Sparse Vertex Cutsets and the Maximum Degree

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Abstract

We show that every graph G of maximum degree Δ and sufficiently large order has a vertex cutset S of order at most Δ that induces a subgraph $G[S]$ of maximum degree at most $\Delta - 3$. For $\Delta \in \{4, 5\}$, we refine this result by considering also the average degree of $G[S]$. If G has no $K_{r,r}$ subgraph, then we show the existence of a vertex cutset that induces a subgraph of maximum degree at most $\left(1-\frac{1}{\binom{r}{2}}\right)\Delta+O(1)$.

Mathematics Subject Classifications: 05C40, 05C69

1 Introduction

Answering a question posed by Caro, Chen and Yu [6] proved the following result.

Theorem 1 (Chen and Yu [6])**.** *Every graph with n vertices and at most $2n - 4$ edges has an independent (vertex) cutset.*

Chen, Faudree, and Jacobson [5] showed that the smallest independent cutsets may be arbitrarily large as the number of edges approaches $2n$ but imposing a slightly stronger bound on the number of edges, one can guarantee the existence of small independent cutsets. Le and Pfender [12] characterized the graphs with n vertices and $2n - 3$ edges that do not have an independent cutset. The algorithmic problem of deciding the existence of independent cutsets in a given graph was considered [10, 11] with a particular focus on line graphs because independent cutsets in the line graph of some graph G correspond to matching (edge) cutsets in G [2, 4, 7, 8]. For subcubic graphs of order at least 8, Theorem 1 implies the existence of an independent cutset, while the complete graph K_4 and the triangular prism are cubic graphs of orders 4 and 6 with no such cutsets. In other

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words, a sufficiently large graph of maximum degree at most 3 has an independent cutset. Motivated by this observation, we consider the existence of sparse cutsets in sufficiently large graphs of bounded maximum degree.

We consider only finite, simple, and undirected graphs, and use standard terminology. Let G be a graph and let S be a set of vertices of G. Let $G[S]$ denote the subgraph of G induced by S and let $G - S = G[V(G) \setminus S]$. The set S is a cutset of G if $G - S$ is disconnected. Let $\Delta_G(S)$ and $\bar{d}_G(S)$ denote the maximum degree and the average degree of $G[S]$, respectively. If S is a minimal cutset in a graph G, then every vertex in S has at least one neighbor in every component of $G - S$. Provided that G has maximum degree at most Δ, this implies the trivial bound

$$\Delta_G(S) \leqslant \Delta - 2. \tag{1}$$

We believe that this can be improved and pose the following.

Question 2. Are there two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ with $\lim_{\Delta \to \infty} f(\Delta) = \infty$ such that every connected graph G of order at least $g(\Delta)$ and maximum degree at most Δ has a cutset S with $\Delta_G(S) \leqslant \Delta - f(\Delta)$?

Our first result shows that $f(\Delta) = O\left(\sqrt{\Delta}\log(\Delta)\right)$; all proofs are given in Section 2.

Proposition 3. *There is a positive constant c such that, for integers Δ and n_0 at least 9, there is a graph G of maximum degree at most Δ and order at least n_0 such that*

$$\Delta_G(S) \geqslant \Delta - (c+3)\sqrt{\Delta}\log(\Delta)$$

for every cutset S of G.

Instead of $\Delta_G(S)$, one may alternatively consider $\bar{d}_G(S)$. Nevertheless, in the setting of Question 2, this only makes sense for cutsets S that are small or minimal. In fact, if G is a graph of order n and maximum degree Δ, and u is some vertex in G, then G has an independent set I of order at least $\frac{n-\Delta^2-1}{\Delta+1}$ that does not contain any vertex at distance at most two from u, and $S = N_G(u) \cup I$ is a cutset of G with $\bar{d}_G(S) \to 0$ for $n \to \infty$ and fixed Δ.

Our next result improves the trivial bound (1).

Theorem 4. *Let Δ be an integer at least 3. If G is a connected graph of order at least $2\Delta + 3$ and maximum degree at most Δ, then G has a cutset S of order at most Δ with*

$$\Delta_G(S) \leqslant \Delta - 3. \tag{2}$$

For $\Delta = 3$, the bound (2) is trivially best possible. The results of [12] allow us to construct arbitrarily large connected 4-regular graphs without independent cutsets, that is, the bound (2) is also best possible for $\Delta = 4$.

See Figure 1 for an illustration.

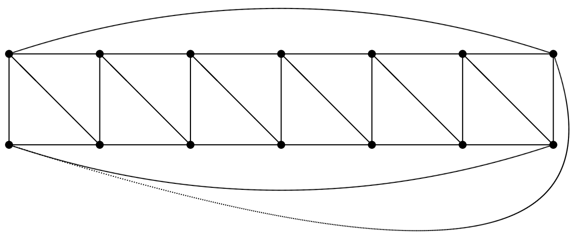

Figure 1: The 4-regular graph C_{14}^2 has no independent cutset.

For $\Delta = 5$, the icosahedron is a 5-regular graph G in which every cutset S satisfies $\Delta_G(S) \geqslant 2$. In fact, note that the neighborhood of every vertex of the icosahedron induces a copy of C_5, that every vertex u in every minimal cutset S of the icosahedron has neighbors in different components of $G - S$, and that the C_5 in the neighborhood of u implies that S contains at least two neighbors of u.

The property of having neighborhoods that induce copies of C_5 can be exploited to construct arbitrarily large connected 5-regular graphs G with no cutset S with $\Delta_G(S) \leqslant 1$. Consider, for instance, a cyclic structure based on the pattern shown in Figure 2; no vertex from the middle path can be contained in a cutset S with $\Delta_G(S) \leqslant 1$, which implies that no such cutset exists. Hence, the bound (2) is also best possible for $\Delta = 5$.

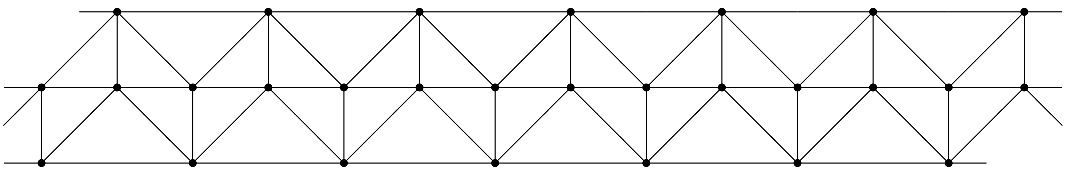

Figure 2: Part of a connected 5-regular G with no cutset S with $\Delta_G(S) \leqslant 1$.

While the bound (2) is best possible for $\Delta \in \{3, 4, 5\}$, we can refine it slightly for 5-regular and 4-regular graphs; next to $\Delta_G(S)$ we also bound $\bar{d}_G(S)$ for a small cutset S.

Theorem 5. *If G is a connected 5-regular graph of sufficiently large order n, then there is a cutset S of order at most 5 with $\Delta_G(S) \leqslant 2$ and $\bar{d}_G(S) < 2$.*

Theorem 6. *If G is a connected 4-regular graph of sufficiently large order n such that $G[N_G(x)]$ is not isomorphic to $2K_2$ for some vertex x of G, then either G is isomorphic to C_n^2 or there is a minimal cutset S of order at most 4 with $\bar{d}_G(S) < 1$.*

The 4-regular graphs in which every neighborhood induces a $2K_2$ form a rich class of graphs; the Cartesian product of K_3 with itself and line graphs of cubic triangle-free graphs are examples.

Le, Mosca, and Müller [11] conjectured that every 3-connected planar graph of maximum degree at most 4 has an independent cutset whose order is bounded by a fixed constant, or it has no independent cutset at all. We show two related statements; in the first one we require 4-regularity instead of maximum degree at most 4 but relax the planarity condition, and in the second one we consider graphs without $K_{r,r}$ as a subgraph; recall that planar graphs do not contain $K_{3,3}$ as a minor.

Theorem 7. *If G is a 4-regular graph with connectivity $\kappa \leqslant 3$, then G has an independent cutset of order at most 3.*

Theorem 8. *Let Δ and r be positive integers such that $c = 3 + \left\lfloor \frac{2(\Delta-3r+2)}{r(r-1)} \right\rfloor > 3$. If G is a graph of maximum degree Δ and order more than $\Delta + (c-3)(r-1) + r$ that does not contain $K_{r,r}$ as a subgraph, then G has a cutset S of order at most $\Delta + (c-3)(r-2)$ with*

$$\Delta_G(S) \leqslant \Delta - c.$$

We conclude with a simple consequence of Theorem 1.

Proposition 9. *If G is a connected graph of order n, size m, and maximum degree Δ with $m \leqslant \left(2 + \frac{1}{\Delta^2+1}\right) n - 4$, then G has a cutset S with $\Delta_G(S) \leqslant 1$.*

2 Proofs

As announced we give the proofs of our results.

Proof of Proposition 3. Axenovich, Sereni, Snyder, and Weber [1] showed the existence of some positive constant c such that, for every positive integers n' and Δ' with $n' \geqslant \Delta'$, there is a bipartite graph $H(n', \Delta')$ of maximum degree at most Δ' whose partite sets A and B both have order n' with the property that

$$\min\left\{|I \cap A|, |I \cap B|\right\} \leqslant c\frac{\log(\Delta')}{\Delta'}n' \text{ for every independent set } I \text{ in } H(n', \Delta').$$

Now, let Δ and n_0 be integers at least 9. Let $n' = \Delta + 1 - 2\left\lceil\sqrt{\Delta}\right\rceil$ and $\Delta' = \left\lceil\sqrt{\Delta}\right\rceil$. Let the graph G arise from a path or cycle G_0 of order at least n_0 by replacing every vertex u of G_0 with a clique K_u of order n' and replacing every edge uv of G_0 with a copy H_{uv} of $H(n', \Delta')$. By construction, the graph G has maximum degree at most Δ and order at least n_0. Furthermore, if S is a cutset of G, then there is some edge uv of G_0 such that $(K_u \cup K_v) \setminus S$ is an independent set in H_{uv}. By the properties of H_{uv}, this implies that

$$\begin{aligned}\Delta_G(S) &\geqslant \max\left\{|K_u \cap S|, |K_v \cap S|\right\} - 1\\ &\geqslant \Delta - 2\left\lceil\sqrt{\Delta}\right\rceil - c\frac{\log(\Delta')}{\Delta'}n'\\ &\geqslant \Delta - (c+3)\sqrt{\Delta}\log(\Delta).\end{aligned}$$

□

We proceed to the proofs of Theorems 4 and 5 that both rely on the following iterative procedure: Let Δ be an integer at least 3, let G be a graph of maximum degree at most Δ, and let u be a vertex of G that is of minimum degree δ. Starting with $U_1 = \{u\}$ and $S_1 = N_G(u)$, we construct two finite sequences

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots \subseteq U_k \quad \text{and} \quad S_1, S_2, S_3, \ldots, S_k$$

of sets of vertices of G such that, for every $i \in [k]$,

(i) $|U_i| = i$, $|S_i| \leqslant \delta$,

(ii) U_i is the vertex set of a component of $G - S_i$, and

(iii) every vertex in S_i has a neighbor in U_i.

Furthermore, either $\Delta_G(S_k) \leqslant \Delta - 3$ or $V(G) = U_k \cup S_k$.

For $i \in [k]$, let $n_i = |S_i|$ and let m_i be the number of edges of G between S_i and U_i.

By construction, (i), (ii), and (iii) hold for $i = 1$.

Now, suppose that S_i and U_i have been constructed such that (i), (ii), and (iii) hold for some i. If $\Delta_G(S_i) \leqslant \Delta - 3$ or $V(G) = U_i \cup S_i$, the procedure terminates and we set $k = i$. Otherwise, we construct S_{i+1} and U_{i+1} as follows:

> *Let $v \in S_i$ have at least $\Delta - 2$ neighbors in S_i. By (iii), v has at least one neighbor in U_i. Hence, the set $N = N_G(v) \setminus (S_i \cup U_i)$ contains at most one vertex. Let $S_{i+1} = (S_i \setminus \{v\}) \cup N$ and $U_{i+1} = U_i \cup \{v\}$.*

By construction, (i), (ii), and (iii) hold for $i + 1$, and $U_i \subseteq U_{i+1}$.

Since G is finite and $|U_i|$ strictly grows with i, this procedure necessarily terminates.

For the proofs of Theorems 4 and 5, we need to analyze the behavior of n_i and m_i: Trivially, $n_1 = m_1 = \delta$, which implies

$$m_1 - 2n_1 \;\; = \;\; \delta - 2\delta \geqslant -\Delta. \tag{3}$$

For $i \in [k-1]$, let v and N be as above. If $|N| = 1$, then $n_{i+1} = n_i$, v has exactly one neighbor in U_i, v has exactly $\Delta - 1$ neighbors in S_{i+1}, and, hence,

$$m_{i+1} = m_i - m_G(\{v\}, U_i) + m_G(\{v\}, S_{i+1}) = m_i + (\Delta - 2),$$

where $m_G(A, B)$ denotes the number of edges of G between disjoint sets A and B of vertices of G.

If $|N| = 0$, then $n_{i+1} = n_i - 1$, v has at most two neighbors in U_i, v has at least $\Delta - 2$ neighbors in S_{i+1}, and, hence,

$$m_{i+1} = m_i - m_G(\{v\}, U_i) + m_G(\{v\}, S_{i+1}) \geqslant m_i - 2 + (\Delta - 2) = m_i + (\Delta - 4).$$

Regardless of the value of $|N|$, we obtain

$$m_{i+1} - 2n_{i+1} \;\; \geqslant \;\; (m_i - 2n_i) + (\Delta - 2) \text{ for every } i \in [k-1]. \tag{4}$$

Combining (3) and (4), we obtain

$$m_k - 2n_k \geqslant -\Delta + (\Delta - 2)(k-1). \tag{5}$$

Since $G[S_{k-1}]$ contains at least $\Delta_G(S_{k-1}) \geqslant \Delta-2$ edges, there are at most $\Delta n_{k-1} - 2(\Delta-2)$ edges between S_{k-1} and U_{k-1}, and we obtain

$$\begin{aligned} m_{k-1} - 2n_{k-1} &\leqslant (\Delta n_{k-1} - 2(\Delta-2)) - 2n_{k-1} \\ &= (\Delta-2)(n_{k-1}-2) \\ &\overset{(i)}{\leqslant} (\Delta-2)(\delta-2) \\ &\leqslant (\Delta-2)^2. \end{aligned} \tag{6}$$

Combining (5) and (6), we obtain $-\Delta + (\Delta-2)(k-1) \leqslant (\Delta-2)^2$, which implies

$$k \leqslant \Delta + \frac{2}{\Delta - 2} \leqslant \Delta + 2. \tag{7}$$

Proof of Theorem 4. By (i) and (7), we have $|U_k| + |S_k| \leqslant k + \Delta \leqslant 2\Delta + 2$, which implies that $V(G) \neq U_k \cup S_k$. Hence, by (ii), S_k is a cutset with $\Delta_G(S_k) \leqslant \Delta - 3$. □

Proof of Theorem 5. Let G be as in the statement. The hypothesis that G is of sufficiently large order n means that there is some fixed n_0 such that the statement holds for $n \geqslant n_0$. A detailed analysis of the following argument shows that $n_0 = 14$ would suffice. For simplicity, we choose $n_0 = 27$. Below, we construct sequences $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots \subseteq U_\ell$ and $S_1, S_2, S_3, \ldots, S_\ell$ with the properties (i), (ii), and (iii) as above such that $|S_i|$ is non-increasing with $|S_1| = 5$ and $m_i = m_G(S_i, U_i)$ is strictly increasing with $m_1 = 5$. Since G is 5-regular, it follows $m_\ell \leqslant 5|S_\ell| \leqslant 25$, which implies $\ell \leqslant 21$, and, hence, we have $|U_\ell \cup S_\ell| \leqslant \ell + 5 \leqslant 26$. It follows that for every such sequence, the set $V(G) \setminus (U_\ell \cup S_\ell)$ is not empty.

Since the only 5-regular connected graph with the property that the neighborhood of every vertex induces a copy of C_5 is the icosahedron [9], which is a graph of order 12, it follows, using $n \geqslant 27$, that there is a vertex u whose neighborhood does not induce a copy of C_5. If $\Delta_G(N_G(u)) \leqslant 2$, then this implies $\bar{d}_G(S) < 2$, and $S = N_G(u)$ has the desired properties. Hence, we may assume that $\Delta_G(N_G(u)) \geqslant 3$. Let $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots \subseteq U_k$ and $S_1, S_2, S_3, \ldots, S_k$ be as above, that is,

- $k \geqslant 2$,
- $|U_i| = i$ for $i \in [k]$,
- $n_i = |S_i|$ satisfies $5 = n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$,
- the number m_i of edges between S_i and U_i satisfies $5 = m_1 < m_2 < \cdots < m_k$, and
- $\Delta_G(S_k) \leqslant 2$.

As explained above, we have that $R_k = V(G) \setminus (U_k \cup S_k)$ is not empty. If $G[S_k]$ is not 2-regular, then $S = S_k$ has the desired properties. Hence, we may assume that $G[S_k]$ is 2-regular, that is, the set S_k induces a cycle. If some vertex x in S_k has no neighbor in R_k, then $S_k \setminus \{x\}$ has the desired properties. Hence, we may assume that every vertex in S_k has at least one neighbor in R_k, and, hence, at most two neighbors in U_k. Since $m_k > n_k$, some vertex v in S_k has exactly two neighbors in U_k, and, hence, exactly one neighbor w in R_k. Let $S_{k+1} = (S_k \setminus \{v\}) \cup \{w\}$ and $U_{k+1} = U_k \cup \{v\}$. Note that $n_{k+1} = n_k$ and $m_{k+1} > m_k$. Repeating the procedure described above, we continue the above sequences with $U_k \subseteq U_{k+1} \subseteq \cdots \subseteq U_{k_2}$ and $S_k, S_{k+1}, \ldots, S_{k_2}$ such that $5 \geqslant n_k \geqslant n_{k+1} \geqslant \cdots \geqslant n_{k_2}$, $5 < m_k < m_{k+1} < \cdots < m_{k_2}$, and $\Delta_G(S_{k_2}) \leqslant 2$. Again, it follows that $V(G) \setminus (U_{k_2} \cup S_{k_2})$ is not empty and that $G[S_{k_2}]$ is 2-regular. Repeating exactly the same arguments, we obtain $U_{k_2+1} \subseteq \cdots \subseteq U_{k_3}$ and $S_{k_2+1}, \ldots, S_{k_3}$. Since m_i strictly increases but $n_i \leqslant 5$, this process can only be repeated a bounded number of times before it returns a cutset with the desired properties. This completes the proof. □

Since Theorem 7 allows a simpler proof of Theorem 6, we prove it first.

Proof of Theorem 7. Let G be a 4-regular graph with connectivity $\kappa \leqslant 3$.

Let S be a cutset of order κ in G that minimizes the order of a smallest component C of $G - S$. If S is independent, then S is the desired cutset. Hence, we may assume that S is not independent, which implies $\kappa \in \{2, 3\}$. Since G is 4-regular, C has order at least 2. Since S is a minimum cutset in G, every vertex in S has a neighbor in every component of $G - S$. If some vertex x in S has exactly one neighbor y in C, then $S' = (S \setminus \{x\}) \cup \{y\}$ is a cutset of order κ in G such that the smallest component of $G - S'$ is strictly smaller than C, which contradicts the choice of S. Hence, every vertex in S has at least two neighbors in C. If $G[S]$ contains two edges, then $|S| \leqslant 3$ implies that some vertex x in S has two neighbors in S as well as at least two neighbors in C, which implies the contradiction that x has no neighbor in components of $G - S$ that are distinct from C. Hence, $G[S]$ contains exactly one edge uv. Note that there are at most $2\kappa - 2$ edges between S and components of $G - S$ that are distinct from C. If $G - S$ has more than two components, then this implies the existence of a component of $G - S$ that is connected to S by at most $\kappa - 1$ edges, which implies the contradiction that the connectivity is at most $\kappa - 1$. Hence, $G - S$ has exactly one component C' that is distinct from C. Since u and v both have exactly one neighbor in C', the order of C' is at least 3. If there is a subset X of S such that the set N_X of vertices of C' that have a neighbor in X satisfies $|X| > |N_X|$, then $(S \setminus X) \cup N_X$ is a cutset of order less than κ in G, which is a contradiction. Hence, $|X| \leqslant |N_X|$ for every subset X of S, which, by Hall's Theorem, implies the existence of a matching M of size κ between S and C'. Let uu' and vv' be two edges in M. Since v has two neighbors in C and is adjacent to u and v', v is not adjacent to u'. Hence, if $\kappa = 2$, then u is the only neighbor of u' in S. Similarly, if $\kappa = 3$ and $S = \{u, v, w\}$, then w has two neighbors in C and one neighbor in C' that is distinct from u' and v'. Since G is 4-regular, we may assume, by symmetry, that u is the only neighbor of u' in S. Now, $(S \setminus \{u\}) \cup \{u'\}$ is an independent cutset, which completes the proof. □

If G is a graph of minimum degree δ, maximum degree Δ, and connectivity κ strictly smaller than δ, then the same argument shows the existence of a cutset S with $\Delta_S(G) \leqslant \Delta - 3$ without a further condition on the order of G.

Proof of Theorem 6. Let G be as in the statement. We call a minimal cutset S of G of order at most 4 with $\bar{d}_G(S) < 1$ *good*, and we assume that G has no good cutset. By Theorem 7, G is 4-connected.

We establish two claims.

Claim 1. *G contains P_6^2 as an induced subgraph.*

Proof of Claim 1. Let x be a vertex of G such that $G[N_G(x)]$ is not isomorphic to $2K_2$. Since the cutset $N_G(x)$ is not good, and G is 4-connected, some vertex y in $N_G(x)$ has exactly two neighbors b and c in $N_G(x)$ and exactly one neighbor d outside of $N_G[x]$. Let a denote the vertex in $N_G(x)$ distinct from y, b, and c. Let $S = (N_G(x) \setminus \{y\}) \cup \{d\} = \{a, b, c, d\}$. Note that there are six edges between the cutset S and the component of $G - S$ that contains x.

Suppose, for a contradiction, that $\Delta_G(S) \geqslant 2$. In this case, proceeding as in the proof of Theorem 4 while using that n is sufficiently large and that G is 4-connected yields a cutset S' of order 4 with $\Delta_G(S') \leqslant 1$ such that there are at least eight edges between S' and the component of $G - S'$ that contains x. Since S' is not good, the graph $G[S']$ is isomorphic to $2K_2$. Let $S' = \{a', b', c', d'\}$ and let $a'b'$ and $c'd'$ be the two edges within S'. Since G is 4-connected, there are exactly two components in $G - S'$, and there are exactly four edges between S' and the vertex set C' of the component of $G - S'$ that does not contain x. Since n is sufficiently large, we may assume that C' contains at least four vertices. Since G is 4-connected, a simple application of Hall's Theorem implies that the four edges between S' and C' form a matching, say $a'a''$, $b'b''$, $c'c''$, and $d'd''$. Now, the set $\{a', b'', c', d''\}$ is a good cutset, which is a contradiction. This contradiction implies that $\Delta_G(S) \leqslant 1$. Since S is not good, there are exactly two edges within S. By symmetry, these two edges are either ad and bc or ab and cd.

Suppose, for a contradiction, that ad and bc are the two edges within S. Since G is 4-connected, a simple application of Hall's Theorem implies the existence of a matching containing four edges between $N_G(x)$ and the component of $G - N_G(x)$ that does not contain x, say $aa^{(3)}$, $bb^{(3)}$, $cc^{(3)}$, and yd. Since a is adjacent to x, $a^{(3)}$, and d, we may assume, by symmetry, that a is not adjacent to $b^{(3)}$. Now, the set $\{a, b^{(3)}, c, y\}$ is a good cutset, which is a contradiction. This contradiction implies that ab and cd are the two edges within S. We obtain that G contains the square of the path $abxycd$ as an induced subgraph, which completes the proof of the claim. □

Claim 2. *If G contains $P_{n'}^2$ as an induced subgraph for some n' with $6 \leqslant n' < n$, then either G contains $P_{n''}^2$ as an induced subgraph for some n'' with $n' < n'' < n$ or G is isomorphic to C_n^2.*

Proof of Claim 2. Let $P : aba_2b_2 \cdots cd$ be a path of order n' such that G contains the induced subgraph P^2 isomorphic to $P_{n'}^2$. Since every vertex in $V(P) \setminus \{a, b, c, d\}$ has all its four neighbors in P^2 and $n' < n$, the set $S = \{a, b, c, d\}$ is a cutset. Since G is 4-connected, the graph $G - S$ has exactly two components and there are exactly six edges between S and the vertex set C of the component of $G - S$ that does not intersect P. Since G is 4-regular, the set C contains more than one vertex.

First, suppose that C contains exactly two vertices u and v, that is, $n' = n - 2$. Since there are six edges between S and C, the vertices u and v are adjacent. Since a and d both have only two neighbors in P^2, they are both adjacent to u and v. By symmetry, we may assume that b is adjacent to u and c is adjacent to v. Now, the graph G is isomorphic to the square of the cycle $aba_2b_2 \cdots cdvua$. Hence, we may assume that C contains more than two vertices.

Next, suppose that C contains exactly three vertices u, v and w, that is, $n' = n - 3$. Since there are six edges between S and C, the vertices u, v, and w form a triangle. If a and d have the same two neighbors in C, say u and v, then w is adjacent to b and c, and $N_G(w)$ is a good cutset, which is a contradiction. Hence, by symmetry, we may assume that a is adjacent to u and v, and that d is adjacent to v and w. If u is adjacent to c and w is adjacent to b, then the set $\{a_2, b_2, u, d\}$ is a good cutset, which is a contradiction. It follows that u is adjacent to b and w is adjacent to c. Now, the graph G is isomorphic to the square of the cycle $aba_2b_2 \cdots cdwvua$. Hence, we may assume that C contains more than three vertices.

Since G is 4-connected, Hall's Theorem implies that there is a matching of size four between S and C, say aa', bb', cc', and dd'. Since the two cutsets $\{a, b', c, d\}$ and $\{a, b, c', d\}$ are not good, it follows that either a is adjacent to c' and d is adjacent to b' or a is adjacent to b' and d is adjacent to c'. If a is adjacent to c' and d is adjacent to b', then $\{a, b', c, d'\}$ is a good cutset, which is a contradiction. It follows that a is adjacent to b' and d is adjacent to c'. Since the cutset $\{a', b', c, d\}$ is not good, the vertex a' is adjacent to b'. Now, the square of the path $P' : a'b'aba_2b_2 \cdots cd$ is an induced subgraph of G and the order of P' is $n' + 2 < n$, which completes the proof of the claim. □

Now, an inductive argument using Claim 1 for the base case and Claim 2 for the inductive step implies that G is isomorphic to C_n^2, which completes the proof. □

As explained in the introduction, Theorem 8 relates to the stated conjecture of Le, Mosca, and Müller [11] concerning planar graphs, which avoid $K_{r,r}$ as a subgraph for $r = 3$. Another reason why $K_{r,r}$ naturally appears within this context is the iterative procedure used for the proofs of Theorems 4 and 5. This procedure repeatedly increases the number of edges in the bipartite subgraph between U_i and S_i, which can be exploited to obtain a copy of $K_{r,r}$, unless a good cutset is found.

Proof of Theorem 8. Let Δ, r, c, and G be as in the statement, and call a cutset S as in the statement *good*. For a proof by contradiction, we assume that G has no good cutset, and deduce the contradiction that G contains $K_{r,r}$ as a subgraph. Therefore, by an inductive argument, for $i \in \{1, \ldots, r\}$, we show the existence of a cutset S_i such that

- S_i has order at most $\Delta + (c - 3)(i - 1)$,

- one component of $G - S_i$ with vertex set C_i has order exactly i,
- every vertex in S_i has a neighbor in C_i, and
- there is a subset T_i of S_i containing at least
$$\Delta - (c-3)\left(\frac{i^2}{2} - \frac{i}{2}\right) - 2(i-1)$$
vertices such that every vertex in C_i is adjacent to every vertex in T_i.

Note that $|S_i| + |C_i| \leqslant \Delta + (c-3)(i-1) + i \leqslant \Delta + (c-3)(r-1) + r$, which is less than the order of G.

For $i = 1$, let $S_1 = T_1$ be the neighborhood of some vertex u_1 of maximum degree, which forms C_1. By construction, S_1, T_1, and C_1 satisfy the desired properties. Now, suppose that S_{i-1}, T_{i-1}, and C_{i-1} with the desired properties have been constructed for some $i \in \{2, \dots, r\}$. Since G has no good cutset, some vertex u_i in S_{i-1} has at least $\Delta - c + 1$ neighbors in S_{i-1}. Since u_i has a neighbor in C_{i-1}, this implies that the set N_i of neighbors of u_i outside of $S_{i-1} \cup C_{i-1}$ has order at most $c - 2$. Let
$$S_i = (S_{i-1} \setminus \{u_i\}) \cup N_i, \quad C_i = C_{i-1} \cup \{u_i\}, \text{ and } \quad T_i = T_{i-1} \cap N_G(u_i).$$
By construction, $|S_i| \leqslant |S_{i-1}| + (c-3) \leqslant \Delta + (c-3)(i-1)$, C_i is the vertex set of some component of $G - S_i$ of order exactly i, and every vertex in S_i has a neighbor in C_i. Since u_i has at most $|S_{i-1}| - |T_{i-1}|$ neighbors in $S_{i-1} \setminus T_{i-1}$, we obtain
$$\begin{aligned} |T_i| &\geqslant \Big(\Delta - c + 1\Big) - \Big(|S_{i-1}| - |T_{i-1}|\Big) \\ &\geqslant \Big(\Delta - c + 1\Big) - \Big(\Delta + (c-3)(i-2)\Big) \\ &\quad + \left(\Delta - (c-3)\left(\frac{(i-1)^2}{2} - \frac{(i-1)}{2}\right) - 2(i-2)\right) \\ &= \Delta - (c-3)\left(\frac{i^2}{2} - \frac{i}{2}\right) - 2(i-1) \end{aligned}$$
Altogether, we obtain S_i, T_i, and C_i with the desired properties. The definition of c implies that $|T_r| \geqslant r$. Hence, $G[C_r \cup T_r]$ contains $K_{r,r}$ as a subgraph, which completes the proof. □

Proof of Proposition 9. Let G be as in the statement. The square of G has maximum degree at most Δ^2, which implies that it has an independent set $\{u_1, \dots, u_\alpha\}$ of order $\alpha \geqslant \frac{n}{\Delta^2+1}$. In view of the desired statement, we may assume that $\Delta_G(N_G(u_i)) \geqslant 2$ for every $i \in [\alpha]$. Hence, for every $i \in [\alpha]$, there is a neighbor v_i of u_i such that u_i and v_i have at least two common neighbors. Let the graph G' with n' vertices and m' edges arise from G by contracting the edges $u_1v_1, \dots, u_\alpha v_\alpha$. Note that
$$m' \;\leqslant\; m - 3\alpha \leqslant \left(2 + \frac{1}{\Delta^2+1}\right) n - 4 - 3\alpha \leqslant 2n - 2\alpha - 4 = 2n' - 4.$$
By Theorem 1, the graph G' has an independent cutset S'. Uncontracting the edges $u_1v_1, \dots, u_\alpha v_\alpha$ yields a cutset S with the desired properties. □

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