On the Hyperdeterminants of Steiner Distance Hypermatrices

Ya-Nan Zheng

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Abstract

Let G be a graph on n vertices. The Steiner distance of a collection of k vertices in G is the fewest number of edges in any connected subgraph containing those vertices. The order k Steiner distance hypermatrix of G is the n-dimensional array indexed by vertices, whose entries are the Steiner distances of their corresponding indices. In this paper, we confirm a conjecture on the Steiner distance hypermatrices proposed by Cooper and Du [Electron. J. Combin. 31(3):#P3.4, 2024]. Furthermore, we also compute the hyperdeterminant of the order k Steiner distance hypermatrix of P_3.

Mathematics Subject Classifications: 05C12, 15A69

1 Introduction

The distance matrix is an important concept in the field of graph theory and combinatorial optimization. In 1971, Graham and Pollak [8] showed a beautiful result that the determinants of the distance matrices of trees depend only on the number of their vertices, but have nothing to do with the structures of the trees. More precisely, let T be a tree on n vertices and the distance matrix $D(T)$ of T is an $n \times n$ matrix whose (u, v)-entry is the distance of u and v, then $\det(D(T)) = (1 - n)(-2)^{n-2}$. This impressive result makes the spectral properties of distance matrices a very interesting research topic. In 1978, Graham and Lovász [7] investigated the coefficients of the characteristic polynomials of distance matrices. In 1990, Merris [12] gave the distance spectrum of a tree.

The Steiner distance is a natural extension of the distance between two vertices, which extends the concept of pairwise distance in a graph to any set of vertices, as proposed in [1]. Given a graph G and a set $S \subseteq V(G)$, the Steiner distance $d(S)$ is defined as the fewest number of edges in any connected subgraph of G containing S. For results on Steiner distance in graphs, we refer the reader to [11] and the references therein.

Recently, Cooper and Tauscheck [3] extended the concept of distance matrices to Steiner distance hypermatrices. The Steiner distance hypermatrix is related to the Steiner

School of Mathematics and Statistics, Henan Normal University, Xinxiang 453007, P.R. China (zh_yn@tju.edu.cn).

distance in the same way as the distance matrix is related to the classical distance. In 2005, Qi [13] defined the hyperdeterminant of a symmetric hypermatrix (also called tensor), which provided a new direction in the study of Steiner distance hypermatrices. Cooper and Tauscheck [3] showed that for a tree T on $n \geqslant 3$ vertices, the hyperdeterminant of the Steiner distance hypermatrix of T is 0 when k is odd. Subsequently, they [4] also showed that the hyperdeterminant for even k is always nonzero. Cooper and Du [2] computed the hyperdeterminant of the Steiner distance hypermatrix of a tree on 2 vertices. They proved that the hyperdeterminant of the Steiner distance hypermatrix of a tree vanishes if and only if (a) $n \geqslant 3$ and k is odd, (b) $n = 1$, or (c) $n = 2$ and $k \equiv 1 \pmod 6$.

For the Steiner distance hypermatrix of a tree on n vertices, Cooper and Du [2], and Cooper and Tauscheck [3], proposed the following conjecture:

Conjecture 1. The hyperdeterminant of the order k Steiner distance hypermatrix of a tree on n vertices only depends on n and k.

In this paper, we confirm the conjecture. Let P_n be the path on n vertices, we also give the hyperdeterminant of the order k Steiner distance hypermatrix of P_3.

2 Preliminaries

In this section, we introduce some notations and basic concepts of hypermatrices. For a positive integer n, we denote $[n] = \{1, \ldots, n\}$. Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^n$ be the n-dimensional complex space.

For positive integers k and n, an order k dimension n *hypermatrix* $\mathcal{A}$ is a multi-dimensional array of n^k entries in $\mathbb{C}$: $\mathcal{A} = (a_{i_1 \cdots i_k}),\ a_{i_1 \cdots i_k} \in \mathbb{C}$, where $i_j \in [n]$ and $j \in [k]$. $\mathcal{A}$ is said to be *symmetric* if the value of $a_{i_1 \cdots i_k}$ is invariant under any permutation of its indices $i_1, \ldots, i_k$. Given a vector $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$, define an n-dimensional vector $\mathcal{A}\mathbf{x}^{k-1}$ whose i-th component is defined as follows:

$$(\mathcal{A}\mathbf{x}^{k-1})_i = \sum_{i_2,\ldots,i_k=1}^{n} a_{ii_2 \cdots i_k} x_{i_2} \cdots x_{i_k}, \quad i \in [n].$$

In [13], Qi defined the *hyperdeterminant* and the *characteristic polynomial* of a hypermatrix. To give the definition of the hyperdeterminant and the characteristic polynomial of a hypermatrix, the resultant theory is needed. For a positive integer n, an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, and an n-tuple $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ of indeterminate variables, denote by $\mathbf{x}^\alpha$ the monomial $\prod_{i=1}^n x_i^{\alpha_i}$. If $F(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$, we use F to denote $F(x_1, \ldots, x_n)$ without causing ambiguity. In the following, we introduce the multipolynomial resultant to study hyperdeterminants, which can be found in [5, 6, 10].

Theorem 2. *Fix degrees $d_1, \ldots, d_n$. For $i \in [n]$, consider all monomials $\mathbf{x}^\alpha$ of total degree d_i in $x_1, \ldots, x_n$. For each such monomial, define a variable $u_{i,\alpha}$. Then there is a unique polynomial* $\mathrm{Res} \in \mathbb{Z}[\{u_{i,\alpha}\}]$ *with the following three properties:*

(1) *If* $F_1, \dots, F_n \in \mathbb{C}[x_1, \dots, x_n]$ *are homogeneous polynomials of degrees* $d_1, \dots, d_n$*, respectively, then the polynomials have a nontrivial common root in* $\mathbb{C}^n$ *exactly when* $\mathrm{Res}(F_1, \dots, F_n) = 0$.

(2) $\mathrm{Res}(x_1^{d_1}, \dots, x_n^{d_n}) = 1$.

(3) Res *is irreducible, even in* $\mathbb{C}[\{u_{i,\alpha}\}]$.

$\mathrm{Res}(F_1, \dots, F_n)$ is called the resultant of $F_1, \dots, F_n$ and it is interpreted as substituting the coefficient of $\mathbf{x}^\alpha$ in F_i for the variable $u_{i,\alpha}$ in Res. Resultant plays an important role in algebraic geometry, algebraic combinatorics, spectral hypergraph theory, etc. The hyperdeterminant of an order k dimension n hypermatrix $\mathcal{A}$ is the resultant of $\mathcal{A}\mathbf{x}^{k-1}$, i.e., $\det(\mathcal{A}) = \mathrm{Res}(\mathcal{A}\mathbf{x}^{k-1})$. The characteristic polynomial of $\mathcal{A}$, denoted $\phi_{\mathcal{A}}(\lambda)$, is the determinant $\det(\lambda\mathcal{I} - \mathcal{A})$, where the unit hypermatrix $\mathcal{I}$ is defined as follows: $\mathcal{I}_{i_1\cdots i_k} = 1$ if $i_1 = i_2 = \cdots = i_k$ and $\mathcal{I}_{i_1\cdots i_k} = 0$ otherwise. Note that when $k = 2$, the hyperdeterminant is exactly the classical determinant.

It is usually difficult to calculate the resultant of a general polynomial system, Hillar and Lim [9] showed that most hypermatrix problems are NP-hard. However, we can study it using some of its properties. In the following, suppose $F_1, \dots, F_n \in \mathbb{C}[x_1, \dots, x_n]$ are homogeneous polynomials of degrees $d_1, \dots, d_n$, respectively. Here we list some useful properties of resultants that will be used in the subsequent discussion, and readers can find them in the reference [10].

Theorem 3. *Fixing some* $i \in [n]$*, suppose* $d_i \geqslant d_j$ *for all* $j \neq i$ *and* $H_{i,j} \in \mathbb{C}[x_1, \dots, x_n]$ *is a homogeneous polynomial of degree* $d_i - d_j$*, then*

$$\mathrm{Res}\left(F_1, \dots, F_i + \sum_{j \neq i} H_{i,j} F_j, \dots, F_n\right) = \mathrm{Res}(F_1, \dots, F_i, \dots, F_n).$$

Theorem 3 is called the elementary transformation property of the resultant.

Theorem 4. *Let* $G = (G_1, \dots, G_n)$*, where* $G_1, \dots, G_n$ *are homogeneous polynomials of degree* d*. For* $i \in [n]$*, denote* $F_i \circ G = F_i(G_1, \dots, G_n)$*. Then*

$$\mathrm{Res}(F_1 \circ G, \dots, F_n \circ G) = \mathrm{Res}(G_1, \dots, G_n)^{d_1 \cdots d_n} \mathrm{Res}(F_1, \dots, F_n)^{d^{n-1}}.$$

Given homogeneous polynomials $F_1, \dots, F_n \in \mathbb{C}[x_1, \dots, x_n]$ of degrees $d_1, \dots, d_n$, let

$$f_i(x_1, \dots, x_{n-1}) = F_i(x_1, \dots, x_{n-1}, 1),$$
$$\overline{F}_i(x_1, \dots, x_{n-1}) = F_i(x_1, \dots, x_{n-1}, 0).$$

Note that $\overline{F}_1, \dots, \overline{F}_{n-1}$ are homogeneous in $\mathbb{C}[x_1, \dots, x_{n-1}]$. The following Poisson formula in [5, 6] gives a recursive method for calculating resultants.

Theorem 5. *Let $F_1, \dots, F_n \in \mathbb{C}[x_1, \dots, x_n]$ are homogeneous polynomials of degrees $d_1, \dots, d_n$, respectively. Let $\mathcal{V}$ be the affine variety defined by the polynomials $f_1, \dots, f_{n-1}$. If $\operatorname{Res}(\overline{F}_1, \dots, \overline{F}_{n-1}) \neq 0$, then*

$$\operatorname{Res}(F_1, \dots, F_n) = \operatorname{Res}(\overline{F}_1, \dots, \overline{F}_{n-1})^{d_n} \prod_{p \in \mathcal{V}} f_n(p)^{\mathrm{m}(p)},$$

where $\mathrm{m}(p)$ is the multiplicity of a point $p \in \mathcal{V}$.

Next, we introduce the concepts of the Steiner distance and the Steiner distance hypermatrix. Given a graph G and a subset $S = \{v_1, \dots, v_k\}$ of the vertices, the *Steiner distance* of S, denoted $d_G(S)$ or $d_G(v_1, \dots, v_k)$, is the number of edges in the smallest connected subgraph of G containing S.

Definition 6. Let G be a graph on n vertices, the order k *Steiner distance hypermatrix* of G is an order k dimension n hypermatrix $\mathcal{S}_k(G)$, whose $(v_1, \dots, v_k)$ entry is $d_G(v_1, \dots, v_k)$.

Given a graph G on n vertices and the vertex set V is labelled as $[n] = \{1, 2 \dots, n\}$. According to the definition, we have

$$(\mathcal{S}_k(G)\mathbf{x}^{k-1})_i = \sum_{i_2,\dots,i_k=1}^{n} d_G(i, i_2, \dots, i_k) x_{i_2} \cdots x_{i_k}, \quad i \in [n]. \tag{1}$$

We can see that for an order k Steiner distance hypermatrix $\mathcal{S}_k(G)$ of a graph G, the hyperdeterminant of $\mathcal{S}_k(G)$, $\det(\mathcal{S}_k(G)) = \operatorname{Res}(\mathcal{S}_k(G)\mathbf{x}^{k-1})$.

3 Hyperdeterminants of Steiner distance hypermatrices of trees on n vertices

In this section, we confirm Conjecture 1 by utilizing the properties of resultants.

Theorem 7. *Let T be a tree on n vertices and $k \geqslant 2$, then $\det(\mathcal{S}_k(T))$ only depends on n and k.*

Proof. Let T be a tree on n vertices with vertex set $V(T) = [n]$, and denote $\mathcal{S}_k(T)\mathbf{x}^{k-1} = (F_1, F_2, \dots, F_n)$. Since we have $\det(\mathcal{S}_k(T)) = 0$ when k is odd, let us assume that k is even. In [4, Proposition 2.2], Cooper and Tauscheck proved the following properties for $\mathcal{S}_k(T)\mathbf{x}^{k-1}$. Let v be a leaf vertex of T and u be its unique neighbour, then

$$F_v - F_u = \left(\sum_{i=1}^{n} x_i - x_v \right)^{k-1} - x_v^{k-1}. \tag{2}$$

Suppose that $\{u_1, u_2\}$ is an edge of T. Let T_1 and T_2 are the trees obtained by adding edges $\{u_1, n+1\}$ and $\{u_2, n+1\}$ to T at vertices u_1 and u_2, respectively. Then we have $\mathcal{S}_k(T_1)\mathbf{x}^{k-1} = (G_1, G_2, \dots, G_{n+1})$ and $\mathcal{S}_k(T_2)\mathbf{x}^{k-1} = (H_1, H_2, \dots, H_{n+1})$.

Considering the equation (2), we obtain that

$$G_{n+1} - G_{u_1} = H_{n+1} - H_{u_2} = \left(\sum_{i=1}^{n} x_i\right)^{k-1} - x_{n+1}^{k-1}. \tag{3}$$

Let $L = (x_1, \ldots, x_{u_1} - x_{n+1}, \ldots, x_{u_2} + x_{n+1}, \ldots, x_{n+1})$, in the following, we show that

$$G_i \circ L = H_i, \text{ for } i \in [n].$$

Note that for any $t \in [k-1]$ and $\alpha \in S^{k-1-t}$, we have $d_T(i, \alpha, u_1, u_2) = d_T(i, \alpha, u_1)$ or $d_T(i, \alpha, u_1, u_2) = d_T(i, \alpha, u_2)$. Let

$$\delta_t(i, \alpha, u_1) = d_T(i, \alpha, u_1, u_2) - d_T(i, \alpha, u_1)$$

and

$$\delta_t(i, \alpha, u_2) = d_T(i, \alpha, u_1, u_2) - d_T(i, \alpha, u_2),$$

then we can see that $\delta_t(i, \alpha, u_1)$ and $\delta_t(i, \alpha, u_2)$ are equal to 1 or 0 for every $\alpha \in S^{k-1-t}$.

For any path P in T_1, we classify it according to whether P contains vertices u_1, u_2 and $n+1$. Note that $\{u_1, n+1\}$ is a pendant edge, then we have the following six cases:

(1) $u_1, u_2, n+1 \notin V(P)$;

(2) $u_1 \in V(P), u_2, n+1 \notin V(P)$;

(3) $u_2 \in V(P), u_1, n+1 \notin V(P)$;

(4) $u_1, u_2 \in V(P), n+1 \notin V(P)$;

(5) $n+1 \in V(P), u_2 \notin V(p)$;

(6) $u_2, n+1 \in V(P)$.

Let $i \in [n] \setminus \{u_1, u_2\}$ and $S = [n] \setminus \{u_1, u_2\}$. Corresponding to these classification cases and equation (1), we have

$$\begin{aligned} G_i = &\sum_{\alpha \in S^{k-1}} d_{T_1}(i, \alpha)\mathbf{x}^\alpha + \sum_{a=1}^{k-1} \binom{k-1}{a} x_{u_1}^a \sum_{\alpha \in S^{k-1-a}} d_{T_1}(i, \alpha, u_1)\mathbf{x}^\alpha \\ &+ \sum_{b=1}^{k-1} \binom{k-1}{b} x_{u_2}^b \sum_{\alpha \in S^{k-1-b}} d_{T_1}(i, \alpha, u_2)\mathbf{x}^\alpha \\ &+ \sum_{\substack{a+b=1 \\ a,b>0}}^{k-1} \binom{k-1}{a, b, k-1-a-b} x_{u_1}^a x_{u_2}^b \sum_{\alpha \in S^{k-1-a-b}} d_{T_1}(i, \alpha, u_1, u_2)\mathbf{x}^\alpha \\ &+ \sum_{\substack{a+c=1 \\ c>0}}^{k-1} \binom{k-1}{a, c, k-1-a-c} x_{u_1}^a x_{n+1}^c \sum_{\alpha \in S^{k-1-a-c}} (d_{T_1}(i, \alpha, u_1) + 1)\mathbf{x}^\alpha \\ &+ \sum_{\substack{a+b+c=1 \\ b,c>0}}^{k-1} \binom{k-1}{a, b, c, k-1-a-b-c} x_{u_1}^a x_{u_2}^b x_{n+1}^c \sum_{\alpha \in S^{k-1-a-b-c}} (d_{T_1}(i, \alpha, u_1, u_2) + 1)\mathbf{x}^\alpha. \end{aligned}$$

Consider the cases (4) and (6), we can see that

$$\sum_{\substack{a+b+c=1\\ b,c>0}}^{k-1}\binom{k-1}{a,b,c,k-1-a-b-c}x_{u_1}^a x_{u_2}^b x_{n+1}^c \sum_{\alpha\in S^{k-1-a-b-c}}(d_{T_1}(i,\alpha,u_1,u_2)+1)\mathbf{x}^\alpha$$

$$=\sum_{a+b+c=1}^{k-1}\binom{k-1}{a,b,c,k-1-a-b-c}x_{u_1}^a x_{u_2}^b x_{n+1}^c \sum_{\alpha\in S^{k-1-a-b-c}}(d_{T_1}(i,\alpha,u_1,u_2)+1)\mathbf{x}^\alpha$$

$$-\sum_{\substack{a+c=1\\ c>0}}^{k-1}\binom{k-1}{a,c,k-1-a-c}x_{u_1}^a x_{n+1}^c \sum_{\alpha\in S^{k-1-a-c}}(d_{T_1}(i,\alpha,u_1,u_2)+1)\mathbf{x}^\alpha$$

$$-\sum_{\substack{a+b=1\\ b>0}}^{k-1}\binom{k-1}{a,b,k-1-a-b}x_{u_1}^a x_{u_2}^b \sum_{\alpha\in S^{k-1-a-b}}(d_{T_1}(i,\alpha,u_1,u_2)+1)\mathbf{x}^\alpha$$

$$-\sum_{a=1}^{k-1}\binom{k-1}{a}x_{u_1}^a \sum_{\alpha\in S^{k-1-a}}(d_{T_1}(i,\alpha,u_1,u_2)+1)\mathbf{x}^\alpha$$

and

$$\sum_{\substack{a+b=1\\ a,b>0}}^{k-1}\binom{k-1}{a,b,k-1-a-b}x_{u_1}^a x_{u_2}^b \sum_{\alpha\in S^{k-1-a-b}}d_{T_1}(i,\alpha,u_1,u_2)\mathbf{x}^\alpha$$

$$=\sum_{a+b=1}^{k-1}\binom{k-1}{a,b,k-1-a-b}x_{u_1}^a x_{u_2}^b \sum_{\alpha\in S^{k-1-a-b}}d_{T_1}(i,\alpha,u_1,u_2)\mathbf{x}^\alpha$$

$$-\sum_{a=1}^{k-1}\binom{k-1}{a}x_{u_1}^a \sum_{\alpha\in S^{k-1-a}}d_{T_1}(i,\alpha,u_1,u_2)\mathbf{x}^\alpha$$

$$-\sum_{b=1}^{k-1}\binom{k-1}{a}x_{u_2}^b \sum_{\alpha\in S^{k-1-b}}d_{T_1}(i,\alpha,u_1,u_2)\mathbf{x}^\alpha.$$

Similarly, we have

$$\sum_{\substack{a+b=1\\ b>0}}^{k-1}\binom{k-1}{a,b,k-1-a-b}x_{u_1}^a x_{u_2}^b \sum_{\alpha\in S^{k-1-a-b}}d_{T_1}(i,\alpha,u_1,u_2)\mathbf{x}^\alpha$$

$$=\sum_{a+b=1}^{k-1}\binom{k-1}{a,b,k-1-a-b}x_{u_1}^a x_{u_2}^b \sum_{\alpha\in S^{k-1-a-b}}d_{T_1}(i,\alpha,u_1,u_2)\mathbf{x}^\alpha$$

$$-\sum_{a=1}^{k-1}\binom{k-1}{a}x_{u_1}^a \sum_{\alpha\in S^{k-1-a}}d_{T_1}(i,\alpha,u_1,u_2)\mathbf{x}^\alpha.$$

Substituting the above equations into G_i, then we can get that

$$
\begin{aligned}
G_i = &\sum_{\alpha\in S^{k-1}} d_{T_1}(i,\alpha)\mathbf{x}^\alpha + \sum_{a+b+c=1}^{k-1}\binom{k-1}{a,b,c,k-1-a-b-c} x_{u_1}^a x_{u_2}^b x_{n+1}^c \\
&\times \sum_{\alpha\in S^{k-1-a-b-c}} (d_{T_1}(i,\alpha,u_1,u_2)+1)\mathbf{x}^\alpha \\
&- \sum_{a+c=1}^{k-1}\binom{k-1}{a,c,k-1-a-c} x_{u_1}^a x_{n+1}^c \sum_{\alpha\in S^{k-1-a-c}} \delta_{a+c}(i,\alpha,u_1)\mathbf{x}^\alpha \\
&- \sum_{a+b=1}^{k-1}\binom{k-1}{a,c,k-1-a-b} x_{u_1}^a x_{u_2}^b \sum_{\alpha\in S^{k-1-a-b}} \mathbf{x}^\alpha \\
&- \sum_{b=1}^{k-1}\binom{k-1}{b} x_{u_2}^b \sum_{\alpha\in S^{k-1-b}} \delta_b(i,\alpha,u_2)\mathbf{x}^\alpha \\
= &\sum_{\alpha\in S^{k-1}} d_{T_1}(i,\alpha)\mathbf{x}^\alpha + \sum_{t=1}^{k-1}\binom{k-1}{t}\sum_{a+b=0}^{t}\binom{t}{a,b,t-a-b} x_{u_1}^a x_{u_2}^b x_{n+1}^{t-a-b} \\
&\times \sum_{\alpha\in S^{k-1-t}} (d_{T_1}(i,\alpha,u_1,u_2)+1)\mathbf{x}^\alpha \\
&- \sum_{t=1}^{k-1}\binom{k-1}{t}\sum_{a=0}^{t}\binom{t}{a} x_{u_1}^a x_{n+1}^{t-a} \sum_{\alpha\in S^{k-1-t}} \delta_t(i,\alpha,u_1)\mathbf{x}^\alpha \\
&- \sum_{t=1}^{k-1}\binom{k-1}{t}\sum_{a=0}^{t}\binom{t}{a} x_{u_1}^a x_{u_2}^{t-a} \sum_{\alpha\in S^{k-1-t}} \mathbf{x}^\alpha \\
&- \sum_{t=1}^{k-1}\binom{k-1}{t} x_{u_2}^t \sum_{\alpha\in S^{k-1-t}} \delta_t(i,\alpha,u_2)\mathbf{x}^\alpha \\
= &\sum_{\alpha\in S^{k-1}} d_{T_1}(i,\alpha)\mathbf{x}^\alpha + \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2}+x_{n+1})^t \\
&\times \sum_{\alpha\in S^{k-1-t}} (d_{T_1}(i,\alpha,u_1,u_2)+1)\mathbf{x}^\alpha \\
&- \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{n+1})^t \sum_{\alpha\in S^{k-1-t}} \delta_t(i,\alpha,u_1)\mathbf{x}^\alpha \\
&- \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2})^t \sum_{\alpha\in S^{k-1-t}} \mathbf{x}^\alpha \\
&- \sum_{t=1}^{k-1}\binom{k-1}{t} x_{u_2}^t \sum_{\alpha\in S^{k-1-t}} \delta_t(i,\alpha,u_2)\mathbf{x}^\alpha
\end{aligned} \tag{4}
$$

Then we consider the vertices u_1 and u_2, and obtain that

$$
\begin{aligned}
G_{u_1} &= \sum_{\alpha\in S^{k-1}} d_{T_1}(u_1,\alpha)\mathbf{x}^\alpha + \sum_{a=1}^{k-1}\binom{k-1}{a} x_{u_1}^a \sum_{\alpha\in S^{k-1-a}} d_{T_1}(u_1,\alpha)\mathbf{x}^\alpha \\
&\quad + \sum_{\substack{a+b=1\\ b>0}}^{k-1} \binom{k-1}{a,b,k-1-a-b} x_{u_1}^a x_{u_2}^b \sum_{\alpha\in S^{k-1-a-b}} d_{T_1}(u_1,\alpha,u_2)\mathbf{x}^\alpha \\
&\quad + \sum_{\substack{a+c=1\\ c>0}}^{k-1} \binom{k-1}{a,c,k-1-a-c} x_{u_1}^a x_{n+1}^c \sum_{\alpha\in S^{k-1-a-c}} (d_{T_1}(u_1,\alpha)+1)\mathbf{x}^\alpha \\
&\quad + \sum_{\substack{a+b+c=1\\ b,c>0}}^{k-1} \binom{k-1}{a,b,c,k-1-a-b-c} x_{u_1}^a x_{u_2}^b x_{n+1}^c \\
&\quad \times \sum_{\alpha\in S^{k-1-a-b-c}} (d_{T_1}(u_1,\alpha,u_2)+1)\mathbf{x}^\alpha \\
&= \sum_{\alpha\in S^{k-1}} d_{T_1}(u_1,\alpha)\mathbf{x}^\alpha + \sum_{a+b+c=1}^{k-1} \binom{k-1}{a,b,c,k-1-a-b-c} x_{u_1}^a x_{u_2}^b x_{n+1}^c \\
&\quad \times \sum_{\alpha\in S^{k-1-a-b-c}} (d_{T_1}(u_1,\alpha,u_2)+1)\mathbf{x}^\alpha \\
&\quad - \sum_{a+c=1}^{k-1} \binom{k-1}{a,c,k-1-a-c} x_{u_1}^a x_{n+1}^c \sum_{\alpha\in S^{k-1-a-c}} \delta_{a+c}(u_1,\alpha,u_1)\mathbf{x}^\alpha \\
&\quad - \sum_{a+b=1}^{k-1} \binom{k-1}{a,c,k-1-a-b} x_{u_1}^a x_{u_2}^b \sum_{\alpha\in S^{k-1-a-b}} \mathbf{x}^\alpha \\
&\quad - \sum_{a=1}^{k-1} \binom{k-1}{a} x_{u_1}^a \sum_{\alpha\in S^{k-1-a}} \delta_a(u_1,\alpha,u_2)\mathbf{x}^\alpha \\
&= \sum_{\alpha\in S^{k-1}} d_{T_1}(u_1,\alpha)\mathbf{x}^\alpha + \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2}+x_{n+1})^t \\
&\quad \times \sum_{\alpha\in S^{k-1-t}} (d_{T_1}(u_1,\alpha,u_2)+1)\mathbf{x}^\alpha \\
&\quad - \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{n+1})^t \sum_{\alpha\in S^{k-1-t}} \delta_t(u_1,\alpha,u_1)\mathbf{x}^\alpha \\
&\quad - \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2})^t \sum_{\alpha\in S^{k-1-t}} \mathbf{x}^\alpha
\end{aligned} \tag{5}
$$

similarly, we have

$$\begin{aligned}G_{u_2} = &\sum_{\alpha\in S^{k-1}} d_{T_1}(u_2,\alpha)\mathbf{x}^\alpha + \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2}+x_{n+1})^t \\ &\times \sum_{\alpha\in S^{k-1-t}}(d_{T_1}(u_1,\alpha,u_2)+1)\mathbf{x}^\alpha \\ &-\sum_{t=1}^{k-1}\binom{k-1}{t}x_{u_2}^t\sum_{\alpha\in S^{k-1-t}}\delta_t(u_2,\alpha,u_2)\mathbf{x}^\alpha \\ &-\sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2})^t\sum_{\alpha\in S^{k-1-t}}\mathbf{x}^\alpha\end{aligned} \tag{6}$$

Since for $t\in[k-1]$ and $\alpha\in[n]^{k-1-t}$, we have $d_{T_1}(\alpha)=d_{T_2}(\alpha)$. Then if $i\in[n]\backslash\{u_1,u_2\}$, according to the last equality in equation (4), we have

$$\begin{aligned}H_i = &\sum_{\alpha\in S^{k-1}} d_{T_1}(i,\alpha)\mathbf{x}^\alpha + \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2}+x_{n+1})^t\sum_{\alpha\in S^{k-1-t}}(d_{T_1}(i,\alpha,u_1,u_2)+1)\mathbf{x}^\alpha \\ &-\sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_2}+x_{n+1})^t\sum_{\alpha\in S^{k-1-t}}\delta_t(i,\alpha,u_2)\mathbf{x}^\alpha \\ &-\sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2})^t\sum_{\alpha\in S^{k-1-t}}\mathbf{x}^\alpha \\ &-\sum_{t=1}^{k-1}\binom{k-1}{t}x_{u_1}^t\sum_{\alpha\in S^{k-1-t}}\delta_t(i,\alpha,u_1)\mathbf{x}^\alpha\end{aligned}$$

Note that linear transformation L transforms the variables x_{u_1} and x_{u_2} into $x_{u_1}-x_{n+1}$ and $x_{u_2}+x_{n+1}$ respectively, while leaving other variables unchanged, and combining it with the last equality in equation (4), we can conclude that $H_i=G_i\circ L$.

For the vertices u_1 and u_2, by direct calculation, we have

$$\begin{aligned}H_{u_1} = &\sum_{\alpha\in S^{k-1}} d_{T_1}(u_1,\alpha)\mathbf{x}^\alpha + \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2}+x_{n+1})^t \\ &\times \sum_{\alpha\in S^{k-1-t}}(d_{T_1}(u_1,\alpha,u_2)+1)\mathbf{x}^\alpha \\ &-\sum_{t=1}^{k-1}\binom{k-1}{t}x_{u_1}^t\sum_{\alpha\in S^{k-1-t}}\delta_t(u_1,\alpha,u_1)\mathbf{x}^\alpha \\ &-\sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2})^t\sum_{\alpha\in S^{k-1-t}}\mathbf{x}^\alpha\end{aligned}$$

and

$$\begin{aligned}H_{u_2} = &\sum_{\alpha\in S^{k-1}} d_{T_1}(u_2,\alpha)\mathbf{x}^\alpha + \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2}+x_{n+1})^t \\ &\times \sum_{\alpha\in S^{k-1-t}} (d_{T_1}(u_1,\alpha,u_2)+1)\mathbf{x}^\alpha \\ &- \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_2}+x_{n+1})^t \sum_{\alpha\in S^{k-1-t}} \delta_t(u_2,\alpha,u_2)\mathbf{x}^\alpha \\ &- \sum_{t=1}^{k-1}\binom{k-1}{t}(x_{u_1}+x_{u_2})^t \sum_{\alpha\in S^{k-1-t}} \mathbf{x}^\alpha.\end{aligned}$$

Similarly, based on the last equality in equation (5) and equation (6), we can conclude that $H_{u_1} = G_{u_1}\circ L$ and $H_{u_2} = G_{u_2}\circ L$.

In summary, for $i \in [n]$, we have $G_i \circ L = H_i$. For $i = n+1$, note by equation (3), we have $(G_{n+1} - G_{u_1})\circ L = H_{n+1} - H_{u_2}$. Then Theorem 4 implies that

$$\begin{aligned}\mathrm{Res}(H_1,H_2,\dots,H_{n+1}-H_{u_2}) =&\mathrm{Res}(x_1,\dots,x_{u_1}-x_{n+1},\dots,x_{u_2}+x_{n+1},\dots,x_{n+1})^{(k-1)^n} \\ &\times \mathrm{Res}(G_1,G_2,\dots,G_{n+1}-G_{u_1}).\end{aligned}$$

It is easy to see that $\mathrm{Res}(x_1,\dots,x_{u_1}-x_{n+1},\dots,x_{u_2}+x_{n+1},\dots,x_{n+1}) = 1$, so

$$\mathrm{Res}(H_1,H_2,\dots,H_{n+1}-H_{u_2}) = \mathrm{Res}(G_1,G_2,\dots,G_{n+1}-G_{u_1}).$$

By Theorem 3, it follows that

$$\begin{aligned}\det(\mathcal{S}_k(T_1)) &= \mathrm{Res}(G_1,G_2,\dots,G_{n+1}) \\ &= \mathrm{Res}(G_1,G_2,\dots,G_{n+1}-G_{u_1})\end{aligned}$$

and

$$\begin{aligned}\det(\mathcal{S}_k(T_2)) &= \mathrm{Res}(H_1,H_2,\dots,H_{n+1}) \\ &= \mathrm{Res}(H_1,H_2,\dots,H_{n+1}-H_{u_2})\end{aligned}$$

Thus we have $\det(\mathcal{S}_k(T_1)) = \det(\mathcal{S}_k(T_2))$.

For any two non-isomorphic trees on n vertices, we can always make them isomorphic by moving the leaf vertices, and we have shown that moving the leaf vertices of an edge does not change the hyperdeterminant of the order k Steiner distance hypermatrix. In this way, we prove that the hyperdeterminant of the order k Steiner distance hypermatrix of any two trees on n vertices is equal, and thus completing the proof. □

Example 8. Let $T = P_3 = ([3],\{\{1,2\},\{2,3\}\})$, we add an edge to vertex 2 and vertex 3, respectively. Then $T_1 = ([4],\{\{1,2\},\{2,3\},\{2,4\}\})$ and $T_2 = ([4],\{\{1,2\},\{2,3\},\{3,4\}\})$. Denote

$$\mathcal{S}_k(T_1)\mathbf{x}^{k-1} = (G_1,G_2,G_3,G_4) \quad\text{and}\quad \mathcal{S}_k(T_2)\mathbf{x}^{k-1} = (H_1,H_2,H_3,H_4).$$

It follows from Definition 6 and equation (1) that we have

$$\begin{aligned}
G_1 =& 3(x_1+x_2+x_3+x_4)^{k-1} - (x_1+x_2+x_3)^{k-1} - (x_1+x_2+x_4)^{k-1} - x_1^{k-1} \\
G_2 =& 3(x_1+x_2+x_3+x_4)^{k-1} - (x_1+x_2+x_3)^{k-1} - (x_1+x_2+x_4)^{k-1} \\
&- (x_2+x_3+x_4)^{k-1} \\
G_3 =& 3(x_1+x_2+x_3+x_4)^{k-1} - (x_1+x_2+x_3)^{k-1} - (x_2+x_3+x_4)^{k-1} - x_3^{k-1} \\
G_4 =& 3(x_1+x_2+x_3+x_4)^{k-1} - (x_1+x_2+x_4)^{k-1} - (x_2+x_3+x_4)^{k-1} - x_4^{k-1}
\end{aligned}$$

and

$$\begin{aligned}
H_1 &= 3(x_1+x_2+x_3+x_4)^{k-1} - (x_1+x_2+x_3)^{k-1} - (x_1+x_2)^{k-1} - x_1^{k-1} \\
H_2 &= 3(x_1+x_2+x_3+x_4)^{k-1} - (x_1+x_2+x_3)^{k-1} - (x_2+x_3+x_4)^{k-1} - (x_1+x_2)^{k-1} \\
H_3 &= 3(x_1+x_2+x_3+x_4)^{k-1} - (x_1+x_2+x_3)^{k-1} - (x_2+x_3+x_4)^{k-1} - (x_3+x_4)^{k-1} \\
H_4 &= 3(x_1+x_2+x_3+x_4)^{k-1} - (x_2+x_3+x_4)^{k-1} - (x_3+x_4)^{k-1} - x_4^{k-1}
\end{aligned}$$

We can see that $G_4 - G_2 = H_4 - H_3 = (x_1+x_2+x_3)^{k-1} - x_4^{k-1}$.

Let $L = (x_1, x_2 - x_4, x_3 + x_4, x_4)$, according to the notations, it is easy to verify that

$$G_1 \circ L = H_1,\ G_2 \circ L = H_2,\ G_3 \circ L = H_3 \text{ and } (G_4 - G_2) \circ L = H_4 - H_3.$$

Since $\mathrm{Res}(x_1, x_2 - x_4, x_3 + x_4, x_4) = 1$, by Theorem 4, we obtain that

$$\mathrm{Res}(G_1, G_2, G_3, G_4 - G_2) = \mathrm{Res}(H_1, H_2, H_3, H_4 - H_3).$$

According to Theorem 3, we have

$$\mathrm{Res}(G_1, G_2, G_3, G_4) = \mathrm{Res}(H_1, H_2, H_3, H_4),$$

i.e., $\det(\mathcal{S}_k(T_1)) = \det(\mathcal{S}_k(T_2))$.

4 Hyperdeterminants of $\mathcal{S}_k(P_3)$

Recently, Cooper and Du [2] showed that for the single-edge graph K_2 (i.e., P_2),

$$\det(\mathcal{S}_k(K_2)) = (-1)^{k-1} \prod_{j=0}^{k-2} \left(\left(1 + e^{\frac{2\pi j}{k-1}\mathbf{i}}\right)^{k-1} - 1 \right), \tag{7}$$

where $\mathbf{i} = \sqrt{-1}$. They found that $\det(\mathcal{S}_k(K_2)) = 0$ if and only if $k \equiv 1 \pmod 6$.

In this section, by using the Poisson formula, we compute the hyperdeterminant of the order k Steiner distance hypermatrix of P_3. Note that Cooper and Tauscheck [3] proved that for a tree T on n vertices, if $n > 2$ and k is odd, then $\det(\mathcal{S}_k(T)) = 0$. Therefore, we only need to consider the case where k is even.

Theorem 9. *Let P_3 be a path on 3 vertices, and let $\mathcal{S}_k(P_3)$ be the order k Steiner distance hypermatrix of P_3, where k is even. Then*

$$\det(\mathcal{S}_k(P_3)) = \prod_{p=0}^{k-2}\prod_{q=0}^{k-2} g(p,q),$$

where $g(p,q) = 2\left(1+\zeta_{k-1}^{p}\right)^{k-1} - 1 - \left(1+\zeta_{k-1}^{p} - \left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}\right)^{k-1}$ *and* $\zeta_{k-1} = e^{\frac{2\pi}{k-1}\mathbf{i}}$.

Proof. Let P_3 on vertices set $\{1,2,3\}$ and denote $\mathcal{S}_k(P_3)\mathbf{x}^{k-1} = (F_1, F_2, F_3)$, it follows from Definition 6 and equation (1) that we have

$$\begin{cases} F_1 = 2(x_1+x_2+x_3)^{k-1} - (x_1+x_2)^{k-1} - x_1^{k-1} \\ F_2 = 2(x_1+x_2+x_3)^{k-1} - (x_1+x_2)^{k-1} - (x_2+x_3)^{k-1} \\ F_3 = 2(x_1+x_2+x_3)^{k-1} - (x_2+x_3)^{k-1} - x_3^{k-1} \end{cases}.$$

According to the notations, let $x_3 = 1$, consider the following polynomial system

$$\begin{cases} f_1 = 2(x_1+x_2+1)^{k-1} - (x_1+x_2)^{k-1} - x_1^{k-1} = 0 \\ f_2 = 2(x_1+x_2+1)^{k-1} - (x_1+x_2)^{k-1} - (x_2+1)^{k-1} = 0 \end{cases},$$

and $f_2 - f_1 = x_1^{k-1} - (x_2+1)^{k-1} = 0$, so we have

$$x_2 + 1 = \zeta_{k-1}^{p} x_1, \quad p \in \{0,1,\dots,k-2\}, \tag{8}$$

where $\zeta_{k-1} = e^{\frac{2\pi}{k-1}\mathbf{i}}$ is a primitive $(k-1)$-th root of unity.

Substituting equation (8) into f_1, we obtain that

$$2\left(\left(1+\zeta_{k-1}^{p}\right)x_1\right)^{k-1} - \left(\left(1+\zeta_{k-1}^{p}\right)x_1 - 1\right)^{k-1} - x_1^{k-1} = 0,$$

i.e.,

$$\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1} - 1\right)x_1^{k-1} = \left(\left(1+\zeta_{k-1}^{p}\right)x_1 - 1\right)^{k-1}.$$

It follows that

$$\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1} - 1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}x_1 = \left(1+\zeta_{k-1}^{p}\right)x_1 - 1, \quad p,q \in \{0,1,\dots,k-2\}.$$

Note that $\zeta_{k-1}^{j} + \zeta_{k-1}^{-j}$ is real for all $j \in \mathbb{Z}$, which implies that

$$\begin{aligned} 2\left(1+\zeta_{k-1}^{p}\right)^{k-1} - 1 =& 2\sum_{j=0}^{k-1}\binom{k-1}{j}\zeta_{k-1}^{pj} - 1 \\ =& 2\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k-1}{j}\left(\zeta_{k-1}^{pj} + \zeta_{k-1}^{-pj}\right) - 1 \end{aligned}$$

is real. Since k is even, we take the value of $\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}$ as a real number. Furthermore, we have

$$x_1 = \frac{1}{1+\zeta_{k-1}^{p}-\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}}, \quad p,q \in \{0,1,\ldots,k-2\},$$

and

$$x_2 = \frac{\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}-1}{1+\zeta_{k-1}^{p}-\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}}, \quad p,q \in \{0,1,\ldots,k-2\}.$$

We will show that for $p',q',p,q \in \{0,1,\ldots,k-1\}$, if $(p',q') \neq (p,q)$, then $(x_1',x_2') \neq (x_1,x_2)$, where $(x_1,x_2),(x_1',x_2') \in \mathcal{V}(f_1,f_2)$.

Suppose that $(x_1',x_2') = (x_1,x_2)$, then

$$1+\zeta_{k-1}^{p'}-\left(2\left(1+\zeta_{k-1}^{p'}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q'} = 1+\zeta_{k-1}^{p}-\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}$$

and

$$\left(2\left(1+\zeta_{k-1}^{p'}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q'}-1 = \left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}-1,$$

so we deduce that

$$\zeta_{k-1}^{p'} = \zeta_{k-1}^{p} \quad \text{and} \quad \zeta_{k-1}^{q'} = \zeta_{k-1}^{q}.$$

Since ζ_{k-1} is a primitive $(k-1)$-th root of unity and $p',q',p,q \in \{0,1,\ldots,k-1\}$, we have $p'=p$ and $q'=q$.

It implies that we get $(k-1)^2$ distinct solutions of f_1,f_2, by Bezout's Theorem the polynomial system f_1,f_2 has at most $(k-1)^2$ distinct solutions. This means that all of the multiplicities $m(p)$ in Theorem 5 are equal to 1.

Substituting these solutions into the polynomial $f_3(x_1,x_2) = 2(x_1+x_2+1)^{k-1}-(x_2+1)^{k-1}-1$, we get that

$$\begin{aligned} f_3(x_1,x_2) =& 2\left(\frac{1+\zeta_{k-1}^{p}}{1+\zeta_{k-1}^{p}-\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}}\right)^{k-1} \\ &-\left(\frac{\zeta_{k-1}^{p}}{1+\zeta_{k-1}^{p}-\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}}\right)^{k-1}-1 \end{aligned}$$

for all $p,q \in \{0,1,\ldots,k-2\}$.

Next, we simplify the expression for $\prod_{p=0}^{k-2}\prod_{q=0}^{k-2} f_3(x_1,x_2)$. Fixed $p \in \{0,1,\ldots,k-2\}$, note that $\zeta_{k-1} = e^{\frac{2\pi}{k-1}\mathbf{i}}$ is a primitive $(k-1)$-th root of unity, then

$$\begin{aligned}
&\prod_{q=0}^{k-2}\left(1+\zeta_{k-1}^{p}-\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}\right)\\
&=\left(1+\zeta_{k-1}^{p}\right)^{k-1}-\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)\\
&=-\left(1+\zeta_{k-1}^{p}\right)^{k-1}+1.
\end{aligned}$$

So we obtain that

$$\prod_{p=0}^{k-2}\prod_{q=0}^{k-2}\left(1+\zeta_{k-1}^{p}-\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}\right)=\det(\mathcal{S}_k(K_2)).$$

Let

$$g(p,q)=2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1-\left(1+\zeta_{k-1}^{p}-\left(2\left(1+\zeta_{k-1}^{p}\right)^{k-1}-1\right)^{\frac{1}{k-1}}\zeta_{k-1}^{q}\right)^{k-1},$$

then

$$\prod_{w\in\mathcal{V}(f_1,f_2)} f_3(w)^{\mathrm{m}(w)}=\frac{\prod_{p=0}^{k-2}\prod_{q=0}^{k-2} g(p,q)}{\left(\det(\mathcal{S}_k(K_2))\right)^{k-1}}.$$

On the other hand, by the result of Cooper and Du [2], when $k \not\equiv 1 \pmod 6$,

$$(-1)^{k-1}\prod_{j=0}^{k-2}\left(\left(1+\zeta_{k-1}^{j}\right)^{k-1}-1\right)\neq 0.$$

Then for even k, $\mathrm{Res}(\overline{F}_1,\overline{F}_2)\neq 0$. By Theorem 5, we have

$$\begin{aligned}
\det(\mathcal{S}_k(P_3)) =&\mathrm{Res}(F_1,F_2,F_3)\\
=&\left(\mathrm{Res}(\overline{F}_1,\overline{F}_2)\right)^{k-1}\frac{\prod_{p=0}^{k-2}\prod_{q=0}^{k-2} g(p,q)}{\left(\det(\mathcal{S}_k(K_2))\right)^{k-1}}\\
=&\prod_{p=0}^{k-2}\prod_{q=0}^{k-2} g(p,q).
\end{aligned}$$

This completes the proof. □

Remark 10. We can see that $g(p,q) = g(k-1-p,q)$ for $p \in \{1,2,\ldots,k-2\}, q \in \{0,1,\ldots,k-2\}$, then

$$\prod_{p=0}^{k-2}\prod_{q=0}^{k-2} g(p,q)=\prod_{q=0}^{k-2} g(0,q)\prod_{p=0}^{k-2}\left(\prod_{p=1}^{\frac{k-2}{2}} g(p,q)\right)^{2}, \tag{9}$$

which can help us simplify the calculations.

In [4], Cooper and Tauscheck listed the hyperdeterminants for $k = 2, 4, 6, 8$. Here, let $k = 10$ in equation (9), with the help of `Maple`, we have

$$\det(\mathcal{S}_{10}(P_3)) = 2^{30} \cdot 5^4 \cdot 7 \cdot 19^{16} \cdot 37^4 \cdot 73 \cdot 271^8 \cdot 307^4 \cdot 449^8 \cdot 739^8 \cdot 26119^8 \cdot 7222694395393^4.$$

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