Fusions of the Tensor Square of a Strongly Regular Graph

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Abstract

In this article, we determine all fusions of the association scheme $\mathcal{A} \otimes \mathcal{A}$, where $\mathcal{A}$ is the symmetric rank 3 association scheme corresponding to a strongly regular graph. This includes both guaranteed fusions, which are fusions for all symmetric rank 3 association schemes $\mathcal{A}$, and specific case fusions, which exist only under restrictions on the parameters of the association scheme. In doing so, we will determine the fusions of wreath products of strongly regular graphs and the fusions of the tensor square of a symmetric rank 3 table algebra. This is an extension of the recent work of the authors and Meagher, where we solved the same problem for the generalized Hamming scheme $H(2, \mathcal{A})$ of the association scheme obtained from a strongly regular graph. The main results of this article prove that the families of strongly regular graphs for which $\mathcal{A} \otimes \mathcal{A}$ has a special case fusion are the same families for which $H(2, \mathcal{A})$ has a special case fusion; and that the imprimitive strongly regular graphs are the only family of strongly regular graphs for which the wreath product $\mathcal{A} \wr \mathcal{A}$ has a special case fusion.

Mathematics Subject Classifications: 05E30, 05C25

1 Introduction

In [12], the authors and Karen Meagher determined all fusions of the generalized Hamming scheme $H(2, \mathcal{A})$, where $\mathcal{A}$ is the symmetric rank 3 association scheme corresponding to a strongly regular graph. In contrast with earlier work of [18] (see [19]) that worked directly with the intersection numbers, the approach in [12] was to use the Bannai-Muzychuk criteria on the character table to establish the existence of each fusion. As noted in [12], the adjacency algebra of $H(2, \mathcal{A})$ is isomorphic to the rank 6 symmetric 2 tensor subalgebra $Sym^2(\mathcal{A})$ of $\mathcal{A} \otimes \mathcal{A}$, so the authors asked if the fusions of other subalgebras of $\mathcal{A} \otimes \mathcal{A}$ could

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be classified in a similar fashion, most importantly for the wreath product $\mathcal{A} \wr \mathcal{A}$ and for the full tensor product $\mathcal{A} \otimes \mathcal{A}$. The main results of this article classify these fusions. As the wreath product has rank 5, the problem is a bit easier than for the full tensor product and does not require the assistance of a computer. For the tensor product, each of the 4140 partitions of the set of 8 nonidentity elements must be considered compared to the set of 52 partitions that had to be considered in [12]. To manage the extra partitions, the authors have implemented a sieve that eliminates partitions whose corresponding fusion would not be consistent with the parameters of a symmetric rank 3 table algebra. For those that are consistent, the Bannai-Muzychuk criterion is applied to verify the partition gives a fusion under the conditions on parameters it imposes.

Our technique applies more generally in the setting where $\mathcal{A}$ is the standard basis of a symmetric rank 3 table algebra, so the main results can be interpreted in that setting. For those unfamiliar with table algebras, the connection between the tensor products of association schemes and table algebras is explained by Xu in [22]. The fact that the wreath product of two association schemes is a fusion of their tensor product was noted by Song in [21]. Our results on rank 3 fusions overlap earlier results of Sankey [8].

We would like to thank the referee for their valuable suggestions, particularly for prompting us to include interesting details about the strongly regular graphs that appear in our fusions. Furthermore, feedback from the referees encouraged us to provide further clarification of the sieving process, which enabled us to discover additional fusions in small cases that had been overlooked in our earlier version.

2 Preliminaries

2.1 Parameters of strongly regular graphs

Let Γ be a strongly regular graph on a set of n vertices. One of the many equivalent definitions of a strongly regular graph is that the set of $n \times n$ 01-matrices $\mathcal{A} = \{A_0, A_1, A_2\}$ is a basis for the adjacency algebra of an association scheme, where $A_0 =$ the $n \times n$ identity matrix, $A_1 =$ the adjacency matrix of Γ, and $A_2 =$ the adjacency matrix of the complement $\bar{\Gamma}$of Γ. In particular, if Γ is a strongly regular graph with parameters (n, k, λ, μ), then these matrices satisfy

$$A_1^2 = kA_0 + \lambda A_1 + \mu A_2, \quad A_1 A_2 = (k - 1 - \lambda)A_1 + (k - \mu)A_2,$$

and

$$A_2^2 = (n - 1 - k)A_0 + (n - 2k + \lambda)A_1 + (n - 2k + \mu - 2)A_2.$$

The algebra spanned by $\mathcal{A} = \{A_0, A_1, A_2\}$ is thus exactly isomorphic to a symmetric rank 3 standard integral table algebra spanned by the set $\mathcal{B} = \{b_0, b_1, b_2\}$ of left regular matrices $b_0 = I_3$,

$$b_1 = \begin{bmatrix} 0 & k & 0 \\ 1 & \lambda & k-1-\lambda \\ 0 & \mu & k-\mu \end{bmatrix}, \text{ and } b_2 = \begin{bmatrix} 0 & 0 & n-k-1 \\ 0 & k-1-\lambda & n-2k+\lambda \\ 1 & k-\mu & n-2k-2+\mu \end{bmatrix}$$

via the linear extension of the map $A_i \mapsto b_i$ for $i = 0, 1, 2$. Of course, k is the valency of a regular graph on n vertices, so k and $\ell := n - k - 1$ is a positive integer. The parameters λ and μ count walks of length 2 in Γ of different types, so they are also nonnegative integers. Furthermore, the eigenvalues of A_1 and A_2 give the entries of the *character table* (a.k.a. *first eigenmatrix*) of the association scheme $\mathcal{A}$:

$$\mathcal{P}(\mathcal{A}) = \begin{bmatrix} 1 & k & \ell \\ 1 & r & -1-r \\ 1 & s & -1-s \end{bmatrix} \begin{matrix} 1 \\ f \\ g \end{matrix} .$$

Each row of $\mathcal{P}$ corresponds to one eigenspace of A_1, with the entries in that row being the eigenvalues taken by A_0, A_1, and A_2, respectively, on that eigenspace. The number on the right is the dimension of this eigenspace, which gives the *multiplicity* of the eigenvalue of A_1. The first common eigenspace for A_1 and A_2 is spanned by the all 1's vector, so it has multiplicity 1 and the eigenvalues of A_1 and A_2 on its eigenvectors are the valencies k and ℓ, respectively. The row orthogonality relations tell us that the following sum is 0

$$1 + \frac{rs}{k} + \frac{(-1-r)(-1-s)}{\ell} = 0,$$

and also give us formulas for the multiplicities f and g:

$$1 + \frac{r^2}{k} + \frac{(-1-r)^2}{\ell} = \frac{n}{f}, \text{ and } 1 + \frac{s^2}{k} + \frac{(-1-s)^2}{\ell} = \frac{n}{g}.$$

The column orthogonality relation on the first two columns tells us

$$k + fr + gs = 0.$$

In fact, our approach does not require the integral table algebra $\mathcal{B}$ to come from a symmetric association scheme of rank 3. If we start with $\mathcal{B} = \{b_0, b_1, b_2\}$ being the basis of any symmetric standard integral table algebra of rank 3, we have that the entries of b_1 and b_2 are nonnegative integers, the degrees k and $\ell := n - k - 1$ of b_1 and b_2 are positive integers, and the character table P of the algebra generated by $\mathcal{B}$ satisfies the same row and column orthogonality relations as those presented above, with r, s real and f and g positive real numbers satisfying $n = 1 + f + g$. Our first lemma gives bounds on our parameters that can be established in the general table algebra setting.

Lemma 1. *Suppose that the last two rows of the character table are ordered so that $r \geqslant s$. Then the character table values k, $\ell(= n - k - 1)$, r, and s satisfy*

1. $\ell = \frac{-k(1+r+s+rs)}{(k+rs)}$;
2. $k, \ell \geqslant 1$ *and* $k \geqslant r \geqslant 0 > -1 \geqslant s = \frac{-(k+kr+k\ell)}{(k+kr+r\ell)}$;
3. $\ell \geqslant -1 - s \geqslant 0 > -1 \geqslant -1 - r$;

4. $(k+rs) \geqslant 0 \geqslant (1+r+s+rs)$; *and*

5. $\ell + (1+r+s+rs) \geqslant 0$ *and* $\ell - 1 + rs \geqslant 0$.

Proof. First, the identities $\ell = \frac{-k(1+r+s+rs)}{(k+rs)}$ and $s = \frac{-(k+kr+k\ell)}{(k+kr+r\ell)}$ are consequences of the orthogonality relations of the rows applied to the last two rows of $\mathcal{P}$.

For the other properties, we use the fact that the columns of $\mathcal{P}$ form the basis of an algebra under entry-wise multiplication that has the same structure constants as the basis $\mathcal{B} = \{b_0, b_1, b_2\}$ (which also matches that of the basis $\mathcal{A}$). This allows us to determine that the regular matrix of b_1 in the basis $\mathcal{B}$ to be

$$\begin{bmatrix} 0 & k & 0 \\ 1 & k+r+s+rs & -(1+r+s+rs) \\ 0 & k+rs & -rs \end{bmatrix}.$$

Since $\mathcal{B}$ is the basis of a table algebra, the entries of this regular matrix are nonnegative real, which implies $1+r+s+rs \leqslant 0$ and $k+rs \geqslant 0$. In addition, k is the Perron-Frobenius eigenvalue of A_1, so $k \geqslant r, s \geqslant -|k|$. Since $r > s$, $-rs \geqslant 0$ implies $r \geqslant 0 > s$ or $r > 0 \geqslant s$. But $r \geqslant 0$ and $1+r+s+rs \leqslant 0$, so we must have $s < 0$. Since $k+r+s+rs \geqslant 0 \geqslant 1+r+s+rs$, we must have $k \geqslant 1$. Since we can interchange columns of $\mathcal{P}$ at the beginning, we can also conclude $\ell \geqslant 1$. From the formula for $s = \frac{-(k+kr+k\ell)}{(k+kr+r\ell)}$, we can now see that having $0 > s > -1$ would imply $r > k$, which is false, so $-1 \geqslant s$.

We can also find the regular matrix of b_2 with respect to the basis $\mathcal{B}$, it is

$$\begin{bmatrix} 0 & 0 & \ell \\ 0 & -(1+r+s+rs) & \ell + (1+r+s+rs) \\ 1 & -rs & \ell - 1 + rs \end{bmatrix}.$$

Since this is a nonnegative matrix when $\mathcal{B}$ is a table algebra, we can conclude $\ell + (1+r+s+rs), \ell - 1 + rs \geqslant 0$. □

The case $k = r$ (which is equivalent to that of $s = -1$) occurs in the case where the strongly regular graph Γ is disconnected and its complement is a complete multipartite graph. When the roles of A_1 and A_2 are interchanged, this is the case for $\ell = -1 - s$ and $r = 0$. We will refer to these cases as *imprimitive*, and to the cases $k > r$ and $-1 > s$ as primitive. In the imprimitive case, if the graph Γ is the disjoint union of $m+1$ copies of the complete graph on $k+1$ vertices, then the character table of the association scheme is

$$\mathcal{P} = \begin{bmatrix} 1 & k & m(1+k) \\ 1 & k & -1-k \\ 1 & -1 & 0 \end{bmatrix} \begin{matrix} 1 \\ m \\ k(1+m) \end{matrix}.$$

As we will see, the imprimitive case will be the one where $\mathcal{A} \otimes \mathcal{A}$ has the most special case fusions.

2.2 The tensor square association scheme

The *tensor product scheme*, or *Kronecker product*, of any two association schemes with sets of adjacency matrices $\mathcal{A} = \{A_0 = I_n, A_1, \ldots, A_d\}$ and $\mathcal{B} = \{B_0 = I_{n'}, B_1, \ldots, B_{d'}\}$ is the association scheme of order nn' and rank $(d+1)(d'+1)$ whose adjacency matrices are $\mathcal{A} \otimes \mathcal{B} = \{A_i \otimes B_j : 0 \leqslant i \leqslant d, 0 \leqslant j \leqslant d'\}$. We will be interested in the tensor square $\mathcal{A} \otimes \mathcal{A}$ when $\mathcal{A} = \{A_0, A_1, A_2\}$, so this association scheme has rank 9.

We will use some convenient shorthand notation for elements and subsets of $\mathcal{A} \otimes \mathcal{A}$. For $i, j \in \{0, 1, 2\}$, we write A_{ij} for the element $A_i \otimes A_j$. We will write $\mathcal{A} \otimes 1$ and $1 \otimes \mathcal{A}$ to identify and distinguish the subsets $\{A_{00}, A_{10}, A_{20}\}$ and $\{A_{00}, A_{01}, A_{02}\}$. Later, we will make use of a single-index notation, which is convenient to identify partitions of $\mathcal{A} \otimes \mathcal{A} - \{A_{00}\}$. If we identify A_{ij} with C_{3j+i+1}, for $i, j \in \{0, 1, 2\}$, then we can identify partitions of $\mathcal{A} \otimes \mathcal{A} - \{A_{00}\}$ with partitions of $\{2, \ldots, 9\}$, which we will also write in shorthand notation. For example, if $\mathcal{T}_{\mathcal{A}} = \{A_0, A_1 + A_2\}$ is the basis of the trivial fusion of $\mathcal{A}$, then the tensor-product subalgebras with basis $\mathcal{T}_{\mathcal{A}} \otimes \mathcal{T}_{\mathcal{A}}$, $\mathcal{T}_{\mathcal{A}} \otimes \mathcal{A}$, and $\mathcal{A} \otimes \mathcal{T}_{\mathcal{A}}$ can be identified with partitions 23|47|5689, 23|4|56|7|89, and 2|3|47|58|69.

What makes such a collection correspond to a *fusion* of $\mathcal{A} \otimes \mathcal{A}$ is that this subset of elements is a basis for the unital subalgebra of $\mathbb{C}[\mathcal{A} \otimes \mathcal{A}]$ that it generates. This is a special situation, as in most cases a collection of disjoint sums of basis elements will generate a subalgebra of larger dimension than the size of the collection.

$\mathcal{A} \otimes \mathcal{A}$ has three types of fusion subalgebras that are guaranteed to exist for all rank 3 table algebra bases A. The first type is the tensor product subalgebra type, which includes $\mathcal{A} \otimes \mathcal{A}$ and the three tensor products involving $\mathcal{T}_{\mathcal{A}}$ previously introduced. The second is the symmetric tensor square subalgebra $Sym^2(\mathcal{A})$, which is known as the generalized Hamming scheme $H(2, \mathcal{A})$ when $\mathcal{A}$ corresponds to a strongly regular graph. Its basis is equal to the set of the elementary symmetric 2-tensors

$$Sym^2(\mathcal{A}) = \{A_{00}, A_{10} + A_{01}, A_{20} + A_{02}, A_{11}, A_{21} + A_{12}, A_{22}\}.$$

Using the indices obtained from $C_{3j+i+1} = A_{ij}$, we identify this fusion of $\mathcal{A} \otimes \mathcal{A}$ with the partition 24|37|5|68|9. The third type consists of wreath products. The full wreath product $\mathcal{A} \wr \mathcal{A}$ occurs in two different ways, as

$$(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A}) = \{A_{00}, A_{10}, A_{20}, A_{01} + A_{11} + A_{21}, A_{02} + A_{12} + A_{22}\}$$

and

$$(1 \otimes \mathcal{A}) \wr (\mathcal{A} \otimes 1) = \{A_{00}, A_{01}, A_{02}, A_{10} + A_{11} + A_{12}, A_{20} + A_{21} + A_{22}\}.$$

These are associated with the partitions 2|3|456|789 and 258|369|4|7, respectively. These wreath products have several proper wreath product fusions, all of which involve $\mathcal{T}_{\mathcal{A}}$: $(\mathcal{T}_{\mathcal{A}} \otimes 1) \wr (1 \otimes \mathcal{T}_{\mathcal{A}})$, $(\mathcal{T}_{\mathcal{A}} \otimes 1) \wr (1 \otimes \mathcal{A})$, $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{T}_{\mathcal{A}})$, $(1 \otimes \mathcal{T}_{\mathcal{A}}) \wr (\mathcal{T}_{\mathcal{A}} \otimes 1)$, $(1 \otimes \mathcal{T}_{\mathcal{A}}) \wr (\mathcal{A} \otimes 1)$, and $(1 \otimes \mathcal{A}) \wr (\mathcal{T}_{\mathcal{A}} \otimes 1)$. These can be identified, in order, with the six partitions

23|456789, 23|456|789, 2|3|456789, 235689|47, 258|369|47, and 235689|4|7.

In Section 4 we will verify that the 15 fusions of $\mathcal{A} \otimes \mathcal{A}$ listed so far are the only fusions that are guaranteed, as any other fusion imposes conditions on the parameters on the symmetric rank 3 table algebra.

2.3 The Bannai-Muzychuk criterion

Given a basis $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ of a table algebra (or adjacency algebra) of rank d, and a partition $\tau = \{T_0 = \{0\}, T_1, \ldots, T_{d'}\}$ of $\{0, 1, \ldots, d\}$, we let $A_{T_j} = \sum_{t \in T_j} A_t$ for all $j \in \{0, 1, \ldots, d'\}$. The partition τ corresponds to a fusion of the table algebra when $A^\tau = \{A_{T_j}\}$ is also the basis of a table algebra.

Let $\mathcal{P}$ be the $(d+1) \times (d+1)$-character table of the table algebra corresponding to $\mathcal{A}$. Given a partition τ of $\{0, 1, \ldots, d\}$, we define $\mathcal{P}_\tau$ to be the order $|\tau| \times (d+1)$ matrix with the columns indexed by the classes $T \in \tau$. The column corresponding to a class $T \in \tau$ is the sum of columns in $\mathcal{P}$ indexed by t for all $t \in T$. Since $\mathcal{P}$ is a non-singular matrix, the rank of $\mathcal{P}_\tau$ is equal to $|\tau|$. Hence, the number of distinct rows in $\mathcal{P}_\tau$ is at least $|\tau|$. The next lemma, known as the *Bannai-Muzychuk criterion* (see Muzychuk [7]), states a necessary and sufficient condition for a partition τ of the rows of $\mathcal{P}$ to produce a fusion.

Lemma 2. *A partition $\tau = \{T_0 = \{0\}, \ldots, T_{d'}\}$ of the index set $\{0, \ldots, d\}$ determines a fusion subalgebra with basis $\mathcal{A}^\tau = \{A_0, A_{T_1}, \ldots, A_{T_{d'}}\}$ if and only if the number of different rows in $\mathcal{P}_\tau$ equals $|\tau|$.*

3 Fusions of the wreath product

If $\mathcal{P}(\mathcal{A})$ is the character table of $\mathcal{A}$, the character table of $\mathcal{A} \otimes \mathcal{A}$ is the Krönecker product $\mathcal{P} \circ \mathcal{P}$. When $\mathcal{A}$ is the basis of adjacency matrices of the association scheme corresponding to a strongly regular graph and the elements of $\mathcal{A} \otimes \mathcal{A}$ are labeled as in the previous section, the wreath product $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$ corresponds to the partition 2|3|456|789 of $\{2, \ldots, 9\}$. When we apply the Bannai-Muzychuk criterion for this partition to the 9×9 character table $P \circ P$, we do indeed get exactly 5 distinct rows, and these rows give us the character table of $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$:

$$\begin{array}{c} \\ \chi_{00} \\ \chi_{01} \\ \chi_{02} \\ \chi_{11} \\ \chi_{21} \end{array}
\begin{array}{c} \begin{array}{ccccc} A_{00} & A_{10} & A_{20} & A_{01}+A_{11}+A_{21} & A_{02}+A_{12}+A_{22} \end{array} \\
\left[\begin{array}{ccccc} 1 & k & \ell & kn & \ell n \\ 1 & k & \ell & rn & n(-1-r) \\ 1 & k & \ell & sn & n(-1-s) \\ 1 & r & -1-r & 0 & 0 \\ 1 & s & -1-s & 0 & 0 \end{array}\right] \end{array}
\begin{array}{c} \\ 1 \\ m_r \\ m_s \\ nm_r \\ nm_s \end{array}$$

Our strategy to find all fusions of $\mathcal{A} \wr \mathcal{A}$ (with the above orientation) is to then apply the Bannai-Muzychuk criterion again to the 15 partitions of 2|3|456|789, and use it to determine the conditions on parameters for a partition to correspond to a special case fusion. (As there are so few partitions, it is straightforward to work through all the possibilities by hand. Nevertheless, we will provide the details here to illustrate what our computer implementation accomplishes for the full tensor product in the next section.)

Theorem 3. *Let $\mathcal{A}$ be the basis of a symmetric rank 3 table algebra (i.e, the set of adjacency matrices for the association scheme corresponding to a strongly regular graph).*

Consider the wreath product $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$ *as the fusion of* $\mathcal{A} \otimes \mathcal{A}$ *corresponding to the partition* $2|3|456|789$ *as described in §2.2. Then*

1. $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$ *has guaranteed fusions corresponding to the partitions* $2|3|456789$, $23|456|789$, *and* $23|456789$. *These correspond to wreath product subalgebras of* $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$ *where either or both of the* $\mathcal{A}$ *s are replaced by the trivial fusion* $\mathcal{T}_{\mathcal{A}}$. *Furthermore, these are the only nontrivial guaranteed fusions of* $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$.

2. $(1 \otimes \mathcal{A}) \wr (\mathcal{A} \otimes 1)$ *has guaranteed fusions corresponding to the partitions* $258|369|47$, $235689|4|7$, *and* $235689|47$, *which correspond to wreath product subalgebras of* $(1 \otimes \mathcal{A}) \wr (\mathcal{A} \otimes 1)$ *where either or both of the* $\mathcal{A}$*'s are replaced by the trivial fusion* $\mathcal{T}_{\mathcal{A}}$. *Furthermore, these are the only nontrivial guaranteed fusions of* $(1 \otimes \mathcal{A}) \wr (\mathcal{A} \otimes 1)$.

3. *The only special case fusions of* $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$ *occur when* $\mathcal{A}$ *corresponds to an imprimitive strongly regular graph. The special case fusions in the case* $k = r$ *and* $s = -1$ *correspond to the partitions* $2|3456789$, $23456|789$, *and* $2|3456|789$. *The special case fusions in the case* $\ell = -1 - s$ *and* $r = 0$ *correspond to the partitions* $2456789|3$, $23789|456$, *and* $2789|3|456$.

4. *The only special case fusions of* $(1 \otimes \mathcal{A}) \wr (\mathcal{A} \otimes 1)$ *occur when* $\mathcal{A}$ *corresponds to an imprimitive strongly regular graph. The special case fusions in the case* $k = r$ *and* $s = -1$ *correspond to the partitions* $2345689|7$, $258|34679$, *and* $258|3469|7$. *The special case fusions in the case* $\ell = -1 - s$ *and* $r = 0$ *correspond to the partitions* $2356789|4$, $24578|369$, *and* $2578|369|4$.

Proof. First, note that since each of the three wreath products does give a table algebra, these do correspond to fusions of $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$ that are guaranteed to exist for all symmetric rank 3 table algebras $\mathcal{A}$.

Now suppose τ' is a partition other than these three for which $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$ has a nontrivial fusion corresponding to τ' for some symmetric rank 3 table algebra $\mathcal{A}$. There are ten possibilities for τ', which we will consider one at a time.

$\tau' = 2|3456789$: Since $\chi_{21}(A_{10}) = s$ is the only negative character value in its column, χ_{21} will be isolated. By the Bannai-Muzychuk criterion, we must have $\chi_{01} = \chi_{02} = \chi_{11}$. This implies $k = r$, which implies $s = -1$. For this to be a rank 3 fusion, we also need $\chi_{00} \neq \chi_{01}$. Since $\ell + n(n-1) > \ell - n$, this is the case. So this gives a fusion only in the imprimitive case $k = r$ and $s = -1$.

$\tau' = 3|2456789$: In this case χ_{11} will be isolated, so we must have $\chi_{01} = \chi_{02} = \chi_{21}$. Equating the values on A_{20} we get $\ell = -1 - s$, which implies $r = 0$. So this will be a fusion in the other imprimitive case when $\ell = -1 - s$ and $r = 0$.

$\tau' = 23789|456$: In this case χ_{02} will be isolated, and we will have $\chi_{11} = \chi_{21}$. Since the values of χ_{00} and χ_{01} on $A_{10} + A_{20} + A_{02} + A_{12} + A_{22}$ are $k + \ell + n\ell > k + \ell + n(-1 - r)$, we can only have a rank 3 fusion when $\chi_{01} = \chi_{11} = \chi_{21}$, which implies $rn = 0$ and $k + \ell + n(-1 - r) = -1$. This implies $r = 0$ and hence $\ell = -1 - s$. So this is a fusion in this imprimitive case.

$\tau' = 23456|789$: This time χ_{01} will be isolated and we will have $\chi_{11} = \chi_{21}$. Since the values of χ_{00} and χ_{02} on $A_{10} + A_{20} + A_{01} + A_{11} + A_{21}$ satisfy $k + \ell + kn > k + \ell + sn$, the only way we can have a rank 3 fusion is for $\chi_{02} = \chi_{11}$. This requires $k + \ell + sn = -1$ and $n(-1 - s) = 0$, which holds when $s = -1$ and $k = r$. So this gives a fusion when $k = r$ and $s = -1$.

$\tau' = 2|3456|789$: Again we will have χ_{21} isolated. Similarly χ_{01} takes only one negative value, $n(-1-r)$, on $A_{02}+A_{12}+A_{22}$, so it is also isolated. So we must have that $\chi_{02} = \chi_{11}$, and that these are not equal to χ_{00}. Comparing the values of these characters on A_{01} gives $k = r$, and on $A_{02} + A_{12} + A_{22}$ gives $n(-1 - s) = 0$. So $k = r$ and $s = -1$. Under these conditions, χ_{11} and χ_{00} agree on A_{10} and $A_{02} + A_{12} + A_{22}$, but they take values $\ell + kn$ and $\ell - n$ on $A_{20} + A_{01} + A_{11} + A_{21}$, which are not equal. So this does give a fusion in the case $k = r$ and $s = -1$.

$\tau' = 2|3789|456$: Again χ_{21} and χ_{02} must be isolated. This implies the values of χ_{01} and χ_{11} will agree. This gives $k = r$ and $kr = 0$. But then $r = 0 = k$, and this is a contradiction.

$\tau' = 2456|3|789$: In this case, χ_{01} and χ_{11} will be isolated, so in order for this partition to produce a fusion we must have $\chi_{02} = \chi_{21}$. Equating their values on A_{20} gives $\ell = -1-s$, and equating their values on $A_{02}+A_{12}+A_{22}$ we get $n(-1-s) = 0$. But this would imply $\ell = 0$, a contradiction.

$\tau' = 2789|3|456$: This time χ_{02} and χ_{11} will be isolated, so in order for this partition to produce a fusion we must have $\chi_{01} = \chi_{21}$. Equating their values on A_{20} we get $\ell = -1-s$, and equating their values on $A_{01} + A_{11} + A_{21}$ gives $rn = 0$, so $r = 0$. So this partition gives a fusion in the imprimitive case where $\ell = -1 - s$ and $r = 0$.

$\tau' = 2456|3789$: The values of χ_{11}, χ_{21}, and χ_{10} on $A_{10} + A_{01} + A_{11} + A_{21}$ satisfy $k + rn > r > s$, and hence we have at least four distinct rows χ_{00}, χ_{01}, χ_{11} and χ_{21} in the modified character table. By the Bannai-Muzychuk criterion, this partition will not produce a fusion of rank 3.

$\tau' = 2789|3456$: This time we see that the values of χ_{20}, χ_{11}, and χ_{21} on $A_{10} + A_{02} + A_{12} + A_{22}$ satisfy $k + n(-1 - s) > r > s$, so as shown above we have at least four distinct rows in the modified character table. Hence,this partition will not result in a rank 3 fusion.

This completes the characterization of fusions of $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$. For the fusions of $(1 \otimes \mathcal{A}) \wr (\mathcal{A} \otimes 1)$, we need only apply the permutation $(24)(37)(68)$ to each partition giving a fusion of $(\mathcal{A} \otimes 1) \wr (1 \otimes \mathcal{A})$. □

4 Classifying fusions of the tensor square in special cases

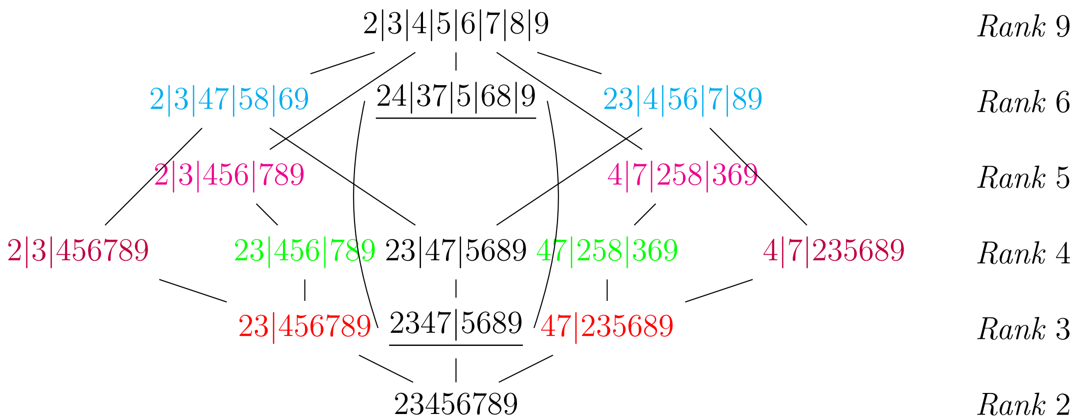
In the previous section, we classified all possible fusions of the wreath products $\mathcal{A} \wr \mathcal{A}$ of symmetric rank 3 association schemes $\mathcal{A}$ with themselves. In this section, aided by a computer program, we will apply the Bannai-Muzychuk criterion to produce the 13 guaranteed fusions of the tensor square $\mathcal{A} \otimes \mathcal{A}$, and all special case fusions of the tensor square for the six families of strongly regular graphs $\mathcal{A}$ where $H(2, \mathcal{A})$ is known from [12] to admit a special case fusion. In Section 5 we will show that there are no other cases where $\mathcal{A} \otimes \mathcal{A}$ has a special case fusion. For each special case fusion, we will describe the graph or association scheme associated with it. Note that these fusions are automatically realized as association schemes when $\mathcal{A}$ is known to be realized by a strongly regular graph, and this has been settled except for one of the six families.

We continue with the notation of §2, so $\mathcal{A} = \{A_0, A_1, A_2\}$, where A_1 is the adjacency matrix of a strongly regular graph Γ of order n with eigenvalues $k \geqslant r \geqslant 0 > -1 \geqslant s$, and A_2 is the adjacency matrix of its complement $\bar{\Gamma}$, with valency $\ell = n - k - 1$ and eigenvalues ℓ, $-1 - s$, and $-1 - r$. Let χ_0 be the valency character, and χ_1 and χ_2 be the other irreducible characters of the adjacency algebra, with $\chi_1(A_1) = r$ and $\chi_2(A_1) = s$, so the rows of the character table of $\mathbb{C}\mathcal{A}$ correspond to χ_0, χ_1, and χ_2. (We remark that the calculations in this section do not depend on A_1 being the adjacency matrix of a graph and we get the same fusions if $\mathcal{A}$ is only assumed to be the basis of a rank 3 symmetric standard integral table algebra.) With our assumptions on eigenvalues, the character table of $\mathcal{A} \otimes \mathcal{A}$, with columns indexed by A_{ij} and columns indexed by $\chi_{ij} := \chi_i \otimes \chi_j$, is the following - with the A_{00} column of all entries equal to 1 is omitted:

	A_{10}	A_{20}	A_{01}	A_{11}	A_{21}	A_{02}	A_{12}	A_{22}	
χ_{00}	k	ℓ	k	k^2	ℓk	ℓ	$k\ell$	ℓ^2	1
χ_{01}	k	ℓ	r	kr	ℓr	$(-1-r)$	$k(-1-r)$	$\ell(-1-r)$	m_r
χ_{02}	k	ℓ	s	$k(-1-r)$	ℓs	$(-1-s)$	$k(-1-s)$	$\ell(-1-s)$	m_s
χ_{10}	r	$(-1-r)$	k	rk	$(-1-r)k$	ℓ	$r\ell$	$(-1-r)\ell$	m_r
χ_{11}	r	$(-1-r)$	r	r^2	$(-1-r)r$	$(-1-r)$	$r(-1-r)$	$(-1-r)^2$	m_r^2
χ_{12}	r	$(-1-r)$	s	$r(-1-r)$	$(-1-r)s$	$(-1-s)$	$r(-1-s)$	$(-1-r)(-1-s)$	$m_r m_s$
χ_{20}	s	$(-1-s)$	k	sk	$(-1-s)k$	ℓ	$s\ell$	$(-1-s)\ell$	m_s
χ_{21}	s	$(-1-s)$	r	sr	$(-1-s)r$	$(-1-r)$	$s(-1-r)$	$(-1-s)(-1-r)$	$m_r m_s$
χ_{22}	s	$(-1-s)$	s	s^2	$(-1-s)(-1-r)$	$(-1-s)$	$s(-1-s)$	$(-1-s)^2$	m_s^2

As defined in §2.2, we relabel A_{ij} as C_{3j+i+1} so that the columns of the character table are labeled with the indices $1, \ldots, 9$. Then each fusion of $\mathcal{A} \otimes \mathcal{A}$ is naturally associated with a partition of $\{2, \ldots, 9\}$. Using a computer, we check the condition of Lemma 2 to find all of the partitions that will give a fusion of $\mathcal{A} \otimes \mathcal{A}$ for all symmetric rank 3 table algebras $\mathcal{A}$.

Lemma 4. *The* 15 *partitions of* $\{2, \ldots, 9\}$ *corresponding to the guaranteed fusions of* $\mathcal{A} \otimes \mathcal{A}$ *are:*



(Here highlighted fusions in the same color at a level indicate a pair of dual fusions obtained from applying the flip map $A_{ij} \mapsto A_{ji}$ to the tensor product, which is obtained when the permutation $(24)(37)(68)$ *is applied to the partition. The underlined partitions give the generalized Hamming scheme and its guaranteed fusion.)*

We remark that, in addition to the duality obtained from the flip map $A_{ij} \leftrightarrow A_{ji}$, each partition corresponding to a fusion of $\mathcal{A} \otimes \mathcal{A}$ above has a *switch partner* that corresponds to switching the order of A_1 and A_2 in both copies of $\mathcal{A}$. The switch partner is obtained by applying the permutation $(23)(47)(59)(68)$ to the partition. The above partitions are all fixed by this operation.

Next, for the other partitions of $\{2, \ldots, 9\}$, after we sum the columns of the character table according to the partition, the equality of any pair of rows imposes conditions on our parameters. For the partition to correspond to a fusion, the number of distinct rows that result has to match the size of the partition. From our earlier results in [12], we know some families of $\mathcal{A}$'s where $H(2, \mathcal{A})$ has a special case fusion, so these will also have special case fusions for their tensor square. It is easier to apply the Bannai-Muzychuk criterion to find the partitions corresponding to the fusions of their tensor squares directly, so we will do that first.

The six families of strongly regular graphs whose generalized Hamming scheme has special case fusions found in [12] are as follows:

(i) *imprimitive case:* $k = r$, $\ell = m(r+1)$, $s = -1$ and its switch partner $k = m(-s)$, $r = 0$ and $\ell = -1 - s$;

(ii) *conference graphs:* $k = \ell = 2(r^2 + r)$, $s = -1 - r$ with $r \geqslant 1$, possibly irrational, and satisfying $r^2 + r \in \mathbb{Z}^+$;

(iii) *the Cartesian product of two complete graphs and its complement:* $k = s^2$, $\ell = -2s$, $r = 1$ and its switch partner $k = 2(r+1)$, $\ell = (1+r)^2$, $s = -2$;

(iv) *three specific graphs of order* 9 *satisfying* $k = 3 - s - r$, $\ell = 5 + s + r$*:* the disjoint union of three triangles ($k = r = 2$, $s = -1$), its complement ($k = 6$, $r = 0$, $s = -3$), and the Paley graph of order 9 ($k = \ell = 2$, $r = 1$, $s = -2$);

(v) a family with $k = r(r+3)$, $\ell = (r+3)$, $s = -2$ and its switch partner $k = -(s-2)$, $\ell = (s-2)(s+1)$, and $r = 1$; and

(vi) a family with $k = r(2r+1)$, $\ell = (r-1)(2r+1)$, $s = -r$ and its switch partner $k = (2+2)(2s+1)$,$\ell = (s+1)(2s+1)$, and $r = -s-2$.

The full list of nontrivial fusions corresponding to each of these families is also available in the `TensorProductFusions.txt` file in our GitHub repository [23].

The first family corresponds to imprimitive strongly regular graphs. In general these produce 45 additional fusions, and some small cases when r or m are at most 2 produce a few extra fusions.

Theorem 5. *(i) In the general imprimitive case, the tensor square has* 45 *additional fusions other than the guaranteed fusions in Lemma 4. (The partitions in red correspond to special case fusions of the generalized Hamming scheme. The partitions in the same color at a level indicate a pair of dual fusions obtained from applying the flip map* $A_{ij} \mapsto A_{ji}$ *to the tensor product, which is obtained when the permutation* (24)(37)(68) *is applied to the partition.)*

	2|3|4|5|6|78|9		2|36|4|5|7|8|9		*rank* 8
2|3|4|5|6|789	2|3|45|6|78|9	2|36|4|5|78|9	25|36|4|7|8|9	2|369|4|5|7|8	*rank* 7
2|3|45|6|789	2|36|45|78|9 2|36|4|5|789	2|3678|4|5|9 24|36|5|78|9	25|36|4|78|9 2|369|4|5|78	25|369|4|7|8	*rank* 6
2|36|45|789 23|4|56|789 2|3678|45|9	24|36|5|789 2|3|4578|69 2|369|45|78	2|36789|4|5 245|36|78|9 24|3678|5|9	24|369|5|78 2356|4|7|89 25|36|4|789	25|369|4|78 2|369|47|58 25|36|4|789	*rank* 5
2|3456|789 2356|4|789	2|36789|45 245|369|78	24|36789|5 245|3678|9	25|36789|4 245|36|789	2578|4|369 4578|369|2	*rank* 4
2|3456789	23456|789	245|36789	24578|369	2356789|4	*rank* 3

(ii) In the case $k = r = 1$ *and* $m = 1$ *(i.e. the strongly regular graph is the complement of a* 4*-cycle), the tensor square has* 46 *extra fusions, in addition to the 15 guaranteed and 45 general imprimitive case fusions:*

2|39|48|57|6	26|35|4|7|89	23|4|56|79|8 2|39|47|58|6	24|38|5|67|9	26|35|4|79|8	*rank* 6
2|3|459|678 23|46|5|789 257|369|4|8	2|378|459|6 2356|4|79|8 2|369|48|57 27|369|48|5	249|38|5|67 249|37|5|68 25|3678|4|9	259|367|4|8 2|39|4578|6 26|35|4|789 26|34|5|789	259|368|4|7 28|369|47|5 2|345|6|789	*rank* 5
23|48|5679 257|369|48	2|3678|459 2478|369|5 26|3579|48	2|34578|69 235679|4|8	259|3678|4 2346|5|789 249|3678|5	26|3589|47 26|345|789	*rank* 4
26|345789 2459|3678	24679|358 2467|3589	2468|3579 23479|568 2346789|5	23489|567 2348|5679	235679|48 24689|357	*rank* 3

(iii) When $k = r = 1$ and $m \geqslant 2$ (i.e. the graph is a perfect matching of order $2(m+1) > 4$), there are 5 extra fusions:

28|369|47|5	25|3678|4|9	23|46|5|789	*rank* 5
2478|369|5		2346|5|789	*rank* 4

(iv) In the cases where $k = r > 1$ and $m = 1$ (i.e. when the strongly regular graph is the complement of a bipartite graph), we see 5 extra fusions:

25|3678|4|9		259|368|4|7	*rank* 5
2|3678|459		259|3678|4	*rank* 4
	2459|3678		*rank* 3

(v) When $k = r = 2$ and $m = 2$ (i.e. when the graph is the disjoint union of three triangles), we find 2 extra fusions:

25|3678|4|9	2|3678|45|9	*rank* 5

If Γ is the complete multipartite graph, so A_1 and A_2 switch roles in the imprimitive association scheme, then we will have $r = 0$ and $k = -s$, and the list of proper nontrivial fusions will be the switch partners of the fusions listed above. These are obtained by applying the permutation (23)(47)(59)(68) to the above partitions.

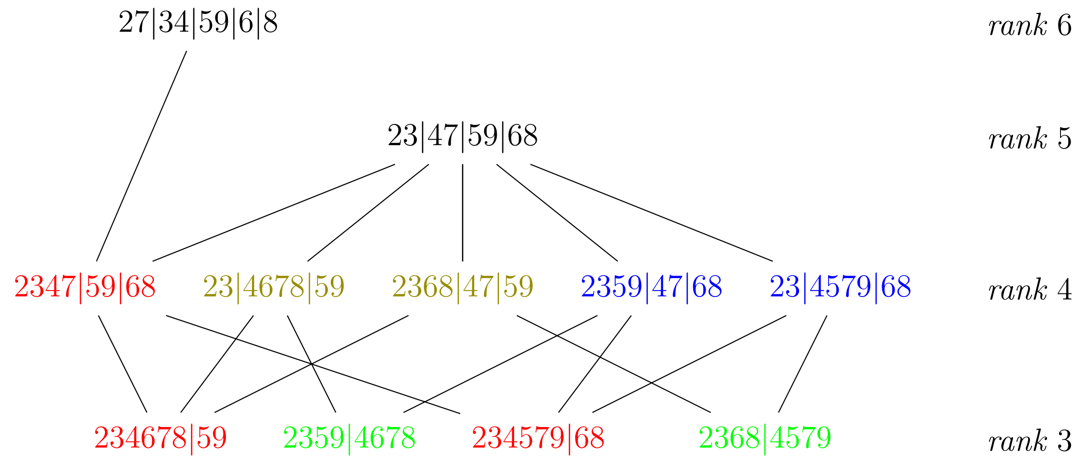
It turns out that all of the association schemes that occur among the 46 fusions in the general imprimitive case also produce imprimitive association schemes. In hindsight, the fusions we see above are all direct consequences of the identities,

$$(A_0 + A_1)^2 = (r+1)(A_0 + A_1) \text{ and } A_2^2 = m(r+1)(A_0 + A_1) + (m-1)(r+1)A_2$$

when A_1 is the adjacency matrix of the disjoint union of m copies of K_{r+1}. In the small special cases the new fusions are equally non-interesting. The rank 3 schemes that appear are all imprimitive, and those of higher rank are not polynomial.

The second family is the family of conference graphs, which includes the symmetric Paley graphs and all their cospectral strongly regular graphs.

Theorem 6. *(i) In the conference graph case, if $r^2 + r > 2$, we get the following 11 additional fusions other than the guaranteed fusions in Lemma 4:*



(Here highlighted fusions in red indicate the self-dual partitions that give fusions of the generalized Hamming scheme. The fusions in the same color at a level indicate a pair of dual fusions obtained by applying the flip map $A_{ij} \mapsto A_{ji}$ to the tensor product, which is obtained when the permutation $(24)(37)(68)$ is applied to the partition.)

When $r^2 + r = 1$ or 2 the tensor square has additional fusions:

(ii) If $r^2 + r = 1$ (the conference graph is a regular 5-gon), the tensor square has two additional fusions:

26|38|49|57 *and its flip partner* 29|35|48|67.

(iii) If $r^2 + r = 2$ (the conference graph is the Paley graph on 9 vertices), the tensor square has 10 extra fusions:

27|348|59|6	249|37|5|68	267|34|59|8	24|357|68|9	*rank* 5
	249|357|68	267|348|59		*rank* 4
267|34589	24689|357	25679|348	249|35768	*rank* 3

The special case fusions we see here are all self-dual. It seems reasonable to expect that fusions of a tensor power of a self-dual table algebra $\mathcal{A}$ will always be self-dual, but we have yet to consider this question fully. The rank 6 fusion 27|34|59|6|8 and its rank 5 fusion 27|34|59|68 correspond to polynomial association schemes that are not P-polynomial. The rank 5 fusions in the case $r^2 + r = 1$ are 6-regular pseudocyclic polynomial association schemes of order 25. All of the rank 4 fusions above turn out to be amorphic association schemes - all of their relations correspond to strongly regular graphs. The rank 3 fusion 234678|59 corresponds to a self-dual primitive strongly regular graph with parameters $(n, k, \lambda, \mu) =$

$$((4r^2+4r+1)^2, 8r(r+1)(r^2+r+1), 2r(r+1)(2r^2+r+3)-1, 2(r^2+r+1)(2r^2+r+1)).$$

The other rank 3 fusions correspond to imprimitive strongly regular graphs.

In the case of the Paley graph of order 9, the extra rank 5 fusions correspond to self-dual polynomial association schemes that are not P-polynomial. The rank 4 fusions are self-dual amorphic association schemes. The extra rank 3 fusions correspond to the strongly regular graph with parameters $(n, k, \lambda, \mu) = (81, 24, 9, 6)$.

Theorem 7. *(i) If Γ is the Cartesian product of two complete graphs with order $-s+1$ with $s < -3$, then the tensor square has one rank 5 fusion other than the guaranteed fusions in Lemma 4:*

249|37|5|68.

(ii) If Γ is the complement of the Cartesian product of two complete graphs with order $r+2$ for $r > 2$, then there is an additional rank 5 fusion of $\mathcal{A} \otimes \mathcal{A}$ which is the switch partner of part one above:

24|375|68|9.

(iii) If Γ is the Cartesian product of two complete graphs of order 2 (i.e. the complement of the 4-cycle), the tensor square has 46 fusions in addition to the 15 guaranteed fusions and the 45 special imprimitive case fusions (see Theorem 5 (ii)).

(iv) If Γ is the Cartesian product of two complete graphs of order 3 (i.e. it is isomorphic to a Paley graph of order 9), then it has the 10 extra fusions in addition to the 15 guaranteed and 11 conference graph fusions (see Theorem 6 (iii)).

(v) If Γ is the Cartesian product of two complete graphs of order 4 (i.e. the complement of the Shrikhande graph) then it has two extra fusions in addition to the 15 guaranteed fusions:

249|37|5|68 *and* 2459|3678.

The fusion in (i) corresponds to a primitive distance-regular graph with intersection array $[-4s, -3s, -2s, -s; 1, 2, 3, 4]$, so it is isomorphic to the Hamming graph $H(4, 1-s)$. The second fusion in (v) corresponds to a strongly regular graph with parameters $(256, 135, 70, 72)$.

In [12], the next 2-parameter family is given in terms of the parameter restrictions $k = 3 - s - r$, $\ell = 5 + s + r$. Only two association schemes are realized with these parameters, both with order 9. The first consists of the disjoint union of three triangles and its multipartite complement, which is the imprimitive scheme has character table

$$\mathcal{P}(\mathcal{A}) = \begin{bmatrix} 1 & 2 & 6 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 6 \end{matrix},$$

and the Paley graph of order 9, which is the conference graph with character table

$$\mathcal{P}(\mathcal{A}) = \begin{bmatrix} 1 & 4 & 4 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{matrix} 1 \\ 4 \\ 4 \end{matrix}.$$

We have encountered both of these association schemes already, the disjoint union of three triangles in Theorem 5 (v) and the Paley graph of order 9 in Theorem 6 (iii).

The next family of strongly regular graphs identified in [12] to have special case fusions for the generalized Hamming scheme $H(2, \mathcal{A})$ are those with parameters $k = r(3 + r)$, $\ell = (3 + r)$, and $s = -2$ and their complement. The Krein condition shows these only exist for $r \leqslant 2$, and for $r = 1$ the table matches that of the Paley graph of order 9. So we need only consider the case where $r = 2$, which corresponds to the complement of the Clebsch graph. The Clebsch graph has parameters $(16, 5, 0, 2)$ and its complement has parameters $(16, 10, 6, 6)$.

Theorem 8. *(i) For the complement of the Clebsch graph, there are two additional fusions in addition to the guaranteed fusions in Lemma 4.*

$$2468|3579 \text{ and } 249|35678$$

(ii) For the Clebsch graph, there are two special case fusions, which are the switch partners of the above:

$$2459|3678 \text{ and } 24689|357.$$

The graphs in the special case fusions above are strongly regular. In the Clebsch graph case, the 2459|3678 partition produces a strongly regular graph with parameters $(256, 135, 70, 72)$, and the 357|24689 partition produces a strongly regular graph with parameters $(256, 45, 16, 6)$.

The last case identified in [12] is the case where $k = r(2r+1)$, $\ell = (r-1)(2r+1)$, $s = -r$, with $r \geqslant 2$. The case $r = 2$ coincides with the complement of the Clebsch graph, which has the two extra fusions given in Theorem 8 (i). Strongly regular graphs with these parameters are known to exist for $2 \leqslant r \leqslant 6$, but their existence in general for $r \geqslant 7$ is an open problem.

Theorem 9. *(i) In the case $k = r(2r+1)$, $\ell = (r-1)(2r+1)$, $s = -r$ with $r > 2$, the tensor square has one rank 3 fusion in addition to the guaranteed fusions in Lemma 4:*

$$2468|3579$$

(ii) The complement of the previous case has parameters $k = (s+2)(2s+1)$, $\ell = (s+1)(2s+1)$, and $r = -2-s$ with $s < -3$, it gives one extra fusion, corresponding to the switch partner of the above:

$$2459|3768$$

The two fusions above are also fusions of the generalized Hamming scheme. The strongly regular graph produced by the fusion 2468|3579 in the first case of the Theorem has parameters

$$(16r^4,\ 8r^4 - 2r^2,\ 4r^4 - 2r^2,\ 4r^4 - 2r^2).$$

The smallest such graph, occurring when $r = 2$, has parameters $(256, 120, 56, 56)$ and is reported in [4, §10.48]. Since we do know $\mathcal{A}$ exists when $2 \leqslant r \leqslant 6$, this fusion also implies the existence of strongly regular graphs with parameters

$$(1296, 630, 306, 306), (4096, 2016, 992, 992), (10000, 4950, 2450, 2450),$$

and

$$(20736, 10296, 5112, 5112).$$

(The graph resulting from $r = 3$ was known to exist (*see* [5]), the others do not appear on Brouwer's tables of known strongly regular graph parameters but are classified in [8].)

5 Our main result

Let $\mathcal{A} = \{A_0 = I, A_1, A_2\}$ be the basis of adjacency matrices of a symmetric rank 3 association scheme, or the standard basis of a rank 3 symmetric table algebra. In this section, we will describe how we have used a computer program to verify that the fusions listed in Theorems 5–9 are all of the possible fusions of $\mathcal{A} \otimes \mathcal{A}$.

Theorem 10. *If $\mathcal{A}$ does not belong to the families of strongly regular graphs listed in Theorems 5–9, then the only fusions of the tensor square of $\mathcal{A}$ are the 15 guaranteed fusions listed in Lemma 4.*

Proof. We know that the rank of a fusion corresponds to the number of columns in the character table which are analyzed case by case and are referenced by their category in the file `AllPartitionsDataCategorized.txt` on our GitHub repository [23]. The special case fusions covered in Theorems 5–9 are also verified there using GAP.

All that is left to show is that the only nontrivial special case fusions occur when $\mathcal{A}$ belongs to one of the families in Theorems 5–9. The first 45 non-trivial fusions (these are the fusions listed in Lines 129-199 of `TensorProductFusions.txt` in [23]) hold only in the rank 3 imprimitive case and hence are ruled out for any other strongly regular graphs or table algebras in our search if we assume that $s < -1$ and $k > r$. Also, their switch partners occur only when $r = 0$ and $\ell = -1 - s$, so we can assume $r > 0$ and $\ell > -1 - s$.

Focusing on the primitive case, we define a set U (given in lines 14-29 of `TensorProductFusions.txt` in [23]) which consists of polynomials that do not meet the parameters of a primitive strongly regular graph necessary to satisfy a row equality, which in turn would fulfill the Bannai-Muzychuk Lemma and result in a fusion. This set is used to sieve out partitions which require one of these polynomial conditions to give a fusion. If an equality of a pair of irreducible characters of the fusion induced by a partition requires a polynomial in U to equal 0, this pair of irreducible characters cannot hold in the fusion. By the Bannai-Muzychuk criterion in Lemma 2, the partition will only induce an actual fusion when the partition it induces on irreducible characters has the same size. Our step-by-step sieving process is explained in lines 5-16 of the `TensorProductFusions.txt` file on [23].

We will give four typical examples here, two examples of partitions that produce fusions and two examples of partitions that do not produce fusions. All of the other partitions of $\{2, \ldots, 9\}$ are dealt with in a similar fashion in [23]. All line numbers are referenced from the file `AllPartitionsDataCategorized.txt` in [23].

1. Consider the partition 27|34|59|6|8 given in lines 263-267 that gives a rank 6 fusion. For this partition to give a nontrivial fusion, the row equalities, $\chi_2 = \chi_7$, $\chi_3 = \chi_4$, and $\chi_5 = \chi_9$ must be satisfied. On further investigation, we can see that $\chi_5 = \chi_9$ implies $r = -1 - s$, and $\chi_2 = \chi_7$, $\chi_3 = \chi_4$ implies $k = \ell = 2r + 2r^2 = 2s + 2s^2$. Hence, it is proved that this rank 6 nontrivial fusion falls under the strongly regular graph family given in Theorem 6.

2. Consider the partition 249|35678 given in lines 391-413 that gives a rank 3 fusion. For this partition to give a nontrivial fusion, the row equalities, $\chi_2 = \chi_6 = \chi_8 = \chi_9$, and $\chi_3 = \chi_4 = \chi_5 = \chi_7$ must be satisfied. On further investigation, we can see that the first set of row equalities implies $r = 1$, $s = -2$, and the latter set of row equalities implies $k = r\ell$, with $\ell = 3 + r$. Hence, it is proved that this rank 3 nontrivial fusion falls under the strongly regular graph family given in Theorem 8.

3. Consider the partition 23489|567 given in lines 521-534. Although initial investigation shows that this partition satisfies the Bannai-Muzychuk criterion and gives a rank 3 fusion with row equalities,

$\chi_2 = \chi_3 = \chi_5 = \chi_6 = \chi_7$, and $\chi_4 = \chi_8 = \chi_9$. On further investigation, we can see that the first set of row equalities implies $kr + \ell r - r + 2s = 0$ with $k - \ell = 1 - 2r$, and the latter set of row equalities implies $ks + \ell s + 2r - s = 0$. Since, all of these conditions must be satisfied, we next check for compatibility of the row equalities. We combine the following two equations

$$kr + \ell r - r + 2s = 0, \text{ and } ks + \ell s + 2r - s = 0,$$

to get $k + \ell = 3$. Since we also have $k - \ell = 1 + 2r$ this implies $\ell = 1 - r$ which is a contradiction. Hence, we prove that that this rank 3 partition does not give a nontrivial fusion.

4. Consider the partition 2678|34|59 given in lines 2118-2124. A quick glance at the row equality conditions does not bring out any obvious contradictions. In this case we can observe that even if all the row equalities are satisfied and compatible with each other, this implies $\chi_2 = \chi_6 = \chi_7$, $\chi_3 = \chi_4$ and $\chi_5 = \chi_9$ we still end up with at least 5 distinct rows in the modified character table. Hence, using Lemma 2 we prove that this rank 4 partition does not give a nontrivial fusion.

After running the sieve, we find that all of the remaining partitions corresponding to special case fusions are either the ones listed for one of the strongly regular graph families in the previous section, or their switch partners. So this verifies the theorem. □

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References

[1] M. E. Muzychuk. Subschemes of Hamming Schemes $H(n, q)$, $q \geqslant 4$. *Acta Appl. Math.*, vol. 29, pp. 119–128, 1992.

[2] C. Godsil. Generalized Hamming scheme. arXiv:1011.1044, Nov. 2010.

[3] P. Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Res. Rep. Suppl.*, vol. 10, pp. vi+97, 1973.

[4] A. E. Brouwer and H. Van Maldeghem, *Strongly Regular Graphs*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2022.

[5] A. E. Brouwer, "Tables of Parameters of Strongly Regular Graphs," 2025, [Online]. Available: https://aeb.win.tue.nl/graphs/srg/srgtab.html. Accessed: Jan. 15, 2025.

[6] M. Muzychuk. On association schemes of the symmetric group S_{2n} acting on partitions of type 2^n. *Bayreuth. Math. Schr.*, vol. 47, pp. 151–164, 1994.

[7] M. Muzychuk. Subschemes of the Johnson Scheme. *European J. Combin.*, vol. 13, no. 5, pp. 187–193, 1992.

[8] A. D. Sankey. Strongly regular fusions of tensor products of strongly regular graphs. *Rocky Mountain J. Math.*, vol. 24, no. 2, pp. 709–718, 1994. [Online]. Available: https://doi.org/10.1216/rmjm/1181072428.

[9] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.1*, 2021, [Online]. Available: https://www.gap-system.org.

[10] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Universitext, Springer, New York, 2012.

[11] E. Bannai. Subschemes of some association schemes, *J. Algebra*, vol. 144, no. 1, pp. 167–188, 1991.

[12] A. Herman, N. Joshi, and K. Meagher. Fusions of the generalized Hamming scheme on a strongly-regular graph, *Graphs Combin.*, vol. 38, no. 5, Paper No. 150, 2022. [Online]. Available: https://doi.org/10.1007/s00373-022-02545-4.

[13] R. A. Bailey, *Association Schemes*, Cambridge Studies in Advanced Mathematics, vol. 84, Cambridge University Press, Cambridge, 2004.

[14] V. M. Mano, E. A. Martins, and L. A. Vieira. Some results on the Krein parameters of an association scheme, in *Dynamics, Games and Science*, CIM Ser. Math. Sci., vol. 1, Springer, Cham, 2015, pp. 441–454.

[15] R. C. Bose and D. M. Mesner. On linear associative algebras corresponding to association schemes of partially balanced designs. *Ann. Math. Statist.*, vol. 30, pp. 21–38, 1959. [Online]. Available: https://doi.org/10.1214/aoms/1177706356.

[16] D. G. Higman. Coherent algebras. *Linear Algebra and its Applications*, vol. 93, pp. 209–239, 1987.

[17] H. I. Blau. Table algebras. *European J. Combin.*, vol. 30, no. 6, pp. 1426–1455, 2009. [Online]. Available: https://doi.org/10.1016/j.ejc.2008.11.008.

[18] I. A. Faradžev. Cellular subrings of the symmetric square of a cellular ring of rank 3. *Investigations in the algebraic theory of combinatorial objects (Russian)*, Vsesoyuz. Nauchno-Issled. Inst. Sistem. Issled., Moscow, 1985, pp. 76–95.

[19] I. A. Faradžev, A. A. Ivanov, M. H. Klin, and A. J. Woldar, Eds., *Investigations in Algebraic Theory of Combinatorial Objects*, Mathematics and its Applications, vol. 84, Kluwer Academic Publishers, Springer, Dordrecht, 1994. [Online]. Available: https://doi.org/10.1007/978-94-017-1972-8.

[20] E. Bannai and S. Y. Song. Character tables of fission schemes and fusion schemes. *European J. Combin.*, vol. 14, no. 5, pp. 385–396, 1993.

[21] S. Y. Song. Fusion relation in products of association schemes. *Graphs Combin.*, vol. 18, no. 3, pp. 655–665, 2002.

[22] B. Xu. Direct products of association schemes and tensor products of table algebras. *Algebra Colloq.*, vol. 20, no. 3, pp. 475–494, 2013.

[23] A. Herman and N. Joshi. Fusions of Tensor Square of rank 9. GitHub repository, 2023, [Online]. Available: https://github.com/nehamainali/Rank9TensorProductFusions.