{|C|}{\binom{n}{2}}\left[a(t)h(t)-3p^2\epsilon\log n\cdot f_{\text{Ava}}-\epsilon\log n\cdot\frac{a(t)f_{\text{Ext}}}{1-t}\right]+O\left(n^{-2+\frac{p-3}{\binom{p}{2}-q+1}+50p^2\epsilon}\right),\end{aligned}\tag{37}$$

where the last line follows from multiplying by the factor $(\text{ava}(t)-|C|f_{\text{Ava}}(t)) = |C|a(t)\left(1-\frac{f_{\text{Ava}}}{a}\right)$ from the line above, using the extra $p^2\epsilon\log n\cdot f_{\text{Ava}}$ to absorb $\frac{f_{\text{Ava}}}{a}\cdot h(t)$ and the f_{Ava}^2 and $f_{\text{Ava}}\cdot f_{\text{Ext}}$ terms (see Observation 21).

Recall that by Taylor's theorem, for any twice differentiable function $g(t)$ we have that for some $\tau\in[t,t+\Delta t]$,

$$g(t+\Delta t)-g(t) = g'(t)\Delta t+\frac{1}{2}g''(\tau)(\Delta t)^2.$$

We will apply the above to the function $\text{ava}(t)+|C|f_{\text{Ava}}(t) = |C|(a(t)+f_{\text{Ava}}(t))$ with $\Delta t = 1/\binom{n}{2}$ (the change in t from step i to $i+1$). Note that

$$a'(t) = -a(t)h(t), \qquad a''(t) = a(t)h(t)^2-a(t)h'(t).$$

Since $a(t) = O(1)$ and $h(t), h'(t) = O(\log n)$ (see (12)), we have $a''(t) = O(\log^2 n)$ for all $0 \leqslant t \leqslant t_{\max}$. Using (19) we have

$$f'_{\text{AVA}}(t) = n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}\left(5p^2(1-t)^{-5p^2-1}e^{10p^4\epsilon t\log n} + 10p^4\epsilon\log n\cdot(1-t)^{-5p^2}e^{10p^4\epsilon t\log n}\right)$$
$$\geqslant n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}\cdot 10p^4\epsilon\log n\cdot(1-t)^{-5p^2}e^{10p^4\epsilon t\log n} \tag{38}$$

Also, for all $1 \leqslant t \leqslant t_{max}$ we have

$$\begin{aligned} f''_{\text{AVA}}(t) = n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}&\Big[5p^2(5p^2+1)(1-t)^{-5p^2-2}e^{10p^4\epsilon t\log n} \\ &+ 100p^6\epsilon\log n(1-t)^{-5p^2-1}e^{10p^4\epsilon t\log n} + 100p^8\epsilon^2\log^2 n(1-t)^{-5p^2}e^{10p^4\epsilon t\log n}\Big] \\ &\leqslant n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon} O\left(n^{(10p^4+5p^2+2)\epsilon}\log^2 n\right) \\ &= o(1), \end{aligned}$$

where on the last line we have used that since $\epsilon = 10^{-3}p^{-6}$ and $\frac{1/2}{\binom{p}{2}-q+1} \geqslant \frac{1}{2}p^{-2}$ we have that the power of n is

$$-\frac{1/2}{\binom{p}{2}-q+1} + 20p^2\epsilon + (10p^4+5p^2+2)\epsilon \leqslant \frac{-490p^4+25p^2+2}{1000p^6} < 0.$$

Thus Taylor's theorem gives us that

$$\Delta\left[|C|(a(t)+f_{\text{AVA}}(t))\right] = \frac{|C|}{\binom{n}{2}}\left[-a(t)h(t)+f'_{\text{AVA}}(t)\right] + O\left(\frac{|C|\log^2 n}{n^4}\right). \tag{39}$$

Using the above and (37) we get

$$\begin{aligned} &\mathbb{E}[\Delta\text{AVA}(e)^+|\mathcal{H}_i] \\ &\quad\leqslant -\frac{|C|}{\binom{n}{2}}\left[a(t)h(t) - 3p^2\epsilon\log n\cdot f_{\text{AVA}} - \epsilon\log n\cdot\frac{a(t)f_{\text{EXT}}}{1-t}\right] - \frac{|C|}{\binom{n}{2}}\left[-a(t)h(t)+f'_{\text{AVA}}(t)\right] \\ &\qquad + O\left(n^{-2+\frac{p-3}{\binom{p}{2}-q+1}+50p^2\epsilon}\right) \\ &\quad= \frac{|C|}{\binom{n}{2}}\left[3p^2\epsilon\log n\cdot f_{\text{AVA}} + \epsilon\log n\cdot\frac{a(t)}{1-t}f_{\text{EXT}} - f'_{\text{AVA}}(t)\right] + O\left(n^{-2+\frac{p-3}{\binom{p}{2}-q+1}+50p^2\epsilon}\right). \end{aligned} \tag{40}$$

Finally, we verify that we have chosen $f_{\text{AVA}}, f_{\text{EXT}}$ that make (40) negative. Indeed, using (19), (20) and (38) we have for all $0 \leqslant t \leqslant t_{max}$,

$$\begin{aligned}
&3p^2\epsilon \log n \cdot f_{\text{AVA}} + \epsilon \log n \cdot \frac{a(t)}{1-t} f_{\text{EXT}} - f'_{\text{AVA}}(t) \\
&\leqslant n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon} \Bigg(3p^2\epsilon \log n \cdot (1-t)^{-5p^2} e^{10p^4\epsilon t \log n} + \epsilon \log n \cdot (1-t)^{-5p^2} e^{10p^4\epsilon t \log n} \\
&\qquad\qquad - 10p^4\epsilon \log n (1-t)^{-5p^2} e^{10p^4\epsilon t \log n} \Bigg) \\
&= \left(-10p^4 + 3p^2 + 1\right) \epsilon n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon} \log n \cdot (1-t)^{-5p^2} e^{10p^4\epsilon t \log n} \\
&= -\tilde{\Omega}\left(n^{-\frac{1/2}{\binom{p}{2}-q+1}+(10p^4+25p^2)\epsilon} \right).
\end{aligned}$$

Therefore (40) is at most

$$-\tilde{\Omega}\left(n^{-2+\frac{p-5/2}{\binom{p}{2}-q+1}+(10p^4+25p^2)\epsilon} \right) + O\left(n^{-2+\frac{p-3}{\binom{p}{2}-q+1}+50p^2\epsilon} \right) < 0.$$

The last inequality follows because the first power of n is larger than the second. Indeed, if we subtract the second exponent from the first, and use the fact that $\binom{p}{2} - q + 1 \leqslant p^2$, we have

$$\frac{1/2}{\binom{p}{2} - q + 1} + (10p^4 - 25p^2)\epsilon \geqslant \frac{1}{2p^2} + \frac{10p^4 - 25p^2}{1000p^6} = \frac{102p^2 - 5}{200p^4} > 0.$$

6.2 Applying Freedman's inequality to $\text{AVA}^+(e)$

First we bound $|\Delta\text{AVA}(e)|$. Suppose at step i that the edge e_i receives the color c_i causing some color to be forbidden at e through S. Then S fits some colors-only type z that is a $(\overline{e}, \overline{c}, \overline{e}', c_i)$-preforbidder (where $\overline{e}, \overline{e}'$ correspond to e, e_i, and $\overline{c}$ corresponds to the color being forbidden). Then by Lemma 39 we have that $\text{MAXPOW}(\overline{S}(z), \overline{e} \cup \overline{e}', z) \leqslant \frac{p-3}{\binom{p}{2}-q+1}$ and so by Corollary 25, if ϕ is the partial embedding on $\overline{e} \cup \overline{e}'$ mapping $\overline{e}$ to e and $\overline{e}'$ to e_i, we have $\text{EXT}(z, \phi) = O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon} \right)$. Since each extension in $\text{EXT}(z, \phi)$ gives a constant number of colors forbidden at e, and since there are a constant number of choices for the preforbidder z, we have $|\Delta\text{AVA}(e)| = O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon} \right)$. Also the one-step change in the deterministic part of $\text{AVA}^+(e)$ is $|\Delta(\text{ava} + |C| f_{\text{AVA}})| = o(1)$ (see (39)) so we have $|\Delta\text{AVA}(e)^+| \leqslant O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon} \right)$. Thus in our application of Lemma 23, we will use $D = O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\right)$. Also we have

$$\begin{aligned}\mathbf{Var}[\Delta\text{AVA}^+(e,k)|\mathcal{H}_k] = \mathbf{Var}[\Delta\text{AVA}(e,k)|\mathcal{H}_k] &\leqslant \mathbb{E}[(\Delta\text{AVA}(e,k))^2\,|\mathcal{H}_k] \\ &\leqslant \mathbb{E}\left[n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\cdot|\Delta\text{AVA}(e,k)||\mathcal{H}_k\right] \\ &\leqslant n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\cdot\frac{2|C|\epsilon\log n}{\binom{n}{2}} \\ &\leqslant \frac{n^{\frac{2p-5}{\binom{p}{2}-q+1}+30p^2\epsilon}}{\binom{n}{2}}\end{aligned}$$

where we have used lines (25) and (36). So for our application of Freedman's inequality we have for all $i \leqslant i_{max} < \binom{n}{2}$ that

$$V(i) = \sum_{0\leqslant k\leqslant i}\mathbf{Var}[\Delta\text{AVA}^+(e,k)|\mathcal{H}_k] \leqslant n^{\frac{2p-5}{\binom{p}{2}-q+1}+30p^2\epsilon},$$

so we will use $b = n^{\frac{2p-5}{\binom{p}{2}-q+1}+30p^2\epsilon}$. Note that at the beginning of the process we have $\text{AVA}^+(e,0) = \text{AVA}(e,0) - \text{ava}(0) - |C|f_{\text{AVA}}(0) = -|C|f_{\text{AVA}}(0)$. For $\text{AVA}^+(e,i)$ to become positive would therefore be a positive change of $\lambda := |C|f_{\text{AVA}}(0) = \tilde{\Theta}\left(n^{\frac{p-5/2}{\binom{p}{2}-q+1}+20p^2\epsilon}\right)$. Freedman's inequality gives us a failure probability of at most

$$\exp\left(-\frac{\lambda^2}{2(b+D\lambda)}\right) \leqslant \exp\left(-\tilde{\Omega}\left(n^{10p^2\epsilon}\right)\right)$$

which beats any polynomial union bound.

7 Dynamic concentration of $\text{EXT}(z,\phi)$

In this section we bound the probability that $\mathcal{E}_{i_{max}}$ fails due to Condition (ii). Let $(z,\overline{e})$ be a rooted trackable type on $\overline{S} = \overline{S}(z)$, let $s := |\overline{S}|$ and let ϕ have domain $\overline{e}$ (so ϕ has order 2), where $\phi(\overline{e}) = e$. Let

$$\begin{aligned}\text{EXT}^{\pm}(z,\phi) &= \text{EXT}^{\pm}(z,\phi,i) \\ &:= \begin{cases}\text{EXT}(z,\phi) - \text{ext}_z \mp n^{s-2}|C|^{-r-k}f_z & \text{if } \mathcal{E}_{i-1} \text{ holds and } e\in U_i,\\ \text{EXT}^{\pm}(z,\phi,i-1) & \text{otherwise.}\end{cases}\end{aligned}$$

We will show that $\textsc{Ext}^+(z,\phi)$ is a supermartingale, i.e. that $\mathbb{E}[\Delta\textsc{Ext}^+(z,\phi)|\mathcal{H}_i] \leqslant 0$. Thus we will need to estimate $\mathbb{E}[\Delta\textsc{Ext}(z,\phi)|\mathcal{H}_i]$. To apply Freedman's inequality we also need to bound $|\Delta\textsc{Ext}(z,\phi)|$ in the good event. We have

$$\Delta\textsc{Ext}(z,\phi) = D_1 - D_2 - D_3,$$

where D_1 is the number of new extensions ϕ' that come into $\textsc{Ext}(z,\phi)$, D_2 is the number of extensions ϕ' that leave $\textsc{Ext}(z,\phi)$ due to edges in $\phi'(\overline{S})$ getting colored, and D_3 is the number of extensions ϕ' that leave $\textsc{Ext}(z,\phi)$ due to colors being forbidden on edges in $\phi'(\overline{S})$ (except for the extensions already counted by D_2). We will handle D_1, D_2, D_3 separately, finding the expected change and maximum possible change for each one. Estimating $\mathbb{E}[\Delta\textsc{Ext}^+(z,\phi)|\mathcal{H}_i]$ will require us to estimate the one-step change in the deterministic function $\mathrm{ext}_z(t)$. Recall from (5) that

$$\mathrm{ext}_z(t) = n^{s-2}|C|^{-r-k}t^\ell(1-t)^{\binom{s}{2}-\ell-1}a(t)^{\binom{s}{2}-\ell-1}$$

and its derivative with respect to t is (using the product rule and $a' = -ah$)

$$\mathrm{ext}_z'(t) = n^{s-2}|C|^{-r-k}(g_1(t) - g_2(t) - g_3(t))$$

where

$$g_1(t) := \ell t^{\ell-1}(1-t)^{\binom{s}{2}-\ell-1}a(t)^{\binom{s}{2}-\ell-1} \tag{41}$$

$$g_2(t) := \left(\binom{s}{2} - \ell - 1\right)t^\ell(1-t)^{\binom{s}{2}-\ell-2}a(t)^{\binom{s}{2}-\ell-1} \tag{42}$$

$$g_3(t) := \left(\binom{s}{2} - \ell - 1\right)t^\ell(1-t)^{\binom{s}{2}-\ell-1}a(t)^{\binom{s}{2}-\ell-1}h(t). \tag{43}$$

We discuss a little motivation for the coming calculations. We will show that the expected one-step change in $\textsc{Ext}(z,\phi)$ is approximately equal to the one-step change in $\mathrm{ext}_z(t)$. Note that

$$\mathbb{E}[\Delta\textsc{Ext}(z,\phi)|\mathcal{H}_i] = \mathbb{E}[D_1|\mathcal{H}_i] - \mathbb{E}[D_2|\mathcal{H}_i] - \mathbb{E}[D_3|\mathcal{H}_i],$$

and the three terms above naturally correspond to the three terms in $\mathrm{ext}_z'(t)$. In particular, we will show that for $1 \leqslant j \leqslant 3$,

$$\mathbb{E}[D_j \mid \mathcal{H}_i] \approx \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}g_j(t), \tag{44}$$

where $\approx$ will be made rigorous. Note that we have

$$g_1(t), g_2(t) \leqslant p^2 a(t)^{\binom{s}{2}-\ell-1}, \quad g_3(t) \leqslant p^2 a(t)^{\binom{s}{2}-\ell-1}h(t) \tag{45}$$

We will now use Taylor's theorem to estimate the one-step change in $\mathrm{ext}_z(t)$. First let us give a crude bound on the second derivative of $\mathrm{ext}_z(t)$. Recall from (12) that $h(t)$ is a polynomial in t, whose coefficients only depend on n in that they have factors of the form $\log^{\frac{s-2}{p-2}} n$ for $s \leqslant p$. Then,

$$\frac{d}{dt}\left[\frac{\mathrm{ext}_z(t)}{n^{s-2}|C|^{-r-k}}\right] = \frac{d}{dt}\left[t^{\ell}(1-t)^{\binom{s}{2}-\ell-1}a(t)^{\binom{s}{2}-\ell-1}\right] = P_1(t)a(t)^{\binom{s}{2}-\ell-1}$$

where $P_1(t)$ is a polynomial in t with coefficients that may have factors $\log^{\frac{s-2}{p-2}} n \leqslant \log n$. Taking another derivative with respect to t, we get

$$\frac{d^2}{dt^2}\left[\frac{\mathrm{ext}_z(t)}{n^{s-2}|C|^{-r-k}}\right] = P_2(t)a(t)^{\binom{s}{2}-\ell-1},$$

where P_2 is again a polynomial in t, and the largest power of $\log n$ that shows up as a coefficient is $\log^2 n$. Therefore, since $P_2/\log^2 n$ and a are bounded when $0 \leqslant t \leqslant 1$,

$$\mathrm{ext}_z''(t) = \tilde{O}\left(n^{s-2}|C|^{-r-k}\right).$$

Taylor's theorem then gives us that from step i to $i+1$, the change in $\mathrm{ext}_z(t)$ is

$$\begin{aligned}\Delta[\mathrm{ext}_z(t)] &= \mathrm{ext}'(t)\frac{1}{\binom{n}{2}} + \tilde{O}\left(\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}^2}\right)\\ &= \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}(g_1(t)-g_2(t)-g_3(t)) + \tilde{O}\left(n^{\mathrm{Pow}(\overline{S},\overline{e},z)-4}\right),\end{aligned} \tag{46}$$

where on the last line we used that by Definition 9,

$$\mathrm{Pow}(\overline{S},\overline{e},z) = |\overline{S}\setminus\overline{e}| - \frac{p-2}{\binom{p}{2}-q+1}\mathrm{Coi}(\overline{S},\overline{e},z) = s-2-\frac{p-2}{\binom{p}{2}-q+1}(r+k). \tag{47}$$

Now we will similarly use Taylor's theorem to estimate $\Delta[f_z(t)]$. Recall from (20) and (21) that

$$f_z(t) = n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}(1-t)^{-5p^2+1}a(t)^{\binom{s}{2}-2-\ell}e^{10p^4\epsilon t\log n}.$$

Then we can see that

$$f_z''(t) = n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}(1-t)^{-5p^2-1}Q(t)a(t)^{\binom{s}{2}-2-\ell}e^{10p^4\epsilon t\log n}$$

where $Q(t)$ is some polynomial in t whose only dependence on n is in the form of powers of $\log n$ up to $\log^2 n$. Thus for all $t \leqslant t_{\max} = 1-n^{-\epsilon}$ we have

$$f_z''(t) = \tilde{O}\left(n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon+(5p^2+1)\epsilon+10p^4\epsilon}\right) = o(1).$$

Taylor's theorem then gives us that the change in $f_z(t)$ is

$$\Delta[f_z(t)] = f_z'(t)\frac{1}{\binom{n}{2}} + o\left(\frac{1}{\binom{n}{2}^2}\right) = \frac{1}{\binom{n}{2}}f_z'(t) + o\left(\frac{1}{n^4}\right) \tag{48}$$

Now putting (46) and (48) together we see that

$$\Delta\left[\mathrm{ext}_z + n^{s-2}|C|^{-r-k} f_z\right] = \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}\left[g_1(t) - g_2(t) - g_3(t) + f_z'(t)\right] + \tilde{O}\left(n^{\mathrm{Pow}(\overline{S},\overline{e},z)-4}\right) \tag{49}$$

Having just estimated the one-step change in the deterministic terms in $\textsc{Ext}^+(z,\phi)$, in the coming subsections we will estimate the expectations of D_1, D_2, D_3, formalizing (44). We will also find absolute bounds on those variables which hold in the good event.

7.1 D_1

7.1.1 Estimating $\mathbb{E}[D_1|\mathcal{H}_i]$

For an extension ϕ' counted by D_1, in the previous step of the process the image $\phi'(\overline{S})$ must have fit some type of the form $\textsc{Pre}(z,\overline{e}')$. Each such extension ϕ' counts exactly one way to create a new extension of type z in this step (the edge $\phi'(\overline{e}')$ corresponding to the edge $\overline{e}'$ in z needs a certain color). Let E_C be the set of edges that are colored under z.

We estimate $\mathbb{E}[D_1|\mathcal{H}_i]$ as follows. For each $\overline{e}' \in E_C$, each ϕ' that is of a type $z' \in \textsc{Pre}(z,\overline{e}')$ has a chance of becoming an extension counted by D_1 if on the ith step, $\phi'(\overline{e}')$ is the edge colored, and the color chosen for this edge is compatible with z. Thus, we will write $D_1 \leqslant \sum X(e',c')$, where the sum first goes over all choices of $\overline{e}'$, then over all $z' \in \textsc{Pre}(z,\overline{e}')$, and finally over all $\phi' \in \textsc{Ext}(z',\phi)$, and where, given one such fixed triple $(\overline{e}', z', \phi')$, the edge $e' = \phi'(\overline{e}')$ and we have that $z'(\overline{e}') = (\textsc{available}, c')$, and $X(e',c')$ is the indicator random variable that is 1 if e' gets colored c' on step i (note that we have $\leqslant$ for this expression for D_1 here because even if $\phi'(\overline{e}')$ is assigned a color compatible with z, this assignment could cause a color to be forbidden on a uncolored edge of ϕ', which may result in the extension not being of type z. We will account for this a little later using an error term D_1^-). Then,

$$\mathbb{E}[D_1|\mathcal{H}_i] \leqslant \sum_{\substack{\overline{e}'\in E_C\\ z'\in\textsc{Pre}(z,\overline{e}')\\ \phi'\in\textsc{Ext}(z',\phi)}} \mathbb{P}[X(e',c') \mid \mathcal{H}_i] = \sum_{\substack{\overline{e}'\in E_C\\ z'\in\textsc{Pre}(z,\overline{e}')\\ \phi'\in\textsc{Ext}(z',\phi)}} \frac{1}{\binom{n}{2}(1-t)|\textsc{Ava}(\phi'(\overline{e}'))|} \tag{50}$$

Now let us estimate the above. For a fixed $\overline{e}'$ colored by z, if $z(\overline{e}') = (\textsc{colored}, c)$ for a real or repeated Platonic color c, then $|\textsc{Pre}(z,\overline{e}')| = 1$ and $z' \in \textsc{Pre}(z,\overline{e}')$ has $\ell' = \ell - 1$ colored edges, and it has r' repeats and k' real colors used on colored edges where $r' + k' = r + k - 1$ (we have either lost a repeat or a real color). Therefore the number of terms in (50) corresponding to this fixed $\overline{e}'$ is

$$\textsc{Ext}(z',\phi) \leqslant \mathrm{ext}_{z'}(t) + n^{s-2}|C|^{-r-k+1} f_{z'}(t) = n^{s-2}|C|^{-r-k+1}\left[t^{\ell-1}(1-t)^{\binom{s}{2}-\ell} a(t)^{\binom{s}{2}-\ell} + f_{z'}(t)\right]. \tag{51}$$

Meanwhile, if $z(\overline{e}') = (\textsc{colored}, c)$ for a nonrepeated Platonic color c, then $|\textsc{Pre}(z, \overline{e}')| \leqslant |C|$ and each $z'' \in \textsc{Pre}(z, \overline{e}')$ has $\ell'' = \ell - 1$ colored edges, $r'' = r$ repeats and $k'' = k$ real colors used on colored edges. Therefore the number of terms in (50) corresponding to this fixed $\overline{e}'$ is

$$\sum_{z'' \in \textsc{Pre}(z,\overline{e}')} \textsc{Ext}(z'', \phi) \leqslant |C|(\mathrm{ext}_{z'} + n^{s-2}|C|^{-r-k} f_{z'}(t))$$
$$= n^{s-2}|C|^{-r-k+1} \left[t^{\ell-1}(1-t)^{\binom{s}{2}-\ell} a(t)^{\binom{s}{2}-\ell} + f_{z'}(t) \right] \tag{52}$$

where z' is any element of $\textsc{Pre}(z, \overline{e}')$, noting that the values of $\mathrm{ext}_{z'}$ and $f_{z'}$ is the same regardless of which one we choose. Note that $f_{z'}$ in line (51) really is the same as $f_{z'}$ in line (52) (see (21)), and so in either case (51) and (52) are equal, so the number of terms in (50) is at most $|E_C| = \ell$ times the expression in (51). Therefore line (50) tells us that

$$\begin{aligned}
\mathbb{E}[D_1|\mathcal{H}_i] &\leqslant \ell \cdot \frac{n^{s-2}|C|^{-r-k+1} \left[t^{\ell-1}(1-t)^{\binom{s}{2}-\ell} a(t)^{\binom{s}{2}-\ell} + f_{z'}(t) \right]}{\binom{n}{2}(1-t)(\mathrm{ava}(t) - |C| f_{\textsc{Ava}}(t))} \\
&= \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} \left[\frac{\ell t^{\ell-1}(1-t)^{\binom{s}{2}-\ell-1} a(t)^{\binom{s}{2}-\ell-1} + \frac{\ell f_{z'}(t)}{(1-t)a(t)}}{1 - \frac{f_{\textsc{Ava}}(t)}{a(t)}} \right] \\
&\leqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} \left[\left(1 + 2\frac{f_{\textsc{Ava}}(t)}{a(t)} \right) \ell t^{\ell-1}(1-t)^{\binom{s}{2}-\ell-1} a(t)^{\binom{s}{2}-\ell-1} + \frac{2\ell f_{z'}(t)}{(1-t)a(t)} \right]
\end{aligned} \tag{53}$$

where the last line follows from the inequalities

$$\frac{1}{1 - \frac{f_{\textsc{Ava}}(t)}{a(t)}} \leqslant 1 + \frac{2 f_{\textsc{Ava}}(t)}{a(t)} \leqslant 2.$$

Now, using (41), we can write (53) as

$$\begin{aligned}
&\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} \left[\left(1 + 2\frac{f_{\textsc{Ava}}(t)}{a(t)} \right) g_1(t) + \frac{2\ell f_{z'}(t)}{(1-t)a(t)} \right] \\
&\leqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} \left[g_1(t) + 2p^2 a(t)^{\binom{s}{2}-\ell-2} f_{\textsc{Ava}}(t) + \frac{2\ell a(t)^{\binom{s}{2}-\ell-1} f_{\textsc{Ext}}(t)}{(1-t)} \right],
\end{aligned} \tag{54}$$

where the above inequality follows from (45) and (21).

While the above work will be enough for us to establish that $\textsc{Ext}^+(z, \phi)$ is a supermartingale, there is some more work required to show that $\textsc{Ext}^-(z, \phi)$ is a submartingale. In particular we would also need a lower bound on $\mathbb{E}[D_1|\mathcal{H}_i]$. The bound given above overcounts situations in which an edge is colored in such a way that a clique would become the right type if it was not for the fact that the newly colored edge also caused a color on one of the uncolored edges of that clique to become forbidden. To deal with this, we will let D_1^- be the number of such extensions that do not become type z for that reason. Note that we have

$$\mathbb{E}[D_1 + D_1^- | \mathcal{H}_i] = \sum_{\substack{\overline{e}' \in E_C \\ z' \in \textsc{Pre}(z, \overline{e}') \\ \phi' \in \textsc{Ext}(z', \phi)}} \frac{1}{\binom{n}{2}(1-t)|A_{\phi'(\overline{e}')}|}$$
$$\geqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} \left[g_1(t) - 2p^2 a(t)^{\binom{s}{2}-\ell-2} f_{\textsc{Ava}}(t) - \frac{2\ell a(t)^{\binom{s}{2}-\ell-1} f_{\textsc{Ext}}(t)}{(1-t)} \right], \tag{55}$$

where the inequality follows from essentially the same work used to derive (54), but applied to a lower bound. Now, to get a lower bound on $E[D_1|\mathcal{H}_i]$, it will suffice to subtract an upper bound on $E[D_1^-|\mathcal{H}_i]$ from (55).

Claim 28. *In the good event we have*

$$\mathbb{E}[D_1^-|\mathcal{H}_i] \leqslant n^{\textsc{Pow}(\overline{S}, \overline{e}, z) - 2 - \frac{1}{\binom{p}{2} - q + 1} + 30p^2\epsilon}. \tag{56}$$

Proof. We intend to use Lemma 42. In order for an extension ϕ' to be counted in D_1^- the set $S_1 = \phi'(\overline{S})$ fits some type $z_1 \in \textsc{Pre}(z, \overline{e}_2)$ which is a predecessor of z rooted at $\overline{e}_1 := \overline{e}$ and $\overline{e}_2$ is some edge colored by z.

Say $z(\overline{e}_2) = (\textsc{colored}, c_2)$, and first consider the case when $|\textsc{Pre}(z, \overline{e}_2)| = 1$, i.e. the case where either c_2 is either real or a Platonic repeated color in z. Then we have only one choice for z_1 and $z_1(\overline{e}_2) = (\textsc{available}, c_2)$. Note that in this case we have $\textsc{Pow}(\overline{S}, \overline{e}, z_1) = \textsc{Pow}(\overline{S}, \overline{e}, z) + \frac{p-2}{\binom{p}{2}-q+1}$ since z_1 has one fewer coincidence than z. There must be some $(\overline{e}_2, c_2, \overline{e}_3, c_3)$-preforbidder z_2 on say $\overline{S}_2$, compatible with z_1, where $z_1(\overline{e}_3) = (\textsc{available}, c_3)$. Indeed, the only real colors that would potentially need to appear on z_2 are c_2, c_3 and possibly one real color that appears on z_1. Note that in this case we must have $\overline{S}_2 \setminus \overline{S}_1 \neq \emptyset$ since if it were the case that $\overline{S}_2 \subseteq \overline{S}_1$, then the fact that $(z_2)_{\overline{e}_2, \overline{e}_3}$ is not legal (see Definition 12 (iii)) would contradict Observation 17 (i).

We bound the contribution to $\mathbb{E}[D_1^-|\mathcal{H}_i]$ from this case as follows. There are a constant number of choices for the predecessor z_1 and the preforbidder z_2. For each fixed choice z_1, z_2 we have by Lemma 42 and Corollary 25 that

$$|\textsc{Ext}(z_1 \cup z_2, \phi)| \leqslant n^{\textsc{Pow}(S, \overline{e}, z_1) - \frac{1}{\binom{p}{2}-q+1} + 20p^2\epsilon} = n^{\textsc{Pow}(S, \overline{e}, z) + \frac{p-3}{\binom{p}{2}-q+1} + 20p^2\epsilon}.$$

Now for a fixed extension $\phi' \in \textsc{Ext}(z_1 \cup z_2, \phi)$ to contribute to D_1^- at step i, note that we need to color a fixed edge a fixed color. Indeed, we are in the case where the predecessor z_1 was formed by uncoloring an edge whose color c_2 was either real or a Platonic repeat. If c_2 is real then the color chosen at step i must be c_2. If c_2 was a Platonic repeat in z then z_1 still has an edge colored c_2, and in this extension ϕ' we must have that c_2 is represented by some fixed real color which must then be the color chosen at step i. Coloring a fixed edge a fixed color has probability at most

$$O\left(\frac{1}{(1-t)\binom{n}{2}|C|a(t)}\right) \leqslant n^{-2-\frac{p-2}{\binom{p}{2}-q+1}+3\epsilon},$$

and so the contribution to $\mathbb{E}[D_1^-|\mathcal{H}_i]$ from this case is at most

$$O\left(n^{\textsc{Pow}(S,\overline{e},z)+\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\cdot n^{-2-\frac{p-2}{\binom{p}{2}-q+1}+3\epsilon}\right) = O\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}\right).$$

Now consider the case where $|\textsc{Pre}(z,\overline{e}_2)| > 1$, i.e. c_2 is a Platonic color that is non-repeated in z. In this case there are $|C| = O\left(n^{\frac{p-2}{\binom{p}{2}-q+1}}\right)$ choices for the predecessor z_1, and for any z_1 we have $\textsc{Pow}(\overline{S},\overline{e},z_1) = \textsc{Pow}(\overline{S},\overline{e},z)$. We still have a constant number of choices for the preforbidder z_2, and for each fixed choice z_1, z_2 we have by Lemma 42 and Corollary 25 that

$$|\textsc{Ext}(z_1\cup z_2,\phi)| \leqslant n^{\textsc{Pow}(S,\overline{e},z_1)-\frac{1}{\binom{p}{2}-q+1}+20p^2\epsilon} = n^{\textsc{Pow}(S,\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+20p^2\epsilon}.$$

Here each extension $\phi' \in \textsc{Ext}(z_1\cup z_2,\phi)$ represents a potential contribution to D_1^- at step i, and for this potential contribution to be realised we need to color a fixed edge a fixed color. Altogether, the contribution to $\mathbb{E}[D_1^-|\mathcal{H}_i]$ from this case is at most

$$\begin{aligned}&O\left(n^{\frac{p-2}{\binom{p}{2}-q+1}}\cdot n^{\textsc{Pow}(S,\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+20p^2\epsilon}\cdot n^{-2-\frac{p-2}{\binom{p}{2}-q+1}+3\epsilon}\right)\\ =&O\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}\right).\end{aligned}$$

Summing the contributions to $\mathbb{E}[D_1^-|\mathcal{H}_i]$ gives the bound (56). □

Subtracting (56) from (55) yields

$$\begin{aligned}\mathbb{E}[D_1|\mathcal{H}_i] \geqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}&\left[g_1(t) - 2p^2a(t)^{\binom{s}{2}-\ell-2}f_{\textsc{Ava}}(t) - \frac{2a(t)^{\binom{s}{2}-\ell-1}f_{\textsc{Ext}}(t)}{(1-t)}\right]\\ &+O\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}\right).\end{aligned}$$

Via a similar calculation, we can show that

$$\begin{aligned}\mathbb{E}[D_1|\mathcal{H}_i] \leqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}&\left[g_1(t) + 2p^2a(t)^{\binom{s}{2}-\ell-2}f_{\textsc{Ava}}(t) + \frac{2a(t)^{\binom{s}{2}-\ell-1}f_{\textsc{Ext}}(t)}{(1-t)}\right]\\ &+O\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}\right).\end{aligned} \tag{57}$$

It will be useful to have a crude upper bound on $\mathbb{E}[D_1|\mathcal{H}_i]$. To that end, using the simple bound $a(t) \leqslant 1$ with (45), we can see that $g_1(t) \leqslant p^2$. Furthermore, using $1-t \geqslant n^{-\epsilon}$, we can see from (19) that $f_{\text{AVA}} \leqslant n^{-\frac{1/2}{\binom{p}{2}-q+1}+(25p^2+10p^4)\epsilon} = o(1)$, and from (20) that $\frac{a(t)\cdot f_{\text{EXT}}}{1-t} \leqslant n^{-\frac{1/2}{\binom{p}{2}-q+1}+(25p^2+10p^4)\epsilon} = o(1)$. Then using (47), we can rewrite (57) as

$$\mathbb{E}[D_1|\mathcal{H}_i] \leqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}\left[p^2+o(1)\right] + O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}\right) \leqslant 2p^2\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}. \tag{58}$$

7.1.2 Bounding D_1

Here we give an absolute bound on how large D_1 can be in the good event. Let $e_i \in \binom{[n]}{2}$. We want to bound D_1 when we color the edge e_i the color c_i. We can bound this by considering predecessors of z.

First let us consider the case where c_i is represented by itself in z (i.e. when $z(\overline{e}') = (\text{COLORED}, c_i)$ where $\overline{e}'$ is the Platonic edge corresponding to e_i). Let z' be the unique element of $\text{Pre}(z, \overline{e}')$. Now, D_1 is bounded by $\text{Ext}(z', \phi')$ where $\overline{D}(\phi') = \overline{e} \cup \overline{e}'$. Towards using Corollary 25, let $\overline{S}'$ be such that $\overline{e} \cup \overline{e}' \subseteq \overline{S}' \subseteq \overline{S}$. Then,

$$\begin{aligned}
\text{Pow}(\overline{S}, \overline{S}', z') &= \text{Pow}(\overline{S}, \overline{e}, z') - \text{Pow}(\overline{S}', \overline{e}, z') \\
&= \text{Pow}(\overline{S}, \overline{e}, z) + \frac{p-2}{\binom{p}{2}-q+1} - \left(\text{Pow}(\overline{S}', \overline{e}, z) + \frac{p-2}{\binom{p}{2}-q+1}\right) \\
&= \text{Pow}(\overline{S}, \overline{e}, z) - \text{Pow}(\overline{S}', \overline{e}, z) \leqslant \text{Pow}(\overline{S}, \overline{e}, z) - \frac{1}{\binom{p}{2}-q+1},
\end{aligned}$$

where the first equality follows from additivity (see Observation 32), then the second follows from the fact that z' is a predecessor of z in which an edge with a real color $\overline{e}' \in \overline{S}' \subseteq \overline{S}$ was uncolored, and then the final inequality since $|\overline{S}'| \geqslant 3$ and z is a trackable type, so $\text{Pow}(\overline{S}', e, z) \geqslant \frac{1}{\binom{p}{2}-q+1}$ by Observation 17 (ii). Thus by Corollary 25, we have that

$$D_1 \leqslant |\text{Ext}(z', \phi')| \leqslant n^{\text{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+20p^2\epsilon}.$$

Now consider the case where $z(\overline{e}') = (\text{COLORED}, \overline{c}')$, where $\overline{c}'$ is a Platonic color corresponding to the real color c_i. Let z'' be the (not necessarily trackable) type that is identical to z except every label of the form $(\text{AVAILABLE}, \overline{c}')$ or $(\text{COLORED}, \overline{c}')$ is replaced with $(\text{AVAILABLE}, c_i)$ or $(\text{COLORED}, c_i)$ respectively. We let z''' be the unique element of $\text{Pre}(z'', \overline{e}')$. Any extension that enters $\text{Ext}(z, \phi)$ at step i must have previously been in $\text{Ext}(z''', \phi')$ where $\overline{D}(\phi') = \overline{e} \cup \overline{e}'$. Thus we want to bound the size of $\text{Ext}(z''', \phi')$, so again, towards using Corollary 25, we let $\overline{S}'$ be such that $\overline{e} \cup \overline{e}' \subseteq \overline{S}' \subseteq \overline{S}$. Then by the construction of z'' and z''' we have

$$\text{Pow}(\overline{S}, \overline{e}, z) = \text{Pow}(\overline{S}, \overline{e}, z'') - \frac{p-2}{\binom{p}{2}-q+1} = \text{Pow}(\overline{S}, \overline{e}, z'''),$$

since going from z to z'' replaces all instances of a Platonic color with a real color that did not appear on z, and that does appear on at least one colored edge of z'' (namely $\overline{e}'$), thus giving us exactly one new coincidence, and then going from z'' to z''' uncolors an edge that was colored a real color, causing exactly one less coincidence to occur. Similarly $\textsc{Pow}(\overline{S}', \overline{e}, z''') = \textsc{Pow}(\overline{S}', \overline{e}, z)$, so

$$\begin{aligned}\textsc{Pow}(\overline{S}, \overline{S}', z''') &= \textsc{Pow}(\overline{S}, \overline{e}, z''') - \textsc{Pow}(\overline{S}', \overline{e}, z''') \\ &= \textsc{Pow}(\overline{S}, \overline{e}, z) - \textsc{Pow}(\overline{S}', \overline{e}, z) \\ &\leqslant \textsc{Pow}(\overline{S}, \overline{e}, z) - \frac{1}{\binom{p}{2} - q + 1},\end{aligned}$$

where the last inequality follows from the fact that $\textsc{Pow}(\overline{S}', \overline{e}, z) \geqslant \frac{1}{\binom{p}{2}-q+1}$ since z is a trackable type and $|\overline{S}'| \geqslant 3$ (see Observation 17 (ii)). Thus we get

$$D_1 \leqslant |\textsc{Ext}(z''', \phi')| \leqslant n^{\textsc{Pow}(\overline{S},\overline{e},z) - \frac{1}{\binom{p}{2}-q+1} + 20p^2\epsilon}.$$

Thus in either case,

$$D_1 \leqslant n^{\textsc{Pow}(\overline{S},\overline{e},z) - \frac{1}{\binom{p}{2}-q+1} + 20p^2\epsilon}. \tag{59}$$

7.2 D_2

7.2.1 Estimating $\mathbb{E}[D_2|\mathcal{H}_i]$

It is easy to see that

$$\begin{aligned}\mathbb{E}[D_2|\mathcal{H}_i] &= \frac{\binom{s}{2} - \ell - 1}{\binom{n}{2}(1-t)} \textsc{Ext}(z, \phi) \\ &\geqslant \frac{\binom{s}{2} - \ell - 1}{\binom{n}{2}(1-t)} \left(n^{s-2}|C|^{-r-k} t^{\ell} (1-t)^{\binom{s}{2}-\ell-1} a(t)^{\binom{s}{2}-\ell-1} - n^{s-2}|C|^{-r-k} f_z \right) \\ &\geqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} \left[g_2(t) - \frac{p^2 a(t)^{\binom{s}{2}-\ell-1} f_{\textsc{Ext}}}{1-t} \right],\end{aligned} \tag{60}$$

where the last line follows from (21), (42) and $\binom{s}{2} - \ell - 1 \leqslant p^2$. Similarly,

$$\mathbb{E}[D_2|\mathcal{H}_i] \leqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} \left[g_2(t) + \frac{p^2 a(t)^{\binom{s}{2}-\ell-1} f_{\textsc{Ext}}}{1-t} \right]. \tag{61}$$

We will need a crude upper bound on $\mathbb{E}[D_2|\mathcal{H}_i]$, so we note that using the simple bound $a(t) \leqslant 1$ and (45), we have that $g_2(t) \leqslant p^2$, and then using $1 - t \geqslant n^{-\epsilon}$, we can see from (20) that $\frac{a(t)\cdot f_{\textsc{Ext}}}{1-t} \leqslant n^{-\frac{1/2}{\binom{p}{2}-q+1}+(25p^2+10p^4)\epsilon} = o(1)$. Then, using this we can rewrite (61) as

$$\mathbb{E}[D_2|\mathcal{H}_i] \leqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}\left[p^2+o(1)\right] \leqslant 2p^2\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}. \tag{62}$$

7.2.2 Bounding D_2

Any extension $\phi' \in \textsc{Ext}(z,\phi)$ that is counted by D_2 must contain both $\phi(\overline{e})$ and the edge e_i that is being colored on this step. Thus, D_2 is at most $\textsc{Ext}(z,\phi'')$, where $\overline{D}(\phi'') = \overline{e}\cup\overline{e}'$ (where $\phi''(\overline{e}') = e_i$). For any such ϕ' and any $\overline{S}'$ with $\overline{D}(\phi'') \subseteq \overline{S}' \subseteq \overline{S}$, we have

$$\textsc{Pow}(\overline{S},\overline{S}',z) = \textsc{Pow}(\overline{S},\overline{e},z) - \textsc{Pow}(\overline{S}',\overline{e},z) \leqslant \textsc{Pow}(\overline{S},\overline{e},z) - \frac{1}{\binom{p}{2}-q+1},$$

where the last inequality follows from Observation 17 (ii) since $\overline{e} \subsetneq \overline{S}'$ and z is a trackable type. Thus, by Corollary 25 we have that

$$D_2 \leqslant |\textsc{Ext}(z,\phi'')| \leqslant n^{\textsc{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+20p^2\epsilon}. \tag{63}$$

7.3 D_3

7.3.1 Estimating $\mathbb{E}[D_3|\mathcal{H}_i]$

Note first that if $\ell = \binom{s}{2}-1$ then $\overline{e}$ is the only edge in $\overline{S}$ that is uncolored, and therefore it is impossible for $\textsc{Ext}(z,\phi)$ to decrease due to a color being forbidden at an edge. In this case we deterministically have $D_3 = 0 = \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}g_3(t)$. For the rest of this subsection assume that $\ell \leqslant \binom{s}{2}-2$.

Recall that $\mathcal{F}(e',c') = \mathcal{F}(e',c',i)$ is the event that the color c' is forbidden at e' at step i. Consider some $\phi' \in \textsc{Ext}(z,\phi)$ and some uncolored edge $e' \neq \phi(\overline{e})$ in $\phi'(\overline{S})$ (where $\overline{S} = \overline{S}(z)$), and let ψ be a color map of ϕ' (see Definition 5). The type z indicates that some color c' is available at $\overline{e}'$ (where $\overline{e}'$ is the Platonic edge such that $\phi'(\overline{e}') = e'$). The color c' might be Platonic, but in that case ϕ' must use some real color $\psi(c')$ in place of c'. To ease the notation, we define the event $\mathcal{F}_{\phi'}(\overline{e}',c') := \mathcal{F}(\phi'(\overline{e}'),\psi(c'))$. Thus, we want to know for which embeddings ϕ' does the event $\mathcal{F}_{\phi'}(\overline{e}',c')$ happen on at least one edge $\overline{e}'$ in $\overline{S}$, but also want to avoid counting any embeddings which end up having an uncolored edge become colored on this step (as those are counted in D_2). We will let D_3^- denote the embeddings ϕ' for which both $\mathcal{F}_{\phi'}(\overline{e}',c')$ happens to at least one edge $\overline{e}' \subseteq \overline{S}$, and an edge in $\phi'(\overline{S})$ is assigned a color on this step. Then $D_3 + D_3^-$ simply counts embeddings removed from $\textsc{Ext}(z,\phi)$ because a color that was originally prescribed as available at an edge was forbidden at an edge (regardless of if an edge in the embedding was also colored).

Thus we have

$$\mathbb{E}[D_3 + D_3^-|\mathcal{H}_i] = \sum_{\phi' \in \textsc{Ext}(z,\phi)} \mathbb{P}\left[\bigcup_{(\overline{e}',c'):z(\overline{e}')=(\textsc{available},c')} \mathcal{F}_{\phi'}(\overline{e}',c')\right].$$

Now we use the simple bounds for any events E_j with $j \in J$ for an index set J,

$$\sum_{j\in J} \mathbb{P}[E_j] \geqslant \mathbb{P}\left[\bigcup_{j\in J} E_j\right] \geqslant \sum_{j\in J} \mathbb{P}[E_j] - \sum_{\substack{j,j'\in J\\ j\neq j'}} \mathbb{P}[E_j \cap E_{j'}],$$

which imply

$$\mathbb{P}\left[\bigcup_{j\in J} E_j\right] = \sum_{j\in J} \mathbb{P}[E_j] + O\left(\sum_{\substack{j,j'\in J\\ j\neq j'}} \mathbb{P}[E_j \cap E_{j'}]\right).$$

Thus we have

$$\mathbb{E}[D_3 + D_3^-|\mathcal{H}_i] = \sum_{\substack{\phi' \in \textsc{Ext}(z,\phi)\\ (\overline{e}',c'):z(\overline{e}')=(\textsc{available},c')}} \mathbb{P}\left[\mathcal{F}_{\phi'}(\overline{e}',c')\right] + O(B_3),$$

where

$$B_3 := \sum_{\substack{\phi' \in \textsc{Ext}(z,\phi)\\ (\overline{e}',c'):z(\overline{e}')=(\textsc{available},c')\\ (\overline{e}'',c''):z(\overline{e}'')=(\textsc{available},c'')\\ \overline{e}'\neq\overline{e}''}} \mathbb{P}\left[\mathcal{F}_{\phi'}(\overline{e}',c') \cap \mathcal{F}_{\phi'}(\overline{e}'',c'')\right].$$

Now we will bound B_3.

Claim 29.

$$B_3 = O\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+40p^2\epsilon}\right).$$

Proof. First we claim that for any e', c', e'', c'' with $e' \neq e''$ there are at most $n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}$ pairs e''', c''' such that assigning c''' to e''' would simultaneously forbid c' at e' and c'' at e''. Indeed, recall Definition 27 and let $S_1 \in S_{e',c'}(e''', c''')$ and consider the following two cases (handled in the following paragraphs): either $e'' \subseteq S_1$, or else $e'' \not\subseteq S_1$ in which case there must be some $S_2 \in S_{e'',c''}(e''', c''')$ with $S_1 \neq S_2$.

Consider first the case where $e'' \subseteq S_1$. The number of possible sets S_1 is bounded by the sum of a constant number of terms of the form $O(\textsc{Ext}(y, \rho))$ where y is a colors-only $(\overline{e}', c', \overline{e}''', c^*)$-preforbidder on $\overline{S}_1$ (where $c^* = c'$ if $c''' = c'$ and otherwise c^* is a Platonic color representing c''') and ρ has domain $\overline{e}' \cup \overline{e}''$. Assume that y is chosen so that no real colors other than possibly c' appear on colored edges. We have by Observation 14 that

$\textsc{Pow}(\overline{S}_1, \overline{e}', y) \leqslant \frac{p-2}{\binom{p}{2}-q+1}$. Also since y is the colored portion of some trackable type rooted at $\overline{e}'$, by Observation 17 (ii) we have $\textsc{Pow}(\overline{S}', \overline{e}', y) \geqslant \frac{1}{\binom{p}{2}-q+1}$ whenever $\overline{e}' \subsetneq \overline{S}' \subseteq \overline{S}_1$. Therefore

$$\textsc{MaxPow}(\overline{S}_1, \overline{e}' \cup \overline{e}'', y) \leqslant \frac{p-3}{\binom{p}{2}-q+1}$$

and so $\textsc{Ext}(y, \rho) = O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\right)$ by Corollary 25. Note that there is a constant number of choices for y, and each element of $\textsc{Ext}(y, \rho)$ represents a constant number of choices for (e''', c'''). Thus the number of choices for (e''', c''') is $O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\right)$.

Now consider the second case, where $e'' \not\subseteq S_1$. We will use Lemma 43. S_1 must fit some colors-only $(\overline{e}', c', \overline{e}''', c^*)$-preforbidder y_1 on $\overline{S}_1$ (where as before, $c^* = c'$ if $c''' = c'$ and otherwise c^* is a Platonic color representing c'''). Likewise S_2 must fit some colors-only $(\overline{e}'', c'', \overline{e}''', c^*)$-preforbidder y_2 on $\overline{S}_2$. Clearly we can choose y_1, y_2 to be compatible. Then the number of possible sets $S_1 \cup S_2$ is bounded by a constant number of terms of the form $\textsc{Ext}(y_1 \cup y_2, \rho)$ where $\overline{D}(\rho) = \overline{e}' \cup \overline{e}''$. Since $e'' \not\subseteq S_1$ we have $\overline{e}'' \not\subseteq \overline{S}_1$. Thus by Lemma 43 and Observation 14 we have

$$\textsc{MaxPow}(\overline{S}_1 \cup \overline{S}_2, \overline{e}' \cup \overline{e}'', y_1 \cup y_2) \leqslant \textsc{Pow}(\overline{S}_1, \overline{e}', y_1) - \frac{1}{\binom{p}{2}-q+1} \leqslant \frac{p-3}{\binom{p}{2}-q+1}.$$

Thus by Corollary 25 we conclude that $\textsc{Ext}(y_1 \cup y_2, \rho) = O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\right)$. Since each possible set $S_1 \cup S_2$ counted by $\textsc{Ext}(y_1 \cup y_2, \rho)$ corresponds to a constant number of choices for (e''', c''') we have that the number of such choices is also $O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\right)$.

Adding together the bounds from the two cases, we get that the total number of choices for pairs (e''', c''') is $O\left(n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}\right)$, so we have

$$\mathbb{P}\left[\mathcal{F}(e', c') \cap \mathcal{F}(e'', c'')\right] \leqslant O\left(\frac{n^{\frac{p-3}{\binom{p}{2}-q+1}+20p^2\epsilon}}{\binom{n}{2}(1-t)|C|a(t)}\right) \leqslant n^{-2-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}.$$

Now since the number of terms in B_3 is at most $O(\textsc{Ext}(z, \phi)) = O\left(n^{s-2}|C|^{-r-k}\right) = \tilde{O}\left(n^{\textsc{Pow}(\overline{S}, \overline{e}, z)}\right)$ (by Condition (ii) from Definition 20 the fact that z is trackable and (47)), we have that

$$B_3 = O\left(n^{\textsc{Pow}(\overline{S}, \overline{e}, z)-2-\frac{1}{\binom{p}{2}-q+1}+40p^2\epsilon}\right).$$

$\square$

Now we bound $\mathbb{E}[D_3^-|\mathcal{H}_i]$.

Claim 30. *In the good event we have*

$$\mathbb{E}[D_3^-|\mathcal{H}_i] = O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{p-2}{\binom{p}{2}-q+1}+30p^2\epsilon}\right).$$

The proof is very similar to that of Claim 28.

Proof. In order for an extension ϕ' to be counted in D_3^- there must be two sets $S_1 = \phi'(\overline{S})$, which fits the type $z_1 = z$ rooted at $\overline{e}_1 := \overline{e}$, and S_2, which fits a preforbidder type. Say S_2 fits the $(\overline{e}_2, c_2, \overline{e}_3, c_3)$-preforbidder type z_2 on $\overline{S}_2$ and compatible with z_1, where $\overline{e}_2$ represents the edge (say e') getting a color forbidden causing ϕ' to be counted in D_3^- and $\overline{e}_3$ represents e_i. First consider the case where $\overline{S}_2 \not\subseteq \overline{S}_1$. For this case we will use Lemma 42. Since $e' \cup e_i \subseteq S_1 \cap S_2$ we have $\overline{e}_2 \cup \overline{e}_3 \subseteq \overline{S}_1 \cap \overline{S}_2$. Thus, if we let $y := z_1 \cup z_2$, applying Lemma 42 we have

$$\text{MaxPow}(\overline{S}_1 \cup \overline{S}_2, \overline{e}, y) \leqslant \text{Pow}(\overline{S}, \overline{e}, z_1) - \frac{1}{\binom{p}{2} - q + 1} = \text{Pow}(\overline{S}, \overline{e}, z) - \frac{1}{\binom{p}{2} - q + 1}.$$

Thus by Corollary 25 we have that

$$|\text{Ext}(y, \phi)| \leqslant O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+20p^2\epsilon}\right).$$

Note there is a constant number of choices for the colored portion of y_2. There is a constant number of relevant types y, and each element of $\text{Ext}(y, \phi)$ represents some constant number of potential ways to get a contribution to D_3^-. Each such way has probability at most $n^{-2-\frac{p-2}{\binom{p}{2}-q+1}+3\epsilon}$. Thus the expected contribution to D_3^- from this case is at most

$$O\left(n^{\text{Pow}(\overline{S},e,z)-\frac{1}{\binom{p}{2}-q+1}+20p^2\epsilon}\right) \cdot n^{-2-\frac{p-2}{\binom{p}{2}-q+1}+3\epsilon} \leqslant n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{p-1}{\binom{p}{2}-q+1}+30p^2\epsilon}$$

which is even smaller than we need.

Now consider the case where $\overline{S}_2 \subseteq \overline{S}_1$. In this case each element of $\text{Ext}(z, \phi)$ represents some constant number of potential ways to get a contribution to D_3^-. Thus the expected contribution to D_3^- from this case is at most

$$O\left(|\text{Ext}(z, \phi)|\right) \cdot n^{-2-\frac{p-2}{\binom{p}{2}-q+1}+3\epsilon} \leqslant n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{p-2}{\binom{p}{2}-q+1}+30p^2\epsilon},$$

where we use Condition (ii) from Definition 20 to bound $|\text{Ext}(z, \phi)|$. ∎

With the claim in hand, we have

$$\mathbb{E}[D_3|\mathcal{H}_i] \geqslant \sum_{\substack{\phi' \in \text{Ext}(z,\phi) \\ (\overline{e}',c'):z(\overline{e}')=(\text{available},c')}} \mathbb{P}\left[\mathcal{F}_{\phi'}(\overline{e}',c')|\mathcal{H}_i\right] + O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+40p^2\epsilon}\right).$$

We then use Condition (ii) from Definition 20 to lower bound the number of terms in the above sum, and (35) to write

$$\begin{aligned}
&\mathbb{E}[D_3|\mathcal{H}_i] \\
&\geqslant \left(\text{ext}_z - n^{s-2}|C|^{-r-k}f_z\right) \cdot \frac{\binom{s}{2}-1-\ell}{\binom{n}{2}} \left[h(t) - 2p^2\epsilon\log n \cdot \frac{f_{\text{Ava}}}{a(t)} - \epsilon\log n \cdot \frac{f_{\text{Ext}}}{1-t}\right] \\
&\qquad + O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+40p^2\epsilon}\right) \\
&= \left(\binom{s}{2}-1-\ell\right)\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} a(t)^{\binom{s}{2}-\ell-1}\left[t^\ell(1-t)^{\binom{s}{2}-\ell-1} - f_{\text{Ext}}\right] \\
&\qquad \cdot \left[h(t) - 2p^2\epsilon a(t)^{-1}\log n \cdot f_{\text{Ava}} - \epsilon\log n \cdot \frac{f_{\text{Ext}}}{1-t}\right] + O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+40p^2\epsilon}\right) \\
&\geqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}\Bigg[g_3(t) - 2p^4\epsilon a(t)^{\binom{s}{2}-\ell-2}(1-t)\log n \cdot f_{\text{Ava}} - 2p^2\epsilon a(t)^{\binom{s}{2}-\ell-1}\log n \cdot f_{\text{Ext}} \\
&\qquad + \tilde{O}\left(f_{\text{Ext}}f_{\text{Ava}} + \frac{f_{\text{Ext}}^2}{1-t}\right)\Bigg] + O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+40p^2\epsilon}\right) \\
&\geqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}\left[g_3(t) - 2p^4\epsilon a(t)^{\binom{s}{2}-\ell-2}(1-t)\log n \cdot f_{\text{Ava}} - 2p^2\epsilon a(t)^{\binom{s}{2}-\ell-1}\log n \cdot f_{\text{Ext}}\right] \\
&\qquad + O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+(20p^4+50p^2+3)\epsilon}\right) \qquad (64)
\end{aligned}$$

On the second-to-last line we used (43), the fact that $h(t) \leqslant \epsilon\log n$, $\binom{s}{2}-1-\ell \leqslant p^2$, and that $\ell \leqslant \binom{s}{2}-2$ so $t^\ell(1-t)^{\binom{s}{2}-\ell-1} \leqslant 1-t$. On the last line we have used that

$$f_{\text{Ext}}f_{\text{Ava}} + \frac{f_{\text{Ext}}^2}{1-t} = O\left(n^{-\frac{1}{\binom{p}{2}-q+1}+(20p^4+50p^2+2)\epsilon}\right)$$

which follows from Observation 21, $a \leqslant 1$ and $1-t \leqslant 1$.

Similarly, using (36) we can write

$$
\begin{aligned}
&\mathbb{E}[D_3|\mathcal{H}_i] \\
&\quad \leqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}\left[g_3(t) + 2p^4\epsilon a(t)^{\binom{s}{2}-\ell-2}(1-t)\log n \cdot f_{\text{Ava}} + 2p^2\epsilon a(t)^{\binom{s}{2}-\ell-1}\log n \cdot f_{\text{Ext}}\right] \\
&\qquad\qquad + O\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+(20p^4+50p^2+3)\epsilon}\right) \qquad (65)
\end{aligned}
$$

We will need a crude upper bound on $\mathbb{E}[D_3|\mathcal{H}_i]$, so using the simple bound $a(t) \leqslant 1$, along with (45) and (16), we have that $g_3(t) \leqslant p^2h(t) \leqslant p^2\epsilon\log n$. Furthermore, using $1-t \geqslant n^{-\epsilon}$, we can see from (19) that $f_{\text{Ava}} \leqslant n^{-\frac{1/2}{\binom{p}{2}-q+1}+(25p^2+10p^4)\epsilon} = o(1)$, and from (20) that $a(t)\cdot f_{\text{Ext}} \leqslant n^{-\frac{1/2}{\binom{p}{2}-q+1}+(25p^2+10p^4)\epsilon} = o(1)$. Using the above bounds, we can rewrite (65) as

$$
\begin{aligned}
\mathbb{E}[D_3|\mathcal{H}_i] &\leqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}\left[p^2\epsilon\log n + o(\log n)\right] + O\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+(20p^4+50p^2+3)\epsilon}\right) \\
&\leqslant 2p^2\epsilon\log n \cdot \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}, \qquad (66)
\end{aligned}
$$

where we have used (47).

7.3.2 Bounding D_3

We describe how to get a contribution to D_3. To get a contribution we must have some extension $\phi' \in \textsc{Ext}(z,\phi)$ whose image is some set $S_1 \supseteq e$ fitting z. Assigning c_i to e_i forbids some color c' at $e' \subseteq S_1$, causing ϕ' to no longer be in $\textsc{Ext}(z,\phi)$. So there is some $S_2 \in \mathcal{S}_{e_i,c_i}(e',c')$ fitting some colors-only $(\overline{e}_2, c_2, \overline{e}_3, c_3)$-preforbidder type y_2 on say $\overline{S}_2$, where $\overline{e}_2, \overline{e}_3$ correspond to e_i and e' respectively, $c_2 = c_i$ and c_3 is either equal to c' if $c' = c_i$ or else c_3 is Platonic representing c'. Assume we choose y_2 compatible with z.

We will use Lemma 43. We use $y_1 = z$, $\overline{S}_1 = \overline{S}$ and $\overline{e}_1 = \overline{e}$ (we have already defined $\overline{S}_2, y_2, \overline{e}_2, \overline{e}_3$). Note that e_i cannot be in S_1 (otherwise we would actually get a contribution to D_3^- instead of D_3), and so $\overline{e}_2 \not\subseteq \overline{S}_1$ as required. Lemma 43 then gives us that

$$
\begin{aligned}
\textsc{MaxPow}(\overline{S}_1 \cup \overline{S}_2, \overline{e}_1 \cup \overline{e}_2, y_1 \cup y_2) &\leqslant \textsc{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \frac{1}{\binom{p}{2}-q+1} \\
&= \textsc{Pow}(\overline{S}, \overline{e}, z) - \frac{1}{\binom{p}{2}-q+1}.
\end{aligned}
$$

Thus, by Corollary 25 we have for ϕ'' with domain $\overline{e}_1 \cup \overline{e}_2$ that

$$
|\textsc{Ext}(y_1 \cup y_2, \phi'')| \leqslant n^{\textsc{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+20p^2\epsilon}.
$$

There are a constant number of choices for y_2 and each element of $\textsc{Ext}(y_1 \cup y_2, \phi'')$ gives a constant sized contribution to D_3. Thus

$$D_3 = O\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+20p^2\epsilon}\right). \tag{67}$$

7.4 Establishing that $\textsc{Ext}^+(z, \phi)$ is a supermartingale

Note that since $f_z(t) = a(t)^{\binom{s}{2}-\ell-1} f_{\textsc{Ext}}(t)$, we have

$$\begin{aligned} f_z'(t) &= a(t)^{\binom{s}{2}-\ell-1} f_{\textsc{Ext}}'(t) - \left(\binom{s}{2} - \ell - 1\right) a(t)^{\binom{s}{2}-\ell-1} h(t) f_{\textsc{Ext}}(t) \\ &\geqslant a(t)^{\binom{s}{2}-\ell-1} \left[f_{\textsc{Ext}}'(t) - p^2\epsilon \log n \cdot f_{\textsc{Ext}}(t)\right], \end{aligned}$$

where we used $\binom{s}{2} - \ell - 1 \leqslant p^2$ and $h(t) \leqslant \epsilon \log n$.

Therefore

$$\begin{aligned} &\mathbb{E}[\Delta\textsc{Ext}^+(z,\phi)|\mathcal{H}_i] \\ &= \mathbb{E}[\Delta\textsc{Ext}(z,\phi)|\mathcal{H}_i] - \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}\left[g_1(t) - g_2(t) - g_3(t) + f_z'(t)\right] + O\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-4+\epsilon}\right) \\ &\leqslant \mathbb{E}[D_1 - D_2 - D_3|\mathcal{H}_i] \\ &\qquad - \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}\left[g_1 - g_2 - g_3 + a(t)^{\binom{s}{2}-\ell-1}\left[f_{\textsc{Ext}}'(t) - p^2\epsilon\log n \cdot f_{\textsc{Ext}}(t)\right]\right] \\ &\qquad + O\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-4+\epsilon}\right), \end{aligned} \tag{68}$$

where the second line uses (49). Now using (54) (and $\ell \leqslant p^2$) we have

$$\begin{aligned} &\mathbb{E}[D_1|\mathcal{H}_i] - \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} g_1 \\ &\qquad \leqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}\left[2p^2 a(t)^{\binom{s}{2}-\ell-2} f_{\textsc{Ava}}(t) + \frac{2p^2 a(t)^{\binom{s}{2}-\ell-1} f_{\textsc{Ext}}(t)}{1-t}\right] \end{aligned} \tag{69}$$

Using (60) we have

$$\mathbb{E}[D_2|\mathcal{H}_i] - \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} g_2 \geqslant -\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} \cdot \frac{p^2 a(t)^{\binom{s}{2}-\ell-1} f_{\textsc{Ext}}}{1-t}. \tag{70}$$

Using (64) we have

$$\begin{aligned}
&\mathbb{E}[D_3|\mathcal{H}_i] - \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} g_3 \\
&\geqslant -\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} \left[2p^4\epsilon a(t)^{\binom{s}{2}-\ell-2}(1-t)\log n \cdot f_{\text{AVA}} + 2p^2\epsilon a(t)^{\binom{s}{2}-\ell-1}\log n \cdot f_{\text{EXT}}\right] \\
&\qquad + O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+(20p^4+50p^2+3)\epsilon}\right). \qquad (71)
\end{aligned}$$

Now to bound line (68) we take (69) minus (70) minus (71) minus

$$\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} a(t)^{\binom{s}{2}-\ell-1}\left[f'_{\text{EXT}}(t) - p^2\epsilon\log n \cdot f_{\text{EXT}}(t)\right] + O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-4+\epsilon}\right).$$

We get

$$\begin{aligned}
\mathbb{E}[\Delta\text{EXT}^+(z,\phi)|\mathcal{H}_i] &\leqslant \frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}} a(t)^{\binom{s}{2}-\ell-1} \\
&\cdot\left[\frac{2p^2+2p^4\epsilon(1-t)\log n}{a(t)} f_{\text{AVA}}(t) + \left(\frac{2p^2}{1-t}+3p^2\epsilon\log n\right) f_{\text{EXT}} - f'_{\text{EXT}}(t)\right] \\
&+ O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+(20p^4+50p^2+3)\epsilon}\right). \qquad (72)
\end{aligned}$$

We verify that we have chosen $f_{\text{AVA}}, f_{\text{EXT}}$ that make the above line negative. Indeed, since $a(t) = e^{-H(t)}$ we have

$$\begin{aligned}
f_{\text{EXT}}(t) &= n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}(1-t)^{-5p^2+1}a(t)^{-1}e^{10p^4\epsilon t\log n} \\
&= n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}(1-t)^{-5p^2+1}e^{10p^4\epsilon t\log n + H(t)}
\end{aligned}$$

and so

$$\begin{aligned}
f'_{\text{EXT}}(t) &= n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}\Big((5p^2-1)(1-t)^{-5p^2}e^{10p^4\epsilon t\log n+H(t)} \\
&\qquad + (1-t)^{-5p^2+1}(10p^4\epsilon\log n + h(t))e^{10p^4\epsilon t\log n+H(t)}\Big) \\
&\geqslant n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}\Big((5p^2-1)(1-t)^{-5p^2}e^{10p^4\epsilon t\log n+H(t)} \\
&\qquad + 10p^4\epsilon\log n\cdot(1-t)^{-5p^2+1}e^{10p^4\epsilon t\log n+H(t)}\Big).
\end{aligned}$$

Thus we have that the expression in brackets on line (72) is

$$\begin{aligned}
&\frac{2p^2+2p^4\epsilon(1-t)\log n}{a(t)} f_{\text{AVA}}(t) + \left(\frac{2p^2}{1-t} + 3p^2\epsilon\log n\right) f_{\text{EXT}} - f'_{\text{EXT}} \\
&\leqslant n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}\Bigg[\Big(2p^2+2p^4\epsilon(1-t)\log n\Big)(1-t)^{-5p^2}e^{10p^4\epsilon t\log n + H(t)} \\
&\qquad + 2p^2(1-t)^{-5p^2}e^{10p^4\epsilon t\log n+H(t)} + 3p^2\epsilon\log n\cdot(1-t)^{-5p^2+1}e^{10p^4\epsilon t\log n+H(t)} \\
&\qquad - (5p^2-1)(1-t)^{-5p^2}e^{10p^4\epsilon t\log n+H(t)} - 10p^4\epsilon\log n\cdot(1-t)^{-5p^2+1}e^{10p^4\epsilon t\log n+H(t)}\Bigg] \\
&= n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}(1-t)^{-5p^2}e^{10p^4\epsilon t\log n+H(t)}\Bigg[2p^2+2p^4\epsilon(1-t)\log n + 2p^2 \\
&\qquad\quad + 3p^2\epsilon\log n\cdot(1-t) - (5p^2-1) - 10p^4\epsilon\log n\cdot(1-t)\Bigg] \\
&= n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}(1-t)^{-5p^2}e^{10p^4\epsilon t\log n+H(t)}\Big[1-p^2+\big(3p^2-8p^4\big)\,\epsilon(1-t)\log n\Big] \\
&= -\Omega\left(n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}\right).
\end{aligned}$$

Thus, using $a \geqslant n^{-\epsilon}$, $\binom{s}{2}-\ell-1 \leqslant p^2$ and (47), line (72) becomes

$$\begin{aligned}
&-\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}a(t)^{\binom{s}{2}-\ell-1}\Omega\left(n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}\right) + O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+(20p^4+50p^2+3)\epsilon}\right) \\
&= -\tilde{\Omega}\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1/2}{\binom{p}{2}-q+1}+10p^2\epsilon}\right) + O\left(n^{\text{Pow}(\overline{S},\overline{e},z)-2-\frac{1}{\binom{p}{2}-q+1}+(20p^4+50p^2+3)\epsilon}\right) \\
&< 0.
\end{aligned}$$

7.5 Applying Freedman's inequality to $\textsc{Ext}^+(z,\phi)$

Note that by (59), (63), and (67) we have

$$|\Delta\textsc{Ext}(z,\phi)| = |D_1 - D_2 - D_3| \leqslant D_1 + D_2 + D_3 \leqslant n^{\text{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}.$$

Thus in our application of Lemma 23, we will use $D = n^{\text{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}$. Also we have

$$\begin{aligned}
\mathbf{Var}[\Delta\textsc{Ext}^+(z,\phi)|\mathcal{H}_k] &= \mathbf{Var}[\Delta\textsc{Ext}(z,\phi)|\mathcal{H}_k]\\
&\leqslant \mathbb{E}[(\Delta\textsc{Ext}(z,\phi))^2\,|\mathcal{H}_k]\\
&\leqslant \mathbb{E}\left[n^{\textsc{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}\cdot|\Delta\textsc{Ext}(z,\phi)|\big|\mathcal{H}_k\right]\\
&\leqslant n^{\textsc{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}\cdot\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}6p^2\epsilon\log n\\
&= \frac{\tilde{O}\left(n^{2\textsc{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}\right)}{\binom{n}{2}},
\end{aligned}$$

where the last inequality follows from using (58), (62) and (66) to write

$$\mathbb{E}\left[|\Delta\textsc{Ext}(z,\phi)|\big|\mathcal{H}_k\right]\leqslant 3\max\{\mathbb{E}[D_1\big|\mathcal{H}_k],\mathbb{E}[D_2\big|\mathcal{H}_k],\mathbb{E}[D_3\big|\mathcal{H}_k]\}\leqslant\frac{n^{s-2}|C|^{-r-k}}{\binom{n}{2}}6p^2\epsilon\log n.$$

and so for our application of Freedman's inequality we have for all $i\leqslant i_{max}<\binom{n}{2}$ that

$$V(i)=\sum_{0\leqslant k\leqslant i}\mathbf{Var}[\Delta\textsc{Ava}^+(e,k)|\mathcal{H}_k]=\tilde{O}\left(n^{2\textsc{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}\right),$$

So, we will use $b=\tilde{O}\left(n^{2\textsc{Pow}(\overline{S},\overline{e},z)-\frac{1}{\binom{p}{2}-q+1}+30p^2\epsilon}\right)$. If z has at least one colored edge then $\textsc{Ext}(z,\phi,0)=\text{ext}_z(0)=0$ and so we have

$$\textsc{Ext}^+(z,\phi,0)=\textsc{Ext}(z,\phi,0)-\text{ext}_z(0)-n^{s-2}|C|^{-r-k}f_z(0)=-n^{s-2}|C|^{-r-k}f_z(0).$$

We use the notation $(n)_m:=n(n-1)\cdots(n-m+1)$. If z has no colored edge then $r=k=0$ and $\textsc{Ext}(z,\phi,0)=(n)_{s-2}$ while $\text{ext}_z(0)=n^{s-2}$ and so

$$\textsc{Ext}^+(z,\phi,0)=\textsc{Ext}(z,\phi,0)-\text{ext}_z(0)-n^{s-2}f_z(0)=-n^{s-2}f_z(0)+O(n^{s-3}).$$

Either way we have $\textsc{Ext}^+(z,\phi,0)=-(1+o(1))n^{s-2}|C|^{-r-k}f_z(0)$. For $\textsc{Ext}^+(z,\phi)$ to become positive would therefore be a positive change of $\lambda:=(1+o(1))n^{s-2}|C|^{-r-k}f_z(0)=\tilde{\Theta}\left(n^{\textsc{Pow}(\overline{S},\overline{e},z)-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}\right)$. Freedman's inequality gives us a failure probability of at most

$$\exp\left(-\frac{\lambda^2}{2(b+D\lambda)}\right)\leqslant\exp\left(-\tilde{\Omega}\left(n^{10p^2\epsilon}\right)\right)$$

which beats any polynomial union bound.

8 Phase 2

In this section we show that we can finish the coloring using a fresh set of colors and a simple random coloring on the remaining uncolored edges. We assume the event $\mathcal{E}_{i_{max}}$ holds. Let C' be a set of colors disjoint from $C \cup \overline{C}$ with $|C'| = |C|$. Then we independently color each edge not colored in Phase 1 with a color chosen uniformly at random from C'. We now show that this simple coloring scheme leaves us with a (p, q)-coloring with positive probability.

We use the asymmetric Lovász local lemma found in [1]:

Lemma 31 (Lovász local lemma)**.** *Let $\mathcal{A}$ be a finite set of events in a probability space Ω and let G be a dependency graph for $\mathcal{A}$. Suppose there is an assignment $x : \mathcal{A} \to [0, 1)$ of real numbers to $\mathcal{A}$ such that for all $A \in \mathcal{A}$ we have*

$$\mathbb{P}(A) \leqslant x(A) \prod_{B \in N(A)} (1 - x(B)). \tag{73}$$

Then we have that the probability none of the events in $\mathcal{A}$ happen is

$$\mathbb{P}\left(\bigwedge_{A \in \mathcal{A}} \overline{A}\right) \geqslant \prod_{A \in \mathcal{A}} (1 - x(A)) > 0.$$

We apply Lemma 31, where we have a bad event A_S for each set S of p vertices, r repeats and $\ell \leqslant q + r - 2$ colored edges. If $\ell \geqslant q + r - 1$ then there is no chance that S could end up with too many repeats. Indeed, even if all of the $\binom{p}{2} - \ell \leqslant \binom{p}{2} - q - r + 1$ uncolored edges of S all got the same color, this would be at most $\binom{p}{2} - q - r$ new repeats for a total of at most $\binom{p}{2} - q$. So we assume $\ell \leqslant q + r - 2$. The bad event A_S is that S gets an additional $\binom{p}{2} - q - r + 1$ repeats. To each such set S we associate the number

$$x(S) := n^{\epsilon - \frac{p-2}{\binom{p}{2} - q + 1}\left(\binom{p}{2} - q - r + 1\right)} = n^{\epsilon - p + 2 + r \cdot \frac{p-2}{\binom{p}{2} - q + 1}}$$

We bound the number of such sets S containing a fixed edge e. Each S must fit some legal type z using only Platonic colors on its colored edges, and we can assume S has at least two uncolored edges (or else there is no chance it would gain any repeats in Phase 2). We claim that z is a trackable type. Indeed, if e, e' are uncolored edges in S, then z is a predecessor of some $(\overline{e}', c, \overline{e}, c)$-preforbidder z', where $\overline{e}, \overline{e}'$ correspond to e, e', c is any real color, and z' is formed by assigning the color c to uncolored edges (other than e, e') of z until it has $\binom{p}{2} - q - 1$ repeats (since there are $\binom{p}{2} - \ell - 2 \geqslant \binom{p}{2} - q - r$ uncolored edges besides e, e', there are enough). Since z only uses Platonic colors on its colored edges, it is determined by choosing the available colors on the $\binom{p}{2} - \ell - 1$ uncolored edges, excluding the root. Thus the number of choices for z is $O\left(|C|^{\binom{p}{2} - \ell - 1}\right)$. For any such type z and a

ϕ of order 2 we have for $i = i_{max}$ and $t = t_{max}$ that

$$\begin{aligned}
&\textsc{Ext}(z, \phi) \leqslant \mathrm{ext}_z(t) + n^{p-2}|C|^{-r} f_z(t) \\
&= n^{p-2}|C|^{-r} a(t)^{\binom{p}{2}-\ell-1} \left[t^{\ell}(1-t)^{\binom{p}{2}-\ell-1} + n^{-\frac{1/2}{\binom{p}{2}-q+1}+20p^2\epsilon}(1-t)^{-5p^2+1} a(t)^{-1} e^{10p^4\epsilon t \log n} \right] \\
&= \tilde{O}\left(n^{p-2-r\cdot\frac{p-2}{\binom{p}{2}-q+1}-(\epsilon+\delta)\left(\binom{p}{2}-\ell-1\right)} \right),
\end{aligned}$$

where on the last line we have used that there exists some constant $\delta > 0$ (depending on our other constants) such that $a(t_{max}) \leqslant n^{-\delta}$. Indeed, from (12), we can see that $h(t) \geqslant h_p(t) \log n \geqslant ct^{\binom{p}{2}-2} \log n$ for some constant $c > 0$, and so

$$\begin{aligned}
H(t_{max}) = \int_0^{t_{max}} h(\tau) d\tau &\geqslant \int_0^{t_{max}} c\tau^{\binom{p}{2}-2} \log n \, d\tau \\
&= \frac{c}{\binom{p}{2}-1}(1-n^{-\epsilon})^{\binom{p}{2}-1} \log n > \frac{c}{p^2} \log n.
\end{aligned}$$

Thus, $a(t_{max}) = \exp\{-H(t_{max})\} \leqslant n^{-\frac{c}{p^2}}$, so $\delta = \frac{c}{p^2}$ suffices.

If we sum the above over all $O\left(|C|^{\binom{p}{2}-\ell-1}\right)$ choices for z and use $a(t) \geqslant n^{-\epsilon}$, each set S is counted $(1+o(1))(|C|a(t))^{\binom{p}{2}-\ell-1} \geqslant (1+o(1))(|C|n^{-\epsilon})^{\binom{p}{2}-\ell-1}$ times. Thus the number of S is

$$\begin{aligned}
\tilde{O}\left(\frac{n^{p-2-r\cdot\frac{p-2}{\binom{p}{2}-q+1}-(\epsilon+\delta)\left(\binom{p}{2}-\ell-1\right)} \cdot |C|^{\binom{p}{2}-\ell-1}}{(|C|n^{-\epsilon})^{\binom{p}{2}-\ell-1}} \right) &= \tilde{O}\left(n^{p-2-r\cdot\frac{p-2}{\binom{p}{2}-q+1}-\delta\left(\binom{p}{2}-\ell-1\right)} \right) \\
&= \tilde{O}\left(n^{p-2-r\cdot\frac{p-2}{\binom{p}{2}-q+1}-\delta} \right),
\end{aligned}$$

where we have used that $\ell \leqslant q+r-2$ and $r \leqslant \binom{p}{2} - q$. We check condition (73). We have

$$\mathbb{P}(A_S) = O\left(\frac{1}{|C|^{\binom{p}{2}-q-r+1}} \right) = \tilde{O}\left(n^{-\frac{p-2}{\binom{p}{2}-q+1}\left(\binom{p}{2}-q-r+1\right)} \right) = \tilde{O}\left(n^{-p+2+r\cdot\frac{p-2}{\binom{p}{2}-q+1}} \right)$$

which is smaller than

$$\begin{aligned}
&x_S \prod_r \left(1 - n^{\epsilon-p+2+r\cdot\frac{p-2}{\binom{p}{2}-q+1}} \right)^{\tilde{O}\left(n^{p-2-r\cdot\frac{p-2}{\binom{p}{2}-q+1}-\delta} \right)} \\
&\geqslant x_S \left(1 - \sum_r n^{\epsilon-p+2+r\cdot\frac{p-2}{\binom{p}{2}-q+1}} \cdot \tilde{O}\left(n^{p-2-r\cdot\frac{p-2}{\binom{p}{2}-q+1}-\delta} \right) \right) \\
&= x_S \left(1 - \tilde{O}\left(n^{\epsilon-\delta} \right) \right) \\
&= (1-o(1))x_S.
\end{aligned}$$

9 Some properties of the functions EdgCoi, Coi and Pow

Our goal of this section is to prove the technical lemmas (namely lemmas 39, 42 and 43) which were used in previous sections. We first will need to build up some machinery involving the functions EdgCoi, Coi and Pow.

We have the following additive property.

Observation 32 (Additive property). *For any type y on $\overline{T}$ and $\overline{E}_3 \subseteq \overline{E}_2 \subseteq \overline{E}_1 \subseteq \binom{\overline{T}}{2}$ we have*

$$\textsc{EdgCoi}(\overline{E}_1, \overline{E}_3, y) = \textsc{EdgCoi}(\overline{E}_1, \overline{E}_2, y) + \textsc{EdgCoi}(\overline{E}_2, \overline{E}_3, y).$$

Also for any $\overline{T}_3 \subseteq \overline{T}_2 \subseteq \overline{T}_1 \subseteq \overline{T}$ we have

$$\textsc{Coi}(\overline{T}_1, \overline{T}_3, y) = \textsc{Coi}(\overline{T}_1, \overline{T}_2, y) + \textsc{Coi}(\overline{T}_2, \overline{T}_3, y)$$

and

$$\textsc{Pow}(\overline{T}_1, \overline{T}_3, y) = \textsc{Pow}(\overline{T}_1, \overline{T}_2, y) + \textsc{Pow}(\overline{T}_2, \overline{T}_3, y).$$

Observation 33 (Monotonicity). *For any type y on $\overline{T}$ and $\overline{E}_4 \subseteq \overline{E}_3 \subseteq \overline{E}_2 \subseteq \overline{E}_1 \subseteq \binom{\overline{T}}{2}$ we have*

$$\textsc{EdgCoi}(\overline{E}_2, \overline{E}_4, y) \leqslant \textsc{EdgCoi}(\overline{E}_1, \overline{E}_4, y)$$

and

$$\textsc{EdgCoi}(\overline{E}_2, \overline{E}_4, y) \geqslant \textsc{EdgCoi}(\overline{E}_2, \overline{E}_3, y).$$

In other words, the function EdgCoi *is increasing in its first argument and decreasing in its second argument. Similarly for any $\overline{T}_4 \subseteq \overline{T}_3 \subseteq \overline{T}_2 \subseteq \overline{T}_1 \subseteq \overline{T}$ we have*

$$\textsc{Coi}(\overline{T}_2, \overline{T}_4, y) \leqslant \textsc{Coi}(\overline{T}_1, \overline{T}_4, y)$$

and

$$\textsc{Coi}(\overline{T}_2, \overline{T}_4, y) \geqslant \textsc{Coi}(\overline{T}_2, \overline{T}_3, y).$$

Definition 34. For any type y on $\overline{T}$ and $\overline{E}_3 \subseteq \overline{E}_2 \subseteq \overline{E}_1 \subseteq \binom{\overline{T}}{2}$ we let

$$\nu(\overline{E}_1, \overline{E}_2, \overline{E}_3, y) := \left| \textsc{Free}(\overline{E}_1 \setminus \overline{E}_3, \overline{E}_2 \setminus \overline{E}_3, y) \setminus \textsc{Free}(\overline{E}_1, \overline{E}_2, y) \right|.$$

In other words, $\nu(\overline{E}_1, \overline{E}_2, \overline{E}_3, y)$ is the number of Platonic colors c such that c appears (on colored edges under the labeling y) in $\overline{E}_1 \setminus \overline{E}_2$ and in $\overline{E}_3$ but not in $\overline{E}_2 \setminus \overline{E}_3$.

Observation 35. *For any type y on $\overline{T}$ and $\overline{E}_3 \subseteq \overline{E}_2 \subseteq \overline{E}_1 \subseteq \binom{\overline{T}}{2}$ we have*

$$\begin{aligned} \textsc{EdgCoi}(\overline{E}_1, \overline{E}_2, y) &= \textsc{EdgCoi}(\overline{E}_1 \setminus \overline{E}_3, \overline{E}_2 \setminus \overline{E}_3, y) + \nu(\overline{E}_1, \overline{E}_2, \overline{E}_3, y) \quad (74) \\ &\geqslant \textsc{EdgCoi}(\overline{E}_1 \setminus \overline{E}_3, \overline{E}_2 \setminus \overline{E}_3, y). \quad (75) \end{aligned}$$

Proof. Obviously (74) implies (75). To see (74), first recall that

$$\textsc{EdgCoi}(\overline{E}_1, \overline{E}_2, y) = |M(\overline{E}_1, \overline{E}_2, y)| - |\textsc{Free}(\overline{E}_1, \overline{E}_2, y)|$$

and

$$\textsc{EdgCoi}(\overline{E}_1 \setminus \overline{E}_3, \overline{E}_2 \setminus \overline{E}_3, y) = |M(\overline{E}_1 \setminus \overline{E}_3, \overline{E}_2 \setminus \overline{E}_3, y)| - |\textsc{Free}(\overline{E}_1 \setminus \overline{E}_3, \overline{E}_2 \setminus \overline{E}_3, y)|.$$

Now observe that $M(\overline{E}_1, \overline{E}_2, y) = M(\overline{E}_1 \setminus \overline{E}_3, \overline{E}_2 \setminus \overline{E}_3, y)$ and $\textsc{Free}(\overline{E}_1, \overline{E}_2, y) \subseteq \textsc{Free}(\overline{E}_1 \setminus \overline{E}_3, \overline{E}_2 \setminus \overline{E}_3, y)$, and (74) follows. □

The next observation looks just like line (75) except we do not assume $\overline{E}_3 \subseteq \overline{E}_2$.

Observation 36. *For any type y on $\overline{T}$ and edge sets $\overline{E}_2, \overline{E}_3 \subseteq \overline{E}_1 \subseteq \binom{\overline{T}}{2}$ we have*

$$\textsc{EdgCoi}\left(\overline{E}_1, \overline{E}_2, y\right) \geqslant \textsc{EdgCoi}\left(\overline{E}_1 \setminus \overline{E}_3, \overline{E}_2 \setminus \overline{E}_3, y\right)$$

Proof. We have

$$\begin{aligned}\textsc{EdgCoi}\left(\overline{E}_1, \overline{E}_2, y\right) &\geqslant \textsc{EdgCoi}\left(\overline{E}_1 \setminus (\overline{E}_2 \cap \overline{E}_3), \overline{E}_2 \setminus (\overline{E}_2 \cap \overline{E}_3), y\right) \\ &\geqslant \textsc{EdgCoi}\left(\overline{E}_1 \setminus \overline{E}_3, \overline{E}_2 \setminus \overline{E}_3, y\right).\end{aligned}$$

The first line follows from Observation 35. The second line follows by from Observation 33 (from the first line to the second, we decreased the first argument and kept the second argument the same).

□

Given two vertex sets A and B, we will denote the collection of edges with one endpoint in A and one in B as $[A, B]$.

Lemma 37 (Excision property)**.** *For any type y on $\overline{T}$ and $\overline{T}_3 \subseteq \overline{T}_2 \subseteq \overline{T}_1 \subseteq \overline{T}$ we have*

$$\textsc{Coi}(\overline{T}_1, \overline{T}_2, y) = \textsc{Coi}(\overline{T}_1 \setminus \overline{T}_3,\ \overline{T}_2 \setminus \overline{T}_3,\ y) + \textsc{EdgCoi}\left(\overline{F}_1, \overline{F}_2, y\right) + \nu\left(\overline{F}_2, \overline{F}_3, \overline{F}_4, y\right) \tag{76}$$

where

$$\overline{F}_1 := \binom{\overline{T}_1}{2}, \quad \overline{F}_2 := \binom{\overline{T}_1}{2} \setminus [\overline{T}_1 \setminus \overline{T}_2, \overline{T}_3], \quad \overline{F}_3 := \binom{\overline{T}_2}{2}, \quad \overline{F}_4 := \binom{\overline{T}_3}{2} \cup [\overline{T}_2 \setminus \overline{T}_3, \overline{T}_3].$$

We also have

$$\begin{aligned}&\textsc{Pow}(\overline{T}_1, \overline{T}_2, y) \\ &= \textsc{Pow}(\overline{T}_1 \setminus \overline{T}_3,\ \overline{T}_2 \setminus \overline{T}_3,\ y) - \frac{p-2}{\binom{p}{2} - q + 1}\left[\textsc{EdgCoi}\left(\overline{F}_1, \overline{F}_2, y\right) + \nu\left(\overline{F}_2, \overline{F}_3, \overline{F}_4, y\right)\right].\end{aligned} \tag{77}$$

We will often use Lemma 37 and the fact that $\textsc{EdgCoi}, \nu$ are nonnegative to get inequalities by dropping the $\textsc{EdgCoi}$ or the ν term from (76) or (77). For example by dropping both terms we get the simpler inequalities

$$\textsc{Coi}(\overline{T}_1, \overline{T}_2, y) \geqslant \textsc{Coi}(\overline{T}_1 \setminus \overline{T}_3,\ \overline{T}_2 \setminus \overline{T}_3,\ y)$$

and

$$\textsc{Pow}(\overline{T}_1,\ \overline{T}_2,\ y) \leqslant \textsc{Pow}(\overline{T}_1 \setminus \overline{T}_3,\ \overline{T}_2 \setminus \overline{T}_3,\ y).$$

Recall that $\nu(\overline{F}_2, \overline{F}_3, \overline{F}_4, y)$ is the number of Platonic colors appearing in $\overline{F}_2 \setminus \overline{F}_3$ and $\overline{F}_4$ but not in $\overline{F}_3 \setminus \overline{F}_4$. Thus it will be helpful for future reference to write

$$\overline{F}_2 \setminus \overline{F}_3 = \binom{\overline{T}_1 \setminus \overline{T}_2}{2} \cup [\overline{T}_1 \setminus \overline{T}_2, \overline{T}_2 \setminus \overline{T}_3], \quad \overline{F}_3 \setminus \overline{F}_4 = \binom{\overline{T}_2 \setminus \overline{T}_3}{2}, \quad \overline{F}_4 = \binom{\overline{T}_3}{2} \cup [\overline{T}_2 \setminus \overline{T}_3, \overline{T}_3]$$

Proof of Lemma 37. Since (76) implies (77) it suffices to prove (76). Note that $\overline{F}_4 \subseteq \overline{F}_3 \subseteq \overline{F}_2 \subseteq \overline{F}_1$. We have

$$\begin{aligned}
\textsc{Coi}(\overline{T}_1, \overline{T}_2, y) &= \textsc{EdgCoi}\left(\binom{\overline{T}_1}{2}, \binom{\overline{T}_2}{2}, y\right) = \textsc{EdgCoi}\left(\overline{F}_1, \overline{F}_3, y\right) \\
&= \textsc{EdgCoi}\left(\overline{F}_1, \overline{F}_2, y\right) + \textsc{EdgCoi}\left(\overline{F}_2, \overline{F}_3, y\right) \\
&= \textsc{EdgCoi}\left(\overline{F}_1, \overline{F}_2, y\right) + \textsc{EdgCoi}\left(\overline{F}_2 \setminus \overline{F}_4, \overline{F}_3 \setminus \overline{F}_4, y\right) + \nu\left(\overline{F}_2, \overline{F}_3, \overline{F}_4, y\right) \\
&= \textsc{EdgCoi}\left(\binom{\overline{T}_1 \setminus \overline{T}_3}{2}, \binom{\overline{T}_2 \setminus \overline{T}_3}{2}, y\right) + \textsc{EdgCoi}\left(\overline{F}_1, \overline{F}_2, y\right) + \nu\left(\overline{F}_2, \overline{F}_3, \overline{F}_4, y\right) \\
&= \textsc{Coi}(\overline{T}_1 \setminus \overline{T}_3, \overline{T}_2 \setminus \overline{T}_3, y) + \textsc{EdgCoi}\left(\overline{F}_1, \overline{F}_2, y\right) + \nu\left(\overline{F}_2, \overline{F}_3, \overline{F}_4, y\right).
\end{aligned}$$

Indeed, the first line is by definition of $\textsc{Coi}$, and the second is by additivity. The third line follows by applying Observation 35 to $\textsc{EdgCoi}\left(\overline{F}_2, \overline{F}_3, y\right)$. On the fourth line we have rearranged terms and used the simple combinatorial identities

$$\overline{F}_2 \setminus \overline{F}_4 = \binom{\overline{T}_1 \setminus \overline{T}_3}{2}, \qquad \overline{F}_3 \setminus \overline{F}_4 = \binom{\overline{T}_2 \setminus \overline{T}_3}{2}.$$

On the last line we use the definition of $\textsc{Coi}$ again. □

Observation 38. *For any type y, and any $\overline{e} \subseteq \overline{T}' \subseteq \overline{T}$,*

$$\textsc{Pow}(\overline{T}, \overline{T}', y) = \textsc{Pow}(\overline{T}, \overline{T}', \mathrm{col}(y)).$$

Proof. The uncolored edges of y and $\mathrm{col}(y)$ do not affect the value of $\textsc{Pow}(\overline{T}, \overline{T}', y)$ or $\textsc{Pow}(\overline{T}, \overline{T}', \mathrm{col}(y))$, and y and $\mathrm{col}(y)$ are identical everywhere except on uncolored edges, so the result follows. □

Lemma 39. *Let y be a $(\overline{e}', c', \overline{e}'', c'')$-preforbidder on $\overline{S}$. Let $\alpha = \alpha(\overline{S}')$ be the number of Platonic colors in $\{c', c''\}$ that appear in $\overline{S}$ but not in $\overline{S}'$. Then for all $\overline{S}'$ with $\overline{e}' \cup \overline{e}'' \subseteq \overline{S}' \subsetneq \overline{S}$,*

$$\textsc{Pow}(\overline{S}, \overline{S}', y) \leqslant \frac{p-2}{\binom{p}{2} - q + 1}\alpha - \frac{1}{\binom{p}{2} - q + 1} \tag{78}$$

Proof. Consider the type $y_{\overline{e}',\overline{e}''}$ on $\overline{S}$ formed from y by replacing the label on $\overline{e}'$ with $(\text{COLORED}, c')$ and the label $\overline{e}''$ with $(\text{COLORED}, c'')$ (and making no other changes). Then by Definition 12 (iii), $y_{\overline{e}',\overline{e}''}$ is not a legal type. In particular, the colored edges of $\overline{S}$ under the labeling $y_{\overline{e}',\overline{e}''}$ have $R(|\overline{S}|)+1$ repeats. Say that x is the number of real colors among the colored edges of $\overline{S}$ under $y_{\overline{e}',\overline{e}''}$. Then using (4) we have

$$\text{Pow}(\overline{S}, \emptyset, y_{\overline{e}',\overline{e}''}) \leqslant |\overline{S}| - 0 - \frac{p-2}{\binom{p}{2}-q+1}\left(\frac{(|\overline{S}|-2)\left(\binom{p}{2}-q+1\right)}{p-2} + x\right) = 2 - \frac{p-2}{\binom{p}{2}-q+1}x.$$

Meanwhile, as a consequence of Definition 12 (iv), the colored edges of $\overline{S}'$ under the labeling $y_{\overline{e}',\overline{e}''}$ have at most $R(|\overline{S}'|)$ repeats. There are at most x real colors among the colored edges of $\overline{S}'$ under $y_{\overline{e}',\overline{e}''}$. Thus we have

$$\begin{aligned}\text{Pow}(\overline{S}', \emptyset, y_{\overline{e}',\overline{e}''}) &\geqslant |\overline{S}'| - 0 - \frac{p-2}{\binom{p}{2}-q+1}\left(\frac{(|\overline{S}'|-2)\left(\binom{p}{2}-q+1\right)}{p-2} - \frac{1}{p-2} + x\right) \\ &= 2 + \frac{1}{\binom{p}{2}-q+1} - \frac{p-2}{\binom{p}{2}-q+1}x.\end{aligned}$$

Thus we have

$$\text{Pow}(\overline{S}, \overline{S}', y_{\overline{e}',\overline{e}''}) = \text{Pow}(\overline{S}, \emptyset, y_{\overline{e}',\overline{e}''}) - \text{Pow}(\overline{S}', \emptyset, y_{\overline{e}',\overline{e}''}) \leqslant -\frac{1}{\binom{p}{2}-q+1}. \tag{79}$$

Now to get (78), we consider the effect of uncoloring the edges $\overline{e}, \overline{e}'$. In particular, recall that

$$\begin{aligned}\text{Coi}(\overline{S}, \overline{S}', y_{\overline{e}',\overline{e}''}) &= \text{EdgCoi}\left(\binom{\overline{S}}{2}, \binom{\overline{S}'}{2}, y_{\overline{e}',\overline{e}''}\right) \\ &= \left|\text{M}\left(\binom{\overline{S}}{2}, \binom{\overline{S}'}{2}, y_{\overline{e}',\overline{e}''}\right)\right| - \left|\text{Free}\left(\binom{\overline{S}}{2}, \binom{\overline{S}'}{2}, y_{\overline{e}',\overline{e}''}\right)\right|,\end{aligned}$$

and

$$\text{M}\left(\binom{\overline{S}}{2}, \binom{\overline{S}'}{2}, y_{\overline{e}',\overline{e}''}\right) = \text{M}\left(\binom{\overline{S}}{2}, \binom{\overline{S}'}{2}, y\right),$$

while

$$\left|\text{Free}\left(\binom{\overline{S}}{2}, \binom{\overline{S}'}{2}, y_{\overline{e}',\overline{e}''}\right)\right| = \left|\text{Free}\left(\binom{\overline{S}}{2}, \binom{\overline{S}'}{2}, y\right)\right| - \alpha,$$

so

$$\text{Pow}(\overline{S}, \overline{S}', y_{\overline{e}',\overline{e}''}) = \text{Pow}(\overline{S}, \overline{S}', y) - \frac{p-2}{\binom{p}{2}-q+1}\alpha,$$

and this combined with (79) implies the result. □

Lemma 40. *Let y be a $(\overline{e}, c, \overline{e}', c')$-preforbidder on $\overline{T}$. Let $\overline{T} = \overline{R}_1 \cup \overline{R}_2 \cup \overline{R}_3$ be a partition such that $\overline{R}_1, \overline{R}_2 \neq \emptyset$ and $|\overline{R}_3| \leqslant 2$ ($\overline{R}_3$ can be empty). Then*

$$\textsc{EdgCoi}\left(\binom{\overline{T}}{2}, \binom{\overline{T}}{2} \setminus [\overline{R}_1, \overline{R}_2], y\right) \geqslant 1.$$

Proof. Let $|\overline{R}_1| = r_1, |\overline{R}_2| = r_2, |\overline{R}_3| = r_3$. Let us assume without loss of generality that $r_1 \geqslant r_2$ and so we have $1 \leqslant r_2 \leqslant p/2$. Note that since y is a preforbidder we have

$$\textsc{Coi}(\overline{T}, \overline{e}, y) \geqslant \frac{(|T| - 2)\left(\binom{p}{2} - q + 1\right)}{p - 2} - 2. \tag{80}$$

On the other hand we have

$$\begin{aligned}
\textsc{Coi}(\overline{T}, \overline{e}, y) &= \textsc{EdgCoi}\left(\binom{\overline{T}}{2}, \emptyset, y\right) \\
&= \textsc{EdgCoi}\left(\binom{\overline{T}}{2}, \binom{\overline{T}}{2} \setminus [\overline{R}_1, \overline{R}_2], y\right) + \textsc{EdgCoi}\left(\binom{\overline{T}}{2} \setminus [\overline{R}_1, \overline{R}_2], \binom{\overline{R}_1 \cup \overline{R}_3}{2}, y\right) \\
&\qquad + \textsc{EdgCoi}\left(\binom{\overline{R}_1 \cup \overline{R}_3}{2}, \emptyset, y\right) \\
&\leqslant \textsc{EdgCoi}\left(\binom{\overline{T}}{2}, \binom{\overline{T}}{2} \setminus [\overline{R}_1, \overline{R}_2], y\right) + \binom{r_2}{2} + r_2 r_3 + \frac{(r_1 + r_3 - 2)\left(\binom{p}{2} - q + 1\right)}{p - 2} + 3.
\end{aligned} \tag{81}$$

Indeed, the first line is by definition and the second line uses Observation 32. The last line follows since $\overline{R}_1 \cup \overline{R}_3$ can contain at most $\frac{(r_1+r_3-2)\left(\binom{p}{2}-q+1\right)}{p-2}$ repeats (see (4)) and 3 real colors (see Definition 12 (v)) and since there are only $\binom{r_2}{2} + r_2 r_3$ many edges in

$\binom{\overline{T}}{2} \setminus [\overline{R}_1, \overline{R}_2]$ that are not in $\binom{\overline{R}_1 \cup \overline{R}_3}{2}$. Now putting (80) and (81) together we get

$$\begin{aligned}
&\textsc{EdgCoi}\left(\binom{\overline{T}}{2}, \binom{\overline{T}}{2} \setminus [\overline{R}_1, \overline{R}_2], y\right) \\
&\quad \geqslant \frac{(|T|-2)\left(\binom{p}{2} - q + 1\right)}{p-2} - 2 - \binom{r_2}{2} - r_2 r_3 - \frac{(r_1 + r_3 - 2)\left(\binom{p}{2} - q + 1\right)}{p-2} - 3 \\
&\quad \geqslant \frac{r_2\left(\binom{p}{2} - q + 1\right)}{p-2} - \binom{r_2}{2} - 2r_2 - 5 \\
&\quad = r_2\left(\frac{\binom{p}{2} - q + 1}{p-2} - \frac{r_2 - 1}{2} - 2\right) - 5 \\
&\quad \geqslant r_2\left(\frac{\binom{p}{2} - q + 1}{p-2} - \frac{p/2 - 1}{2} - 2\right) - 5 \\
&\quad = r_2\left(\frac{p^2 - 6p + 16 - 4q}{4(p-2)}\right) - 5 \\
&\quad \geqslant \frac{p^2 - 6p + 16 - 4q}{4(p-2)} - 5 = \frac{p^2 - 26p + 56 - 4q}{4(p-2)} > 0.
\end{aligned}$$

In the second line we used that $|\overline{T}| = r_1 + r_2 + r_3$ and that $r_3 \leqslant 2$. On the fourth line we use $1 \leqslant r_2 \leqslant p/2$, and on the last line we use the same again, along with the bound $q \leqslant \frac{p^2 - 26p + 55}{4}$ in the hypothesis of Theorem 1. Since EdgCoi only returns integer values we are done. ☐

Lemma 41. *Let y_1 be a trackable type on $\overline{S}_1$ with root $\overline{e}_1$, and let y_2 be a $(\overline{e}_2, c_2, \overline{e}_3, c_3)$-preforbidder on $\overline{S}_2$ that is compatible with y_1. Let $\overline{S}' \subseteq \overline{S}_1 \cup \overline{S}_2$. Assume the following:*

- *(i)* $\overline{S}_2 \not\subseteq \overline{S}_1$
- *(ii)* $\overline{e}_1 \subseteq \overline{S}'$
- *(iii)* $\overline{e}_2 \subseteq \overline{S}' \cup \overline{S}_1$
- *(iv) either* $\overline{e}_2 \subseteq \overline{S}_1$ *or* c_2 *is a real color*
- *(v)* $\overline{e}_3 \subseteq \overline{S}_1$

Then

$$\textsc{Pow}(\overline{S}_1 \cup \overline{S}_2, \overline{S}', y_1 \cup y_2) \leqslant \textsc{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \frac{1}{\binom{p}{2} - q + 1}. \tag{82}$$

Proof. We will consider two cases.

Case 1: $\overline{S}_2 \setminus \overline{S}_1 \not\subseteq \overline{S}'$. Equivalently, $(\overline{S}' \cup \overline{S}_1) \cap \overline{S}_2 \neq \overline{S}_2$. In this case we have by additivity that

$$\textsc{Pow}(\overline{S}_1 \cup \overline{S}_2, \overline{S}', y_1 \cup y_2) = \textsc{Pow}(\overline{S}' \cup \overline{S}_1, \overline{S}', y_1 \cup y_2) + \textsc{Pow}(\overline{S}_1 \cup \overline{S}_2, \overline{S}' \cup \overline{S}_1, y_1 \cup y_2) \tag{83}$$

Now using Lemma 37 with $\overline{T}_3 = \overline{S}' \setminus \overline{S}_1$, we have

$$\text{Pow}(\overline{S}' \cup \overline{S}_1, \overline{S}', y_1 \cup y_2) \leqslant \text{Pow}(\overline{S}_1, \overline{S}' \cap \overline{S}_1, y_1 \cup y_2).$$

Using Lemma 37 with $\overline{T}_1 = \overline{S}_1 \cup \overline{S}_2$, $\overline{T}_2 = \overline{S}' \cup \overline{S}_1$ and $\overline{T}_3 = \overline{S}_1 \setminus \overline{S}_2$ we have

$$\begin{aligned}\text{Pow}(&\overline{S}_1 \cup \overline{S}_2, \overline{S}' \cup \overline{S}_1, y_1 \cup y_2) \\ &\leqslant \text{Pow}(\overline{S}_2, (\overline{S}' \cup \overline{S}_1) \cap \overline{S}_2, y_2) - \frac{p-2}{\binom{p}{2} - q + 1} \nu\left(\overline{F}_2, \overline{F}_3, \overline{F}_4, y_1 \cup y_2\right), \quad (84)\end{aligned}$$

where

$$\overline{F}_2 = \binom{\overline{T}_1}{2} \setminus [\overline{T}_1 \setminus \overline{T}_2, \overline{T}_3], \qquad \overline{F}_3 = \binom{\overline{T}_2}{2}, \qquad \overline{F}_4 = \binom{\overline{T}_3}{2} \cup [\overline{T}_2 \setminus \overline{T}_3, \overline{T}_3].$$

Let α be the number of Platonic colors in $\{c_2, c_3\}$ that appear in $\overline{S}_2$, but which do not appear in $(\overline{S}' \cup \overline{S}_1) \cap \overline{S}_2$. Note that this is precisely the value $\alpha = \alpha((\overline{S}' \cup \overline{S}_1) \cap \overline{S}_2)$ we need to bound $\text{Pow}(\overline{S}_2, (\overline{S}' \cup \overline{S}_1) \cap \overline{S}_2, y_2)$ using Lemma 39. We claim that $\nu\left(\overline{F}_2, \overline{F}_3, \overline{F}_4, y_1 \cup y_2\right) \geqslant \alpha$. To justify this claim, first note that by Definition 34 this ν value counts the number of Platonic colors appearing in $\overline{F}_2 \setminus \overline{F}_3$ and in $\overline{F}_4$ but not in $\overline{F}_3 \setminus \overline{F}_4$. Suppose a Platonic color $c^* \in \{c_2, c_3\}$ is counted by α. Then c^* does not appear in $\binom{\overline{S}_2 \cap (\overline{S}' \cup \overline{S}_1)}{2} = \overline{F}_3 \setminus \overline{F}_4$. c^* also appears in $\binom{\overline{S}_2}{2}$, so it must appear in $\binom{\overline{S}_2}{2} \setminus \binom{\overline{S}_2 \cap (\overline{S}' \cup \overline{S}_1)}{2} = \overline{F}_2 \setminus \overline{F}_3$. Furthermore, by conditions (iv) (if $c^* = c_2$) and (v) (if $c^* = c_3$), y_1 assigns $(\text{AVAILABLE}, c^*)$ to some edge, and so since y_1 is a trackable type we know c^* must appear in $\binom{\overline{S}_1}{2} \setminus \binom{\overline{S}_2 \cap (\overline{S}' \cup \overline{S}_1)}{2} \subseteq \overline{F}_4$. Thus c^* is also counted by $\nu\left(\overline{F}_2, \overline{F}_3, \overline{F}_4, y_1 \cup y_2\right)$, and so $\nu\left(\overline{F}_2, \overline{F}_3, \overline{F}_4, y_1 \cup y_2\right) \geqslant \alpha$.

Using (84) we now have

$$\text{Pow}(\overline{S}_1 \cup \overline{S}_2, \overline{S}' \cup \overline{S}_1, y_1 \cup y_2) \leqslant \text{Pow}(\overline{S}_2, (\overline{S}' \cup \overline{S}_1) \cap \overline{S}_2, y_1 \cup y_2) - \frac{p-2}{\binom{p}{2} - q + 1}\alpha$$

and so (83) becomes

$$\begin{aligned}\text{Pow}(&\overline{S}_1 \cup \overline{S}_2, \overline{S}', y_1 \cup y_2) \\ &\leqslant \text{Pow}(\overline{S}_1, \overline{S}' \cap \overline{S}_1, y_1) + \text{Pow}(\overline{S}_2, (\overline{S}' \cup \overline{S}_1) \cap \overline{S}_2, y_2) - \frac{p-2}{\binom{p}{2} - q + 1}\alpha \\ &\leqslant \text{Pow}(\overline{S}_1, \overline{S}' \cap \overline{S}_1, y_1) + \frac{p-2}{\binom{p}{2} - q + 1}\alpha - \frac{1}{\binom{p}{2} - q + 1} - \frac{p-2}{\binom{p}{2} - q + 1}\alpha \\ &= \text{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \text{Pow}(\overline{S}' \cap \overline{S}_1, \overline{e}_1, y_1) - \frac{1}{\binom{p}{2} - q + 1} \\ &\leqslant \text{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \frac{1}{\binom{p}{2} - q + 1}\end{aligned}$$

where on the second line we have used Lemma 39 (note that $(\overline{S}' \cup \overline{S}_1) \cap \overline{S}_2 \neq \overline{S}_2$ follows from the case assumption) and the last inequality follows from $\text{Pow}(\overline{S}' \cap \overline{S}_1, \overline{e}_1, y_1) \geqslant 0$

(since y_1 is trackable, see Observation 17 (ii) and note that $\textsc{Pow}(\overline{e}_1, \overline{e}_1, y_1) = 0$). So, (82) holds and we are done in this case.

Case 2: $\overline{S}_2 \setminus \overline{S}_1 \subseteq \overline{S}'$. In this case we have $\overline{S}' \cup \overline{S}_1 = \overline{S}_1 \cup \overline{S}_2$. We have

$$\begin{aligned} &\textsc{Pow}(\overline{S}_1 \cup \overline{S}_2, \overline{S}', y_1 \cup y_2) \\ &\quad = \textsc{Pow}(\overline{S}' \cup \overline{S}_1, \overline{S}', y_1 \cup y_2) \\ &\quad \leqslant \textsc{Pow}(\overline{S}_1, \overline{S}' \cap \overline{S}_1, y_1) - \frac{p-2}{\binom{p}{2} - q + 1} \textsc{EdgCoi}\left(\overline{F}_1, \overline{F}_2, y_1 \cup y_2\right) \\ &\quad = \textsc{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \textsc{Pow}(\overline{S}' \cap \overline{S}_1, \overline{e}_1, y_1) - \frac{p-2}{\binom{p}{2} - q + 1} \textsc{EdgCoi}\left(\overline{F}_1, \overline{F}_2, y_1 \cup y_2\right), \end{aligned} \tag{85}$$

where on the second line we have applied Lemma 37 using

$$\overline{T}_1 = \overline{S}' \cup \overline{S}_1, \qquad \overline{T}_2 = \overline{S}', \qquad \overline{T}_3 = \overline{S}' \setminus \overline{S}_1,$$

and where

$$\overline{F}_1 = \binom{\overline{T}_1}{2}, \qquad \overline{F}_2 = \binom{\overline{T}_1}{2} \setminus [\overline{T}_1 \setminus \overline{T}_2, \overline{T}_3].$$

Now if $\overline{S}' \cap \overline{S}_1 \neq \overline{e}_1$ then $\textsc{Pow}(\overline{S}' \cap \overline{S}_1, \overline{e}_1, y_1) \geqslant \frac{1}{\binom{p}{2}-q+1}$ by Observation 17 (ii) since y_1 is a trackable type, so (85) would imply (82) and we would be done. Otherwise we have $\overline{S}' \cap \overline{S}_1 = \overline{e}_1$ and since we are also assuming $\overline{S}_2 \setminus \overline{S}_1 \subseteq \overline{S}'$ we have that $\overline{S}' = \overline{e}_1 \cup (\overline{S}_2 \setminus \overline{S}_1)$. Thus we have

$$\overline{T}_1 = \overline{S}_1 \cup \overline{S}_2, \qquad \overline{T}_2 = \overline{e}_1 \cup (\overline{S}_2 \setminus \overline{S}_1), \qquad \overline{T}_3 = \overline{S}_2 \setminus \overline{S}_1,$$

and

$$\overline{F}_1 = \binom{\overline{S}_1 \cup \overline{S}_2}{2}, \qquad \overline{F}_2 = \binom{\overline{S}_1 \cup \overline{S}_2}{2} \setminus [\overline{S}_1 \setminus \overline{e}_1, \overline{S}_2 \setminus \overline{S}_1].$$

Therefore

$$\begin{aligned} &\textsc{EdgCoi}\left(\overline{F}_1, \overline{F}_2, y_1 \cup y_2\right) \\ &\qquad = \textsc{EdgCoi}\left(\binom{\overline{S}_1 \cup \overline{S}_2}{2}, \binom{\overline{S}_1 \cup \overline{S}_2}{2} \setminus [\overline{S}_1 \setminus \overline{e}_1, \overline{S}_2 \setminus \overline{S}_1], y_1 \cup y_2\right) \\ &\qquad \geqslant \textsc{EdgCoi}\left(\binom{\overline{S}_2}{2}, \binom{\overline{S}_2}{2} \setminus [(\overline{S}_1 \cap \overline{S}_2) \setminus \overline{e}_1, \overline{S}_2 \setminus \overline{S}_1], y_1 \cup y_2\right) \end{aligned}$$

where the second line follows from Observation 36 by removing all edges outside of $\binom{\overline{S}_2}{2}$. But now by Lemma 40 with $\overline{T} = \overline{S}_2, \overline{R}_1 = (\overline{S}_1 \cap \overline{S}_2) \setminus \overline{e}_1, \overline{R}_2 = \overline{S}_2 \setminus \overline{S}_1, \overline{R}_3 = \overline{e}_1 \cap \overline{S}_2$ we have
$\textsc{EdgCoi}\left(\binom{\overline{S}_2}{2}, \binom{\overline{S}_2}{2} \setminus [(\overline{S}_1 \cap \overline{S}_2) \setminus \overline{e}_1, \overline{S}_2 \setminus \overline{S}_1], y_1 \cup y_2\right) \geqslant 1$, and again (85) implies (82), so we are done. □

Lemma 42. *Let y_1 be a trackable type on $\overline{S}_1$ with root $\overline{e}_1$, and let y_2 be a $(\overline{e}_2, c_2, \overline{e}_3, c_3)$-preforbidder on $\overline{S}_2$ that is compatible with y_1. Assume $\overline{e}_2, \overline{e}_3 \subseteq \overline{S}_1$ and that $y_1(\overline{e}_2) = (\text{AVAILABLE}, c_2), y_1(\overline{e}_3) = (\text{AVAILABLE}, c_3)$. Assume $\overline{S}_2 \setminus \overline{S}_1 \neq \emptyset$. Then*

$$\textsc{MaxPow}(\overline{S}_1 \cup \overline{S}_2, \overline{e}_1, y_1 \cup y_2) \leqslant \textsc{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \frac{1}{\binom{p}{2} - q + 1}.$$

Proof. Consider $\overline{S}'$ such that $\overline{e}_1 \subseteq \overline{S}' \subseteq \overline{S}_1 \cup \overline{S}_2$. We apply Lemma 41. It is easy to see that by our assumptions, Conditions (i) and (ii) hold. Conditions (iii), (iv) and (v) hold since $\overline{e}_2, \overline{e}_3$ are contained in both $\overline{S}_1$ and $\overline{S}_2$. Thus by Lemma 41 we have

$$\textsc{Pow}(\overline{S}_1 \cup \overline{S}_2, \overline{S}', y_1 \cup y_2) \leqslant \textsc{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \frac{1}{\binom{p}{2} - q + 1}.$$

This completes the proof. □

Lemma 43. *Let y_1 be a trackable type on $\overline{S}_1$ with root $\overline{e}_1$, and let y_2 be a $(\overline{e}_2, c_2, \overline{e}_3, c_3)$-preforbidder on $\overline{S}_2$ that is compatible with y_1. Assume $\overline{e}_3 \subseteq \overline{S}_1$, $\overline{e}_3 \neq \overline{e}_1$, and $\overline{e}_2 \nsubseteq \overline{S}_1$. Assume $y_1(\overline{e}_3) = (\text{AVAILABLE}, c_3)$. Assume c_2 is a real color. Then*

$$\textsc{MaxPow}(\overline{S}_1 \cup \overline{S}_2, \overline{e}_1 \cup \overline{e}_2, y_1 \cup y_2) \leqslant \textsc{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \frac{1}{\binom{p}{2} - q + 1}.$$

Proof. Consider $\overline{S}'$ such that $\overline{e}_1 \cup \overline{e}_2 \subseteq \overline{S}' \subseteq \overline{S}_1 \cup \overline{S}_2$. We apply Lemma 41. Condition (i) holds since $\overline{S}_2 \setminus \overline{S}_1$ contains $\overline{e}_2 \setminus \overline{S}_1 \neq \emptyset$. Condition (ii) holds by assumption. Condition (iii) holds since $\overline{e}_2 \subseteq \overline{S}'$. Condition (iv) holds since c_2 is real. Condition (v) holds by assumption. Thus by Lemma 41 we have

$$\textsc{Pow}(\overline{S}_1 \cup \overline{S}_2, \overline{S}', y_1 \cup y_2) \leqslant \textsc{Pow}(\overline{S}_1, \overline{e}_1, y_1) - \frac{1}{\binom{p}{2} - q + 1}.$$

This completes the proof. □

10 The Forbidden Submatching Method

Shortly after the authors uploaded the first draft of this paper, Delcourt, Li and Postle [4] (see version 1) extended Theorem 1 to all q below the linear threshold, i.e. $q < \binom{p}{2} - p + 3$, and gave a generalization of the result to list-coloring problems on hypergraphs. Shortly after that (see version 2 of [4]), the first author joined that paper to extend Theorem 1 to all q between the linear and quadratic thresholds, i.e. $\binom{p}{2} - p + 3 < q < \binom{p}{2} - \lfloor \frac{p}{2} \rfloor + 2$. We summarize the differences and similarities between the two approaches.

Delcourt and Postle [18] proved several very general and powerful results that show the existence (under suitable technical conditions) of matchings in certain hypergraphs that do not contain any member of some family of forbidden submatchings. These results and their applications are called the **forbidden submatchings method** (alternatively called **conflict-free matchings** by Glock, Joos, Kim, Kühn and Lichev [26] who independently proved similar results). In particular one of the results takes as input a hypergraph $\mathcal{H}$, a family of forbidden submatchings, and a subset $A \subseteq V(\mathcal{H})$, and guarantees that there is a matching that covers all of A and avoids all the forbidden submatchings. Such a result can be used for (p, q)-coloring as follows. The hypergraph $\mathcal{H}$ will just be a bipartite graph with bipartition $A \cup B$, where $A = E(K_n)$ and $B = A \times C$ where C is the set of colors. A vertex $a \in A$ is adjacent to all of the vertices $(a, c) \in B$ (and no other vertices). A matching $M \subseteq \mathcal{H}$ which covers A corresponds to a coloring of $E(K_n)$ in the obvious way. To ensure that this coloring is a (p, q)-coloring, one must forbid our matching M from containing certain submatchings that correspond to a p-clique having fewer than q colors. Moreover, to ensure that $\mathcal{H}$ satisfies the necessary technical conditions to apply the forbidden submatchings method, one must forbid even more submatchings. Indeed, in [4] (as it applies to (p, q)-colorings) they forbid any submatching that would correspond to a coloring with some set of $s \leqslant p$ vertices with more than $R(s)$ repeats, i.e. exactly the same number of repeats that we forbid on an s-clique in our process in this paper.

The process we analyzed for phase 1 in this paper is equivalent to choosing one edge of $\mathcal{H}$ at each step uniformly at random from all edges that can be chosen without intersecting previous edges and without creating a forbidden submatching with the previous edges. The approach in [4] instead applies Theorem 1.16 from [18] as a black box. The proof of this theorem first uses a random sparsification trick to delete many of the edges of $\mathcal{H}$, and then uses a nibble method, sometimes called a *semi-random* method, to find a matching using only the remaining edges. It is widely accepted that nibble processes are approximately equivalent to random greedy processes, but often the nibble version of a random greedy process turns out to be more amenable to analysis. Adding to the list of tools available for analyzing nibble processes, Delcourt and Postle [18] proved and utilized a new version of Talagrand's inequality which was crucial in order to cover the sparse regime.

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