A new family of (q^4+1)-tight sets with an automorphism group $F_4(q)$

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Abstract

In this paper, we construct a new family of (q^4+1)-tight sets in $Q(24,q)$ or $Q^-(25,q)$ depending on whether $q=3^f$ or $q \equiv 2 \pmod 3$. The novelty of the construction is the use of the action of the exceptional simple group $F_4(q)$ on its minimal module over $\mathbb{F}_q$, and the construction has a close connection with the $F_4(q)$ geometry and metasymplectic spaces.

Keywords: Exceptional simple group, intriguing set, polar space, tight set

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1 Introduction

Let q be a prime power, $\mathbb{F}_q$ the finite field of order q, and V a finite-dimensional vector space over $\mathbb{F}_q$. Let κ be a non-degenerate reflexive sesquilinear form or a non-singular quadratic form on V. The *finite classical polar space* $\mathcal{P}$ associated with (V,κ) is the geometry consisting of the totally singular or totally isotropic subspaces with respect to κ of the ambient projective space $\mathrm{PG}(V)$, depending on whether κ is a quadratic or sesquilinear form. The totally singular or totally isotropic one-dimensional subspaces are the *points* of $\mathcal{P}$. The totally singular or totally isotropic subspaces of maximum dimension are called the *generators* of $\mathcal{P}$. The *rank* of $\mathcal{P}$ is the vector space dimension of its generators. A finite classical polar space of rank 2 is a point-line geometry, and it is a *finite generalized quadrangle*. A *spread* of $\mathcal{P}$ is a collection of pairwise disjoint generators that partition the point set of $\mathcal{P}$.

Tight sets in generalized quadrangles were first introduced by Payne [22], and this definition has been generalized to finite classical polar spaces by Drudge [10]. There are several different but equivalent definitions of a tight set in a finite classical polar space. We will be using the following definition in this paper.

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Definition 1. Let $\mathcal{P}_r$ be a finite classical polar space of rank $r \geqslant 2$ over $\mathbb{F}_q$. A subset $\mathcal{M}$ of points of $\mathcal{P}_r$ is said to be *tight* if for all points P of $\mathcal{P}_r$, there is an integer $i > 0$ such that

$$|P^\perp \cap \mathcal{M}| = \begin{cases} \dfrac{i(q^{r-1}-1)}{q-1} + q^{r-1}, & \text{if } P \in \mathcal{M}, \\ \dfrac{i(q^{r-1}-1)}{q-1}, & \text{if } P \notin \mathcal{M}, \end{cases} \tag{1.1}$$

where $P^\perp$ is the set of points in $\mathcal{P}_r$ that are collinear with P. The integer i is called the *parameter* of the tight set; a tight set with parameter i is usually called an *i-tight set*.

Parallel to the notion of a tight set is the concept of an m-ovoid of $\mathcal{P}_r$. The notions of tight sets and m-ovoids were unified under the umbrella of intriguing sets in [2]. During the past two decades, intriguing sets have been extensively investigated because of their close connections with many other combinatorial/geometric objects such as strongly regular graphs, partial difference sets, Boolean degree one functions, and Cameron-Liebler line classes, cf. [2, 11, 12].

Nontrivial tight sets with large parameters are rare, and they tend to exist in finite classical polar spaces of low ranks. Trivially 1-tight sets exist in finite classical polar space $\mathcal{P}_r$ for all ranks $r \geqslant 2$ since each generator of $\mathcal{P}_r$ is a 1-tight set. In fact, Drudge [10] proved that any 1-tight set of $\mathcal{P}_r$ must be a generator. Tight sets with large parameters are much more complicated. One of the main problems concerning tight sets is to determine for which values of i there exist i-tight sets in $\mathcal{P}_r$. A second problem is to characterize all the i-tight sets in $\mathcal{P}_r$ for a specific parameter i.

As we saw before, each generator of $\mathcal{P}_r$ is a 1-tight set. Suppose that A and B are an i-tight set and a j-tight set in $\mathcal{P}_r$, respectively. If A, B are disjoint, then $A \cup B$ is an $(i+j)$-tight set; if $A \subseteq B$, then $B \setminus A$ is a $(j-i)$-tight set. Thus, the union of any i pairwise disjoint generators of $\mathcal{P}_r$ forms an i-tight set. For this reason, the first question has a simple and complete answer if the classical polar space $\mathcal{P}_r$ admits a spread of generators. Thus, we will only consider the problem of constructing tight sets for polar spaces which do not admit a spread; and for such a polar space, it is interesting to construct tight sets with large parameters i, where i is greater than the maximum size of a partial spread. Some results on existence of spreads of classical polar spaces can be found in [14, Table 7.4].

In this paper, we construct a new family of (q^4+1)-tight sets in $Q(24, q)$ or $Q^-(25, q)$ depending on whether $q = 3^f$ or $q \equiv 2 \pmod 3$. Our main theorem in this paper is the following.

Theorem 1.1. Let $q = p^f$, where p is prime and $f \geqslant 1$. If $q \equiv 2 \pmod 3$, then the classical polar space $Q^-(25, q)$ admits a (q^4+1)-tight set with an automorphism group isomorphic to the exceptional group $F_4(q)$; if $q = 3^f$, then the classical polar space $Q(24, q)$ admits a (q^4+1)-tight set with an automorphism group isomorphic to the exceptional group $F_4(q)$.

There are few constructions of intriguing sets in high dimensional orthogonal polar spaces, which do not follow from field reduction or nondegenerate hyperplane sections, cf. [2, 16]. The tight sets in Theorem 1.1 follow from neither of those two construction methods. The novelty of our construction is the use of the action of the exceptional simple group $F_4(q)$ on its minimal module over $\mathbb{F}_q$, and the construction has a close connection with the $F_4(q)$ geometry and metasymplectic spaces, cf. Section 4. We remark that the result in the characteristic 3 case of Theorem 1.1 is essentially due to Cohen and Cooperstein. They showed in [8, Table 2] that the group $F_4(q)$ has two orbits on the singular points of the polar space $Q(24, q)$ associated with its 25-dimensional minimal module, and from this fact we deduce that each orbit is a tight set.

The paper is organized as follows. In Section 2, we introduce some preliminary results on intriguing sets, octonions and the minimal module of $F_4(q)$. In particular, we give a more effective way to decide whether a subset $\mathcal{M}$ of points of a finite classical polar space $\mathcal{P}_r$ is an intriguing set, see Lemma 2. In Section 3, we present the proof of Theorem 1.1.

2 Preliminary

2.1 Intriguing sets on finite classical polar spaces

Let $\mathbb{F}_q$ be the finite field of order q, where $q = p^f$, p is a prime and $f \geqslant 1$. Let V be a d-dimensional vector space over $\mathbb{F}_q$ equipped with a nondegenerate reflexive sesquilinear form or quadratic form κ, and let $\mathcal{P}_r$ be the associated polar space. A point of $\mathcal{P}_r$ is defined as a 1-dimensional totally isotropic/singular subspace of V. A maximal totally isotropic/singular subspace of $\mathcal{P}_r$ is called a *generator* of $\mathcal{P}_r$. The vector space dimension of a generator, denoted by r, is called the *rank* of $\mathcal{P}_r$. A generator of $\mathcal{P}_r$ has $\frac{q^r-1}{q-1}$ points. An *ovoid* of $\mathcal{P}_r$ is a set of points which meets each generator in exactly one point. We use θ_r to denote the size of a putative ovoid, which we call *the ovoid number* of $\mathcal{P}_r$. A simple counting argument shows that $|\mathcal{P}_r| = \theta_r \cdot \frac{q^r-1}{q-1}$. We list the ranks and the ovoid numbers of the six classes of classical polar spaces in Table 2.1.

Table 2.1: The parameters r and θ_r

	d	f	polar space $\mathcal{P}_r$	rank r	ovoid number θ_r
S	even	-	$W(d-1,q)$	$d/2$	$q^{d/2}+1$
	even	-	$Q^+(d-1,q)$	$d/2$	$q^{d/2-1}+1$
O	even	-	$Q^-(d-1,q)$	$d/2-1$	$q^{d/2}+1$
	odd	-	$Q(d-1,q)$	$(d-1)/2$	$q^{(d-1)/2}+1$
U	odd	even	$H(d-1,q)$	$(d-1)/2$	$q^{d/2}+1$
	even	even	$H(d-1,q)$	$d/2$	$q^{(d-1)/2}+1$

Suppose that $r \geqslant 2$, and let $\mathcal{M}$ be a nonempty set of points of $\mathcal{P}_r$. The set $\mathcal{M}$ is called an *intriguing set* if there exist some constants $h_1 \neq h_2$ such that $|P^\perp \cap \mathcal{M}| = h_1$ or h_2 depending on whether $P \in \mathcal{M}$ or not, where P ranges over all the points of $\mathcal{P}_r$, cf. [2]. An intriguing set $\mathcal{M}$ is proper if $\mathcal{M} \neq \mathcal{P}_r$. There are exactly two types of intriguing sets:

(1) i-tight sets: $|\mathcal{M}| = \frac{i(q^r-1)}{q-1}$, $h_1 = q^{r-1} + \frac{i(q^{r-1}-1)}{q-1}$, $h_2 = \frac{i(q^{r-1}-1)}{q-1}$, and

(2) m-ovoids: $|\mathcal{M}| = m\theta_r$, $h_1 = (m-1)\theta_{r-1} + 1$, $h_2 = m\theta_{r-1}$.

We refer the reader to [2] for more properties of intriguing sets. In particular, if H is a subgroup of semisimilarities that has exactly two orbits O_1, O_2 on the points of $\mathcal{P}_r$, then both O_1 and O_2 are intriguing sets of the same type.

To prove that a candidate subset $\mathcal{M}$ of points of $\mathcal{P}_r$ is an intriguing set, one needs to show that $\mathcal{M}$ is a two-intersection set with respect to the perp of singular points, cf. [2]. However, there is one shortcut method which seems to have gone unnoticed (see Lemma 2.1 below). We observe that $\theta_r - 1 = q(\theta_{r-1} - 1)$ by Table 2.1.

Lemma 2. *Let $\mathcal{M}$ be a subset of size $\frac{i(q^r-1)}{q-1}$ or $m\theta_r$ in $\mathcal{P}_r$ for some positive integers i or m, and let h_1, h_2 be the corresponding parameters determined by $|\mathcal{M}|$ (see above). Then the following are equivalent:*

(1) $\mathcal{M}$ is an intriguing set in $\mathcal{P}_r$;

(2) $|P^\perp \cap \mathcal{M}| = h_1$ for all $P \in \mathcal{M}$;

(3) $|P^\perp \cap \mathcal{M}| = h_2$ for all $P \in \mathcal{P}_r \setminus \mathcal{M}$.

Proof. We observe that $\mathcal{M}$ is an i-tight set if and only if the complement $\mathcal{P}_r \setminus \mathcal{M}$ is a $(\theta_r - i)$-tight set, and $\mathcal{M}$ is an m-ovoid if and only if $\mathcal{P}_r \setminus \mathcal{M}$ is a $\left(\frac{q^r-1}{q-1} - m\right)$-ovoid. We deduce that the equivalence of (1) and (3) for $\mathcal{M}$ is the same as the equivalence of (1) and (2) for $\mathcal{P}_r \setminus \mathcal{M}$. Therefore, it suffices to establish the equivalence of (1) and (2). It is clear that (1) implies (2). So we only need to prove that (2) implies (1). We will use $\mathcal{P}_j$ to denote the polar space of rank j and of the same type as $\mathcal{P}_r$. We compute $\sum_{P\in\mathcal{P}_r} |P^\perp \cap \mathcal{M}|$ and $\sum_{P\in\mathcal{P}_r} |P^\perp \cap \mathcal{M}|^2$, and will show that $\sum_{P\in\mathcal{P}_r\setminus\mathcal{M}} \left(|P^\perp \cap \mathcal{M}| - h_2\right)^2 = 0$. First we have

$$\begin{aligned}\sum_{P\in\mathcal{P}_r} |P^\perp \cap \mathcal{M}| &= |\{(P, z) : P \in \mathcal{P}_r, z \in \mathcal{M}, P \sim z\}| \\ &= \sum_{z\in\mathcal{M}} |z^\perp \cap \mathcal{P}_r| = q|\mathcal{M}| \cdot |\mathcal{P}_{r-1}| + |\mathcal{M}|.\end{aligned}$$

For the last equality, we observe that for a point $z \in \mathcal{M}$, $z^\perp \cap \mathcal{P}_r$ is a cone with vertex z and base $\mathcal{P}_{r-1}$. Furthermore, we have

$$\begin{aligned}\sum_{P\in\mathcal{P}_r} |P^\perp \cap \mathcal{M}|^2 &= \sum_{P\in\mathcal{P}_r} |\{(P, z_1, z_2) :\ z_1, z_2 \in \mathcal{M},\ z_1 \sim P,\ z_2 \sim P\}| \\ &= \sum_{z\in\mathcal{M}} |z^\perp \cap \mathcal{P}_r| + \sum_{\substack{z_1,z_2\in\mathcal{M},\\ z_1\neq z_2}} |\langle z_1, z_2\rangle^\perp \cap \mathcal{P}_r|. \end{aligned} \tag{2.1}$$

For the last summation in (2.1), each summand takes value either $q^2 \cdot |\mathcal{P}_{r-2}| + q + 1$ or $|\mathcal{P}_{r-1}|$ depending on whether z_1, z_2 are perpendicular or not: $\langle z_1, z_2\rangle^\perp \cap \mathcal{P}_r$ is a cone with vertex $\langle z_1, z_2\rangle$ and base $\mathcal{P}_{r-2}$ if $\langle z_1, z_2\rangle$ is totally singular or isotropic, and it is $\mathcal{P}_{r-1}$ otherwise. Next we compute directly

$$\sum_{P\in\mathcal{P}_r} \left(|P^\perp \cap \mathcal{M}| - h_2\right)^2 = \sum_{P\in\mathcal{P}_r} \left(|P^\perp \cap \mathcal{M}|^2 - 2h_2|P^\perp \cap \mathcal{M}| + h_2^2\right)$$

$$= |\mathcal{M}| \cdot \Big(|\mathcal{P}_{r-1}|(|\mathcal{M}| + q - h_1 - 2h_2q) + (qh_1 - q + h_1 - 2h_2) + q^2(h_1 - 1)|\mathcal{P}_{r-2}| + \frac{h_2^2|\mathcal{P}_r|}{|\mathcal{M}|}\Big). \tag{2.2}$$

First consider the case where $\mathcal{M}$ is an i-tight set, so that $|\mathcal{M}| = i\frac{q^r-1}{q-1}, h_1 = i\frac{q^{r-1}-1}{q-1} + q^{r-1}, h_2 = i\frac{q^{r-1}-1}{q-1}$. Observe that the right hand side of (2.2) divided by $|\mathcal{M}|$ can be viewed as a degree-one polynomial in variable i. Then some tedious but routine computations show that the coefficient of i equals 0. Hence we are only concerned with the constant term of the aforementioned polynomial, which equals

$$\begin{aligned}&|\mathcal{M}|\big(|\mathcal{P}_{r-1}|(q - q^{r-1}) + (q+1)q^{r-1} - q + q^2(q^{r-1} - 1)|\mathcal{P}_{r-2}|\big)\\ =&|\mathcal{M}| \cdot q^{2r-2} = |\mathcal{M}| \cdot (h_1 - h_2)^2.\end{aligned}$$

From $\sum_{P\in\mathcal{P}_r} \big(|P^\perp \cap \mathcal{M}| - h_2\big)^2 = |\mathcal{M}|(h_1 - h_2)^2$ we deduce that

$$\sum_{P\in\mathcal{P}_r\setminus\mathcal{M}} \big(|P^\perp \cap \mathcal{M}| - h_2\big)^2 = 0.$$

This proves that (2) implies (1) when $\mathcal{M}$ is an i-tight set. When $\mathcal{M}$ is an m-ovoid, we can use the same method to obtain the result. This completes the proof. $\square$

3 A Construction of $(q^4 + 1)$-Tight Sets

3.1 Octonions and the minimal module of $F_4(q)$

The split octonion algebra $\mathbb{O}_{\mathbb{F}_q}$ is an 8-dimensional non-commutative and non-associative algebra over $\mathbb{F}_q$. A detailed introduction to octonions can be found in the book [25]. Let $\{x_1, \dots, x_8\}$ be the basis of the octonion algebra $\mathbb{O}$ (here in order to simplify notation we omit the subscript) as defined in [28, (4.26)]. With respect to this basis, the multiplication of $\mathbb{O}$ is defined as follows:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1					x_1	x_2	$-x_3$	$-x_4$
x_2			$-x_1$	x_2			$-x_5$	x_6
x_3		x_1		x_3		$-x_5$		x_7
x_4	x_1			x_4		x_6	x_7	
x_5		x_2	x_3		x_5			x_8
x_6	$-x_2$		$-x_4$		x_6		x_8	
x_7	x_3	$-x_4$			x_7	$-x_8$		
x_8	$-x_5$	$-x_6$	x_7	x_8				

(3.1)

where the blank entries are 0, cf. [28, (4.27)]. It is clear that $\mathbf{1} := x_4 + x_5$ is the identity of $\mathbb{O}$. The octonion conjugation $^-$ swaps x_4, x_5 and maps the other x_i's to their negations, and it is an anti-isomorphism of $\mathbb{O}$. For an octonion $x = \sum_{i=1}^{8} \lambda_i x_i$, we define its trace as $\mathrm{Tr}(x) = x + \overline{x} = \lambda_4 + \lambda_5$ and its norm as $N(x) = x\overline{x}$. It is routine to show that $N(x) = \sum_{i=1}^{4} \lambda_i \lambda_{9-i}$ which takes value in $\mathbb{F}_q$, and defines a nondegenerate hyperbolic quadric $Q^+(7, q)$. Its associated bilinear form is $B(x, y) := N(x+y) - N(x) - N(y)$, and we have $B(x, y) = \mathrm{Tr}(x\overline{y})$ for $x, y \in \mathbb{O}$. In particular, we have $\mathrm{Tr}(x) = B(x, 1)$. The split octonion algebra $\mathbb{O}$ is an alternative ring by [27, Lemma 3], so that the subalgebra generated by any two elements is associative, cf. [25, Chapter 1.4]. It follows that $\mathbb{O}$ is a composition algebra, i.e., $N(xy) = N(x)N(y)$ for $x, y \in \mathbb{O}$. The automorphism group of the octonion algebra $\mathbb{O}$ is known as $G_2(q)$, and each of its elements fixes the vector $\mathbf{1} := x_4 + x_5$. It holds that $x^2 - \mathrm{Tr}(x)x + N(x) = 0$ for $x \in \mathbb{O}$ by [25, Proposition 1.2.3], so $G_2(q)$ lies in the isometry group of the quadratic form N. We write $\mathbf{1}^\perp$ for the perp of $\mathbf{1}$ with respect to B, which is stabilized by $G_2(q)$.

The $G_2(q)$-orbits on the nonzero octonions are implicitly known by the results in [28, Chapter 4.3]. We list them explicitly in the next lemma for future use.

Lemma 3. *Suppose that $q > 2$. For q odd, let α be a nonsquare of $\mathbb{F}_q^*$ and T be a complete set of coset representatives of $\{1, -1\}$ in $\mathbb{F}_q^*$; for q even, let β be an element of $\mathbb{F}_q^*$ with absolute trace 1 and S be a complete set of coset representatives of $\{0, 1\}$ in $\mathbb{F}_q$. Then the $G_2(q)$-orbits on the octonions are as listed in Table 3.1.*

Table 3.1: The $G_2(q)$-orbits on the octonions

Representative	Orbit size	Stabilizer	Trace	Norm	Condition
k, $k \in \mathbb{F}_q$	1	$G_2(q)$	$2k$	k^2	
$x_1 + k$, $k \in \mathbb{F}_q$	$q^6 - 1$	$q^{2+1+2} : \mathrm{SL}_2(q)$	$2k$	k^2	
$a(x_4 - x_5) + k$, $a \in T, k \in \mathbb{F}_q$	$q^6 + q^3$	$\mathrm{SL}_3(q)$	$2k$	$k^2 - a^2$	q odd
$a(x_1 - \alpha x_8) + k$, $a \in T, k \in \mathbb{F}_q$	$q^6 - q^3$	$\mathrm{SU}_3(q)$	$2k$	$k^2 - \alpha a^2$	q odd
$a(x_4 + k)$, $a \in \mathbb{F}_q^*, k \in S$	$q^6 + q^3$	$\mathrm{SL}_3(q)$	a	$a^2(k^2 + k)$	q even
$a(x_4 + x_1 + \beta x_8 + k)$, $a \in \mathbb{F}_q^*$, $k \in S$	$q^6 - q^3$	$\mathrm{SU}_3(q)$	a	$a^2(k^2 + k + \beta)$	q even

Proof. We give a sketch of the proof by quoting the relevant results in [28, Chapter 4.3]. Since each element of $G_2(q)$ fixes $\mathbf{1}$, each element $k \in \mathbb{F}_q$ is stabilized by $G_2(q)$. Take an element $k \in \mathbb{F}_q$ and set $v = x_1 + k$. For an element $g \in G_2(q)$, it stabilizes v if and only if it stabilizes x_1. The stabilizer of $\langle x_1 \rangle$ is a maximal parabolic subgroup $q^{2+1+2} : \mathrm{GL}_2(q)$, and the stabilizer of x_1 in the latter subgroup is $q^{2+1+2} : \mathrm{SL}_2(q)$. Hence the $G_2(q)$-orbit of $x_1 + k$ has size $\frac{|G_2(q)|}{q^5 \cdot |\mathrm{SL}_2(q)|} = q^6 - 1$ for each $k \in \mathbb{F}_q$. The trace and norm of $x_1 + k$ are respectively $2k, k^2$, so we obtain the representatives for q distinct $G_2(q)$-orbits as k varies. The arguments so far work for both even and odd q.

Suppose that q is odd and set $u = x_4 - x_5$, $v = x_1 - \alpha x_8$, where α is a nonsquare in $\mathbb{F}_q^*$. The stabilizer of $\langle u \rangle$ in $G_2(q)$ is a maximal subgroup $\mathrm{SL}_3(q) : \langle s \rangle$, where s has order 2 and swaps x_4 and x_5. The stabilizer of u in the latter subgroup is $\mathrm{SL}_3(q)$. As in the previous paragraph, we deduce that the $G_2(q)$-orbit of $au + k$ has size $\frac{|G_2(q)|}{|\mathrm{SL}_3(q)|} = q^6 + q^3$ for each $a \in \mathbb{F}_q^*$ and $k \in \mathbb{F}_q$. The trace and norm of $au + k$ are respectively $2k, -a^2 + k^2$, so $au + k$ and $a'u + k'$ are in the same $G_2(q)$-orbit if and only if $a' = \pm a$ and $k' = k$. The subalgebra $C := \langle \mathbf{1}, v \rangle$ is isomorphic to $\mathbb{F}_{q^2}$, and the stabilizer of v in $G_2(q)$ is $\mathrm{SU}_3(q)$. There is an involution $r \in G_2(q)$ that maps x_1, x_8 to $-x_1, -x_8$ and thus v to $-v$, and $\mathrm{SU}_3(q) : \langle r \rangle$ is a maximal subgroup of $G_2(q)$ that stabilizes $\langle v \rangle$. Similarly, we deduce that the $G_2(q)$-orbit of $av + k$ has size $\frac{|G_2(q)|}{|\mathrm{SU}_3(q)|} = q^6 - q^3$ for each $a \in \mathbb{F}_q^*$ and $k \in \mathbb{F}_q$. The trace and norm of $av + k$ are respectively $2k, k^2 - a^2$, so $av + k$ and $a'v + k'$ are in the same $G_2(q)$-orbit if and only if $a' = \pm a$ and $k' = k$. Since

$$q + q \cdot (q^6 - 1) + \frac{1}{2}(q-1)q \cdot (q^6 + q^3) + \frac{1}{2}(q-1)q \cdot (q^6 - q^3) = q^8,$$

we conclude that we have obtained all the $G_2(q)$-orbits and their relevant information is as listed in Table 3.1 for q odd.

Suppose that q is even and set $u = x_4$, $v = x_1 + \beta x_8 + x_4$, where β has absolute trace 1, i.e., $X^2 + X + \beta$ is irreducible over $\mathbb{F}_q$. The element v has trace 1 and norm β, and there are $q^6 - q^3$ octonions with those properties. The stabilizer of u in $G_2(q)$ is $\mathrm{SL}_3(q)$, and there is an order 2 element s that swaps u_4, u_5 such that $\mathrm{SL}_3(q) : \langle s \rangle$ is a maximal subgroup. We have $v^2 + v + \beta = 0$, so the subalgebra $\langle \mathbf{1}, v \rangle$ is a field with q^2 element. There is an involution $r \in G_2(q)$ that fixes x_1, x_8 and swaps x_4 and x_5, so that r maps v to $v + \mathbf{1}$. The stabilizer of v in $G_2(q)$ is $\mathrm{SU}_3(q)$, and $\mathrm{SU}_3(q) : \langle r \rangle$ is a maximal subgroup. The remaining arguments are exactly the same as in the q odd case, and we omit the details. This completes the proof. □

For a nonzero element a in $\mathbb{O}$, we define its left and right annihilators as follows

$$\mathrm{ann}_L(a) = \{x \in \mathbb{O} :\, xa = 0\}, \quad \mathrm{ann}_R(a) = \{x \in \mathbb{O} :\, ax = 0\}.$$

By [25, Lemma 1.3.3], we have $\overline{x}(xa) = N(x)a$ and $(ax)\overline{x} = N(x)a$, so if $N(a) = 0$ then both $\mathrm{ann}_L(a)$ and $\mathrm{ann}_R(a)$ are totally singular subspaces for the quadratic form N. If $N(a) \neq 0$, then both of its annihilators are trivial by a similar argument. The following result is [3, Theorem 5], and the interested reader can verify it by using Lemma 3.

Lemma 4. *Suppose that D, E are nonzero elements in $\mathbb{O}$ such that $DE = 0$. Then $D\overline{D} = E\overline{E} = 0$, $\mathrm{ann}_L(D)$ and $\mathrm{ann}_R(D)$ are totally isotropic subspaces of dimension 4, and*

(1) $\mathrm{ann}_L(D) \cap \mathrm{ann}_R(D)$ *has size q^3 or q depending on whether $D \in \mathbf{1}^\perp$ or not;*

(2) $\mathrm{ann}_L(D) \cap \mathrm{ann}_R(E)$ *has size q^3.*

The following result is well known, and we include a short proof here. We refer the reader to [13, Proposition 9.10] for the case where q is odd. For a property $\mathbf{P}$, we use the Iverson bracket $[\![\mathbf{P}]\!]$ which takes the value either 1 or 0 depending on whether $\mathbf{P}$ holds or not.

Lemma 5. *For a natural number k and an element $a \in \mathbb{F}_q$, write $N_k(a)$ for the number of tuples $(a_1, \ldots, a_{2k}) \in \mathbb{F}_q^{2k}$ such that $a_1a_2 + \cdots + a_{2k-1}a_{2k} = a$. Then*

$$N_k(a) = q^{2k-1} - q^{k-1} + q^k \cdot [\![a = 0]\!] \quad \text{ for } a \in \mathbb{F}_q.$$

Proof. It is easy to see that the claim holds for $k = 1$. We have

$$\begin{aligned} N_k(a) &= \sum_{a_{2k-1} \in \mathbb{F}_q} \sum_{a_{2k} \in \mathbb{F}_q} N_{k-1}(a - a_{2k-1}a_{2k}) \\ &= q^2 N_{k-1}(1) + (N_{k-1}(0) - N_{k-1}(1)) \cdot |\{a_{2k-1}, a_{2k} \in \mathbb{F}_q : a_{2k-1}a_{2k} = a\}| \\ &= q^2 N_{k-1}(1) + (N_{k-1}(0) - N_{k-1}(1)) \left(q - 1 + q \cdot [\![a = 0]\!]\right), \end{aligned}$$

and the claim follows by induction on k. $\square$

Lemma 6. *For $a \in \mathbb{F}_q$ there are $q^7 - q^3 + q^4 \cdot [\![a = 0]\!]$ octonions of norm a, and among them there are $q^6 - q^3 + q^4 \cdot [\![a = 0]\!]$ such octonions α that $\mathrm{Tr}(x_1\overline{\alpha}) = 0$.*

Proof. The first claim immediately follows by specifying $k = 4$ in Lemma 5. For an octonion $\alpha = \sum_{i=1}^8 \lambda_i x_i$, we have $\mathrm{Tr}(x_1\overline{\alpha}) = \lambda_8$ and $N(\alpha) = \sum_{i=1}^4 \lambda_i\lambda_{9-i}$. If $\lambda_8 = 0$, then $N(\alpha) = \sum_{i=2}^4 \lambda_i\lambda_{9-i}$. We deduce that there are $q \cdot N_3(a)$ such octonions α that $\lambda_8 = 0$ and $N(\alpha) = a$, where $N_3(a)$ is as in Lemma 5. This completes the proof. $\square$

Lemma 7. *There are $(q^6 - q^3 + q^4 - 1)(q^4 - 1)$ pairs (D, E) of nonzero octonions such that $DE = 0$ and $\mathrm{Tr}(x_1\overline{D}) = 0$.*

Proof. If $DE = 0$ for some nonzero D, E, then $D\overline{D} = 0$ by Lemma 4. There are $q^6 - q^3 + q^4 - 1$ nonzero octonions D such that $D\overline{D} = 0$ and $\mathrm{Tr}(x_1\overline{D}) = 0$ by Lemma 6, and for each such D there are $q^4 - 1$ nonzero octonions E such that $DE = 0$ by Lemma 4. The claim then follows. $\square$

Let $\mathcal{A}$ be the algebra of 3×3 Hermitian matrices over the octonions $\mathbb{O}$. To be specific, a 3×3 matrix x over $\mathbb{O}$ is a Hermitian matrix if $x^\top = \overline{x}$ and the diagonal entries of x lie in $\mathbb{F}_q$. Here, $\overline{x}$ is the matrix obtained by applying the octonion conjugate to the entries of x. For $d, e, f \in \mathbb{F}_q$ and $D, E, F \in \mathbb{O}$, we define

$$(\lambda_0, \lambda_0', \lambda_0'' \mid D, E, F) := \begin{pmatrix} \lambda_0 & F & \overline{E} \\ \overline{F} & \lambda_0' & D \\ E & \overline{D} & \lambda_0'' \end{pmatrix}$$

which is in $\mathcal{A}$. For $u, v \in \mathcal{A}$, their product in the algebra $\mathcal{A}$ is $u \circ v := uv + vu$, where uv and vu are matrix multiplications. We set $I := (1, 1, 1 \mid 0, 0, 0)$, so that $I \circ a = 2a$ for any element a in $\mathcal{A}$. The algebra $\mathcal{A}$ has close connections with Albert algebras and Jordan algebras, cf. [19, 24]. We choose a basis $\{w_i, w_i', w_i'' : 0 \leqslant i \leqslant 8\}$ of $\mathcal{A}$ as follows:

$$\begin{aligned} w_0 &= (1,0,0 \mid 0,0,0) \text{ and } w_i = (0,0,0 \mid x_i,0,0) \text{ for } i>0, \\ w_0' &= (0,1,0 \mid 0,0,0) \text{ and } w_i' = (0,0,0 \mid 0,x_i,0) \text{ for } i>0, \\ w_0'' &= (0,0,1 \mid 0,0,0) \text{ and } w_i'' = (0,0,0 \mid 0,0,x_i) \text{ for } i>0. \end{aligned}$$

The multiplication table of $\mathcal{A}$ with respect to the basis $\{w_i, w_i', w_i'' : 0 \leqslant i \leqslant 8\}$ can be written down explicitly as in [28, (4.90)-(4.92)]. The similarity group of the Dickson-Freudenthal determinant of $\mathcal{A}$, which we denote by $\tilde{E}_6(q)$, is the universal covering group of the simple group $E_6(q)$, cf. [4, 27]. It has three orbits on the nonzero vectors of $\mathcal{A}$, which are called the white, gray and black vectors, respectively, cf. [1, 8]. They correspond to vectors of rank 1, rank 2 and rank 3 in [17]. The 1-dimensional subspace spanned by a white, gray or black vector is called a white, gray or black point, respectively. The stabilizer of I in $\tilde{E}_6(q)$ is $F_4(q)$.

For an element $v = \sum_{t=0}^{8}(\lambda_t w_t + \lambda_t' w_t' + \lambda_t'' w_t'') \in \mathcal{A}$, we define the trace of v as $\mathrm{Tr}_{\mathcal{A}}(v) = \lambda_0 + \lambda_0' + \lambda_0''$. We define the $F_4(q)$-invariant subspaces $U = \{v \in \mathcal{A} : \mathrm{Tr}_{\mathcal{A}}(v) = 0\}$, $U' = \langle I \rangle_{\mathbb{F}_q}$, and set $W := U/(U \cap U')$. For $v \in U$, we define

$$Q_0(v) = \lambda_0^2 + \lambda_0\lambda_0' + \lambda_0'^2 + \sum_{t=1}^{4}(\lambda_t\lambda_{9-t} + \lambda_t'\lambda_{9-t}' + \lambda_t''\lambda_{9-t}'').$$

In particular, if $v = (\lambda_0, \lambda_0', \lambda_0'' \mid D, E, F)$, then $Q_0(v) = \lambda_0^2 + \lambda_0\lambda_0' + \lambda_0'^2 + D\overline{D} + E\overline{E} + F\overline{F}$. The quadratic form Q_0 on U is $F_4(q)$-invariant and has $U \cap U'$ as its radical, so it induces a nondegenerate quadratic form Q on W. The space W is the minimal module of $F_4(q)$, and $\dim(W) = 25$ or 26 depending on whether the characteristic of $\mathbb{F}_q$ is 3 or not. The form Q on W is hyperbolic if $q \equiv 1 \pmod 3$ and is elliptic if $q \equiv 2 \pmod 3$. The polar spaces in Theorem 1.1 are defined by the quadratic space (W, Q).

3.2 The proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. Take the same notation as in the previous section, and we consider the polar space defined by the quadratic space (W, Q). If the characteristic of $\mathbb{F}_q$ is 3, then the associated polar space is $Q(24, q)$ and the group $F_4(q)$ has two orbits on the singular points by [8, Table 2]. One orbit $\mathcal{M}_1$ has size $(q^4+1)\frac{q^{12}-1}{q-1}$ which is not divisible by the ovoid number $q^{12}+1$; thus $\mathcal{M}_1$ is a (q^4+1)-tight set of $Q(24, q)$ as desired. We suppose that $q \equiv 2 \pmod 3$ in the following, so that $W = U$ and the associated polar space is $Q^-(25, q)$. Let M_1 be the set of white vectors in U, and write $\mathcal{M}_1$ for the corresponding set of projective points. The set M_1 has size $(q^4+1)(q^{12}-1)$ and forms a single $F_4(q)$-orbit by [8, (W.3)]. By [27, Lemma 5] or [4, Lemma 7.1], we enumerate the white vectors in U as follows:

(I) $v = f \cdot (A\overline{A}, B\overline{B}, 1 \mid \overline{B}, A, \overline{A}B)$ for some $f \in \mathbb{F}_q^*$ and $A, B \in \mathbb{O}$ such that $A\overline{A} + B\overline{B} + 1 = 0$,

(II) $v = e \cdot (C\overline{C}, 1, 0 \mid A, \overline{CA}, C)$ for some $e \in \mathbb{F}_q^*$ and octonions A, C such that $A\overline{A} = 0$, $C\overline{C} + 1 = 0$,

(III) $v = (0, 0, 0 \mid D, E, F)$, where D, E, F are octonions such that

$$D\overline{D} = E\overline{E} = F\overline{F} = 0, \quad DE = EF = FD = 0. \tag{3.2}$$

In particular, M_1 is $\mathbb{F}_q^*$-invariant and it contains the singular vector $(0, 0, 0 \mid x_1, 0, 0)$. It follows that $\mathcal{M}_1$ has size $(q^4 + 1)\frac{q^{12}-1}{q-1}$ and consists of singular points of $Q^-(25, q)$.

We claim that $\mathcal{M}_1$ is a $(q^4 + 1)$-tight set of $Q^-(25, q)$. By Lemma 2 and the fact that $\mathcal{M}_1$ is a single $F_4(q)$-orbit, it suffices to show that there are $q^{11} + (q^4 + 1)\frac{q^{11}-1}{q-1}$ points in $\mathcal{M}_1$ that are perpendicular to $\langle(0, 0, 0 \mid x_1, 0, 0)\rangle$. Suppose that $\langle v\rangle$ is such a point in $\mathcal{M}_1$, and we examine the three types (I)-(III) one by one. If $v = (\lambda, \lambda', \lambda'' \mid D, E, F)$, then the condition that $\langle v\rangle$ is perpendicular to $\langle(0, 0, 0 \mid x_1, 0, 0)\rangle$ translates into $\mathrm{Tr}(x_1\overline{D}) = 0$.

First consider the case where $\langle v\rangle$ is of type (I). The number of such points in $\mathcal{M}_1$ that are perpendicular to $\langle(0, 0, 0 \mid x_1, 0, 0)\rangle$ equals the number of (A, B) pairs such that $A\overline{A} + B\overline{B} + 1 = 0$, $\mathrm{Tr}(x_1B) = 0$. If we write $B = \sum_{i=1}^{8} b_i x_i$, then $\mathrm{Tr}(x_1B) = -b_8 = 0$ and $B\overline{B} = b_2b_7 + b_3b_6 + b_4b_5$. By Lemma 5, we deduce that the number of such pairs (A, B) is $q \cdot N_7(-1) = q^7(q^7 - 1)$. Here, q is contributed by the choices for b_1 and $N_7(-1)$ is as defined in Lemma 5.

Next consider the case where $\langle v\rangle$ is of type (II). The number of such points in $\mathcal{M}_1$ perpendicular to $\langle(0, 0, 0 \mid x_1, 0, 0)\rangle$ equals the number of (A, C) pairs such that $C\overline{C} + 1 = 0$, $A\overline{A} = 0$ and $\mathrm{Tr}(x_1\overline{A}) = 0$. By Lemma 6, it equals $(q^7 - q^3)(q^6 - q^3 + q^4)$.

Finally, consider the case where $\langle v\rangle$ is of type (III), i.e., $v = (0, 0, 0 \mid D, E, F)$ such that (3.2) holds and $\mathrm{Tr}(x_1\overline{D}) = 0$. We observe that the conditions in (3.2) are invariant under the cyclic shift of (D, E, F). We divide into the following three cases.

(1) Suppose that D, E, F are all nonzero. There are $(q^6 - q^3 + q^4 - 1)(q^4 - 1)$ pairs (D, E) of nonzero octonions such that $DE = 0$, $\mathrm{Tr}(x_1\overline{D}) = 0$ by Lemma 7. For a pair (D, E) of nonzero octonions such that $DE = 0$, there are $q^3 - 1$ nonzero octonions F such that $EF = FD = 0$ by Lemma 4, and we deduce that $D\overline{D} = E\overline{E} = F\overline{F} = 0$ for such D, E, F by the same lemma. This case contributes $\frac{q^4-1}{q-1}(q^6 - q^3 + q^4 - 1)(q^3 - 1)$.

(2) Suppose that exactly one of D, E, F is zero. If $F = 0$, then (3.2) reduces to $DE = 0$ by Lemma 4. There are $(q^6 - q^3 + q^4 - 1)(q^4 - 1)$ pairs (D, E) of nonzero octonions such that $DE = 0$, $\mathrm{Tr}(x_1\overline{D}) = 0$ by Lemma 7. By symmetry, we obtain the same number if $E = 0$. If $D = 0$, then (3.2) reduces to $EF = 0$ and similarly there are $(q^3 + 1)(q^4 - 1)^2$ such pairs (E, F) of nonzero octonions that $EF = 0$ by Lemmas 4 and 6. To sum up, this case contributes $\frac{q^4-1}{q-1}\big(2(q^6 - q^3 + q^4 - 1) + (q^3 + 1)(q^4 - 1)\big)$.

(3) Suppose that exactly one of D, E, F is nonzero. If $E = F = 0$, then (3.2) reduces to $D\overline{D} = 0$. There are $q^6 - q^3 + q^4 - 1$ nonzero octonions D such that $D\overline{D} = 0$ and $\mathrm{Tr}(x_1\overline{D}) = 0$ by Lemma 6. If $D = E = 0$, then (3.2) reduces to

$F\overline{F} = 0$ and there are $(q^3+1)(q^4-1)$ such nonzero octonions F. By symmetry, we obtain the same number if $D = F = 0$. To sum up, this case contributes $\frac{1}{q-1}\left(q^6 - q^3 + q^4 - 1 + 2(q^3+1)(q^4-1)\right)$.

By adding up all the numbers, we deduce that there are exactly $q^{11} + (q^4+1)\frac{q^{11}-1}{q-1}$ points in $\mathcal{M}_1$ that are perpendicular to $\langle(0,0,0 \mid x_1,0,0)\rangle$. This completes the proof of Theorem 1.1.

Remark 8. Suppose that the characteristic of $\mathbb{F}_q$ is not 3. Take the same notation as in Section 2, and let K be the normalizer of $F_4(q)$ in the normalizer of the quadratic form Q_0 on U. The vectors $x = w_1$, $y = w_4 + w_4'$, $z = w_0 - w_0'' + w_4 - w_5$ are singular vectors of Q_0 in U, and they are stabilized by the Frobenius automorphism that raises the coordinates of a vector with respect to the basis $\{w_i, w_i', w_i'' : 0 \leqslant i \leqslant 8\}$ to their p-th powers respectively. Moreover, they are white, gray and black vectors, respectively, and lie in distinct $\tilde{E}_6(q)$-orbits. We conclude that K has at least three orbits on the singular points of the polar space associated with (U, Q_0). When $q = 2$, we verify with Magma [5] that there are exactly three $F_4(q)$-orbits on the singular point. They are tight sets with parameters 17, $2^{12} - 2^4$, 2^{12} in $Q^-(25,2)$, respectively. It is not clear whether these will lead to new infinite families of tight sets in $Q^-(25,q)$ with $q \equiv 2 \pmod 3$, and we leave it as an open problem.

4 Further discussions

Let $\mathcal{A}$ be the algebra of 3×3 Hermitian matrices over the octonions $\mathbb{O}$ and let U, W and Q_0 be as introduced in Section 3.1. Let $\mathcal{W}_0$ be the set of white points in U, and for each white vector v we define $\sigma(v) = \{v \circ x : x \in V\}$. We shall make use of the facts on $F_4(q)$ geometry and metasymplectic spaces in [7, 20, 23] freely in the following. By [9] the transitive action of $F_4(q)$ on $\mathcal{W}_0$ has five orbitals, which we describe now. For $u = (0,0,0 \mid x_1, 0, 0)$, we have

$$\sigma(u) = \langle(0,a,a \mid kx_1, B, C) : a,k \in \mathbb{F}_q, B \in \langle x_1,x_2,x_3,x_4\rangle, C \in \langle x_1,x_2,x_3,x_5\rangle\rangle.$$

A nonzero vector $v = (0,a,a \mid kx_1, B, C)$ in $U \cap \sigma(u)$ is white if and only if $a = 0$ and $BC = 0$, and it holds that $v \circ u = 0$ for such a v. It holds that $u \in \sigma(y_1)$, and $\langle u, y_1\rangle \subseteq \cap_{z\in\langle u,y_1\rangle}\sigma(z)$ for $y_1 = (0,0,0 \mid 0, x_1, 0)$. Let $y_2 = (0,0,0 \mid x_5, 0, 0)$, which satisfies $x \circ y_2 = 0$ but is not in $\sigma(u)$. We have $\sigma(u)\cap\sigma(y_2) = \{(0,a,a \mid 0, B, C) : a \in \mathbb{F}_q, B \in \langle x_1, x_4\rangle, C \in \langle x_2, x_3\rangle\}$ which contains $(q+1)(q^2+1)$ white points in $\mathcal{W}_0$. Let $y_3 = (0,0,0 \mid 0, x_5, 0)$, so that $x \circ y_3 = (0,0,0 \mid 0,0,x_1)$. We similarly deduce that $\sigma(u)\cap\sigma(y_2) = \langle(0,0,0 \mid 0,0,cx_1)\rangle$, which contains exactly an element of $\mathcal{W}_0$. Let $y_4 = (0,0,0 \mid x_8, 0, 0)$, and we have $\sigma(u)\cap\sigma(y_4) = \{0\}$. The points $\langle u\rangle$ and $\langle y_1\rangle, \ldots, \langle y_4\rangle$ are representatives for the five orbits of $F_4(q)_{\langle u\rangle}$ on $\mathcal{W}_0$, and their orbit sizes are 1, $\frac{q^4-1}{q-1}q(q^3+1)$, $\frac{q^6-1}{q-1}q^5$, $\frac{q^4-1}{q-1}q^8(q^3+1)$ and q^{15}, respectively. The first four orbit lengths add up to $q^{11} + (q^4+1)\frac{q^{11}-1}{q-1}$, and the quotient images of $\langle u\rangle$, $\langle y_1\rangle, \ldots, \langle y_4\rangle$ in $W = U/(U \cap \langle I\rangle)$ are singular points of Q_0

perpendicular to $\langle u\rangle$. This provides an alternative proof of Theorem 1.1, and it also explains why the same construction fails for $q \equiv 1 \pmod 3$.

Suppose that $q \equiv 0$ or $2 \pmod 3$, and let $\mathcal{M}_1$ be the tight set corresponding to $\mathcal{W}_0$ in the polar space $\mathcal{Q}$ associated with (U, Q_0) as in Theorem 1.1. Let Γ be the collinearity graph of $\mathcal{Q}$. We define a graph Γ_0 as follows: the vertex set is $\mathcal{W}_0$, and two vertices $\langle x\rangle, \langle y\rangle$ are adjacent if and only if $y \in \sigma(x)$. It is an undirected graph of diameter 3, since $x \in \sigma(y)$ implies $y \in \sigma(x)$. Moreover, $\langle x\rangle$, $\langle y\rangle$ are adjacent if and only if $\langle x, y\rangle \subseteq \cap_{z\in\langle x,y\rangle}\sigma(z)$, cf. [20]. For two vertices $\langle x\rangle, \langle y\rangle$ at distance 2 in Γ_0, they have either 1 or $(q+1)(q^2+1)$ common neighbors. By the previous paragraph, we deduce that the distance-3 graph of Γ_0 embeds naturally in the distance-2 graph of Γ as an induced subgraph. This provides more geometric insights into the tight sets that we have constructed in Theorem 1.1, and it is of theoretical interest to study this phenomenon in a more general context.

There is a striking similarity between our construction and the construction of Kantor's unitary ovoids in [15]; and there is a general pattern behind these constructions. If we use an associative split subalgebra of $\mathbb{O}$ instead of $\mathbb{O}$ itself, then we obtain more intriguing sets by the same procedure. To be specific, we take an associative subalgebra $\mathbb{D}$ of $\mathbb{O}$, and let $\mathcal{A}_{\mathbb{D}} = \{(d, e, f \mid D, E, F) : d, e, f \in \mathbb{F}_q, D, E, F \in \mathbb{D}\}$, $U_{\mathbb{D}} = U \cap \mathcal{A}_{\mathbb{D}}$. The stabilizer of $\mathcal{A}_{\mathbb{D}}$ in $F_4(q)$ has been determined in [18, 5.12] when the characteristic is not 2 or 3. We set $W_{\mathbb{D}} = U_{\mathbb{D}}/(U_{\mathbb{D}} \cap \langle I\rangle)$, and let $\mathcal{M}_{\mathbb{D}}$ be the projective points corresponding to white vectors in $W_{\mathbb{D}}$. If $\mathbb{D}$ is properly chosen, then the restriction of Q_0 to $W_{\mathbb{D}}$ is a nondegenerate quadratic form $Q_{\mathbb{D}}$ and the white points in $W_{\mathbb{D}}$ yield a tight set $\mathcal{M}_{\mathbb{D}}$ in the polar space associated with $(W_{\mathbb{D}}, Q_{\mathbb{D}})$. First suppose that $\mathbb{D}$ is a split quaternion. The stabilizer of $U_{\mathbb{D}}$ in $F_4(q)$ contains a copy of $\mathrm{Sp}_6(q)$ by [28, 4.8.9], so $\mathcal{M}_{\mathbb{D}}$ is invariant under $\mathrm{Sp}_6(q)$. If q is a power of 3, then we obtain a (q^2+1)-tight set in $Q(12, q)$. If $q \equiv 5, 11 \pmod{12}$ or q is an odd power of 2, then we obtain a (q^2+1)-tight set in $Q^-(13, q)$. Next suppose that $\mathbb{D}$ is a split quadratic extension of $\mathbb{F}_q$. The stabilizer of $U_{\mathbb{D}}$ in $F_4(q)$ contains a copy of $\mathrm{SL}_3(q)$ by [28, 4.8.9], so $\mathcal{M}_{\mathbb{D}}$ is invariant under $\mathrm{SL}_3(q)$. The $\mathrm{Sp}_6(q)$-invariant (q^2+1)-tight sets in $Q(12, q)$ (for $q = 3^f$) and $Q^-(13, q)$ for $q \equiv 5, 11 \pmod{12}$ have been constructed in [6] by using the geometry of the line Grassmannian of the symplectic polar space $W(5, q)$. We remark that the conclusion in [6, Proposition 14] also holds for $q \equiv 5 \pmod{12}$, so [6, Theorem 15] also holds for such q's. The $(q+1)$-tight sets in $Q(6, q)$ (for $q = 3^f$) and in $Q^-(7, q)$ (for $q \equiv 2 \pmod 3$) have been obtained by Ran and collaborators [26], and they informed us that the proof makes use of weak generalized hexagons of order $(q, 1)$ and is a slight modification of those in [21].

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