Frozen Colourings in $2K_2$-Free Graphs

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Abstract

The *reconfiguration graph of the k-colourings* of a graph G, denoted $\mathcal{R}_k(G)$, is the graph whose vertices are the k-colourings of G and two vertices of $\mathcal{R}_k(G)$ are joined by an edge if the colourings of G they correspond to differ in colour on exactly one vertex. A k-colouring of a graph G is called *frozen* if for every vertex $v \in V(G)$, v is adjacent to a vertex of every colour different from its colour.

A clique partition is a partition of the vertices of a graph into cliques. A clique partition is called a k-clique-partition if it contains at most k cliques. Clearly, a k-colouring of a graph G corresponds precisely to a k-clique-partition of its complement, $\overline{G}$. A k-clique-partition $\mathcal{Q}$ of a graph H is called *frozen* if for every vertex $v \in V(H)$, v has a non-neighbour in each of the cliques of $\mathcal{Q}$ other than the one containing v.

The complement of the cycle on four vertices, C_4, is called $2K_2$. We give several infinite classes of $2K_2$-free graphs with frozen colourings. We give an operation that transforms a k-chromatic graph with a frozen $(k+1)$-colouring into a $(k+1)$-chromatic graph with a frozen $(k+2)$-colouring. The operation requires some restrictions on the graph, the colouring, and the frozen colouring. The operation preserves being $2K_2$-free. Using this we prove that for all $k \geqslant 4$, there is a k-chromatic $2K_2$-free graph with a frozen $(k+1)$-colouring. We prove these results by studying frozen clique partitions in C_4-free graphs.

We say a graph G is *recolourable* if $R_\ell(G)$ is connected for all ℓ greater than the chromatic number of G. We prove that every 3-chromatic $2K_2$-free graph G is recolourable and that for all ℓ greater than the chromatic number of G, the diameter of $R_\ell(G)$ is at most $14n$ where n is the number of vertices of G.

Mathematics Subject Classifications: 0C15

1 Introduction

All graphs in this paper are finite and simple. For a simple graph G, the *complement* $\overline{G}$ of G is the simple graph with vertex-set $V(G)$ and where uv is an edge of $\overline{G}$ if and only

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if uv is not an edge of G. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. We use $n = |V(G)|$ to denote the number of vertices of G when the context is clear. An *independent set* in a graph G is a set of vertices no two of which are joined by an edge; a *clique* is a set of vertices every pair of which are joined by an edge. For a positive integer k, a *k-colouring* of G is a partition $\mathcal{C}$ of the vertices into at most k independent sets, called *colour classes*. A *k-clique-partition* is a partition $\mathcal{Q}$ of the vertices into at most k cliques. Clearly, $\mathcal{C}$ is a k-colouring of G if and only if $\mathcal{C}$ is a k-clique-partition of $\overline{G}$.

We say that G is *k-colourable* if it admits a k-colouring and is *q-clique-partitionable* if it admits a q-clique-partition. The *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k such that G is k-colourable and the *clique partition number* of G, denoted $\theta(G)$, is the smallest integer q such that G is q-clique-partitionable. Clearly, $\chi(G) = \theta(\overline{G})$. A graph G whose chromatic number is k is called k-chromatic.

The *reconfiguration graph of the k-colourings*, denoted $\mathcal{R}_k(G)$, is the graph whose vertices are the k-colourings of G and two vertices are joined by an edge in $\mathcal{R}_k(G)$ if the colourings they correspond to differ in colour on exactly one vertex. Equivalently, two k-colourings are adjacent in $\mathcal{R}_k(G)$ if some vertex v can be moved from the part of the partition it is in (that is, from the colour class it is in) to another part, say U, of the partition so that the new partition is a colouring. This can be done exactly when v is not adjacent to any vertex of U. We say that G is *k-mixing* if $\mathcal{R}_k(G)$ is connected, and that G is *recolourable* if G is k-mixing for all $k > \chi(G)$.

We can also consider the reconfiguration graph of the q-clique-partitions of a graph G. The vertices of the reconfiguration graph are the q-clique-partitions of G and two vertices are joined by an edge in the reconfiguration graph if some vertex v can be moved from the part of the partition it is in (that is, from the clique it is in) to another part, say U, of the partition so that the new partition is a clique partition. This can be done exactly when v is adjacent to every vertex of U.

Considering colourings and clique partitions as partitions of the vertex-set of a graph, the reconfiguration graph of the k-clique-partitions of $\overline{G}$ is precisely $\mathcal{R}_k(G)$. (We comment that normally in mathematics, a partition is thought of as a set of non-empty sets. In reconfiguration of graph colourings, two colourings of a graph are considered different if some vertex has a different colour in the two colourings. So the sets in the partition are really ordered: interchanging the colours of the vertices in two colour classes gives a different colouring. The same concept of order applies to reconfiguration of clique partitions. Also, some of the sets of a colouring or a clique partition can be empty.)

A k-colouring of a graph G is called *frozen* if it is an isolated vertex in $\mathcal{R}_k(G)$; in other words, for every vertex $v \in V(G)$, each of the k colours appears in the closed neighbourhood of v, or equivalently, if v has a neighbour in each of the colour classes different from the colour class it is in. One way to show that a graph G is not k-mixing is to exhibit a frozen k-colouring of G. Since every k-colouring of K_k is frozen, it is common to study $\mathcal{R}_{k+1}(G)$ for a k-colourable graph G.

A q-clique-partition of a graph G is called *frozen* if for every vertex $v \in V(G)$, v has a non-neighbour in each clique of the partition different from the clique it is in. Note that when considering colourings and clique partitions as partitions of the same set V of vertices, a partition corresponding to a colouring of G is frozen if and only if the same partition, considered as a clique partition of $\overline{G}$, is frozen.

Dunbar et al. [8] used the term *fall colouring* for frozen colouring, and proved that for each $k \geqslant 3$, the problem of deciding whether an input graph admits a frozen k-colouring is NP-complete. Cockayne and Hedetniemi [7] used the term *indominable graph* for a graph which admits a frozen colouring.

The cycle on six vertices, C_6, admits a frozen 3-colouring, and has the smallest number of vertices of a graph G which admits a frozen k-colouring where $k > \chi(G)$. In fact, a cycle C_n admits a frozen 3-colouring if and only if $n \equiv 0 (mod\ 3)$.

2 Preliminaries

For a vertex $v \in V(G)$, the *open neighbourhood*, $N(v)$, of v is the set of vertices adjacent to v in G. The *closed neighbourhood, $N[v]$, of* v is the set of vertices adjacent to v in G together with v.

As usual, let P_n, C_n, and K_n denote the path, cycle, and complete graph on n vertices, respectively. We sometimes refer to K_3 as a *triangle* and C_4 as a *square*.

For two vertex-disjoint graphs G and H, the *disjoint union* of G and H, denoted by $G + H$, is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. For a positive integer t, we use tG to denote the disjoint union of t copies of G. In particular, the graph $2K_2$ consists of the disjoint union of two copies of K_2. The complement of $2K_2$ is C_4. The *paw* is the graph on four vertices consisting of a K_3 together with another vertex adjacent to exactly one vertex of the K_3. The *diamond* is K_4 with one edge deleted (often referred to as $K_4 - e$). The edge of the diamond whose end-vertices are of degree 3 is called the *middle edge*.

The subgraph of a graph G *induced* by a subset $S \subseteq V(G)$ is the graph whose vertex-set is S and whose edge-set is all edges of G with both ends in S. For a fixed graph H, graph G is H-free if no induced subgraph of G is isomorphic to H. For a set $\mathcal{H}$ of graphs, G is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$.

A *universal vertex* in a graph G is a vertex which is adjacent to every other vertex of G. An *isolated vertex* in a graph G is a vertex which is not adjacent to any vertex of G. The *join* of two vertex-disjoint graphs G and H is obtained by adding all edges between a vertex of G and a vertex of H. In particular, adding a universal vertex v to a graph G is the same as taking the join of G and a graph consisting of one vertex. Two sets of vertices are called *anticomplete (to eachother)* if there is no edge with one end in one set and the other end in the other set. Two sets of vertices are called *complete (to each other)* if there are all possible edges with one end in one set and the other end in the other set.

It is quite easy to see and is used in several papers (see, for example, [9]) that:

Proposition 1. *If G is a k-chromatic graph which admits a frozen ℓ-colouring and if H is an r-chromatic graph which admits a frozen s-colouring, then the join of G and H is a $(k+r)$-chromatic graph which admits a frozen $(\ell+s)$-colouring. In particular, adding a universal vertex to k-chromatic graph which admits a frozen ℓ-colouring results in a $(k+1)$-chromatic graph which admits a frozen $(\ell+1)$-colouring.*

Further, note that if G and H are $2K_2$-free, then so is their join.

A *perfect matching* M in a graph G is a set of edges such that each vertex of G is incident to exactly one edge of M. For an integer $s \geqslant 2$, let $K_{s,s}$ denote the complete bipartite graph with s vertices in each part, and let B_s denote $K_{s,s}$ with a perfect matching removed. In [8] and [6], it was proved that B_s has a frozen s-colouring: give each vertex in one paritite set and its non-neigbhour in the other partite set the same colour. Note that B_s is P_6-free.

3 Our contributions

A question that has received some attention (see for example, [3], [12] and [9]) is:

Question 2. Given positive integers k and t, does there exist a k-colourable P_t-free graph which admits a frozen $(k+1)$-colouring?

Several authors have contributed to the solution of this problem, resulting in the following theorem.

Theorem 3. *([3, 4, 6, 8, 9, 11], Theorem 4, Theorem 5) There exists a k-colourable P_t-free graph with a frozen $(k+1)$-colouring if and only if ($t \geqslant 6$ and $k \geqslant 2$) or ($t = 5$ and $k \geqslant 4$).*

The graphs B_s show that for all $t \geqslant 6$ and $k \geqslant 2$, the answer to the question is yes [6, 8].

Recall that a graph G is *recolourable* if $R_\ell(G)$ is connected for all $\ell \geqslant \chi(G)+1$. Bonamy and Bousquet [3] proved that every P_4-free graph G is recolourable, thus for $t \leqslant 4$, the answer to the question is no.

Bonamy et al. [4] proved that every 2-colourable P_5-free graph is recolourable, so for $t = 5$ and $k = 2$, the answer to the question is no.

Feghali and Merkel [9] gave a 7-chromatic $2K_2$-free graph on 16 vertices which admits a frozen 8-colouring. By adding universal vertices, it follows that the answer to Question 2 is yes for $t = 5$ and $k \geqslant 7$.

Feghali and Merkel [9] asked about the remaining cases. We answer this in the positive for $k \in \{4, 5, 6\}$ with the following theorem.

Theorem 4. *For every $k \geqslant 4$, there is a k-chromatic $2K_2$-free graph with a frozen $(k+1)$-colouring.*

In Section 4, we prove:

Theorem 5. *Every 3-chromatic $2K_2$-free graph G is recolourable with ℓ-recolouring diameter at most $14n$, for all $\ell \geqslant \chi(G)+1$.*

Thus the only remaining case of Question 2 is when $t = 5$ and $k = 3$, and the graph contains a $2K_2$. Recently, Lei et al. [11] proved that every 3-chromatic P_5-free graph is recolourable, thus for $t = 5$ and $k = 3$, the answer to the question is no.

It follows that Theorem 3 holds.

In [1], it was proved that for a fixed graph H, every H-free graph is recolourable if and only if H is an induced subgraph of P_4 or of $K_3 + K_1$. Where H_1 and H_2 are two fixed graphs on four vertices, it was determined in [2] whether or not all (H_1, H_2)-free graphs were recolourable except for $(2K_2, K_4)$-free graphs. This class of graphs is known to be 4-colourable [10]. Further in [2], it was proved that every $(2K_2, K_3)$-free graph is recolourable. Thus Theorem 5 brings us close to a dichotomy theorem for recolourability when two graphs on four vertices are forbidden as induced subgraphs. The only open case remaining is whether all 4-chromatic $(2K_2, K_4)$-free graphs are recolourable.

The first and third authors did a computer search on all graphs with at most ten vertices to find k-colourable $2K_2$-free graphs which admit a frozen $(k + 1)$-colouring. Only two graphs were found. One was the graph we call D_2. See Figure 1 for the complementary graph, $\overline{D_2}$. Note that what is shown in Figure 1 is actually a 4-clique-partition and a frozen 5-clique-partition of $\overline{D_2}$; numbers are used to indicate which clique a vertex is in. The other graph they found is one we call F_2, which is D_2 with one edge added (the edge we will later call u_1u_2). See Figure 3 for $\overline{F_2}$. Both graphs are 4-chromatic and admit a frozen 5-colouring. We believe that these two graphs are the smallest k-colourable $2K_2$-free graphs which admit a frozen $(k + 1)$-colouring.

In Section 5, we give four infinite classes of k-colourable $2K_2$-free graphs which admit frozen $(k + p)$-colourings for various values of k and p. These graphs are connected and not decomposable by the join operation. The graphs we construct are dense, so we study clique-partitions in their complements.

In Section 6, we give an operation which transforms a k-chromatic graph with a frozen $(k + 1)$-colouring into a $(k + 1)$-chromatic graph with a frozen $(k + 2)$-colouring. The operation consists of carefully choosing an edge in the complement and subdividing it twice. Note that the operation requires some restrictions on the graph, the colouring, and the frozen colouring, and an appropriate edge may not exist. See Figure 7. Further, the operation preserves being $2K_2$-free and does not add universal vertices or use the join operation. Again, our approach is to study clique partitions.

In Section 7, we combine the results of Sections 5 and 6 to prove Theorem 4.

Section 8 has some concluding remarks.

4 $2K_2$-free graphs with chromatic number 3 are recolourable

We use the following result of Bonamy and Bousquet [3].

Lemma 6 (Renaming Lemma [3]). *Let β' and γ' be two k-colourings of G that induce the same partition of vertices into colour classes and let $\ell \geqslant k + 1$. Then β' can be recoloured into γ' in $\mathcal{R}_\ell(G)$ by recolouring each vertex at most 2 times.*

For graph G and a positive integer k, we can think of a k-colouring of G as a function $\beta \colon V(G) \to \{1, 2, \ldots, k\}$ such that for each edge $uv \in E(G)$, $\beta(u) \neq \beta(v)$. We use $[k]$ to denote $\{1, 2, \ldots, k\}$.

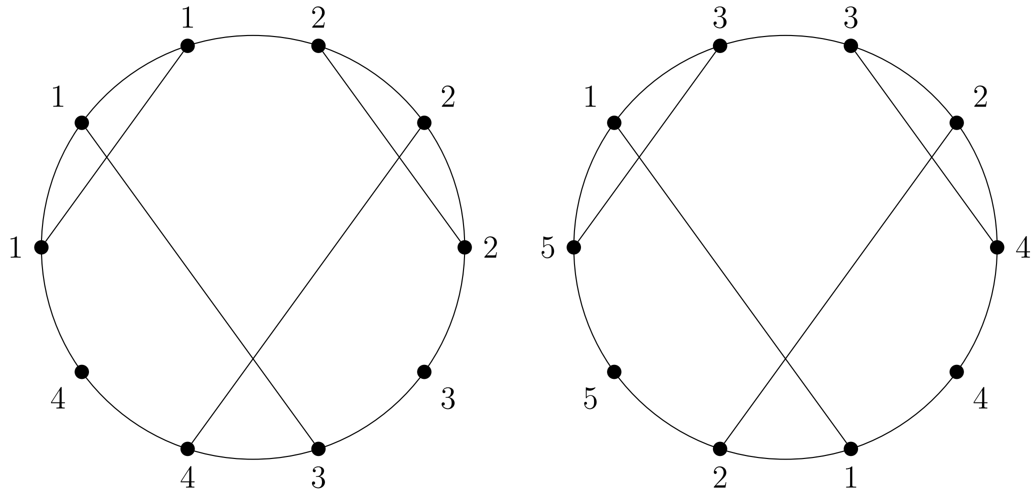

Figure 1: A square-free graph $\overline{D_2}$ with a 4-clique-partition (left) and a frozen 5-clique-partition (right). The numbers indicate which clique a vertex is in. Equivalently, the numbers indicate a 4-colouring of the complement D_2 of the graph shown (left) and a frozen 5-colouring of D_2 (right).

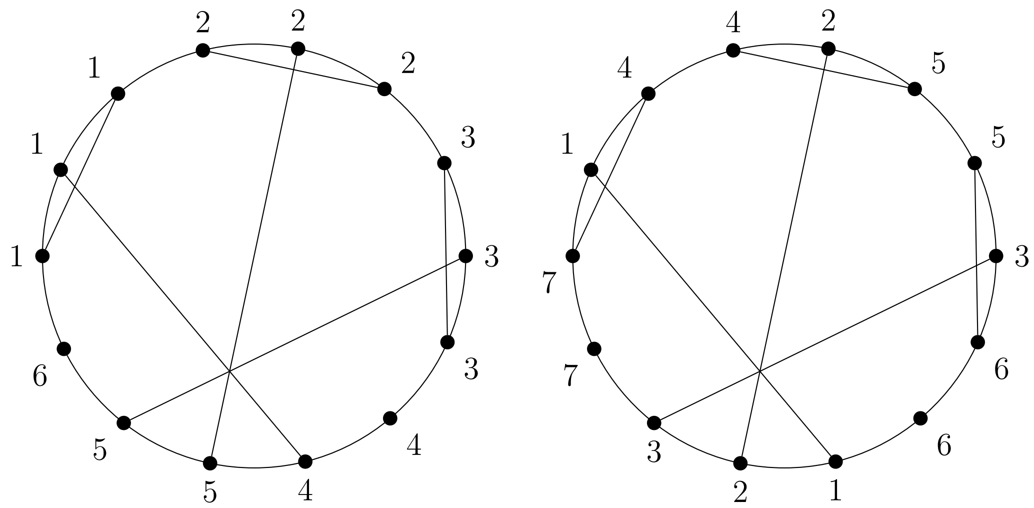

Figure 2: A square-free graph $\overline{D_3}$ with a 6-clique-partition (left) and a frozen 7-clique-partition (right). The numbers indicate which clique a vertex is in. Equivalently, the numbers indicate a 6-colouring of the complement D_3 of the graph shown (left) and a frozen 7-colouring of D_3 (right).

The *diameter* of a graph is the length of a longest shortest path between any two distinct vertices of the graph. The *k-recolouring diameter* of G is the diameter of $\mathcal{R}_k(G)$.

A bipartite graph G is *chordal bipartite* if it does not contain an induced cycle of length more than four. Note that every $2K_2$-free bipartite graph G is a chordal bipartite graph and hence recolourable with ℓ-recolouring diameter at most $2n^2$, for all $\ell \geqslant \chi(G)+1$ [4]. This also follows from the fact that every ($2K_2$, triangle)-free graph G is recolourable with ℓ-recolouring diameter at most $2n^2$, for all $\ell \geqslant \chi(G)+1$ [2]. Here we improve the upper bound on the ℓ-recolouring diameter of $2K_2$-free bipartite graphs.

Lemma 7. *Let G be a $2K_2$-free graph. Suppose $V(G)$ can be partitioned into independent sets $A_1, A_2, \dots, A_i$ such that A_1 is (inclusion-wise) maximal. Then for each $j \in \{2, \dots, i\}$, A_1 contains a vertex complete to A_j.*

Proof. Let G be a $2K_2$-free graph. Partition $V(G)$ into independent sets $A_1, A_2, \dots, A_i$ such that A_1 is (inclusion-wise) maximal. For each $j \in \{2, \dots, i\}$, choose a vertex in A_1, say x_j, such that $N(x_j) \cap A_j$ is maximized. If x_j is not complete to A_j, there is a vertex y in A_j non-adjacent to x_j. By the maximality of A_1, y has a neighbour u in A_1. By the choice of x_j, there is a vertex v in A_j adjacent to x_j but non-adjacent to u. Then $\{x_j, v, y, u\}$ induces a $2K_2$, a contradiction. Thus x_j is complete to A_j. □

Theorem 8. *Every $2K_2$-free bipartite graph G is recolourable with ℓ-recolouring diameter at most $4n$, for all $\ell \geqslant \chi(G)+1$.*

Proof. Let G be a $2K_2$-free bipartite graph. Let $\ell \geqslant 3$. Partition $V(G)$ into independent sets A_1 and A_2 such that A_1 is (inclusion-wise) maximal. Given any ℓ-colouring of G we prove that we can reach a 2-colouring of G that partitions the vertex-set into A_1 and A_2 by recolouring each vertex at most once. By the Renaming Lemma, there is a path between any two 2-colourings of G that partition the vertex-set into A_1 and A_2, where each vertex is recoloured at most twice. Starting from any two ℓ-colourings of G, β and γ, we can reach 2-colourings β' and γ' in $R_\ell(G)$, respectively, which partition the vertex-set into A_1 and A_2. Then we can obtain γ from β by recolouring vertices starting from β to β' to γ' to γ. Each vertex will be recoloured at most 4 times to go from β to γ in $R_\ell(G)$.

By Lemma 7, A_1 contains a vertex, say x, complete to A_2. Let β be any ℓ-colouring of G. There is no vertex in A_2 coloured $\beta(x)$. Recolour each vertex in A_1 with the colour $\beta(x)$ and recolour each vertex in A_2 with a colour $c \neq \beta(x)$. Starting from β, we have reached a colouring which partitions the vertex-set into A_1 and A_2, by recolouring each vertex at most once. □

We now prove:

Theorem 5. *Every 3-chromatic $2K_2$-free graph G is recolourable with ℓ-recolouring diameter at most $14n$, for all $\ell \geqslant \chi(G)+1$.*

Proof. Let G be a 3-chromatic $2K_2$-free graph. Let $\ell \geqslant 4$ and let $[\ell]$ be the set of available colours. Partition $V(G)$ into independent sets A_1, A_2, and A_3 such that A_1 is (inclusion-wise) maximal. We need some $\chi(G)$-colourings of G to act as anchor points. We say a $\chi(G)$-colouring of G is *canonical* if it partitions the vertex-set into A_1, A_2, and A_3. By the

Renaming Lemma, for all $\ell \geqslant 4$, there is a path between any two canonical colourings in $R_\ell(G)$ where each vertex is recoloured at most twice. Starting from any two ℓ-colourings of G, β and γ, we prove that we can reach canonical colourings β' and γ' in $R_\ell(G)$, respectively, by recolouring each vertex at most 6 times. Then we can obtain γ from β by recolouring vertices starting from β to β' to γ' to γ. Each vertex will be recoloured at most 14 (= 6+2+6) times to go from β to γ in $R_\ell(G)$.

Claim 9. *Any ℓ-colouring of G which assigns only one colour to some part A_i, $i \in [3]$, can be recoloured to a canonical colouring by recolouring each vertex in $V(G) \setminus A_i$ at most 4 times and without recolouring any vertex of A_i.*

Proof. Let ψ be any ℓ-colouring of G which, for some $i \in [3]$, assigns only one colour, say c_i, to the part A_i. Let $A \subseteq V(G)$ be the set of all vertices coloured c_i under ψ. Clearly $A_i \subseteq A$. Let j and k be distinct integers in $[3] \setminus \{i\}$. Since G-A is a $2K_2$-free bipartite graph, as in the proof of Theorem 8, we can recolour each vertex of $V(G) \setminus A$ at most 4 times to obtain a colouring of G where every vertex of $A_j \setminus A$ is coloured some colour $c_j \neq c_i$ and every vertex of $A_k \setminus A$ is coloured some colour $c_k \notin \{c_i, c_j\}$ without using the colour c_i. Recolour each vertex in $A_j \cap A$ with the colour c_j and recolour each vertex in $A_k \cap A$ with the colour c_k to obtain a canonical colouring of G. Thus there is a path from ψ to a canonical colouring of G in $R_\ell(G)$, for all $\ell \geqslant \chi(G)+1$, where each vertex of $A_j \cup A_k$ is recoloured at most 4 times. ◇

Claim 10. *If there is a vertex in some A_i, $i \in [3]$, adjacent to every vertex not in A_i, then any ℓ-colouring of G can be recoloured to a canonical colouring by recolouring each vertex at most 4 times.*

Proof. For some $i \in [3]$, let x in A_i be adjacent to every vertex outside A_i. Let ψ be any ℓ-colouring of G. Recolour each vertex in A_i with the colour $\psi(x)$. Now, by Claim 9, we can reach a canonical colouring of G by recolouring each vertex in $V(G) \setminus A_i$ at most 4 times and without recolouring any vertex in A_i. Therefore, we can reach a canonical colouring of G by recolouring each vertex at most 4 times. ◇

By Lemma 7, there are vertices x_2 and x_3 in A_1 complete to A_2 and A_3, respectively. By Claim 10, we may assume that x_2 and x_3 are distinct. Let β be any ℓ-colouring of G.

Suppose $\beta(x_2) = \beta(x_3) = c_1$, then there is no vertex outside A_1 coloured c_1. Recolour each vertex in A_1 with colour c_1. Now, by Claim 9, we can reach a canonical colouring of G by recolouring each vertex in $A_2 \cup A_3$ at most 4 times and without recolouring any vertex in A_1. Therefore, we can reach a canonical colouring of G by recolouring each vertex at most 4 times.

Suppose $\beta(x_2) \neq \beta(x_3)$. Let $\beta(x_2) = 1$ and let $\beta(x_3) = 2$. Note that no vertex of A_2 received colour 1 and no vertex of A_3 received colour 2. Recolour as many vertices as possible in A_2 with colour 2; that is, recolour with colour 2 every vertex of A_2 which does not have a neighbour of colour 2 in A_1. Recolour as many vertices as possible in A_3 with colour 1; that is, recolour with colour 1 every vertex of A_3 which does not have a neighbour of colour 1 in A_1. Recolour as many vertices as possible in A_1 with either colour 1 or 2; that is, for vertex v of A_1 which is non-adjacent to a vertex coloured 1 or 2, recolour v with colour 1 if v is non-adjacent to a vertex coloured 1 in A_3 or recolour v with colour 2 if v is non-adjacent to a vertex coloured 2 in A_2. This new colouring, say ζ, is obtained from β by recolouring each vertex at most once.

Now a vertex in A_1 is coloured neither colour 1 nor colour 2 if and only if it is adjacent to a vertex coloured 1 in A_3 and adjacent to a vertex coloured 2 in A_2.

Claim 11. *If there is a vertex in A_1 coloured $c \in \{3, 4\}$ under ζ, then there are no vertices outside A_1 coloured c under ζ.*

Proof. We prove the claim for $c = 3$. Let $x \in A_1$ and $y \in A_2 \cup A_3$ be coloured 3 under ζ. The vertex x was not recoloured with colour either 1 or 2, because it is adjacent to a vertex u coloured 2 in A_2 and adjacent to a vertex v coloured 1 in A_3. If $y \in A_2$, then by the choice of ζ, it is adjacent to a vertex w coloured 2 in A_1. Then $\{y, w, x, u\}$ induces a $2K_2$, a contradiction. The proof is similar if y is in A_3. ◇

We have two cases.

Case 1: A_1 contains a vertex coloured either 3 or 4 under ζ.

Let there be a vertex x coloured either 3 or 4 in A_1. Then by Claim 11 there are no vertices coloured $\zeta(x)$ outside A_1. Recolour each vertex in A_1 with the colour $\zeta(x)$. Now, by Claim 9, we can reach a canonical colouring of G by recolouring each vertex in $A_2 \cup A_3$ at most 4 times and without recolouring any vertex in A_1. Therefore, starting from β we recoloured each vertex at most once to reach ζ and then recoloured each vertex at most 4 times to reach a canonical colouring of G. This completes the proof for Case 1.

Case 2: A_1 does not contain any vertex coloured 3 and does not contain any vertex coloured 4, under ζ.

Case 2 (a): For some $j \in \{2, 3\}$, A_j does not contain any vertex coloured 3 or does not contain any vertex coloured 4, under ζ.

Let $i \in \{2, 3\} \setminus \{j\}$. If colour 3 does not appear on A_j, then recolour each vertex in A_i with colour 3. Otherwise, colour 4 does not appear on A_j, then recolour each vertex in A_i with colour 4. Now, by Claim 9, we can reach a canonical colouring of G by recolouring each vertex in $V(G) \setminus A_i$ at most 4 times and without recolouring any vertex in A_i. Therefore, starting from β we recoloured each vertex at most once to reach ζ and then recoloured each vertex at most 4 times to reach a canonical colouring of G. This completes the proof for Case 2(a).

Case 2 (b): Colours 3 and 4 appear on both A_2 and A_3 under ζ.

Suppose there are two vertices u and v in A_2 coloured 3 and 4, respectively, such that u has a neighbour u' coloured 4 and v has a neighbour v' coloured 3. Then u' and v' must be in A_3. This implies that $\{u, u', v, v'\}$ induces a $2K_2$, a contradiction. Therefore there are no two vertices u and v coloured 3 and 4, respectively, in A_2 such that u has a neighbour coloured 4 and v has a neighbour coloured 3. Without loss of generality, assume that there is no vertex in A_2 coloured 4 which is a adjacent to a vertex coloured 3.

Recolour each vertex coloured 4 in A_2 with colour 3. Now there is no vertex in $A_1 \cup A_2$ coloured 4. Recolour each vertex in A_3 with colour 4. Now, by Claim 9, we can reach a canonical colouring of G by recolouring each vertex in $A_1 \cup A_2$ at most 4 times and without recolouring any vertex in A_3. Therefore, starting from β we recoloured each vertex at most once to reach ζ and then recoloured each vertex at most 5 times to reach a canonical colouring of G. This completes the proof for Case 2(b). □

In a recent paper that appeared on arXiv [5], Cambie, Cames van Batenburg and Cranston gave an improved version of the Renaming Lemma, which they call the Optimal Renaming Lemma. We can use their Optimal Renaming Lemma, instead of the Renaming Lemma, and reduce the upper bound on the ℓ-recolouring diameter by $0.5n$ each time the lemma is used. Thus, in Theorems 8 and 5, we can reduce the upper bound on the ℓ-recolouring diameter by $0.5n$.

5 Four infinite classes of $2K_2$-free graphs which admit frozen colourings

A *Hamiltonian cycle* in a graph G is a cycle which contains all the vertices of G.

For an integer $q \geqslant 2$, $\overline{D_q}$ is the graph with $4q+2$ vertices
$\{u_i : i = 0, 1, \ldots, q+1\} \cup \ \{\cup\{v_{i1}, v_{i2}, v_{i3}\} : i = 1, 2, \ldots, q\}$
whose edges are:

- the edges of a Hamiltonian cycle C: u_0, u_1, …, u_{q+1}, v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{33},…, v_{q1}, v_{q2}, v_{q3}, u_0
- edges $u_i v_{i2}$ for $i = 1, 2, \ldots, q$
- edges $v_{i1} v_{i3}$ for $i = 1, 2, \ldots, q$

See Figure 1 for $\overline{D_2}$ and Figure 2 for $\overline{D_3}$.

We refer to $\{v_{i1}, v_{i2}, v_{i3}\}$ as *triangle* i. Note that $\overline{D_q}$ consists of a Hamiltonian cycle C together with q edges which induce q vertex-disjoint triangles with consecutive pairs of edges of C, and q more edges $u_i v_{i2}$ each of which induces a paw with triangle i. Also note that the only neighbours of vertices u_0 and u_{q+1} are their neighbours on C. The number of edges of $\overline{D_q}$ is $(4q+2) + 2q = 6q + 2$.

Lemma 12. *For $q \geqslant 2$, $\overline{D_q}$ is C_4-free.*

Proof. Consider the graph $\overline{D_q}$ where $q \geqslant 2$. Edge $v_{i1}v_{i3}$ cannot be part of an induced 4-cycle in $\overline{D_q}$ because clearly v_{i2} can't be part of such a cycle and v_{i1}'s only other neighbour is either $v_{i-1\,3}$ if $i \geqslant 2$ or u_{q+1} if $i = 1$, and v_{i3}'s only other neighbour is either $v_{i+1\,1}$ if $i \leqslant q-1$ or u_0 if $i = q$, and these neighbours are not adjacent.

Edge $u_i v_{i2}$ makes two cycles with C. The two cycles are generally not induced cycles; a shorter cycle can be obtained by replacing any occurrence of v_{j1}, v_{j2}, v_{j3} by v_{j1}, v_{j3}. We first consider cycles containing only one edge of the type $u_i v_{i2}$. The shortest such cycles occur when $i = 1$ or $i = q$, and are $v_{12}, u_1, u_2, \ldots, u_{q+1}, v_{11}, v_{12}$ and $v_{q2}, v_{q3}, u_0, u_1, \ldots, u_q, v_{q2}$, respectively, and each has length $q + 3 \geqslant 5$.

A shortest cycle containing two edges $u_i v_{i2}$ and $u_j v_{j2}$ is when $j = i + 1$, and is the 6-cycle:
$u_i, v_{i2}, v_{i3}, v_{i+1\,1}, v_{i+1\,2}, u_{i+1}, u_i$.

Thus $\overline{D_q}$ is C_4-free.

□

Corollary 13. *For $q \geqslant 2$, D_q is $2K_2$-free.*

For a graph G, $\alpha(G)$ denotes the size of a largest independent set in G and $\omega(G)$ denotes the size of a largest clique in G.

Theorem 14. *For $q \geqslant 2$,*

$$\theta(\overline{D_q}) = \alpha(\overline{D_q}) = \begin{cases} (3q{+}2)/2 & \text{if } q \text{ is even} \\ (3q{+}3)/2 & \text{if } q \text{ is odd} \end{cases}$$

Proof. Let $q \geqslant 2$ be even. Create a clique partition of $\overline{D_q}$ consisting of the following cliques:

- For $i = 1, 2, \ldots, q$, let the vertices of triangle i be a clique in the clique partition
- Divide the vertices of the path $u_0, u_1, \ldots, u_{q+1}$ into $(q+2)/2$ cliques as follows: $\{u_0, u_1\}, \{u_2, u_3\}, \ldots, \{u_q, u_{q+1}\}$ and put these cliques into the clique partition

Vertices $u_0, u_2, \ldots, u_q, v_{11}, v_{21}, v_{31}, \ldots, v_{q1}$ form an independent set of size $(q+2)/2+q$ in $\overline{D_q}$.

Since we have a clique partition and an independent set of the same size, we know they are a minimum clique partition and a maximum independent set.

Now let $q \geqslant 3$ be odd. The proof is similar to the even case, except that the path $u_0, u_1, \ldots, u_{q+1}$ in $\overline{D_q}$ has an odd number of vertices. Create a clique partition of $\overline{D_q}$ consisting of the following cliques:

- For $i = 1, 2, \ldots, q$, let the vertices of triangle i be a clique in the clique partition
- Divide the vertices of the path $u_0, u_1, \ldots, u_{q+1}$ into $(q+3)/2$ cliques as follows: $\{u_0, u_1\}, \{u_2, u_3\}, \ldots, \{u_{q-1}, u_q\}, \{u_{q+1}\}$ and put these cliques into the clique partition

Vertices $u_0, u_2, \ldots, u_{q+1}, v_{12}, v_{21}, v_{31}, \ldots, v_{q1}$ form an independent set of size $(q+1)/2 + 1 + q$ in $\overline{D_q}$.

Again, since we have a clique partition and an independent set of the same size, we know they are a minimum clique partition and a maximum independent set. □

Corollary 15. *For $q \geqslant 2$,*

$$\chi(D_q) = \omega(D_q) = \begin{cases} (3q+2)/2 & \text{if } q \text{ is even,} \\ (3q+3)/2 & \text{if } q \text{ is odd.} \end{cases}$$

Lemma 16. *Let $\mathcal{Q}$ be a partition of the vertex-set $V(G)$ of graph G into cliques of size 2. Then $\mathcal{Q}$ is a frozen clique partition if and only if every triangle of G intersects three distinct cliques of $\mathcal{Q}$.*

Proof. Let $\mathcal{Q}$ be a partition of the vertex-set $V(G)$ of graph G into cliques of size 2. Then every triangle of G intersects at least two cliques of $\mathcal{Q}$.

By definition, $\mathcal{Q}$ is not a frozen clique partition if and only if there is some clique $Q = \{q_1, q_2\} \in \mathcal{Q}$ and some vertex $v \notin Q$ such that v is adjacent to both q_1 and q_2, which means that triangle $\{v, q_1, q_2\}$ intersects exactly two cliques of $\mathcal{Q}$, namely Q and the clique containing v. □

Theorem 17. *For $q \geqslant 2$, $\overline{D_q}$ has a frozen $(2q+1)$-clique-partition.*

Proof. Create a clique partition $\mathcal{Q}^*$ of $\overline{D_q}$ consisting of the following cliques:

- For $i = 1, 2, \ldots, q$, let $\{u_i, v_{i2}\}$ be a clique of the clique partition.
- For $i = 1, 2, \ldots, q-1$, let $\{v_{i3}, v_{i+1\,1}\}$ be a clique of the clique partition.
- Let $\{v_{q3}, u_0\}$ and $\{u_{q+1}, v_{11}\}$ be cliques of the clique partition.

In $\overline{D_q}$, the only triangles are triangles 1 to q. It is easily seen that each triangle i intersects three different cliques of $\mathcal{Q}^*$. Thus the result follows from Lemma 16. □

Corollary 18. *For $q \geqslant 2$, D_q has a frozen $(2q+1)$-colouring.*

As noted in Section 3, the graph D_2 was found by a computer search as was the graph we will call F_2 which is D_2 with edge u_1u_2 added.

We now define a second class of graphs, D_q^* where $q \geqslant 2$. We obtain D_q^* from D_q by deleting the edge u_0u_{q+1}. Equivalently, we obtain $\overline{D_q^*}$ from $\overline{D_q}$ by adding the edge u_0u_{q+1}.

Theorem 19. *For $q \geqslant 3$, $\overline{D_q^*}$ is a C_4-free graph with the same clique partition number and frozen clique partition as $\overline{D_q}$.*

The proof of Theorem 19 is similar to the proofs for $\overline{D_q}$. One needs to check that adding the edge u_0u_{q+1} does not create a 4-cycle. To see that the clique partition is minimum when q is odd, note that the vertex-set of $\overline{D_q^*}$ can be partitioned into q triangles and one induced odd cycle C_{q+2}; the odd cycle C_{q+2} requires at least $(q+3)/2$ cliques in any clique partition. Thus the size of a smallest clique partition of $\overline{D_q^*}$ when q is odd is $q+(q+3)/2 = (3q+3)/2$. Note that when q is odd, $\alpha(\overline{D_q^*}) = \theta(\overline{D_q^*}) - 1$.

Corollary 20. *For $q \geqslant 3$, D_q^* is a $2K_2$-free graph with the same chromatic number and frozen colouring as D_q.*

Remark 21. By deleting the edge $u_0u_{q+1} = u_0u_3$ from D_2, we will obtain a $2K_2$ induced by $\{u_0, u_1, u_2, u_3\}$. The resulting graph D_2^* is thus not $2K_2$-free, but is P_5-free.

We now define a third class of graphs, F_q where $q \geqslant 2$. We obtain $\overline{F_q}$ from $\overline{D_q}$ by removing the edges of the path $u_1, u_2, \ldots, u_q$. Equivalently, we obtain F_q from D_q by adding the edges $u_1u_2, u_2u_3, \ldots, u_{q-1}u_q$ to D_q. See Figure 3 for $\overline{F_2}$ and Figure 4 for $\overline{F_3}$.

Theorem 22. *For $q \geqslant 2$, $\overline{F_q}$ is a C_4-free graph with $\theta(\overline{F_q}) = \alpha(\overline{F_q}) = 2q$, which admits a frozen $(2q+1)$-clique-partition.*

The proof of Theorem 22 is similar to the proofs for $\overline{D_q}$. The clique partition of $\overline{F_q}$ consists of the following cliques:

- For $i = 1, 2, \ldots, q$, let the vertices of triangle i be a clique in the clique partition
- Let $\{u_0, u_1\}$ and $\{u_q, u_{q+1}\}$ be cliques of the clique partition
- Let $\{u_2\}, \{u_3\}, \ldots, \{u_{q-1}\}$ be cliques of the clique partition.

This clique partition has size $q + 2 + (q-2) = 2q$.

Vertices $u_1, u_2, u_3, \ldots, u_q, v_{11}, v_{21}, v_{31}, \ldots, v_{q1}$ form an independent set of size $2q$ in $\overline{F_q}$.

The frozen clique partition of $\overline{D_q}$ given in Theorem 17 is a frozen clique partition of $\overline{F_q}$.

Corollary 23. *For $q \geqslant 2$, F_q is a $2K_2$-free graph with $\chi(F_q) = \omega(F_q) = 2q$, which admits a frozen $(2q+1)$-colouring.*

Remark 24. For any $q \geqslant 2$, one can obtain a $(2q+1)$-clique-partitionable graph which admits a frozen clique-partition by modifying the construction of $\overline{D_q}$ or $\overline{F_q}$ as follows. The edges $\{u_iv_{i2}, 1 \leqslant i \leqslant q\}$ can be replaced by any set of edges which forms a matching between $\{u_i : 1 \leqslant i \leqslant 1\}$ and $\{v_{i2} : 1 \leqslant i \leqslant q\}$; to avoid a C_4, in $\overline{D_q}$, u_qv_{12} and u_1v_{q2} should not be edges.

We now define a fourth class of graphs. For $r \geqslant 1$, $\overline{Y_r}$ is the graph with $6r$ vertices
$\{\{\cup\{v_{i1}, v_{i2}, v_{i3}\} : i = 1, 2, \ldots, 2r\}$
whose edges are:

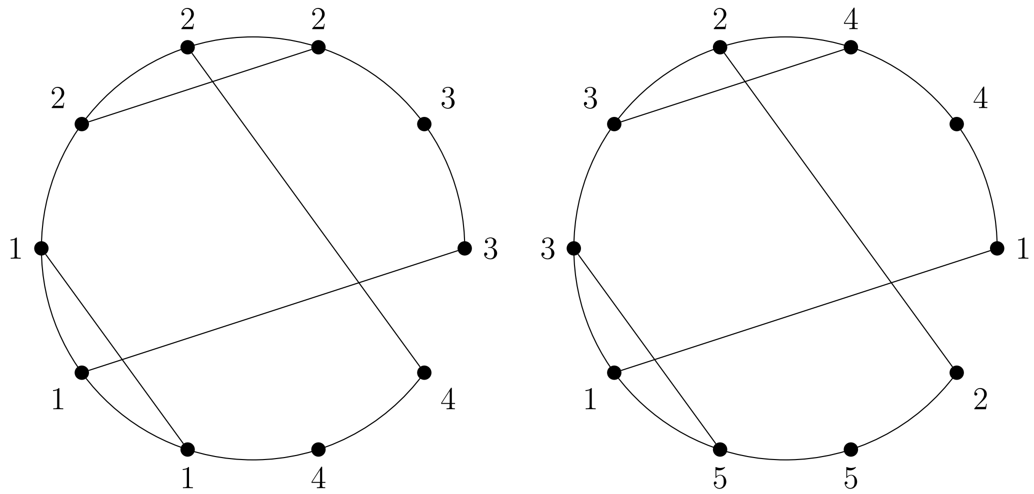

Figure 3: A square-free graph $\overline{F_2}$ with a 4-clique-partition (left) and a frozen 5-clique-partition (right). Equivalently, a 4-colouring of the complement F_2 of the graph shown (left) and a frozen 5-colouring of F_2 (right).

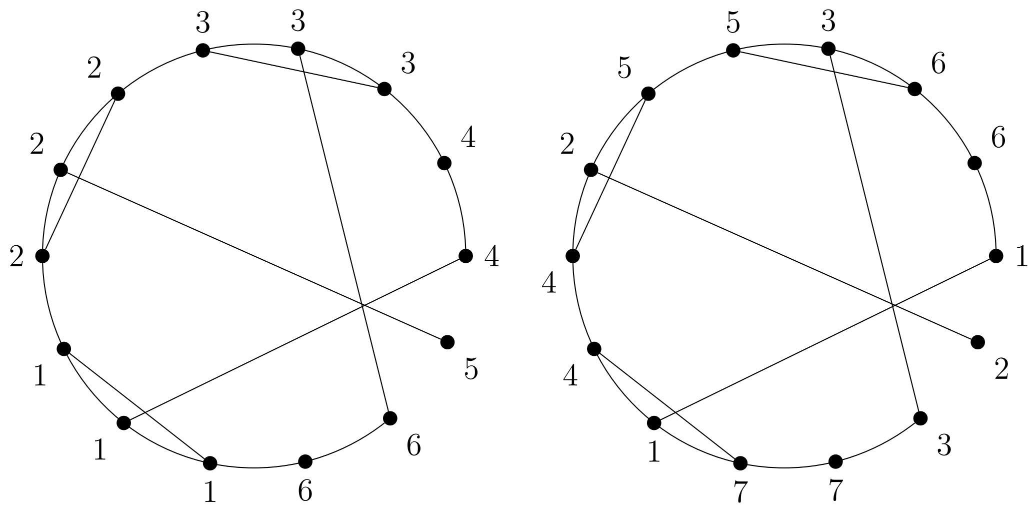

Figure 4: A square-free graph $\overline{F_3}$ with a 6-clique-partition (left) and a frozen 7-clique-partition (right). Equivalently, a 6-colouring of the complement F_3 of the graph shown (left) and a frozen 7-colouring of F_3 (right).

- the edges of a Hamiltonian cycle C: $v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{33}, \ldots, v_{2r\,1}, v_{2r\,2}, v_{2r\,3}$
- edges $v_{i2}v_{i+r\,2}$ for $i = 1, 2, \ldots, r$

As above, we refer to $\{v_{i1}, v_{i2}, v_{i3}\}$ as *triangle* i. Note that $\overline{Y_r}$ consists of a Hamiltonian cycle C together with $2r$ edges which induce $2r$ vertex-disjoint triangles with consecutive pairs of edges of C, and r more edges pairing the middle vertices v_{i2} of "opposite" triangles. The number of edges of $\overline{Y_r}$ is $9r$.

See Figure 5 for $\overline{Y_2}$ and Figure 6 for $\overline{Y_3}$. Note that $\overline{Y_1}$ is $\overline{C_6}$.

Theorem 25. *For $r \geqslant 2$, $\overline{Y_r}$ is a C_4-free graph with $\theta(\overline{Y_r}) = \alpha(\overline{Y_r}) = 2r$, which admits a frozen $3r$-clique-partition.*

To prove Theorem 25 note that triangles $1, 2, \ldots, 2r$ form a clique partition of $\overline{Y_r}$ and vertices $v_{11}, v_{21}, \ldots, v_{2r\,1}$ form an independent set. Note also that the following $3r$-clique-partition is frozen:
$\mathcal{Q} = \{\{v_{12}, v_{r+1\,2}\}, \{v_{22}, v_{r+2\,2}\}, \ldots, \{v_{r2}, v_{2r\,2}\}, \{v_{13}, v_{21}\}, \{v_{23}, v_{31}\}, \ldots, \{v_{2r\,3}, v_{11}\}\}$.

Corollary 26. *For $r \geqslant 2$, Y_r is a $2K_2$-free graph with $\chi(Y_q) = \omega(Y_q) = 2r$, which admits a frozen $3r$-colouring.*

Remark 27. For any $r \geqslant 2$, one can obtain a $2r$-clique-partitionable graph which admits a frozen $3r$-clique-partition by modifying the construction of $\overline{Y_r}$ as follows. Pair the vertices $\{v_{i2} : 1 \leqslant i \leqslant r\}$ in any way, and then join the members of each pair by an edge (rather than joining v_{i2} to $v_{i+r\,2}$ as in the construction). To avoid creating a C_4, do not pair v_{i2} with $v_{i+1\,2}$ for $1 \leqslant i \leqslant 2r-1$ and do not pair $v_{2r\,2}$ with v_{12}.

6 An operation which preserves being $2K_2$-free and admitting a frozen colouring

Operation 1. Given a graph H and adjacent vertices x and y in H, we subdivide the edge xy to obtain a new graph H' by deleting the edge xy, adding two vertices u and v, and adding edges xu, uv, and vy; that is, the edge xy is replaced by a path on four vertices: x, u, v, y.

Theorem 28. *Let H be a graph with a k-clique-partition $\mathcal{Q}$ and with a frozen $(k+1)$-clique-partition $\mathcal{F}$, and let x and y be adjacent vertices of H which are in different cliques of $\mathcal{Q}$ such that either*

(1) x and y are in different cliques of $\mathcal{F}$ or

(2) $\{x, y\}$ is a clique of $\mathcal{F}$.

Then the graph H' obtained by subdividing edge xy as in Operation 1 is $(k+1)$-clique-partitionable and admits a frozen $(k+2)$-clique-partition.
Furthermore,

class	$q/r/s$	n	min degree	max degree	# edges	χ (= ω except for D_q^*, q odd)	# colours in frozen colouring	(# colours in in frozen colouring) - χ
D_q	$q \geqslant 2$	$4q+2$	$4q-2$	$4q-1$	$8q^2-1$	$(3q+2)/2$ for even q; $(3q+3)/2$ for odd q	$n/2 = 2q+1$	$q/2$ for even q; $(q-1)/2$ for odd q
D_q^*	$q \geqslant 3$	$4q+2$	$4q-2$	$4q-2$	$8q^2-2$	$(3q+2)/2$ for even q; $(3q+3)/2$ for odd q	$n/2 = 2q+1$	$q/2$ for even q; $(q-1)/2$ for odd q
F_q	$q \geqslant 2$	$4q+2$	$4q-2$	$4q$ for $q > 2$; $4q-1$ for $q = 2$	$8q^2+q-2$	$2q$	$2q+1$	1
Y_r	$r \geqslant 2$ For comparison: $r = \frac{2q+1}{3}$ $q \equiv 1(mod\ 3)$	$6r$ $4q+2$	$6r-4$ $4q-2$	$6r-4$ $4q-2$	$18r^2-12r$ $8q^2-2$	$2r$ $(4q+2)/3$	$3r$ $2q+1$	r $(2q+1)/3$
H_s	$s \geqslant 3$ For comparison: $s = q+1$	$4s-2$ $4q+2$	$3s-3$ $3q$	$4s-5$ $4q-1$	$7s^2-12s+5$ $7q^2+2q$	$s+1$ $q+2$	$2s-1$ $2q+1$	$s-2$ $q-1$

Table 1: Parameters of $2K_2$-free graph classes.

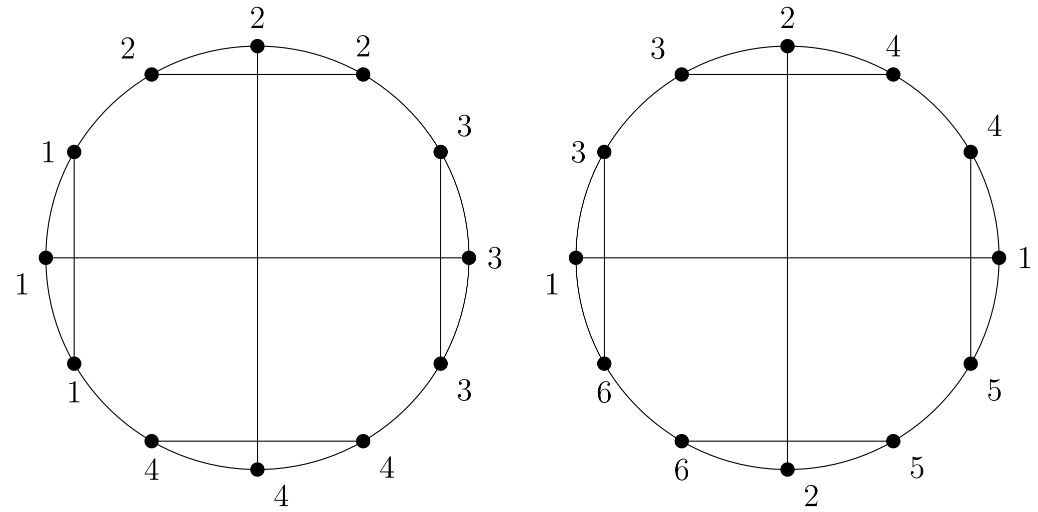

Figure 5: A C_4-free graph $\overline{Y_2}$ with a 4-clique-partition (left) and a frozen 6-clique-partition (right). Equivalently, a 4-colouring of the complement Y_2 (left) and a frozen 6-colouring (right).

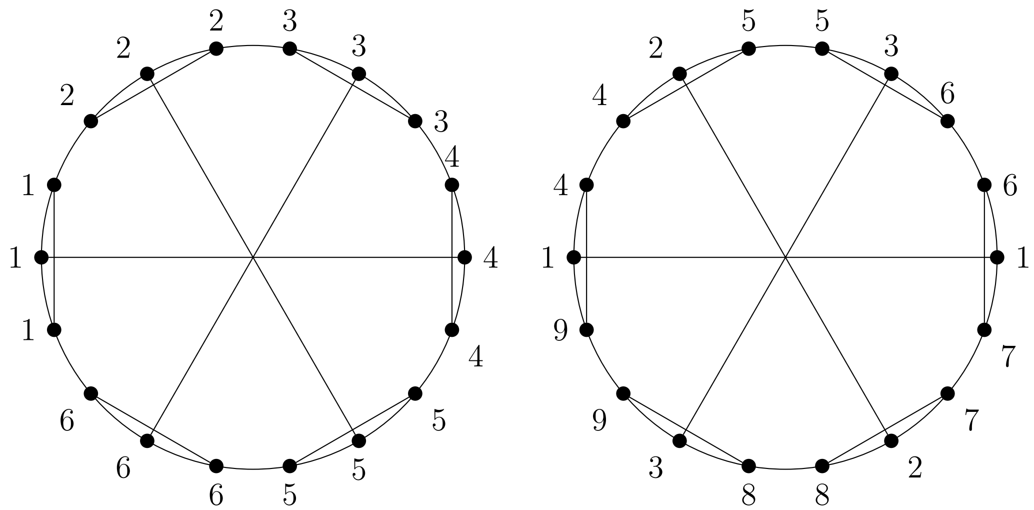

Figure 6: A C_4-free graph $\overline{Y_3}$ with a 6-clique-partition (left) and a frozen 9-clique-partition (right). Equivalently, a 6-colouring of the complement Y_3 (left) and a frozen 9-colouring (right).

(3) if $\theta(H) = k$, then $\theta(H') = k + 1$.

(4) if H is C_4-free and if in case (1), xy is not the middle edge of a diamond, then H' is C_4-free.

Proof. Let H be a graph with a k-clique-partition $\mathcal{Q}$ and with a frozen $(k+1)$-clique-partition $\mathcal{F}$, and let x and y be adjacent vertices of H which are in different cliques of $\mathcal{Q}$. Let H' be the graph obtained by subdividing edge xy.

Claim 29. *By adding $\{u,v\}$ to $\mathcal{Q}$ we obtain a $(k+1)$-clique-partition $\mathcal{Q}'$ of H'.*

Claim 30. *We can modify $\mathcal{F}$ to be a frozen clique partition $\mathcal{F}'$ of H' as follows.*

> *In Case (1): By adding $\{u,v\}$ to $\mathcal{F}$ we obtain a $(k+1)$-clique-partition $\mathcal{F}'$ of H'.*
>
> *In Case (2): Remove $\{x,y\}$ from $\mathcal{F}$ and add $\{x,u\}$ and $\{v,y\}$ to obtain a $(k+1)$-clique-partition $\mathcal{F}'$ of H'.*

Proof. It is easy to see that $\mathcal{F}'$ is a clique partition of H'. We now prove that $\mathcal{F}'$ is frozen.

In Case (1): In H', every vertex is nonadjacent to either u or v or both, so every vertex not in clique $\{u,v\}$ is nonadjacent to a vertex of $\{u,v\}$.

Since $\mathcal{F}$ is a frozen clique partition of H, every vertex of H is nonadjacent to some vertex of every clique of $\mathcal{F}$ other than the clique containing it, and this remains true when the edge xy is deleted.

Thus, for every vertex z of H and every clique Q of $\mathcal{F}'$ other than the clique containing z, z is nonadjacent to some vertex of Q.

In any frozen clique partition, if there is a clique consisting of a single vertex, say w, then w must be an isolated vertex. In H, x and y are adjacent, so neither is an isolated vertex, and thus there is vertex x' of H different from x in the clique of $\mathcal{F}$ containing x and a vertex y' different from y in the clique of $\mathcal{F}$ containing y.

Since u is nonadjacent to every vertex of H other than x, and in particular, is non-adjacent to x', it follows that u is nonadjacent to some vertex of every clique of $\mathcal{F}'$ other than $\{u,v\}$. Similarly, v is nonadjacent to some vertex of every clique of $\mathcal{F}'$ other than $\{u,v\}$.

In Case (2): In H', vertex u is nonadjacent to every vertex other than x and v. Thus u is nonadjacent to some vertex of every clique of $\mathcal{F}'$ other than $\{x,u\}$. Analogously, v is nonadjacent to some vertex of every clique of $\mathcal{F}'$ other than $\{v,y\}$.

Since $\mathcal{F}$ is a frozen clique partition of H, every vertex of H is nonadjacent to some vertex of every clique of $\mathcal{F}$ other than the clique containing it. In particular, every vertex w in $V(H)-\{x,y\}$ is nonadjacent to a vertex of each clique of $\mathcal{F}\setminus\{x,y\}$. Since w is nonadjacent to u and v, it follows w is nonadjacent to some vertex of each clique of $\mathcal{F}'=(\mathcal{F}\setminus\{x,y\})\cup\{\{x,u\},\{v,y\}\}$.

Since $\mathcal{F}$ is a frozen clique partition of H, x is nonadjacent to a vertex of every clique of $\mathcal{F}$ other than $\{x,y\}$. Vertex x is nonadjacent to $v\in\{v,y\}\in\mathcal{F}'$. Thus vertex x is nonadjacent to some vertex of every clique of $\mathcal{F}'$ other than $\{x,u\}$. Analogously, vertex y is nonadjacent to some vertex of every clique of $\mathcal{F}'$ other than $\{v,y\}$. ◇

Claim 31. *If $\theta(H)=k$, then $\theta(H')=k+1$.*

Proof. Assume $\theta(H)=k$.

If there were a $(k-2)$-clique partition of $H-\{x,y\}$, then by adding $\{x,y\}$ to the clique partition, we would obtain a $(k-1)$-clique-partition of H, which is a contradiction. So $\theta(H-\{x,y\})\geqslant k-1$.

By Claim 29, $\theta(H') \leqslant k+1$. We need to show that there is no k-clique-partition of H'. First, consider a clique partition of H' where u and v are in different cliques. Since u and v are each anticomplete to $H - \{x, y\}$ and $\theta(H - \{x, y\}) \geqslant k-1$, a total of at least $k+1$ cliques would be required. Now consider a clique partition of H' where u and v are in the same clique. This clique must then be $\{u, v\}$, and thus the clique partition must have at least $\theta(H) + 1 = k + 1$ cliques. ◇

Claim 32. *If H is C_4-free, then H' is C_4-free.*

Proof. Assume H is C_4-free.

In H', u and v are adjacent and each have degree 2, so any C_4 containing one of them, must contain the other, and then also contain u's only other neighbour, which is x, and v's only other neighbour, which is y, but xy is not an edge of H', so no such C_4 exists.

In constructing H' from H, the edge xy is removed. This could create a C_4 if xy was the middle edge of a diamond in H. This is excluded by hypothesis in Case (1). In Case (2), $\{x, y\}$ is a clique in the frozen clique partition $\mathcal{F}$. If there were a vertex w adjacent to both x and y in H, then $\mathcal{F}$ would not be frozen. Thus xy cannot be the middle edge of a diamond in H. ◇

Thus, the proof is completed. □

Here is the same operation described directly for colourings.

Operation 2. Given a graph G and nonadjacent vertices x and y in G, we define the following operation to create a new graph G'. Define G' to be the graph G together with two additional vertices u and v and with edges vx, xy and yu; join u and v to all vertices of $G - \{x, y\}$.

Corollary 33. *Let G be a k-colourable graph with a k-colouring β and a frozen $(k+1)$-colouring γ, and let x and y be nonadjacent vertices of G such that $\beta(x) \neq \beta(y)$ and such that either*

(1) $\gamma(x) \neq \gamma(y)$, or

(2) $\{x, y\}$ is a colour class of γ.

Then the graph G' of Operation 2 is $(k+1)$-colourable and admits a frozen $(k+2)$-colouring.
Furthermore,

(3) if G is k-chromatic, then G' is $(k+1)$-chromatic.

(4) if G is $2K_2$-free and if in case (1), there is no edge rs such that $\{r, s\}$ is anticomplete to $\{x, y\}$, then G' is $2K_2$-free.

A $(k+1)$-colouring of G' is the colouring β of G extended by making vertices u and v a new colour class. A new frozen $(k+2)$-colouring of G' is obtained from the frozen colouring γ of G by

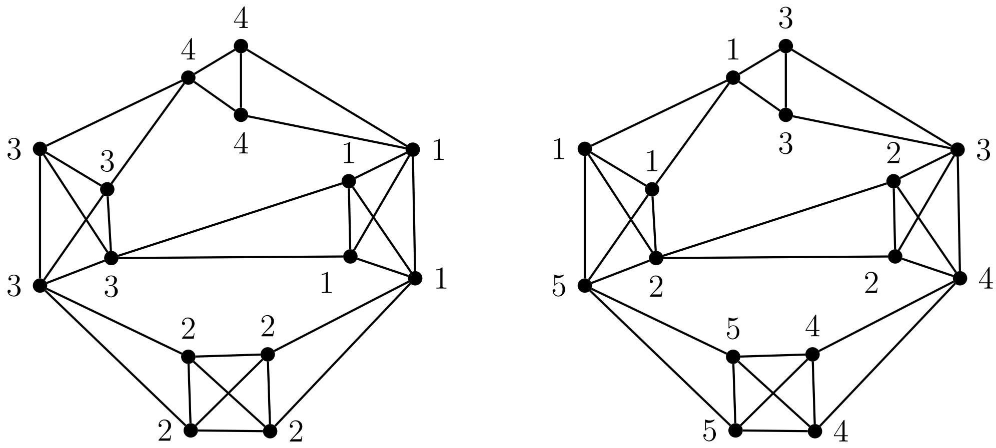

Figure 7: A C_4-free graph with a 4-clique-partition (left) and a frozen 5-clique-partition (right) with no edge satisfying the conditions of Theorem 28.

- In Case (1), making vertices u and v a new colour class.
- In Case (2), give x a new colour and assign this colour to u as well; give v the colour $\gamma(y)$.

Remark 34. Note that a graph with a k-clique-partition and a frozen $(k+1)$-clique partition may not have an edge as required by Theorem 28. See Figure 7 for an example. This graph was obtained by adding a true twin to one vertex from each of the cliques in our frozen clique partition given of $\overline{F_2}$. Adding a true twin w of a vertex v means adding vertex w and joining it to v and all neighbours of v.

7 k-chromatic $2K_2$-free graphs which admit a frozen (k+1)-colouring for all k $\geqslant$4

Theorem 35. *For every $k \geqslant 4$, there is a C_4-free graph with clique partition number k which admits a frozen $(k+1)$-clique partition.*

Proof. One way to construct the graphs described in the theorem is to start with $\overline{D_2}$ which is a C_4-free graph with clique partition number 4 and with a frozen 5-clique-partition and then apply Operation 1 with $x = u_1$ and $y = u_2$. These two vertices are in different cliques in both the 4-clique-partition and in the frozen 5-clique-partition, so Case (1) will be applied. The additional hypothesis holds in this case. The result is a C_4-free graph with clique partition number 5 and with a frozen 6-clique-partition. Note that the two added vertices are a clique of size 2 in both the 5-clique-partition and the frozen 6-clique-partition. One can then apply the operation again, with $x = u_1$ and y being the vertex u of the previous operation to obtain a C_4-free graph with clique partition number 6 and with a frozen 7-clique-partition. One can continue this process, always choosing $x = u_1$ and y being the vertex u of the previous operation. This class of graphs is illustrated in

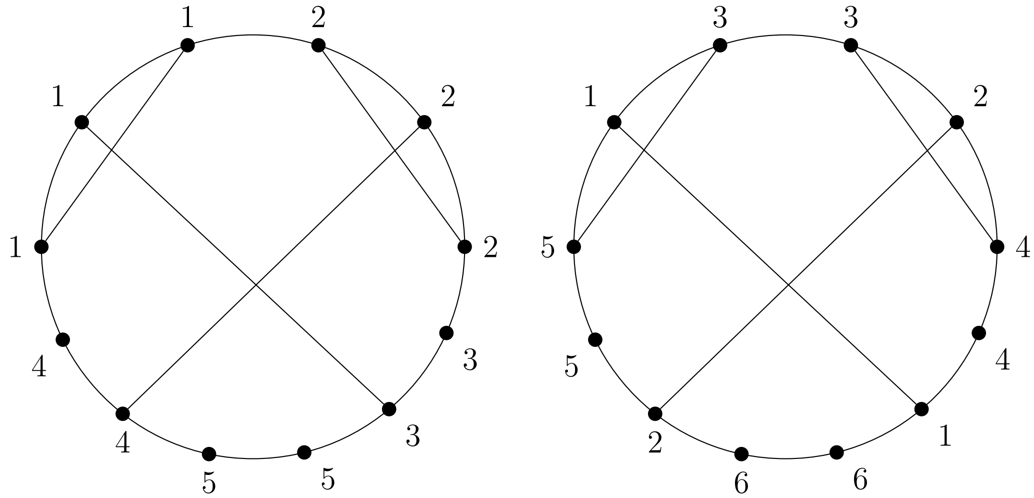

Figure 8: A C_4-free graph with a 5-clique-partition (left) and a frozen 6-clique-partition (right). Equivalently, a 5-colouring of the complement (left) and a frozen 6-colouring (right).

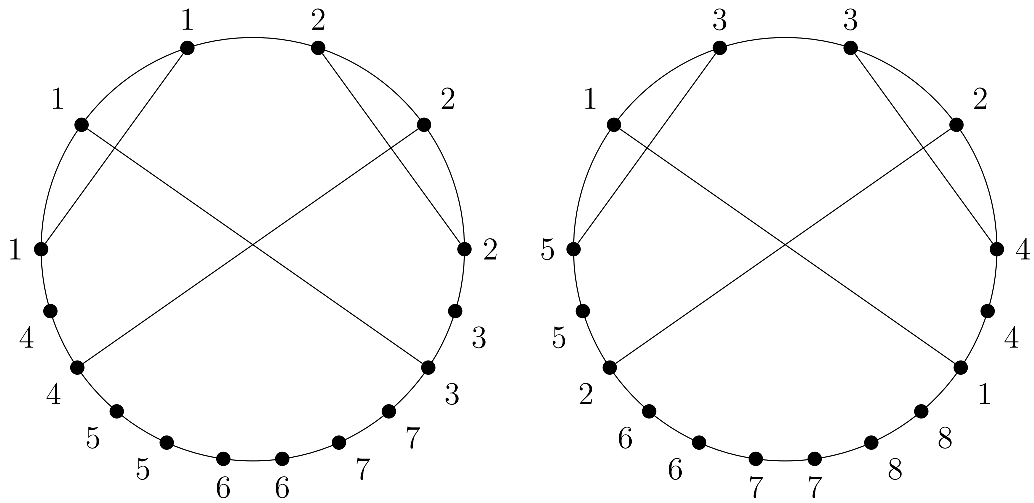

Figure 9: A C_4-free graph with a 7-clique-partition (left) and a frozen 8-clique-partition (right). Equivalently, a 7-colouring of the complement (left) and a frozen 8-colouring (right).

Figures 8 and 9 and can be described as follows: For $t \geqslant 4$, to obtain a C_4-free graph with clique partition number t and with a frozen $(t+1)$-clique-partition, start with $\overline{D_2}$ and subdivide the edge u_1u_2 by $2(t-4)$ vertices (in other words, replace the edge u_1u_2 by a path $u_1, w_1, w_2, \ldots, w_{2t-9}, w_{2t-8}, u_2$). □

We now obtain Theorem 2 as a corollary.

Theorem 4. *For every $k \geqslant 4$, there is a k-chromatic $2K_2$-free graph with a frozen $(k+1)$-colouring.*

Remark 36. There are many other ways to apply Operation 1 to prove Theorem 35 - it is not necessary to choose the same vertices as x and y as above.

8 Some curiosities and open problems

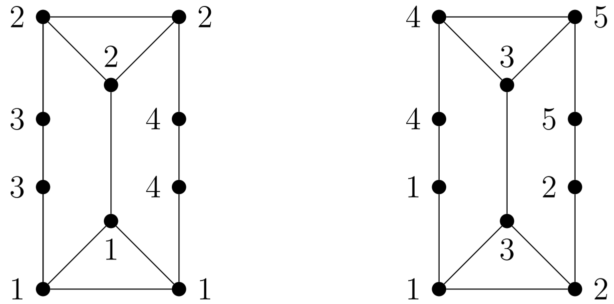

Figure 10: A 4-clique-partition of $\overline{F_2} \cong \overline{H_3}$ (left) and a frozen 5-clique-partition (right).

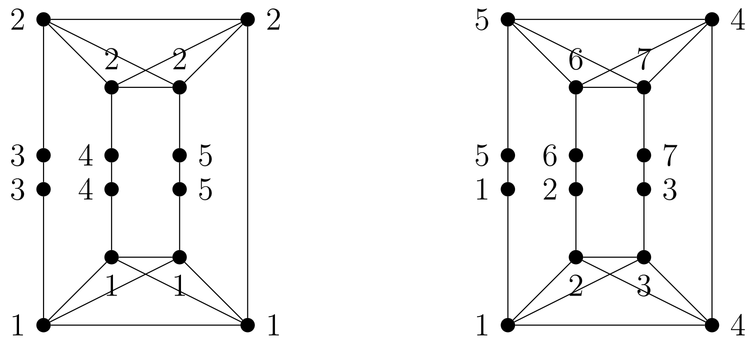

Figure 11: A 5-clique-partition of $\overline{H_4}$ (left) and a frozen 7-clique-partition (right).

Subdividing an edge of a C_4 gets rid of that C_4. The complement $\overline{C_6}$ of C_6 contains three C_4s; each pair of C_4s intersect in a distinct edge. By applying Operation 1 to two of these three edges, we obtain $\overline{F_2}$ which is C_4-free. See Figure 10. Thus besides preserving $2K_2$-freeness of a graph, Operation 2 can transform a graph containing $2K_2$s into a $2K_2$-free graph.

Recall that for $s \geqslant 2$, the graph B_s is $K_{s,s}$ with a perfect matching removed, and is 2-chromatic and admits a frozen s-colouring. The complement of B_s consists of two copies of K_s with a perfect matching M_s joining each vertex of one copy to a distinct vertex of the other copy. Note that B_s contains many $2K_2$s and (equivalently) $\overline{B_s}$ contains many C_4s. By applying Operation 1 to all but one edge of M_s in $\overline{B_s}$ where $s \geqslant 3$, we obtain a C_4-free graph $\overline{H_s}$ which is $(s+1)$-clique-partitionable and admits a frozen $(2s-1)$-clique partition. Note that $\overline{H_3}$ is isomorphic to $\overline{F_2}$. See Figure 10 for $\overline{H_3}$ and Figure 11 for $\overline{H_4}$.

In Figure 12 is the complement of the $2K_2$-free graph given by Feghali and Merkel in [9] with their 7-colouring (shown as a 7-clique-partition) and their frozen 8-colouring (shown as a frozen 8-clique-partition). The complement of their graph is very similar to our $\overline{F_3}$. In fact, the complement of their graph is $\overline{F_3}$ with Operation 1 applied once (to get the vertices in clique 5 of the 7-clique-partition).

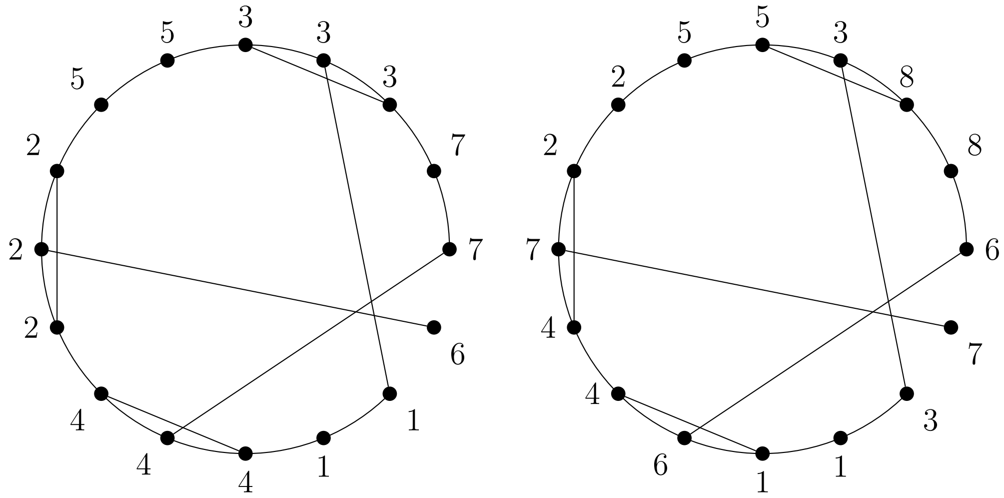

Figure 12: A 7-clique-partition (left) and a frozen 8-clique-partition (right) of a C_4-free graph. Equivalently, a 7-colouring (left) and a frozen 8-colouring (right) of the complement [9].

We conclude with an open problem, the remaining case for a dichotomy theorem for recolourability of graphs where two 4-vertex graphs are forbidden as induced subgraphs: Is the class of 4-chromatic $(2K_2, K_4)$-free graphs recolourable?

Acknowledgements

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