Edge Cover Through Edge Coloring

Guantao Chen[a] Songling Shan[b]

Submitted: Jun 29, 2024; Accepted: Feb 19, 2025; Published: May 13, 2025

Abstract

Let G be a multigraph. A subset F of $E(G)$ is an edge cover of G if every vertex of G is incident to an edge of F. The cover index, $\xi(G)$, is the largest number of edge covers into which the edges of G can be partitioned. Clearly $\xi(G) \leqslant \delta(G)$, the minimum degree of G. For $U \subseteq V(G)$, denote by $E^+(U)$ the set of edges incident to a vertex of U. When $|U|$ is odd, to cover all the vertices of U, any edge cover needs to contain at least $(|U|+1)/2$ edges from $E^+(U)$, indicating $\xi(G) \leqslant |E^+(U)|/((|U|+1)/2)$. Let $\rho_c(G)$, the co-density of G, be defined as the minimum of $|E^+(U)|/((|U|+1)/2)$ ranging over all $U \subseteq V(G)$, where $|U| \geqslant 3$ and $|U|$ is odd. Then $\rho_c(G)$ provides another upper bound on $\xi(G)$. Thus $\xi(G) \leqslant \min\{\delta(G), \lfloor \rho_c(G) \rfloor\}$. For a lower bound on $\xi(G)$, in 1978, Gupta conjectured that $\xi(G) \geqslant \min\{\delta(G)-1, \lfloor \rho_c(G) \rfloor\}$. Gupta himself verified the conjecture for simple graphs, and Cao et al. recently verified this conjecture when $\rho_c(G)$ is not an integer, assuming the Goldberg-Seymour Conjecture. (Proofs of the Goldberg-Seymour Conjecture have been announced in three arXiv manuscripts (1901.10316, 2308.15588, and 2407.09403), but have not yet been appeared for publication in peer-reviewed journals.) In this paper, also assuming the Goldberg-Seymour Conjecture, we confirm Gupta's conjecture when the maximum multiplicity of G is at most two or $\min\{\delta(G)-1, \lfloor \rho_c(G) \rfloor\} \leqslant 6$. The proof relies on a newly established result on edge colorings. The result holds independent interest and has the potential to significantly contribute towards resolving the conjecture entirely.

Mathematics Subject Classifications: 05C38

Keywords. edge cover; cover index; co-density; chromatic index.

1 Introduction

Graphs in this paper have no isolated vertex, may contain multiple edges but contain no loop. Let G be a graph. Denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For $v \in V(G)$, $d_G(v)$, the degree of v, is the number of edges of G

[a]Georgia State University, Atlanta, GA 30302, USA (gchen@gsu.edu).
[b]Auburn University, Auburn, AL 36849, USA (szs0398@auburn.edu).

that are incident with v. For $S \subseteq V(G)$, the subgraph of G induced on S is denoted by $G[S]$, and on $V(G) \setminus S$ is denoted by $G - S$. For notational simplicity we write $G - x$ for $G - \{x\}$. For $e \in E(G)$, $G - e$ is obtained from G by deleting the edge e. For an edge $e \notin E(G)$, $G + e$ is obtained by adding the edge e to G. Let $A, B \subseteq V(G)$ be disjoint. We denote by $E_G(A)$ the set of edges with both endvertices in A, $E_G(A, B)$ the set of edges with one endvertex in A and the other endvertex in B, and by $E_G^+(A)$ the set of edges of G incident with a vertex of A. Note that $E_G^+(A)$ is the union of $E_G(A)$ and $E_G(A, V(G) \setminus A)$. When $A = \{x\}$, we simply write $E_G(\{x\}, B)$ as $E_G(x, B)$. Let $e_G(A) = |E_G(A)|$, $e_G(A, B) = |E_G(A, B)|$, and $e_G^+(A) = |E_G^+(A)|$. When G is clear from the context, we skip the subscript G from the corresponding notation.

Let $F \subseteq E(G)$. The set F *saturates* $v \in V(G)$ if v is incident in G with an edge from F; otherwise F *misses* v. For $S \subseteq V(G)$, we say F saturates S if F saturates every vertex of S. We call F *an edge cover* of G if F saturates $V(G)$. The cover index, $\xi(G)$, is the largest number of edge covers into which the edges of G can be partitioned. Clearly $\xi(G) \leqslant \delta(G)$, the minimum degree of G. For any $U \subseteq V(G)$ such that $|U|$ is odd, every edge cover of G contains at least $(|U| + 1)/2$ edges from $E^+(U)$. Therefore, we have $\xi(G) \leqslant e^+(U)/((|U| + 1)/2)$. Let $\rho_c(G)$, the *co-density* of G, be defined as the minimum of $e^+(U)/((|U|+1)/2)$ ranging over all $U \subseteq V(G)$, where $|U| \geqslant 3$ and $|U|$ is odd. Then $\rho_c(G)$ provides another upper bound on $\xi(G)$. Thus $\xi(G) \leqslant \min\{\delta(G), \lfloor \rho_c(G) \rfloor\}$. For a lower bound on $\xi(G)$, in 1978, Gupta [6] conjectured that $\xi(G) \geqslant \min\{\delta(G) - 1, \lfloor \rho_c(G) \rfloor\}$, and he proved the conjecture when G is simple [5]. This conjecture can be viewed as a counterpart to the Goldberg-Seymour Conjecture, which concerns edge coloring of multigraphs. (While proofs of the Goldberg-Seymour Conjecture have been announced, see, e.g., [2, 7, 3], they are not yet published in peer-reviewed journals.)

A deeper connection exists between these two conjectures. Assuming the Goldberg-Seymour Conjecture, in 2023, Cao, Chen, Ding, Jing and Zang [1] verified Gupta's conjecture when $\rho_c(G)$ is not an integer. Here, again, assuming the validity of the Goldberg-Seymour Conjecture, we generalize Gupta's result from simple graphs to graphs with maximum multiplicity at most two and confirm the conjecture for graphs G with small $\delta(G)$ and $\rho_c(G)$ as stated below.

Theorem 1. *Let G be a graph and $k = \min\{\delta(G) - 1, \lfloor \rho_c(G) \rfloor\}$. If the maximum multiplicity of G is at most 2 or $k \leqslant 6$, then G has at least k edge-disjoint edge covers.*

As long as there exist k edge-disjoint edge covers, then the rest edges of G not included in the edge covers can be arbitrarily assigned to the edge covers to get a partition of $E(G)$. Thus, Theorem 1 implies Gupta's conjecture for the described classes of graphs. The proof of Theorem 1 relies on a newly established result on edge colorings, which might be of independent interest. We introduce some notation in order to state the result.

For two integers p and q, let $[p, q] = \{i \in \mathbb{Z} : p \leqslant i \leqslant q\}$. Let G be a graph and $m \geqslant 0$ be an integer. An *edge m-coloring* of G is a map φ: $E(G) \to [1, m]$ that assigns to every edge e of G a color $\varphi(e) \in [1, m]$ such that no two adjacent edges receive the same color. Denote by $\mathcal{C}^m(G)$ the set of all edge m-colorings of G. The *chromatic index* $\chi'(G)$ is the least integer $m \geqslant 0$ such that $\mathcal{C}^m(G) \neq \emptyset$. For a vertex $v \in V(G)$ and a coloring $\varphi \in \mathcal{C}^m(G)$

for some integer $m \geqslant 1$, define the two color sets $\varphi(v) = \{\varphi(f) : f \text{ is incident to } v \text{ in } G\}$ and $\overline{\varphi}(v) = [1, m] \setminus \varphi(v)$. We call $\varphi(v)$ the set of colors *presenting* at v and $\overline{\varphi}(v)$ the set of colors *missing* at v. For a color α, the edge set $E_\alpha = \{f \in E(G) \mid \varphi(f) = \alpha\}$ is called a *color class*. Clearly, E_α is a *matching* of G (possibly empty). For two distinct colors α, β, the subgraph of G induced by $E_\alpha \cup E_\beta$ is a union of disjoint paths and even cycles. Each nontrivial component of $E_\alpha \cup E_\beta$ is called an (α, β)-*chain* of G with respect to φ. For a vertex x and two distinct colors α, β such that exactly one of them is missing at x, we use $P_x(\alpha, \beta, \varphi)$ to denote the (α, β)-chain containing the vertex x.

Theorem 2. *Let G be a graph and $k \geqslant 1$ be an integer. Suppose $\Delta(G) \leqslant k+1$ and $\chi'(G) \leqslant k+2$. Let S be the set of vertices of G with degree at most $k/2$. Then there exists an edge $(k+2)$-coloring of G satisfying the following properties:*

(1) The color $k+2$ is missing at every vertex of S;

(2) If $k+1 \in \varphi(x)$ for some $x \in S$, then $P_x(k+1, k+2, \varphi)$ ends at a vertex of $V(G) \setminus S$.

These constraints on the graph G in Theorem 1 allow us to construct a special edge coloring, as defined in Theorem 2, for a graph H_1 derived from the original graph G. If this special coloring were achievable without the constraints on G, then Gupta's conjecture would be proven already. This highlights the potential of Theorem 2 to significantly advance the resolution of the conjecture.

The remainder of this paper is organized as follows. In the next section, we prove Theorem 2; in Section 3, we provide further preliminaries that are necessary for proving Theorem 1; and in the last section, we prove Theorem 1.

2 Proof of Theorem 2

Let G be a graph and $\varphi \in \mathcal{C}^m(G)$ for some integer $m \geqslant 1$. For $x, y \in V(G)$, if x and y are contained in the same (α, β)-chain with respect to φ, we say x and y are (α, β)-*linked*. Otherwise, they are (α, β)-*unlinked*.

For a vertex v, let $C_v(\alpha, \beta, \varphi)$ denote the unique (α, β)-chain containing v. If $C_v(\alpha, \beta, \varphi)$ is a path, we just write it as $P_v(\alpha, \beta, \varphi)$. The notation $P_v(\alpha, \beta, \varphi)$ is commonly used when we know $|\overline{\varphi}(v) \cap \{\alpha, \beta\}| = 1$. If we interchange the colors α and β on an (α, β)-chain C of G, we briefly say that the new coloring is obtained from φ by an (α, β)-*swap* on C, and we write it as φ/C. This operation is called a *Kempe-change*.

Proof of Theorem 2. For any $\varphi \in \mathcal{C}^{k+2}(G)$, we define

$$\begin{aligned} s_\varphi &= |\{x \in S : k+2 \in \varphi(x)\}|, \quad \text{and} \\ c_\varphi &= |\{P_x(k+1, k+2, \varphi) : P_x(k+1, k+2, \varphi) = P_y(k+1, k+2, \varphi) \\ &\qquad \text{for distinct } x, y \in S\}|, \end{aligned}$$

to be respectively the number of vertices of S at which the color $k+2$ presents and the number of $(k+1, k+2)$-chains (path-chain) with both endvertices in S under φ. We choose

$\varphi \in \mathcal{C}^{k+2}(G)$ with s_φ minimum and subject to this, with c_φ minimum. If $s_\varphi = c_\varphi = 0$, then we are done. Thus we assume $s_\varphi + c_\varphi > 0$. We consider two cases in finishing the proof.

Case 1: $s_\varphi > 0$.

Let $x \in S$ such that $k+2 \in \varphi(x)$. Since $d(x) \leqslant k/2$, there exists $\alpha \in [1,k]$ such that $\alpha \in \overline{\varphi}(x)$. We consider $P_x(\alpha, k+2, \varphi)$. If $P_x(\alpha, k+2, \varphi)$ ends at a vertex not in S or ends at a vertex from S that presents $k+2$, then $\psi := \varphi/P_x(\alpha, k+2, \varphi)$ is an edge $(k+2)$-coloring of G with $s_\psi < s_\varphi$. Thus we assume that $P_x(\alpha, k+2, \varphi)$ ends at a vertex $y \in S \setminus \{x\}$ such that $\alpha \in \varphi(y)$ and $k+2 \in \overline{\varphi}(y)$. Let

$$P_x(\alpha, k+2, \varphi) = v_0 v_1 \ldots v_{2t-1} v_{2t},$$

for some integer $t \geqslant 1$, where $v_0 := x$ and $v_{2t} := y$.

Since $|\varphi(x) \cup \varphi(y)| \leqslant d(x)+d(y) \leqslant k$, we have $\overline{\varphi}(x) \cap \overline{\varphi}(y) = [1, k+2] \setminus (\varphi(x) \cup \varphi(y)) \neq \emptyset$. Let $i \in [1, 2t]$ be the smallest index such that $\overline{\varphi}(v_i) \cap \overline{\varphi}(x) \neq \emptyset$. As $k+2 \in \varphi(x)$, $k+2 \notin \overline{\varphi}(v_i) \cap \overline{\varphi}(x)$. Among all the edge $(k+2)$-colorings ξ with $s_\xi = s_\varphi$, $c_\xi = c_\varphi$, and $P_x(\alpha, k+2, \xi) = P_x(\alpha, k+2, \varphi)$, we may assume φ is the one such that the index i is smallest.

If $i = 1$, then simply recoloring xv_1 by a color from $\overline{\varphi}(v_1) \cap \overline{\varphi}(x)$ gives a new coloring ψ with $s_\psi < s_\varphi$. Thus $i \geqslant 2$. Let $\beta \in \overline{\varphi}(v_i) \cap \overline{\varphi}(x) \subseteq [1, k+1]$. By the minimality of i, we have $\beta \in \varphi(v_{i-1})$. As $d(v_{i-1}) \leqslant k+1$ and $\alpha, \beta, k+2 \in \varphi(v_{i-1})$, there exists $\gamma \in \overline{\varphi}(v_{i-1}) \subseteq [1, k+2] \setminus \{\alpha, \beta, k+2\}$.

If v_i and v_{i-1} are not (β, γ)-linked with respect to φ, then let ψ be obtained by doing a Kemple-change on $P_{v_i}(\beta, \gamma, \varphi)$ and then recoloring the edge $v_{i-1}v_i$ on $P_x(\alpha, k+2, \varphi)$ by γ. Note that $s_\psi \leqslant s_\varphi$, and we have that $\alpha \in \overline{\psi}(v_{i-1})$ or $k+2 \in \overline{\psi}(v_{i-1})$, and $P_x(\alpha, k+2, \psi) = P_{v_{i-1}}(\alpha, k+2, \psi)$. If $\alpha \in \overline{\psi}(v_{i-1})$, then we can do a Kempe-change on $P_x(\alpha, k+2, \psi)$ to decrease s_ψ and so to decrease s_φ. Thus we assume that $k+2 \in \overline{\psi}(v_{i-1})$. If $v_{i-1} \in S$, then we have $s_\psi < s_\varphi$ already. Thus we assume $v_{i-1} \notin S$. Then we can do a Kempe-change on $P_x(\alpha, k+2, \psi)$ to decrease s_ψ and so to decrease s_φ.

Thus we assume now that v_i and v_{i-1} are (β, γ)-linked with respect to φ. Then let $\psi = \varphi/P_{v_i}(\beta, \gamma, \varphi)$. We have $s_\psi = s_\varphi$, $c_\psi = c_\varphi$, and $P_x(\alpha, k+2, \psi) = P_y(\alpha, k+2, \varphi)$. However, we have $\beta \in \overline{\psi}(v_{i-1}) \cap \overline{\psi}(x) \neq \emptyset$, contradicting the choice of φ.

Case 2: $s_\varphi = 0$ and $c_\varphi > 0$.

Then there exist distinct $x, y \in S$ such that $P_x(k+1, k+2, \varphi) = P_y(k+1, k+2, \varphi)$. Note that $k+2 \in \overline{\varphi}(x) \cap \overline{\varphi}(y)$, and $P_x(k+1, k+2, \varphi)$ is internally disjoint from S as $s_\varphi = 0$. Let

$$P_x(k+1, k+2, \varphi) = v_0 v_1 \ldots v_{2t} v_{2t+1},$$

for some integer $t \geqslant 0$, where $v_0 := x$ and $v_{2t+1} := y$. Since $d(x) + d(y) \leqslant k$ and $k+1 \in \varphi(x) \cap \varphi(y)$, we have $(\overline{\varphi}(x) \cap \overline{\varphi}(y)) \cap [1,k] \neq \emptyset$.

Let $i \in [1, 2t+1]$ be the smallest index such that $(\overline{\varphi}(v_i) \cap \overline{\varphi}(x)) \cap [1,k] \neq \emptyset$. Among all the edge $(k+2)$-colorings ξ with $s_\xi = 0$, $c_\xi = c_\varphi$ and $P_x(k+1, k+2, \xi) = P_x(k+1, k+2, \varphi)$, we may assume φ is the one such that the index i is smallest.

If $i = 1$, then recoloring xv_1 by a color from $(\overline{\varphi}(v_1) \cap \overline{\varphi}(x)) \cap [1,k]$ gives a new coloring ψ with $c_\psi < c_\varphi$. Furthermore, we still have $s_\psi = s_\varphi = 0$ as the new color is from $[1,k]$. This gives a contradiction to the choice of φ. Thus we assume $i \geqslant 2$. Let $\beta \in (\overline{\varphi}(v_i) \cap \overline{\varphi}(x)) \cap [1,k]$. By the minimality of i, we have $\beta \in \varphi(v_{i-1})$. As $d(v_{i-1}) \leqslant k+1$ and $\beta, k+1, k+2 \in \varphi(v_{i-1})$, there exists $\gamma \in \overline{\varphi}(v_{i-1}) \subseteq [1,k] \setminus \{\beta\}$.

If v_i and v_{i-1} are not (β,γ)-linked with respect to φ, then let ψ be obtained by doing a Kemple-change on $P_{v_i}(\beta,\gamma,\varphi)$ and then recoloring the edge $v_{i-1}v_i$ on $P_x(k+1,k+2,\varphi)$ by γ. Then $c_\psi < c_\varphi$. Furthermore, we still have $s_\psi = s_\varphi = 0$ as $\beta,\gamma \in [1,k]$. This gives a contradiction to the choice of φ. Thus we assume that v_i and v_{i-1} are (β,γ)-linked with respect to φ. Let $\psi = \varphi/P_{v_i}(\beta,\gamma,\varphi)$. We have $s_\psi = s_\varphi = 0$, $c_\psi = c_\varphi$, and $P_x(k+1,k+2,\psi) = P_x(k+1,k+2,\varphi)$. However, we have $\beta \in (\overline{\psi}(v_{i-1}) \cap \overline{\psi}(x)) \cap [1,k] \neq \emptyset$, contradicting the choice of φ. □

3 Further Preliminaries

For an integer $s \geqslant 1$, a graph G is *s-dense* if $|V(G)| \geqslant 3$ is odd and $|E(G)| = s(|V(G)|-1)/2$. As a maximum matching in G can have size at most $(|V(G)|-1)/2$, the lemma below is a consequence of G being s-dense, where a matching is *near perfect* in G if it misses only one vertex of G.

Lemma 3. *Let G be an s-dense graph with $\chi'(G) = s$ for some integer $s \geqslant 1$, and let $\varphi \in \mathcal{C}^s(G)$. Then for any two distinct $u, v \in V(G)$, we have $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$. In particular, each color class of φ is a near perfect matching of G, and each vertex $v \in V(G)$ is missed by exactly $s - d(v)$ of the color classes of φ.*

Let $\rho(G)$, the *density* of G, be defined as the maximum of $e(U)/((|U|-1)/2)$ ranging over all $U \subseteq V(G)$, where $|U| \geqslant 3$ and $|U|$ is odd. In the 1970s, Goldberg [4] and Seymour [8] independently conjectured that every graph G satisfies $\chi'(G) \leqslant \max\{\Delta(G)+1, \lceil \rho(G) \rceil\}$. Over the past four decades this conjecture has been a subject of extensive research. In 2019, Chen, Jing, and Zang [2] announced a proof of the Conjecture. An edge e of G is *critical* if $\chi'(G-e) < \chi'(G)$. As every graph G contains a connected subgraph H with $\chi'(H) = \chi'(G)$ such that every edge of H is critical, the lemma below is a consequence of Theorem 2.2(ii) from [2].

Lemma 4. *Let G be a graph with $\chi'(G) = s+1 \geqslant \Delta(G)+2$. Then G has a subgraph H and an edge $e \in E(H)$ such that $H - e$ is s-dense .*

Let G be a graph and $k = \min\{\delta(G)-1, \lfloor \rho_c(G) \rfloor\}$. A subset U of $V(G)$ is odd if $|U| \geqslant 3$ and $|U|$ is odd. An odd set U of G is *optimal* (with respect to k) if $e^+(U) = k(|U|+1)/2$. For an optimal set U of G, since $2e^+(U) = \sum_{v \in U} d(v) + e(U, V(G) \setminus U)$, we get $k(|U|+1) = \sum_{v \in U} d(v) + e(U, V(G) \setminus U) \geqslant (k+1)|U| + e(U, V(G) \setminus U)$ with equality holds if $\sum_{v \in U} d(v) = (k+1)|U|$. Thus

$$\begin{aligned} k &\geqslant |U| + e(U, V(G) \setminus U) \quad \text{and} \\ k &= |U| + e(U, V(G) \setminus U) \quad \text{if} \sum_{v \in U} d(v) = (k+1)|U|. \end{aligned} \tag{1}$$

We have the following property for optimal sets of G.

Lemma 5. *Let G be a graph with $k = \min\{\delta(G) - 1, \lfloor \rho_c(G) \rfloor\}$. Suppose that U is a minimal optimal set of G. Then for any optimal set U' of G with $U \not\subseteq U'$, we have $U \cap U' = \emptyset$.*

Proof. Suppose to the contrary that $U \cap U' \neq \emptyset$. Let

$$L = U \setminus U', \quad M = U \cap U', \quad R = U' \setminus U, \quad \text{and} \quad W = V(G) \setminus (U \cup U').$$

Since $U \not\subseteq U'$ and $U \cap U' \neq \emptyset$, we have $L, M \neq \emptyset$. As U is a minimal optimal set with $U \not\subseteq U'$, it follows that $U' \not\subseteq U$. Thus $R \neq \emptyset$ as well. By counting the edges within distinct parts, we have

$$\begin{aligned} e^+(U \cup U') &= e(L) + e(M) + e(R) + e(L, M) + e(M, R) + e(L, R) + e(L, W) + \\ &\quad\ e(M, W) + e(R, W), \\ e^+(U) &= e(L) + e(M) + e(L, M) + e(M, R) + e(L, R) + e(L, W) + e(M, W), \\ e^+(U') &= e(R) + e(M) + e(L, M) + e(M, R) + e(L, R) + e(R, W) + e(M, W), \\ e^+(M) &= e(M) + e(L, M) + e(M, R) + e(M, W), \\ e^+(L) &= e(L) + e(L, M) + e(L, R) + e(L, W), \\ e^+(R) &= e(R) + e(M, R) + e(L, R) + e(R, W). \end{aligned}$$

Therefore,

$$e^+(U \cup U') = e^+(U) + e^+(U') - e^+(M) - e(L, R).$$

If $|M| = 1$, then $e^+(M) \geqslant \delta(G) \geqslant k + 1 = k(|M| + 1)/2 + 1$. If $|M| \geqslant 3$ and $|M|$ is odd, then since $|M| < |U|$ and $\emptyset \neq M \subseteq U$, we know that M is not optimal by the choice of U. Thus $e^+(M) \geqslant k(|M| + 1)/2 + 1$.

Suppose first that $|M|$ is odd and so $|U \cup U'|$ is odd. Then

$$\begin{aligned} e^+(U \cup U') &= e^+(U) + e^+(U') - e^+(M) - e^+(L, R) \\ &\leqslant k(|U| + 1)/2 + k(|U'| + 1)/2 - (k(|M| + 1)/2 + 1) - e^+(L, R) \\ &= k(|U \cup U'| + 1)/2 - 1 - e^+(L, R) < k(|U \cup U'| + 1)/2, \end{aligned}$$

a contradiction to the assumption that $\lfloor \rho_c(G) \rfloor \geqslant k$.

Thus we assume that $|M|$ is even. Then $|L|$ and $|R|$ are odd. Again we have $e^+(L) \geqslant k(|L|+1)/2$ and $e^+(R) \geqslant k(|R|+1)/2$ by the assumption that $k = \min\{\delta(G)-1, \lfloor \rho_c(G) \rfloor\}$. As $2e(M) + e(L, M) + e(M, R) + e(M, W) = \sum_{x \in M} d(x) \geqslant (k + 1)|M|$, we get

$$\begin{aligned} e^+(U) + e^+(U') &= e^+(L) + e^+(R) + 2e(M) + e(L, M) + e(M, R) + 2e(M, W) \\ &\geqslant k(|L| + 1)/2 + k(|R| + 1)/2 + (k + 1)|M| + e(M, W) \\ &\geqslant k(|L| + 1)/2 + k(|R| + 1)/2 + k|M|/2 + k|M|/2 + |M| \\ &= k(|U| + 1)/2 + k(|U'| + 1)/2 + |M| \\ &\geqslant \frac{k(|U| + 1)}{2} + \frac{k(|U'| + 1)}{2} + 1, \end{aligned}$$

a contradiction to the assumption that both U and U' are optimal. $\square$

Let G be a graph and $k = \min\{\delta(G) - 1, \lfloor \rho_c(G) \rfloor\}$. We will show that when we are working with edge covers, in some sense, we can assume $\Delta(G) = k + 1$. For this, we introduce an operation called *edge-splitting*. Let $xy \in E(G)$. An edge-splitting at x with respect to xy gives a new graph G', which is obtained from G by deleting xy, adding a new vertex x', and adding the edge $x'y$. It is clear that $d_{G'}(x) = d_G(x) - 1$ and $d_{G'}(v) = d_G(v)$ for all $v \in V(G)$ with $v \neq x$.

Lemma 6. *Let G be a graph, $k = \min\{\delta(G) - 1, \lfloor \rho_c(G) \rfloor\}$. and $x \in V(G)$ with $d_G(x) \geqslant k + 2$. Let H be obtained through the following operation:*

- *If x is not contained in any optimal set of G, then we apply an edge-splitting at x with respect to an arbitrary edge incident with x, say xy;*
- *If x is contained in an optimal set of G, we let U be a minimal optimal set containing x. Let $y \in U$ with $xy \in E(G)$, and then we apply an edge-splitting at x with respect to xy. (Such a vertex y exists as $e_G(x, V(G) \setminus U) \leqslant k - |U|$ by Equation* (**1**)*.)*

Then $e_H^+(U) \geqslant k(|U| + 1)/2$ for any odd set $U \subseteq V(G)$.

Proof. Suppose to the contrary that there exists $U' \subseteq V(G)$ such that $e_H^+(U') \leqslant k(|U'| + 1)/2 - 1$. As we only applied one edge-splitting at x with respect to xy in getting H, it follows that $x \in U'$ and $y \notin U'$, $e_H^+(U') = k(|U'| + 1)/2 - 1$, and $e_G^+(U') = k(|U'| + 1)/2$. Thus U' is optimal in G.

As x is contained in the optimal set U' of G, the second operation in Lemma 6 was applied to get H from G. Thus there exists an optimal set U of G such that $x, y \in U$ and U is a minimal. Now we have $U \not\subseteq U'$ (since $y \in U \setminus U'$) and $x \in U'$. This shows a contradiction to Lemma 5. □

4 Proof of Theorem 1

In this section we complete the proof of Theorem 1.

Proof of Theorem 1. Let $V = V(G)$ and $E = E(G)$, and $k = \min\{\delta(G) - 1, \lfloor \rho_c(G) \rfloor\}$. Then $\delta(G) \geqslant k + 1$ and for any odd $U \subseteq V(G)$, we have $e_G^+(U) \geqslant k(|U| + 1)/2$. Recall that an odd $U \subseteq V(G)$ is optimal if $e_G^+(U) = k(|U| + 1)/2$. The general idea is first to iteratively apply the edge-splitting operations starting from G to produce a graph H with $V \subseteq V(H)$ such that $d_H(v) = k+1$ for each $v \in V$, and that $e_H^+(U) \geqslant k(|U|+1)/2$ for any odd $U \subseteq V(G)$. The graph H has chromatic index at least $k + 3$. However, by deleting one edge from each minimal optimal set U of H with $U \subseteq V$, the resulting graph H_1 is edge $(k+2)$-colorable. In particular, we can partition the edges of H_1 into $(k+2)$ disjoint matchings $M_1, \ldots, M_{k+2}$ with some good properties. Finally k disjoint edge covers of G is constructed based the $(k+2)$ matchings by adding edges of $M_{k+1} \cup M_{k+2}$ and the deleted edges in $E(H) \setminus E(H_1)$ to each of $M_1, \ldots, M_k$ if necessary to make each of them into an edge set that saturates V.

Algorithm 1 Edge-Splitting Algorithm

while there exists $x \in V$ with $d_G(x) \geqslant k+2$ **do**

Apply an edge-splitting at x using the operation defined in Lemma 6, and set G to be the resulting graph.

end while

We first apply the operation stated in Lemma 6 iteratively to get a graph H through the following algorithm.

Denote the graph resulting from Algorithm 1 by H. Now we have $d_H(v) = k+1$ for any $v \in V$ and $d_H(v) = 1$ for any $v \in V(H) \setminus V$. Furthermore, by Lemma 6, we have $e_H^+(U) \geqslant k(|U|+1)/2$ for any odd $U \subseteq V$. As $E(H[V]) \subseteq E$ and every edge from $e_H(V, V(H) \setminus V)$ corresponds to an edge of E, it suffices to show that H has k disjoint edge sets that each saturate V.

For any odd $U \subseteq V$ of H, we have $e_H(U) + e_H^+(U) = 2e_H(U) + e_H(U, V(H) \setminus U) = (k+1)|U|$. Thus

$$e_H(U) \begin{cases} \leqslant (k+1)|U| - k(|U|+1)/2 - 1 & \text{if } U \text{ is not optimal;} \\ = k(|U|-1)/2 + |U| - 1 = (k+2)(|U|-1)/2 & \\ & \\ = k(|U|-1)/2 + |U| = (k+2)(|U|-1)/2 + 1 & \text{if } U \text{ is optimal.} \end{cases} \quad (\mathbf{2})$$

By (**2**), any odd set U with $e_H(U) \geqslant (k+2)(|U|-1)/2 + 1$ must have $e_H(U) = (k+2)(|U|-1)/2+1$ and so U is an optimal set in H. By Lemma 5, all minimal optimal sets contained in V are vertex-disjoint. If exist, let $U_1, U_2, \ldots, U_t$ be all the minimal optimal sets of H that are contained in V, where $t \geqslant 1$ is an integer. As each U_i is odd and $e(U_i) = (k+2)(|U_i|-1)/2+1$, if they exist, then we know that $\chi'(H) \geqslant k+3$. However, we will show that after deleting one edge within each U_i, the resulting graph has smaller chromatic index. For each $i \in [1,t]$, we delete an edge x_iy_i from $H[U_i]$. Denote the resulting graph by H_1.

Claim 7. *We have* $\chi'(H_1) = k+2$.

Proof. As vertices of $V(H_1) \setminus V$ have degree 1 in H_1, it suffices to show that $\chi'(H_1[V]) = k+2$. Since $e(H_1[U_i]) = (k+2)(|U_i|-1)/2$ and U_i is an odd set, we know that $\chi'(H_1[V]) \geqslant k+2$. We show that $\chi'(H_1[V]) \leqslant k+2$. Suppose for a contradiction that $\chi'(H_1[V]) = s+1 \geqslant k+3 = \Delta(H)+2$ for some integer s. Applying Lemma 4, there is a subgraph $J \subseteq H_1[V]$ and an edge $e \in E(J)$ such that $J - e$ is s-dense. Thus $|E(J-e)| = s(|V(J)|-1)/2 \geqslant (k+2)(|V(J)|-1)/2$ and so $e_{H_1}(V(J)) \geqslant (k+2)(|V(J)|-1)/2+1$.

If $U_i \subseteq V(J)$ for some $i \in [1,t]$, then we have $e_H(V(J)) \geqslant e_{H_1}(V(J)) + 1 \geqslant (k+2)(|V(J)|-1)/2+2$. This gives a contradiction to (**2**) since $V(J)$ is an odd set. Thus $U_i \not\subseteq V(J)$ for any $i \in [1,t]$. Again, as $V(J)$ is an odd set and $e_H(V(J)) \geqslant e_{H_1}(V(J)) \geqslant (k+2)(|V(J)|-1)/2+1$, it follows from (**2**) that $V(J)$ is an optimal set of H. We let $U^* \subseteq V(J)$ be a minimal optimal set of H. By Lemma 5, we must have $U^* = U_i$ for some $i \in [1,t]$. However, this contradicts our previous assumption that $U_i \not\subseteq V(J)$ for any $i \in [1,t]$. Thus we must have $\chi'(H_1) \leqslant k+2$, as desired. $\square$

Let H_2 be obtained from H_1 by contracting each U_i into a single vertex u_i for each $i \in [1,t]$.

Claim 8. *We have $d_{H_2}(u_i) \leqslant k/2$ for each $i \in [1,t]$.*

Proof. Suppose, without loss of generality, that $|U_1| \leqslant |U_2| \leqslant \ldots \leqslant |U_t|$. Then by (**1**), we have $e_H(U_1, V(H) \setminus U_1) \geqslant e_H(U_2, V(H) \setminus U_2) \geqslant \ldots \geqslant e_H(U_t, V(H) \setminus U_t)$. Since H_1 was obtained from H by deleting one edge within each U_i, we have $d_{H_2}(u_i) = e_H(U_i, V(H)\setminus U_i)$. Thus $d_{H_2}(u_1) \geqslant d_{H_2}(u_2) \geqslant \ldots \geqslant d_{H_2}(u_t)$. It then suffices to show that $d_{H_2}(u_1) \leqslant k/2$, or equivalently $e_H(U_1, V(H) \setminus U_1) \leqslant k/2$. As $(k+2)(|U_1|-1)+2 = 2e_H(U_1)$ by (**2**), when the maximum multiplicity of G is at most 2, we have $2e_H(U_1) \leqslant 2(|U_1|-1)|U_1|$ and so $k+2 \leqslant 2|U_1|$. This gives $|U_1| \geqslant (k+2)/2$. Now by (**1**) that $k = |U_1| + e_H(U_1, V(H)\setminus U_1)$, we get $e_H(U_1, V(H)\setminus U_1) = k - |U_1| \leqslant k-(k+2)/2 < k/2$. When $k \leqslant 6$, then as $|U_1| \geqslant 3$, $k = |U_1| + e_H(U_1, V(H) \setminus U_1)$ from (**1**) implies that $e_H(U_1, V(H) \setminus U_1) \leqslant k/2$. Therefore $d_{H_2}(u_1) \leqslant k/2$ and thus $d_{H_2}(u_i) \leqslant k/2$ for each $i \in [1,t]$. $\square$

For each $i \in [1,t]$, as $e_{H_1}(U_i) = e_H(U_i) - 1 = (k+2)(|U_i|-1)/2$, by Lemma 3, we know that for any $\varphi \in \mathcal{C}^{k+2}(H_1)$, the colors on the edges in $E_{H_1}(U_i, V(H_1) \setminus U_i)$ under φ are all distinct. Thus the graph H_2 is edge $(k+2)$-colorable. By Theorem 2, H_2 has an edge $(k+2)$-coloring φ satisfying the following two properties: (1) the color $k+2$ is missing at every vertex in $\{u_1, \ldots, u_t\}$; and (2) if $k+1 \in \varphi(u_i)$ for some $i \in [1,t]$, then $P_{u_i}(k+1, k+2, \varphi)$ does not end at any vertex from $\{u_1, \ldots, u_t\} \setminus \{u_i\}$. We extend the coloring φ of H_2 into a coloring ψ of H_1 using $(k+2)$ colors. We claim that such an extension is possible.

Claim 9. *For each $i \in [1,t]$, there is an edge $(k+2)$-coloring φ_i of $H_1[U_i]$ that satisfies the following two properties:*

(i) The coloring φ_i coincides with φ: for any $uw \in E_{H_1}(U_i, V(H_1) \setminus U_i)$ with $u \in U_i$, the color $\varphi(u_i w)$ is missing at u under φ_i;

(ii) The color $k+2$ is missing at x_i.

Proof. By Claim 7, $H_1[U_i]$ is edge $(k+2)$-colorbale. Since $e(H_1[U_i]) = (k+2)(|U_i|-1)/2$ and U_i is an odd set, it follows that edges of $H_1[U_i]$ can be partitioned into $k+2$ near perfect matchings of $H_1[U_i]$. Let $F_1, \ldots, F_{k+2}$ be a partition of edges of $H_1[U_i]$ into near perfect matchings. Since $d_{H_1[U_i]}(u) = k+1-e_{H_1}(u, V(H_1) \setminus U_i)$ for $u \in U_i \setminus \{x_i, y_i\}$ and $d_{H_1[U_i]}(u) = k - e_{H_1}(u, V(H_1)\setminus U_i)$ for $u \in \{x_i, y_i\}$, by Lemma 3, we know that each vertex $u \in U_i \setminus \{x_i, y_i\}$ is missed by exactly $(k+2)-(k+1-e_{H_1}(u, V(H_1)\setminus U_i)) = 1 + e_{H_1}(u, V(H_1) \setminus U_i)$ of those matchings, and each $u \in \{x_i, y_i\}$ is missed by exactly $2 + e_{H_1}(u, V(H_1) \setminus U_i)$ of those matchings. For each $u \in U_i$, we let $\varphi(u) = \{\varphi(u_i w) : u_i w \in E(H_2), uw \in E(H_1)\}$ be the set of colors presenting on edges of H_2 incident with u_i which are corresponding to edges incident with u in H_1. We now define an edge $(k+2)$-coloring φ_i of $H_1[U_i]$ by assigning appropriate colors to edges of these $(k+2)$ matchings as followings:

- For one matching, without loss of generality say F_{k+2}, that misses x_i, we assign color $k+2$ to each of its edges: This assignment coincides with φ as we have $k+2 \in \overline{\varphi}(u_i)$.

- For each vertex $u \in U_i$ and $|\varphi(u)|$ of F_j's with $j \in [1, k+1]$ such that F_j misses u, we assign a distinct color from $\varphi(u)$ to F_j. Since there are $1 + e_{H_1}(u, V(H_1) \setminus U_i)$ of the matchings missing u and $|\varphi(u)| = e_{H_1}(u, V(H_1) \setminus U_i)$, all the colors in $\varphi(u)$ are used. Under this assignment: for any edge $uw \in E_{H_1}(U_i, V(H_1) \setminus U_i)$, the color $\varphi(u_i w)$ on the edge $u_i w$ of H_2 is missing at u.

- After the above two procedures, all colors in $\varphi(u_i) \cup \{k+2\}$ are used on $|\varphi(u_i) \cup \{k+2\}|$ of the matchings in $F_1, \dots, F_{k+2}$. Thus there are $k+2-|\varphi(u_i) \cup \{k+2\}|$ of the matchings that have not assigned a color so far. We assign each color from $[1, k+2] \setminus (\varphi(u_i) \cup \{k+2\})$ to all the edges of exactly one of the rest uncolored matchings.

By the construction above, φ_i is an edge $(k+2)$-coloring of $H_1[U_i]$ that satisfies Properties (i) and (ii). □

By Claim 9, we find an edge $(k+2)$-coloring ψ of H_1 with the following properties:

(1) The color $(k+2)$ is missing at each vertex from $\{x_1, \dots, x_t\}$;

(2) If the color $k+1$ presents on an edge from $E_{H_1}(U_i, V(H_1) \setminus U_i)$ for some $i \in [1, t]$, then the corresponding $(k+1, k+2)$-chain including that edge ends has one of its endvertex from $V(H_1) \setminus (\bigcup_{i=1}^{t} U_i)$.

We let $M_1, \dots, M_{k+2}$ be the color classes of ψ corresponding to the colors $1, \dots, k+2$ respectively. We will add edges from $M_{k+1} \cup M_{k+2} \cup \{x_1y_1, \dots, x_ty_t\}$ to each of $M_1, \dots, M_k$ if necessary to modify them into k disjoint edge sets of H that each saturate V. To do so, let D^* be the subgraph of H_1 induced on $M_{k+1} \cup M_{k+2}$. As $\Delta(D^*) \leqslant 2$, each component of D^* is either a cycle or a path. We orient D^* such that each of its component is either a directed cycle or a directed path. In particular, if for some $i \in [1, t]$, x_i is an endvertex of a path-component of D^*, then the path is oriented towards x_i. Note that by Property (2) of ψ, if a path has x_i as one of its endvertex, then its another endvertex is a vertex from $V(H_1) \setminus (\bigcup_{i=1}^{t} U_i)$. Let D be the orientation of D^*.

Each vertex w from $V \setminus \{x_i, y_i : i \in [1, t]\}$ has degree $k+1$ in H_1, and so it is missed by at most one of $M_1, \dots, M_k$. If w is missed by exactly one of $M_1, \dots, M_k$, then it has degree two in D^* and so has indegree one in D. As the color $k+2$ is missing at x_i for each $i \in [1, t]$, x_i has degree at most one in D^*, and if x_i has degree one in D^*, then x_i also has indegree one in D by our orientation of D^*. Vertex y_i for $i \in [1, t]$ has degree k in H_1 and so can be missed by at most two of $M_1, \dots, M_k$. If y_i is missed by exactly two of $M_1, \dots, M_k$, then it has degree two in D^* and so it has indegree one in D.

Now for each vertex $w \in V \setminus \{y_1, \dots, y_t\}$, if w is missed by exactly one matching M_i for some $i \in [1, k]$, we let zw be the arc of D with w as head. We add to M_i the edge zw. Now let $w \in \{y_1, \dots, y_t\}$, say $w = y_j$ for some $j \in [1, t]$. If y_j is missed by exactly one matching M_i for some $i \in [1, k]$, we add the edge x_jy_j to M_i. If y_j is missed by exactly two matchings from $M_1, \dots, M_k$, we let zy_j be the arc of D with y_j as head. Then we add zy_j to one of the matchings from $M_1, \dots, M_k$ that misses y_j, and we add x_jy_j to the other matching from $M_1, \dots, M_k$ that misses y_j. Denote by $M_1^*, \dots, M_k^*$ is corresponding modifications of $M_1, \dots, M_k$, respectively. Now each vertex $w \in V$ is saturated by each of $M_1^*, \dots, M_k^*$, and so $M_1^*, \dots, M_k^*$ are k disjoint edge sets of H that each saturate V. The proof of Theorem 1 is now complete. □

Acknowledgment

The authors thank the referees for their careful reading and insightful comments. Guantao Chen was partially supported by NSF grant DMS-2154331 and Songling Shan was partially supported by NSF grant DMS-2345869.

References

[1] Y. Cao, G. Chen, G. Ding, G. Jing, and W. Zang. On Gupta's codensity conjecture. *SIAM J. Discrete Math.*, 37(3):1666–1673, 2023.

[2] G. Chen, G. Jing, and W. Zang. Proof of the Goldberg-Seymour Conjecture on Edge-Colorings of Multigraphs. arXiv:1901.10316, 2019.

[3] G. Chen, Y. Hao, X. Yu, and W. Zang. A short proof of the goldberg-seymour conjecture. arXiv:2407.09403, 2024.

[4] M. K. Goldberg. Multigraphs with a chromatic index that is nearly maximal. *Diskret. Analiz*, (23):3–7, 72, 1973.

[5] R. P. Gupta. On decompositions of a multigraph into spanning subgraphs. *Bull. Amer. Math. Soc.*, (80):500–502, 1974.

[6] R. P. Gupta. On the chromatic index and the cover index of a multigraph. In *Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976)*, Lecture Notes in Math., Vol. 642, pages 204–215. Springer, Berlin, 1978.

[7] G. Jing. On edge coloring of multigraphs. arXiv:2308.15588, 2023.

[8] P. D. Seymour. On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc. (3)*, 38(3):423–460, 1979.