A Max-Min Problem on Spectral Radius and Connectedness of Graphs

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Submitted: Jan 29, 2024; Accepted: Apr 16, 2025; Published: May 23, 2025

Abstract

In the past decades, many scholars have been concerned with the question of which edge-extremal problems have spectral analogues. Recently, Wang, Kang, and Xue established an interesting result on F-free graphs [*J. Combin. Theory Ser. B* 159 (2023) 20–41]. In this paper, we investigate this problem in the context of critical graphs. Let P be a property defined on a family $\mathbb{G}$ of graphs. A graph $G \in \mathbb{G}$ is said to be P-critical if it satisfies P but $G - e$ does not satisfy P for any edge $e \in E(G)$. Specifically, a graph is *minimally k-(edge)-connected* if it is k-connected (respectively, k-edge-connected) and the deletion of any edge results in a graph that is not k-connected (respectively, k-edge-connected). A natural max-min problem is to determine the maximum spectral radius of minimally k-(edge)-connected graphs with n vertice. In 2019, Chen and Guo [*Discrete Math.* 342 (2019) 2092–2099] resolved the case $k = 2$. In 2021, Fan, Goryainov, and Lin [*Discrete Appl. Math.* 305 (2021) 154–163] determined the extremal spectral radius for minimally 3-connected graphs. In this paper, we establish structural properties of minimally k-(edge)-connected graphs. Furthermore, we solve the max-min problem for the case $k \geqslant 3$, demonstrating that any minimally k-(edge)-connected graph attaining the maximum spectral radius simultaneously achieves the maximum number of edges.

Mathematics Subject Classifications: 05C50, 05C75

1 Introduction

Perhaps the most basic property a graph may blue posses is that of being connected. At a more refined level, there are various functions that may be said to measure the connectedness of a connected graph [2]. A graph is said to be *connected* if for every pair of vertices there is a path joining them. Otherwise the graph is disconnected. The *connectivity* (or *vertex-connectivity*) $\kappa(G)$ of a graph G is the minimum number of vertices

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 https://doi.org/10.37236/12747

whose removal results in a disconnected graph or in a trivial graph. The *edge-connectivity* $\kappa'(G)$ is defined analogously, only instead of vertices we remove edges. A graph is k-*connected* if its connectivity is at least k and k-*edge-connected* if its edge-connectivity is at least k. It is almost as simple to check that the minimal degree $\delta(G)$, the edge-connectivity and *vertex-connectivity* satisfy the following inequality:

$$\delta(G) \geqslant \kappa'(G) \geqslant \kappa(G).$$

A number of extremal problems related to graph connectivity have been studied in recent years. One of the most important tasks for characterization of k-connected graphs is to give a certain operation such that they can be produced from simple k-connected graphs by repeatedly applying this operation [2]. This goal has been accomplished by Tutte [27] for 3-connected graphs, by Dirac [12] and Plummer [25] for 2-connected graphs, and by Slater [26] for 4-connected graphs.

A graph is said to be *minimally* k*-(edge)-connected* if it is k-(edge)-connected but omitting any of edges the resulting graph is no longer k-(edge)-connected. Clearly, a k-(edge)-connected graph whose every edge is incident with one vertex of degree k is minimally k-(edge)-connected. Especially, a k-regular and k-(edge)-connected graph is minimally k-(edge)-connected.

One of the central problems in this area is to determine the number of vertices of degree k in a minimally k-edge-connected graph. In 1972, Lick [16] showed that every minimally k-edge-connected finite graph has at least two vertices of degree k (see also Lemma 13 in [20]), which is clearly best possible. But for simple graphs, this was improved in [17] as follows: every minimally k-edge-connected finite simple graph has at least $k+1$ vertices of degree k. It was proved in [19] that for every $k \notin \{1,3\}$ there exists a $c_k > 0$ such that every minimally k-edge-connected finite simple graph G has at least $c_k|G|$ vertices of degree k. The value of the constant c_k was improved in [3] and [5], and a rather good estimate for c_k was given by Cai [6]. In 1995, Mader [21] further improved the value c_k and gave the best possible linear bound for $k \equiv 3(\mod 4)$.

Another interesting problem is to determine the maximum number of edges in a minimally k-(edge)-connected graph. Mader [18] proved that $e(G) \leqslant kn - \binom{k+1}{2}$ for every minimally k-connected graph G of order n, and if $n \geqslant 3k-2$ then $e(G) \leqslant k(n-k)$, where the equality is uniquely attained by the complete bipartite graph $K_{k,n-k}$ provided that $k \geqslant 2$ and $n \geqslant 3k-1$. Cai [4] proved that $e(G) \leqslant \lfloor \frac{(n+k)^2}{8} \rfloor$ for every minimally k-connected graph G of order $n < 3k-2$. Mader [18] also proved that every minimally k-edge-connected graph on n vertices has at most $k(n-k)$ edges provided $n \geqslant 3k-2$. The complete bipartite graph $K_{k,n-k}$ shows that this bound is tight. Dalmazzo [11] proved that every minimally k-edge-connected multidigraph on n vertices has at most $2k(n-1)$ edges. In 2005, Berg and Jordán [1] showed that if multiple edges are not allowed then Dalmazzo's bound can be improved to $2k(n-k)$ for n sufficiently large. In this paper, we first obtain an extremal result for every subgraph of a minimally k-(edge)-connected graph.

Theorem 1. *Let G be a minimally k-(edge)-connected graph and let H be a subgraph of G. Then $e(H) \leqslant k(|H|-1)$. Moreover, if $|H| \geqslant \frac{1}{2}k(k+5)$, then $e(H) \leqslant k(|H|-k)$, where the equality holds if and only if $H \cong K_{k,|H|-k}$.*

Let $A(G)$ be the adjacency matrix of a graph G. The largest eigenvalue of $A(G)$ is called the *spectral radius* of G, and denoted by $\rho(G)$. In classical theory of graph spectra, many scholars are interested in an extremal problem, that is, what is the maximal spectral radius of a family $\mathbb{G}$ of graphs, where graphs in $\mathbb{G}$ have a common property P. A graph is said to be *P-saturated*, if it has the property P but adding an edge between an arbitrary pair of non-adjacent vertices results in a graph which does not have the property. It is known that $A(G)$ is a non-negative matrix, and adding an edge in G always increases the spectral radius provided that G is connected. Therefore, most of the spectral extremal problems have saturated extremal graphs (see for example, [8, 9, 23, 15, 28, 22, 30, 31, 32]). Particularly, we have the following problem.

Problem 2. What is the maximal spectral radius among all n-vertex saturated graphs with fixed vertex-connectivity or edge-connectivity?

Ye, Fan and Wang [29] showed that among all graphs of order n with vertex (edge)-connectivity k, $K(n-1,k)$ has the maximal spectral radius, where $K(n-1,k)$ is obtained from the complete graph K_{n-1} by adding a new vertex of degree k. Clearly, $K(n-1,k)$ has the same vertex-connectivity, edge-connectivity and minimum degree. Ning, Lu and Wang [24] proved that for all graphs of order n with minimum degree δ and edge connectivity $\kappa' < \delta$, the maximal spectral radius is attained by joining κ' edges between two disjoint complete graphs $K_{\delta+1}$ and $K_{n-\delta-1}$, and they also determined the unique extremal graph with minimum degree δ and edge-connectivity $\kappa' \in \{0,1,2,3\}$. Very recently, Fan, Gu and Lin [14] determined the unique spectral extremal graph over all n-vertex graphs with minimum degree δ and edge connectivity $\kappa' \in \{4,\dots,\delta-1\}$.

A graph G is said to be *P-critical*, if it admits a property P but $G-e$ does not have it for any edge $e \in E(G)$. Clearly, every minimally k-(edge)-connected graph is a connectivity-critical graph. Comparing with Problem 2, the following problem also attracts interest of scholars.

Problem 3. What is the maximal spectral radius among all n-vertex critical graphs with fixed vertex-connectivity or edge-connectivity?

Obviously, every minimally 1-(edge)-connected graph is a tree. Furthermore, it is known that the maximal spectral radius of a tree is attained uniquely by a star (see [10]). In 2019, Chen and Guo [7] showed that $K_{2,n-2}$ attains the maximal spectral radius among all minimally 2-connected graphs and minimally 2-edge-connected graphs, respectively. Subsequently, Fan, Goryainov and Lin [13] proved that $K_{3,n-3}$ attains the largest spectral radius over all minimally 3-connected graphs.

Now let $k \geqslant 3$ be a fixed integer and $\alpha = \frac{1}{24k(k+1)}$. Let $X = (x_1, x_2, \dots, x_n)^T$ be a non-negative eigenvector with respect to $\rho(G)$. We may assume that $x_{u^*} = \max_{1 \leqslant i \leqslant n} x_i$ for some $u^* \in V(G)$. In this paper, we prove the following result, which implies that every minimally k-(edge)-connected graph with large spectral radius contains a certain number of vertices of high degrees.

Theorem 4. *Let G be an n-vertex minimally k-(edge)-connected graph, where $n \geqslant \frac{18k}{\alpha^2}$. If $\rho^2(G) \geqslant k(n-k)$, then G contains a k-vertex subset L such that $x_v \geqslant (1-\frac{1}{2k})x_{u^*}$ and $d_G(v) \geqslant (1-\frac{2}{3k})n$ for each vertex $v \in L$.*

The main result of the paper is the following Max-Min theorem, which implies that every minimally k-(edge)-connected graph with maximal spectral radius also has maximal number of edges.

Theorem 5. *For $n \geqslant \frac{18k}{\alpha^2}$, the maximal spectral radius of an n-vertex minimally k-(edge)-connected graph is attained uniquely by the complete bipartite graph $K_{k,n-k}$.*

Finally, we present the following problem.

Problem 6. Consider a given property P. Does an edge-extremal problem on P-critical graphs possess a spectral analogue?

The rest of the paper is organized as follows. In Section 2, we give some structural properties of a minimally k-(edge)-connected graph as well as the proof of Theorem 1. In Section 3, we use Theorem 1 to show Theorems 4 and 5.

2 Structural properties

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the number of vertices and $e(G)$ the number of edges in G. For a vertex $v \in V(G)$, let $N_G(v)$ be the neighborhood of v. For $S \subseteq V(G)$, we denote $N_S(v) = N(v) \cap S$ and $d_S(v) = |N_S(v)|$. The subgraph of G induced by S and $V(G) \setminus S$ are denoted by $G[S]$ and $G - S$, respectively. Let $e_G(S)$ be the number of edges within S, and let $e_G(S, V(G) \setminus S)$ be the number of edges between S and $V(G) \setminus S$. All the subscripts defined here will be omitted if it is clear from the context. We start with the following lemma.

Lemma 7. *Every k-(edge)-connected subgraph of a minimally k-(edge)-connected graph is minimally k-(edge)-connected.*

Proof. We first prove that for every subgraph of a minimally k-edge-connected graph, if it is k-edge-connected then it is minimally k-edge-connected. Let G be a minimally k-edge-connected graph. Suppose to the contrary that H is a k-edge-connected subgraph of G but it is not minimally k-edge-connected. Then there exists an edge, say u_1u_2, of H such that $H - u_1u_2$ is also k-edge-connected.

Notice that G is a minimally k-edge-connected graph. Hence, $G - u_1u_2$ is $(k-1)$-edge-connected. Thus, there exists a partition $V(G) = V_1 \cup V_2$ such that $u_1 \in V_1$, $u_2 \in V_2$ and $e(V_1, V_2) = k$. Now, let $V_i(H) = V(H) \cap V_i$ for $i \in \{1, 2\}$. Clearly,

$$e\big(V_1(H), V_2(H)\big) \leqslant e\big(V_1, V_2\big) = k,$$

and thus $e\big(V_1(H), V_2(H)\big) \leqslant k-1$ in $H - u_1u_2$, which contradicts the fact that $H - u_1u_2$ is k-edge-connected. Therefore, the result follows.

The vertex-connected case of the lemma is an exercise of Chapter one in [2]. Hence, we omit its proof here. □

Next, we give the maximal number of edges in every subgraph of a minimally k-edge-connected graph. Before proceeding, we need two more lemmas due to Mader [17].

Lemma 8 ([17]). *Let G be a graph of order $n \geqslant k$. If G does not contain any $(k+1)$-edge-connected subgraph, then*

$$e(G) \leqslant k(n-k) + \tbinom{k}{2}.$$

Furthermore, this bound is best possible.

Lemma 9 ([17]). *Let G be a minimally k-edge-connected graph of order $n \geqslant 3k$. Then*

$$e(G) \leqslant k(n-k),$$

with equality if and only if $G \cong K_{k,n-k}$.

Theorem 10. *Let G be a minimally k-edge-connected graph and let H be a subgraph of G. Then $e(H) \leqslant k(|H|-1)$. Moreover, if $|H| \geqslant \frac{1}{2}k(k+5)$, then $e(H) \leqslant k(|H|-k)$, where the equality holds if and only if $H \cong K_{k,|H|-k}$.*

Proof. Firstly, we will show that $e(H) \leqslant k(|H|-1)$. If $|H| < k$, then $e(H) \leqslant \frac{1}{2}|H|(|H|-1) \leqslant k(|H|-1)$, as desired. Now assume that $|H| \geqslant k$. It suffices to show $e(H) \leqslant k(|H| - \frac{k+1}{2})$. By Lemma 7, every k-edge-connected subgraph of G is minimally k-edge-connected, and thus has edge-connectivity k. Hence, G contains no $(k+1)$-edge-connected subgraphs. By Lemma 8, we have $e(H) \leqslant k(|H|-k) + \binom{k}{2} = k(|H| - \frac{k+1}{2})$, as required.

In the following, we prove that $e(H) \leqslant k(|H|-k)$ for $|H| \geqslant \frac{1}{2}k(k+5)$. The proof should be distinguished into two cases.

Case 1: H contains no k-edge-connected subgraphs. By Lemma 8, we know that $e(H) \leqslant (k-1)(|H| - \frac{k}{2})$. Note that $|H| \geqslant \frac{1}{2}k(k+5)$. It is easy to see that $(k-1)(|H|-\frac{k}{2}) < k(|H|-k)$, and the result follows.

Case 2: H contains k-edge-connected subgraphs. Let H_0 be a maximal k-edge-connected subgraph of H. Then H_0 is a vertex-induced subgraph with $|H_0| \geqslant k+1$. If $H = H_0$, then by Lemma 7, H is minimally k-edge-connected. Since $|H| \geqslant \frac{1}{2}k(k+5) \geqslant 3k$, by Lemma 9 we have $e(H) \leqslant k(|H|-k)$, with equality if and only if $H \cong K_{k,|H|-k}$.

Now we may assume that H_0 is a proper induced subgraph of H. Then $\kappa'(H) \leqslant k-1$, and thus we can find a partition $V(H) = V_0 \cup V_1$ such that $e(H) \leqslant e(V_0) + e(V_1) + (k-1)$. One can observe that H_0 is a subgraph of $H[V_0]$ or $H[V_1]$ (otherwise, write $U_i = V(H_0) \cap V_i$ for $i \in \{0,1\}$, then $e(U_0, U_1) \geqslant k$ as H_0 is k-edge-connected, consequently, $e(V_0, V_1) \geqslant k$, a contradiction). For $i \in \{0,1\}$, if $\kappa'(H[V_i]) \leqslant k-1$ and $|V_i| \geqslant 2$, then we can find a partition $V_i = V_i' \cup V_i''$ such that $e(V_i) \leqslant e(V_i') + e(V_i'') + (k-1)$. Similarly, every k-edge-connected subgraph of $H[V_i]$ can only be a subgraph of $H[V_i']$ or $H[V_i'']$.

By a series of above iterative operations (say s steps), we can obtain a partition $V(H) = \cup_{i=0}^{s} V_i$ satisfying that

$$e(H) \leqslant \sum_{i=0}^{s} e(V_i) + (k-1)s \tag{1}$$

and every $H[V_i]$ is either k-edge-connected or a single vertex. Recall that G contains no $(k+1)$-edge-connected subgraphs. If $H[V_i]$ is k-edge-connected, then $|V_i| \geqslant k+1$ and $e(H[V_i]) \leqslant k(|V_i| - \frac{k+1}{2})$ by Lemma 8. Let $S_1 = \{i \mid |V_i| = 1\}$ and $S_2 = \{0, \ldots, s\} \setminus S_1$. Then $s = |S_1| + |S_2| - 1$ and $|H| = \sum_{i \in S_2} |V_i| + |S_1|$. In view of (1), we have

$$\begin{aligned} e(H) &\leqslant \sum_{i \in S_2} k(|V_i| - \tfrac{k+1}{2}) + (k-1)(|S_1| + |S_2| - 1) \\ &= k|H| - \tfrac{1}{2}(k^2 - k + 2)|S_2| - |S_1| - (k-1). \end{aligned} \tag{2}$$

If $|S_2| \geqslant 2$, then $\frac{1}{2}(k^2 - k + 2)|S_2| + (k-1) > k^2$, and so $e(H) < k(|H| - k)$, as desired. Now assume that $|S_2| = 1$ (say $S_2 = \{0\}$ and $H[V_0] = H_0$). Then $S_1 \neq \varnothing$ as H_0 is a proper induced subgraph of H. By Lemma 7, H_0 is minimally k-edge-connected. If $|H_0| \geqslant 3k$, then by Lemma 9, we have $e(H_0) \leqslant k(|V_0| - k)$. Combining (1), we obtain $e(H) \leqslant k(|V_0| - k) + (k-1)|S_1| = k(|H| - k) - |S_1|$. The result follows. If $|H_0| < 3k$, then $|S_1| = |H| - |H_0| > \frac{1}{2}k(k-1)$, and by (2) we have $e(H) \leqslant k|H| - \frac{1}{2}(k^2 - k + 2) - |S_1| - (k-1) < k(|H| - k)$. This completes the proof. $\square$

Now we give a vertex-connected version of Theorem 10, which will be proved by a different approach.

Lemma 11 ([2]). *Let G be a minimally k-connected graph and let S be the set of vertices of degree k in G. Then $G - S$ is empty or a forest.*

Recall that $e(G) \leqslant k(n-k)$ for $n \geqslant 3k-2$ and every n-vertex minimally k-connected graph G. We also want to know the maximal number of edges in every subgraph of a minimally k-connected graph.

Theorem 12. *Let G be a minimally k-connected graph and let H be a subgraph of G. Then $e(H) \leqslant k(|H| - 1)$. Moreover, if $|H| \geqslant 5k - 4$, then $e(H) \leqslant k(|H| - k)$, where the equality holds if and only if $H \cong K_{k,|H|-k}$.*

Proof. Firstly, we show $e(H) \leqslant k(|H| - 1)$. We partition $V(H)$ into two parts: $V(H) = V_1 \cup V_2$, where V_1 is the set of vertices of degree k in G. If $|V_2| = 0$, then $e(H) \leqslant \frac{k|H|}{2} \leqslant k(|H| - 1)$, as desired. If $|V_2| \geqslant 1$, from Lemma 11 we know that $G[V_2]$ is a forest, and so $e(V_2) \leqslant |V_2| - 1$. Thus, we can get an upper bound of $e(H)$ as below:

$$e(H) = e(V_1) + e(V_1, V_2) + e(V_2) \leqslant k|V_1| + (|V_2| - 1), \tag{3}$$

where the equality holds if and only if $G[V_2]$ is a tree and $N_G(v) \subseteq V_2$ for each $v \in V_1$. It is clear that $k|V_1| + |V_2| - 1 \leqslant k(|V_1| + |V_2| - 1)$, and hence $e(H) \leqslant k(|H| - 1)$.

Next, we shall distinguish three cases to show $e(H) \leqslant k(|H|-k)$ for $|H| \geqslant 5k-4$. If $k=1$, then G is a tree. Clearly, the result holds. In the following, we may assume $k \geqslant 2$.

Case 1: $|V_2| \geqslant k+1$. From (3) we have

$$e(H) \leqslant k|V_1| + |V_2| - 1 < k(|V_1| + |V_2| - k) = k(|H| - k).$$

The result follows.

Case 2: $|V_2| = k$. Then $|V_1| \geqslant 4(k-1)$. If $e(V_2) = 0$, then by (3), we have $e(H) \leqslant k|V_1| = k(|V_1| + |V_2| - k) = k(|H| - k)$, with equality if and only if $H \cong K_{k,|H|-k}$.

Now, assume that $e(V_2) \geqslant 1$, and let $V_1' = \{v \in V_1 \mid N_G(v) = V_2\}$. Then $K_{|V_1'|,|V_2|} \subseteq G[V_1' \cup V_2]$. We will see that $|V_1'| \leqslant k-1$. Otherwise, if $|V_1'| \geqslant k$, then $G[V_1' \cup V_2]$ is k-connected. By Lemma 7, $G[V_1' \cup V_2]$ is minimally k-connected, which implies that $G[V_1' \cup V_2] \cong K_{|V_1'|,|V_2|}$ and so $e(V_2) = 0$, a contradiction. Hence, $|V_1'| \leqslant k-1$.

On the other hand, let $V_1'' = V_1 \setminus V_1'$, then

$$e\big(V_1''\big) + e\big(V_1'', V_2\big) \leqslant \big(|V_2| - 1\big)|V_1''| + \tfrac{1}{2}|V_1''| = \big(k - \tfrac{1}{2}\big)|V_1''|.$$

Since $|V_1'| \leqslant k-1$ and $|V_1'| + |V_1''| = |V_1|$, we further obtain

$$e\big(V_1\big) + e\big(V_1, V_2\big) \leqslant k|V_1'| + e\big(V_1''\big) + e\big(V_1'', V_2\big) \leqslant \big(k - \tfrac{1}{2}\big)|V_1| + \tfrac{1}{2}\big(k-1\big).$$

Recall that $|V_1| \geqslant 4(k-1)$ and $e(V_2) \leqslant k-1$. Thus we also have

$$e\big(H\big) \leqslant \big(k - \tfrac{1}{2}\big)|V_1| + \tfrac{3}{2}\big(k-1\big) < k|V_1| = k\big(|H| - k\big).$$

Case 3: $|V_2| \leqslant k-1$. Then $|V_1| \geqslant 4k-3$. Let $|V_1| = x$ and $|V_2| = y$. Then

$$\begin{aligned} e(H) &= e(V_1, V_2) + e(V_1) + e(V_2) \\ &\leqslant |V_1||V_2| + \tfrac{1}{2}|V_1|(k - |V_2|) + (|V_2| - 1) \\ &= \tfrac{1}{2}xy + \tfrac{1}{2}kx + (y-1), \end{aligned}$$

Notice that $k(|H|-k) = k(x+y-k)$. Let

$$f(x,y) = \tfrac{1}{2}xy + \tfrac{1}{2}kx + (y-1) - k(x+y-k).$$

It suffices to show $f(x,y) < 0$ for $x \geqslant 4k-3$ and $y \leqslant k-1$. Note that $\frac{\partial f(x,y)}{\partial x} = \frac{1}{2}(y-k) < 0$ and $\frac{\partial f(x,y)}{\partial y} = \frac{1}{2}x + 1 - k > 0$. Hence, $f(x,y)$ is decreasing with respect to x and increasing with respect to y. Therefore, $f(x,y)\mid_{max} = f(4k-3, k-1) = -\frac{1}{2}$, as desired. □

Observe that $\frac{1}{2}k(k+5) \geqslant 5k-4$ for every positive integer k. Combining Theorems 10 and 12, we immediately obtain Theorem 1.

3 Spectral extremal results

Let G be a minimally k-(edge)-connected graph of order n. By Perron-Frobenius theorem, there exists a positive unit eigenvector with respect to $\rho(G)$, which is called the *Perron vector* of G. Let $X = (x_1, x_2, \dots, x_n)^T$ be the Perron vector with coordinate $x_{u^*} = \max\{x_i \mid i \in V(G)\}$. In this section, we first show Theorem 4, that is, if $\rho^2(G) \geqslant k(n-k)$, then G contains a k-vertex subset L such that $x_v \geqslant (1 - \frac{1}{2k})x_{u^*}$ and $d(v) \geqslant (1 - \frac{2}{3k})n$ for each vertex $v \in L$. Before proceeding, we define three subsets of $V(G)$.

$$\begin{aligned} L_\alpha &= \{v \in V(G) \mid x_v > \alpha x_{u^*}\}, \quad \text{where } 0 < \alpha \leqslant \tfrac{1}{24k(k+1)}; \\ L_\beta &= \{v \in V(G) \mid x_v > \beta x_{u^*}\}, \quad \text{where } \tfrac{5}{3}\alpha \leqslant \beta \leqslant \tfrac{1}{6k^2}; \\ L_\gamma &= \{v \in V(G) \mid x_v \geqslant \gamma x_{u^*}\}, \quad \text{where } \tfrac{1}{2k} \leqslant \gamma \leqslant 1. \end{aligned}$$

Clearly, $L_\gamma \subseteq L_\beta \subseteq L_\alpha$. In the following, assume that $k \geqslant 3$ and $n \geqslant \frac{18k}{\alpha^2}$. We shall prove some lemmas on these three subsets.

Lemma 13. $|L_\alpha| < \sqrt{4kn}$.

Proof. For every $v \in L_\alpha$, we have $\rho x_v = \sum_{u \in N(v)} x_u$, and thus

$$\rho x_v = \sum_{u \in N(v) \cap L_\alpha} x_u + \sum_{u \in N(v) \setminus L_\alpha} x_u \leqslant \Big(d_{L_\alpha}(v) + \alpha \cdot d_{V(G) \setminus L_\alpha}(v)\Big) x_{u^*}. \tag{4}$$

Since $\rho x_v \geqslant \sqrt{k(n-k)}\alpha x_{u^*}$ for $v \in L_\alpha$, from (4) we have

$$\sqrt{k(n-k)}\alpha \leqslant d_{L_\alpha}(v) + \alpha \cdot d_{V(G) \setminus L_\alpha}(v). \tag{5}$$

Summing (5) over all $v \in L_\alpha$, we have

$$|L_\alpha|\sqrt{k(n-k)}\alpha \leqslant 2e(L_\alpha) + \alpha \cdot e(L_\alpha, V(G) \setminus L_\alpha). \tag{6}$$

By Theorem 1, we have $e(L_\alpha) \leqslant k|L_\alpha|$ and $e(L_\alpha, V(G) \setminus L_\alpha) \leqslant e(G) \leqslant k(n-k)$. Combining (6), we get that

$$|L_\alpha|\sqrt{k(n-k)} \leqslant \frac{2k}{\alpha}|L_\alpha| + k(n-k). \tag{7}$$

Since $n \geqslant \frac{18k}{\alpha^2}$, we have $n - k > \frac{16k}{\alpha^2}$, and hence $\frac{2k}{\alpha} < \frac{1}{2}\sqrt{k(n-k)}$. Combining (7), we obtain that $|L_\alpha| < 2\sqrt{k(n-k)}$, and thus $|L_\alpha| < \sqrt{4kn}$, as desired. □

For a vertex $v \in V(G)$, let $N[v] = N(v) \cup \{v\}$ and $N^2(v)$ denote the set of vertices at distance two from v.

Lemma 14. $|L_\beta| < \frac{12k}{\alpha}$.

Proof. We proceed the proof by contradiction. Suppose that $|L_\beta| \geqslant \frac{12k}{\alpha}$. Recall that $L_\beta \subseteq L_\alpha$ and $\alpha \leqslant \frac{1}{24k(k+1)}$. Then $|L_\alpha| \geqslant \frac{12k}{\alpha} \geqslant \max\{5k-4, \frac{1}{2}k(k+5)\}$. We first prove that $d(v) > \frac{\alpha}{12}n + k$ for each vertex $v \in L_\beta$.

By Theorem 1, we get that $e(G) \leqslant kn$, $e(N[v]) \leqslant k(|N[v]|-1) = kd(v)$ and $e(N(v) \cup L_\alpha) \leqslant k(d(v) + |L_\alpha| - k)$. Since $v \in L_\beta$, we can easily see that $v \in L_\alpha$. Let $S = N(v) \cup (L_\alpha \setminus \{v\})$. Then $e(S) = e(N(v) \cup L_\alpha) - d(v) \leqslant (k-1)d(v) + k|L_\alpha| - k^2$, where $|L_\alpha| < \sqrt{4kn} < \frac{\alpha}{2}n$ by Lemma 13 and the assumption that $n \geqslant \frac{18k}{\alpha^2}$.

It is easy to see that

$$d(v)x_v + \sum_{u \in N(v)} d_{N(v)}(u)x_u \leqslant \Big(d(v) + 2e(N(v))\Big)x_{u^*} = \Big(e(N[v]) + e(N(v))\Big)x_{u^*}.$$

Note that $S = N(v) \cup (L_\alpha \setminus \{v\})$. Then $e\big(N^2(v) \cap L_\alpha, N(v)\big) \leqslant e(S) - e\big(N(v)\big)$ and

$$\begin{aligned}\sum_{u \in N^2(v)} d_{N(v)}(u)x_u &= \sum_{u \in N^2(v) \cap L_\alpha} d_{N(v)}(u)x_u + \sum_{u \in N^2(v) \setminus L_\alpha} d_{N(v)}(u)x_u \\ &\leqslant \Big(e(S) - e(N(v)) + \alpha \cdot e(G)\Big)x_{u^*}.\end{aligned}$$

Combining the above two inequalities, we obtain

$$\begin{aligned}\rho^2 x_v &= d(v)x_v + \sum_{u \in N(v)} d_{N(v)}(u)x_u + \sum_{u \in N^2(v)} d_{N(v)}(u)x_u \\ &\leqslant \Big(e(N[v]) + e(S) + \alpha \cdot e(G)\Big)x_{u^*}. \\ &\leqslant \Big((2k-1)d(v) + \tfrac{3\alpha}{2}kn - k^2\Big)x_{u^*}.\end{aligned}$$

Notice that $\frac{5}{3}\alpha \leqslant \beta < 1$ and $\rho^2 x_v \geqslant k(n-k)\beta x_{u^*} > (\beta kn - k^2)x_{u^*}$ for each vertex $v \in L_\beta$. In view of the above inequality, we have $(\beta - \frac{3}{2}\alpha)kn < (2k-1)d(v)$, which yields that $d(v) > \frac{k}{2k-1}(\beta - \frac{3}{2}\alpha)n > \frac{\alpha}{12}n + k$ for each vertex $v \in L_\beta$.

By Theorem 1, we also have $e(L_\beta) \leqslant k|L_\beta|$. Observe that $\sum_{u \in V(G) \setminus L_\beta} d(u) \geqslant e(L_\beta, V(G) \setminus L_\beta) = \sum_{v \in L_\beta} d(v) - 2e(L_\beta)$. Therefore,

$$2e(G) = \sum_{v \in L_\beta} d(v) + \sum_{u \in V(G) \setminus L_\beta} d(u) \geqslant 2\sum_{v \in L_\beta} d(v) - 2e(L_\beta) > |L_\beta|\frac{\alpha}{6}n.$$

Combining $e(G) \leqslant kn$, we obtain $|L_\beta| < \frac{12k}{\alpha}$. This completes the proof. □

Lemma 15. *$d(v) > \big(\gamma - \frac{1}{6k}\big)n$ for each $v \in L_\gamma$.*

Proof. Suppose to the contrary that there exists a vertex $v_0 \in L_\gamma$ with $d(v_0) \leqslant (\gamma - \frac{1}{6k})n$. We may assume that $x_{v_0} = \gamma_0 x_{u^*}$. By the definition of L_γ, we know that $\frac{1}{2k} \leqslant \gamma \leqslant \gamma_0 \leqslant 1$, and thus $d(v_0) \leqslant (\gamma_0 - \frac{1}{6k})n$.

Set $R = N(v_0) \cup N^2(v_0)$. Then $x_v \leqslant \beta x_{u^*}$ for each $v \in R \setminus L_\beta$. Therefore,

$$\begin{aligned}\rho^2 x_{v_0} &= d(v_0)x_{v_0} + \sum_{v\in R} d_{N(v_0)}(v)x_v \\ &= d(v_0)x_{v_0} + \sum_{v\in R\setminus L_\beta} d_{N(v_0)}(v)x_v + \sum_{v\in R\cap L_\beta} d_{N(v_0)}(v)x_v \\ &\leqslant \Big(\gamma_0 d(v_0) + \beta \sum_{v\in R\setminus L_\beta} d_{N(v_0)}(v) + \sum_{v\in R\cap L_\beta} d_{N(v_0)}(v)\Big)x_{u^*}.\end{aligned} \tag{8}$$

Since $N(v_0) \subseteq R$, we can see that

$$\sum_{v\in R\setminus L_\beta} d_{N(v_0)}(v) \leqslant \sum_{v\in R} d_R(v) = 2e(R) \leqslant 2e(G) \leqslant 2kn. \tag{9}$$

Observe that $R \cap L_\beta \subseteq L_\beta \setminus \{v_0\}$. We also have

$$\begin{aligned}\sum_{v\in R\cap L_\beta} d_{N(v_0)}(v) &\leqslant \sum_{v\in L_\beta\setminus\{v_0\}} d_{N(v_0)\cap L_\beta}(v) + \sum_{v\in L_\beta\setminus\{v_0\}} d_{N(v_0)\setminus L_\beta}(v) \\ &\leqslant 2e\big(L_\beta\big) + e\big(L_\beta, N(v_0)\setminus L_\beta\big) - |N(v_0)\setminus L_\beta|.\end{aligned} \tag{10}$$

Furthermore, $e\big(L_\beta, N(v_0)\setminus L_\beta\big) \leqslant e\big(L_\beta \cup N(v_0)\big) - e\big(L_\beta\big)$. Notice that $e(L_\beta) \leqslant k|L_\beta|$ and $e\big(L_\beta \cup N(v_0)\big) \leqslant k\big(|L_\beta| + d(v_0)\big)$. Combining (10), we obtain

$$\begin{aligned}\sum_{v\in R\cap L_\beta} d_{N(v_0)}(v) &\leqslant e\big(L_\beta \cup N(v_0)\big) - |N(v_0)\setminus L_\beta| + e(L_\beta) \\ &\leqslant (k-1)d(v_0) + (k+1)|L_\beta| + e(L_\beta) \\ &\leqslant (k-1)d(v_0) + (2k+1)|L_\beta|.\end{aligned} \tag{11}$$

Substituting (9) and (11) into (8), we get that

$$\begin{aligned}\rho^2 x_{v_0} &\leqslant \Big(\gamma_0 d(v_0) + 2k\beta n + (k-1)d(v_0) + (2k+1)|L_\beta|\Big)x_{u^*} \\ &= \Big((\gamma_0 + k - 1)d(v_0) + 2k\beta n + (2k+1)|L_\beta|\Big)x_{u^*}.\end{aligned} \tag{12}$$

Since $n \geqslant \frac{18k}{\alpha^2}$ and $\alpha < \frac{1}{24k^2}$, we have $\frac{12k}{\alpha} \leqslant \frac{2}{3}\alpha n < \frac{n}{(6k)^2}$. Moreover, by Lemma 14, we have $|L_\beta| < \frac{12k}{\alpha}$. Thus, we can check that $(2k+1)|L_\beta| < \frac{n}{6k} - k^2 \leqslant \frac{n}{6k} - k^2\gamma_0$. Recall that $\rho^2 x_{v_0} \geqslant k(n-k)\gamma_0 x_{u^*}$ and $d(v_0) \leqslant (\gamma_0 - \frac{1}{6k})n$. Combining (12), we obtain that

$$k(n-k)\gamma_0 < \big(\gamma_0 + k - 1\big)\big(\gamma_0 - \tfrac{1}{6k}\big)n + 2k\beta n + \tfrac{n}{6k} - k^2\gamma_0,$$

which gives $k\gamma_0 < (\gamma_0 + k - 1)(\gamma_0 - \frac{1}{6k}) + 2k\beta + \frac{1}{6k}$. Recall that $\beta \leqslant \frac{1}{6k^2}$. It follows that

$$\big(\gamma_0 - 1\big)\big(\gamma_0 - \tfrac{1}{6k}\big) > \tfrac{k-1}{6k} - 2k\beta \geqslant \tfrac{k-3}{6k} \geqslant 0. \tag{13}$$

Now let $f(\gamma) = (\gamma - 1)\big(\gamma - \frac{1}{6k}\big)$, where $\frac{1}{2k} \leqslant \gamma \leqslant 1$. Obviously, $f(\gamma)|_{\max} = f(1) = 0$, which contradicts (13). The proof is completed. $\square$

Recall that $L_\gamma = \{u \in V(G) \mid x_u \geqslant \gamma x_{u^*}\}$, where $\frac{1}{2k} \leqslant \gamma \leqslant 1$. Let $\gamma_0 := \frac{1}{2k}$. Clearly, $L_{1-\gamma_0} \subseteq L_{\gamma_0}$. We will see that every vertex $u \in L_{\gamma_0}$ has a larger value x_u.

Lemma 16. $L_{\gamma_0} = L_{1-\gamma_0}$.

Proof. Suppose to the contrary that there exists a vertex $u_0 \in L_{\gamma_0} \setminus L_{1-\gamma_0}$. Assume that $x_{u_0} = \gamma x_{u^*}$. Then $\gamma_0 \leqslant \gamma < 1 - \gamma_0$. Set $R = N[u^*] \cup N^2(u^*)$. Then we have

$$\rho^2 x_{u^*} = \sum_{u \in R} d_{N(u^*)}(u) x_u = \sum_{u \in R \setminus L_\beta} d_{N(u^*)}(u) x_u + \sum_{u \in R \cap L_\beta} d_{N(u^*)}(u) x_u. \tag{14}$$

Recall that $e(G) \leqslant kn$ and $x_u \leqslant \beta x_{u^*}$ for each $u \in R \setminus L_\beta$. Then

$$\sum_{u \in R \setminus L_\beta} d_{N(u^*)}(u) x_u \leqslant \sum_{u \in R} d_R(u) \beta x_{u^*} \leqslant 2e(G)\beta x_{u^*} \leqslant 2\beta kn x_{u^*}. \tag{15}$$

On the other hand, since $u_0 \in L_{\gamma_0}$ and $L_{\gamma_0} \subseteq L_\beta$, we have $u_0 \in L_\beta$, and thus

$$\sum_{u \in R \cap L_\beta} d_{N(u^*)}(u) x_u \leqslant \sum_{u \in L_\beta} d_{N(u^*)}(u) x_{u^*} + d_{N(u^*)}(u_0)(x_{u_0} - x_{u^*}), \tag{16}$$

where $x_{u_0} - x_{u^*} = (\gamma - 1)x_{u^*}$ and

$$\begin{aligned} \sum_{u \in L_\beta} d_{N(u^*)}(u) &= \sum_{u \in L_\beta} d_{N(u^*) \setminus L_\beta}(u) + \sum_{u \in L_\beta} d_{N(u^*) \cap L_\beta}(u) \\ &\leqslant e(L_\beta, N(u^*) \setminus L_\beta) + 2e(L_\beta) \\ &\leqslant e(G) + e(L_\beta). \end{aligned}$$

Recall that $e(G) \leqslant k(n-k)$ and $e(L_\beta) \leqslant k|L_\beta| < \frac{12}{\alpha}k^2$. Consequently, $\sum_{u \in L_\beta} d_{N(u^*)}(u) \leqslant k(n-k) + \frac{12}{\alpha}k^2$. Combining (14)-(16), we obtain

$$\rho^2 x_{u^*} \leqslant \Big(2\beta kn + k(n-k) + \tfrac{12}{\alpha}k^2 + (\gamma - 1) d_{N(u^*)}(u_0)\Big) x_{u^*}. \tag{17}$$

By Lemma 15, we have $d(u^*) \geqslant (1 - \frac{1}{6k})n$ and $d(u_0) \geqslant (\gamma - \frac{1}{6k})n$. Thus, $|V(G) \setminus N(u^*)| \leqslant \frac{n}{6k}$ and $d_{N(u^*)}(u_0) \geqslant (\gamma - \frac{1}{3k})n$. Notice that $\rho^2 \geqslant k(n-k)$. It follows from (17) that

$$\big(\gamma - 1\big)\big(\gamma - \tfrac{1}{3k}\big)n \geqslant -\big(2\beta kn + \tfrac{12}{\alpha}k^2\big).$$

Recall that $\alpha \leqslant \frac{1}{24k(k+1)}$, $\beta \geqslant \frac{5}{3}\alpha$ and $n \geqslant \frac{18k}{\alpha^2}$. Then $\frac{12}{\alpha}k^2 \leqslant \frac{2}{3}\alpha kn$. Now choose $\beta = \frac{5}{3}\alpha$. Then we have $2\beta kn + \frac{12}{\alpha}k^2 \leqslant 4\alpha kn$, and hence $(\gamma - 1)(\gamma - \frac{1}{3k}) \geqslant -4\alpha k \geqslant -\frac{1}{6(k+1)}$. Let $f(\gamma) = (\gamma - 1)(\gamma - \frac{1}{3k})$, where $\gamma_0 \leqslant \gamma \leqslant 1 - \gamma_0$ and $\gamma_0 = \frac{1}{2k}$. Obviously, $f(\gamma)|_{\max} = f(\gamma_0) = -\frac{2k-1}{12k^2} < -\frac{1}{6(k+1)}$ for $k \geqslant 3$, a contradiction. $\square$

With the above lemmas in hand, we now provide the proof of Theorem 4.

Proof. Choose $L = L_{\gamma_0}$ in Theorem 4. Given an arbitrary vertex $v \in L$. By Lemma 16, we have $v \in L_{1-\gamma_0}$, and thus $x_v \geqslant (1-\gamma_0)x_{u^*} = (1-\frac{1}{2k})x_{u^*}$. Furthermore, by Lemma 15 we have $d(v) \geqslant (1-\gamma_0-\frac{1}{6k})n = (1-\frac{2}{3k})n$.

In the following, it remains to show that $|L| = k$. Firstly, suppose that $|L| \geqslant k+1$. Taking $v_1, v_2, \ldots, v_{k+1}$ from L, we have

$$\Big|\bigcap_{i=1}^{k+1} N(v_i)\Big| \geqslant \sum_{i=1}^{k+1} \big|N(v_i)\big| - k\Big|\bigcup_{i=1}^{k+1} N(v_i)\Big| \geqslant \big(k+1\big)\big(1-\tfrac{2}{3k}\big)n - kn = \tfrac{k-2}{3k}n \geqslant k+1.$$

Thus, G contains a copy of $K_{k+1,k+1}$, which is clearly a $(k+1)$-(edge)-connected subgraph. However, by Lemma 7, every k-(edge)-connected subgraph of G is minimally k-(edge)-connected, which implies that G contains no any $(k+1)$-(edge)-connected subgraph. We get a contradiction. Therefore, $|L| \leqslant k$.

Finally, suppose that $|L| \leqslant k-1$. Since $L = L_{\gamma_0}$, we have $x_v < \gamma_0 x_{u^*} = \frac{1}{2k}x_{u^*}$ for every $v \in V(G) \setminus L$. Setting $R = N[u^*] \cup N^2(u^*)$, we have

$$\rho^2 x_{u^*} = \sum_{u\in R} d_{N(u^*)}(u)x_u \leqslant \Big(\sum_{u\in R\cap L} d_{N(u^*)}(u) + \tfrac{1}{2k}\sum_{u\in R\setminus L} d_{N(u^*)}(u)\Big)x_{u^*}, \tag{18}$$

Let E_0 be the set of edges incident to vertices of L. Then, every edge in E_0 can not be counted twice in $\sum_{u\in R\setminus L} d_{N(u^*)}(u)$. Moreover, it is easy to see that $u^* \in L$ and every edge incident to u^* can not be counted in $\sum_{u\in R\setminus L} d_{N(u^*)}(u)$. Consequently, $\sum_{u\in R\setminus L} d_{N(u^*)}(u) \leqslant 2e(G) - d(u^*) - |E_0|$. Note that $e(G) \leqslant kn$ and

$$d(u^*) + |E_0| = d(u^*) + \sum_{v\in L} d(v) - e(L) \geqslant \big(|L|+1\big)\big(1-\tfrac{2}{3k}\big)n - \tfrac{1}{2}k^2 \geqslant |L|n.$$

It follows that $\sum_{u\in R\setminus L} d_{N(u^*)}(u) \leqslant (2k-|L|)n$. Observe that $\sum_{u\in R\cap L} d_{N(u^*)}(u) \leqslant |L|n$. Combining (18) and $|L| \leqslant k-1$, we obtain

$$\rho^2 \leqslant |L|n + \tfrac{1}{2k}\big(2k-|L|\big)n \leqslant (k-1)n + \tfrac{(k+1)}{2k}n = kn - \tfrac{k-1}{2k}n,$$

which contradicts $\rho^2 \geqslant k(n-k)$. Therefore, $|L| = k$. This completes the proof. □

At the end of this section, we give the proof of Theorem 5.

Proof. Let G^* be a graph that has the maximal spectral radius among all minimally k-(edge)-connected graphs of order n, where $n \geqslant \frac{18k}{\alpha^2}$ and $\alpha = \frac{1}{24k(k+1)}$. Since $K_{k,n-k}$ is also minimally k-(edge)-connected, we have $\rho^2(G^*) \geqslant \rho^2(K_{k,n-k}) = k(n-k)$. Furthermore, by Theorem 4, G^* contains a k-vertex subset L such that $x_v \geqslant (1-\frac{1}{2k})x_{u^*}$ and $d(v) \geqslant (1-\frac{2}{3k})n$ for each vertex $v \in L$, where $L = L_{\frac{1}{2k}}$.

Denote by V the common neighbourhood of vertices in L, and let $U = V(G^*)\setminus(L\cup V)$. Since $|L| = k$ and every vertex in L has at most $\frac{2}{3k}n$ non-neighbors, we can see that

$$|L \cup V| \geqslant n - k \cdot \tfrac{2}{3k}n = \tfrac{n}{3} > \tfrac{1}{2}k(k+5).$$

The key point is to show that $U = \varnothing$. Suppose to the contrary that $|U| = t \neq 0$. By Theorem 1, we have $e(G^*) \leqslant k(n-k) = k(|V| + |U|)$.

Now, define $G_0 = G^*$ and $U_0 = U$. Moreover, let E_0 be the subset of $E(G_0)$ in which every edge is incident to at least one vertex from U_0. Then $|E_0| \leqslant e(G^*) - e(L, V) \leqslant k|U_0|$, as $e(L, V) = |L||V| = k|V|$. It follows that $\sum_{u \in U_0} d_{G_0}(u) \leqslant 2|E_0| \leqslant 2k|U_0|$, which implies that there exists a vertex $u_0 \in U_0$ such that $d_{G_0}(u_0) \leqslant 2k$.

Then, let $G_1 = G_0 - \{u_0\}$, $U_1 = U_0 \setminus \{u_0\}$ and E_1 be the subset of $E(G_1)$ in which every edge is incident to some vertices from U_1. Similarly as above, we have $e(G_1) \leqslant k(|V| + |U_1|)$ and $|E_1| \leqslant e(G_1) - e(L, V) \leqslant k|U_1|$. Thus, we can find a vertex $u_1 \in U_1$ such that $d_{G_1}(u_1) \leqslant 2k$. Consequently, we can obtain a vertex ordering $u_0, u_1, \ldots, u_{t-1}$ such that $G_i = G_{i-1} - \{u_{i-1}\}$, $U_i = U_{i-1} \setminus \{u_{i-1}\}$ and $d_{G_i}(u_i) \leqslant 2k$ for each $i \in \{1, \ldots, t-1\}$. For simplicity, we denote $d_L(u_i) = d_i$ and $d_{G_i - L}(u_i) = d_i'$. Then $d_i \leqslant k-1$ by the definition of U, and $d_i + d_i' = d_{G_i}(u_i) \leqslant 2k$ for $i \in \{0, \ldots, t-1\}$.

We shall construct a new graph G from G^* in the following way. For each vertex u_i $(0 \leqslant i \leqslant t-1)$, we delete all d_i' edges from u_i to $V(G_i - L)$, and then add all possible $k - d_i$ edges from u_i to L. Denote $\overline{N}_L(u_i) = L \setminus N_L(u_i)$. Then, we can see that

$$\rho(G) - \rho(G^*) \geqslant \sum_{uv \in E(G)} x_u x_v - \sum_{uv \in E(G^*)} x_u x_v = \sum_{i=0}^{t-1} x_{u_i} \Big(\sum_{v \in \overline{N}_L(u_i)} x_v - \sum_{v \in N_{G_i - L}(u_i)} x_v \Big). \qquad (19)$$

Recall that $x_v \geqslant \big(1 - \frac{1}{2k}\big) x_{u^*}$ for each $v \in L$. Moreover, since we choose $L = L_{\frac{1}{2k}}$, it is obvious that $x_v < \frac{1}{2k} x_{u^*}$ for each $v \notin L$. In view of (19), we obtain

$$\rho(G) - \rho(G^*) \geqslant \sum_{i=0}^{t-1} x_{u_i} x_{u^*} \Big((k - d_i)(1 - \tfrac{1}{2k}) - d_i' \cdot \tfrac{1}{2k} \Big).$$

Recall that $d_i + d_i' \leqslant 2k$ and $d_i \leqslant k-1$ for each $i \in \{0, \ldots, t-1\}$. Thus,

$$(k - d_i)(1 - \tfrac{1}{2k}) - d_i' \cdot \tfrac{1}{2k} \geqslant (k - d_i)(1 - \tfrac{1}{2k}) - (2k - d_i)\tfrac{1}{2k} \geqslant 1 - \tfrac{k+2}{2k} > 0.$$

It follows that $\rho(G) > \rho(G^*)$.

Observe that $N_G(u_i) = L$ for each $u_i \in U$. We will further see that $G \cong K_{k,n-k}$. Indeed, otherwise, $G \ncong K_{k,n-k}$, then either $e_G(L) \neq 0$ or $e_G(V) \neq 0$. However, $G^*[L \cup V]$ contains a spanning subgraph $K_{|L|,|V|}$, where $|L| = k$ and $|L \cup V| \geqslant \frac{n}{3}$. Hence, $G^*[L \cup V]$ is clearly k-(edge)-connected. By Lemma 7, $G^*[L \cup V]$ is minimally k-(edge)-connected, which implies that $G^*[L \cup V] \cong K_{|L|,|V|}$. Since $G[L \cup V] = G^*[L \cup V]$, we have $G[L \cup V] \cong K_{|L|,|V|}$, and thus $e_G(L) = e_G(V) = 0$, a contradiction. Hence, $G \cong K_{k,n-k}$. But now, the inequality $\rho(G^*) < \rho(G)$ contradicts the assumption that G^* has maximal spectral radius. Therefore, $U = \varnothing$ and $G^* \cong K_{k,n-k}$. This completes the proof. □

Acknowledgments

The authors would like to express their sincere gratitude to the anonymous referees for their careful reading and invaluable suggestions. The authors were supported by National Natural Science Foundation of China (Nos. 12271362 and 12171154).

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